Exponential mixing for automorphisms on compact Kähler manifolds

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July 22, 2009

Abstract

Let f be a holomorphic automorphism of positive entropy on a compact Kähler surface. We show that the equilibrium measure of f is exponentially mixing. The proof uses some recent development on the pluripotential theory. The result also holds for automorphisms on compact Kähler manifolds of higher dimension under a natural condition on their dynamical degrees.

AMS classification : 37F, 32H.
Key-words : dynamical degree, equilibrium measure, exponential mixing.

1 Introduction

Let (X, ω) be a compact Kähler manifold of dimension k and f a holomorphic automorphism of X. The *dynamical degree of order* q of f is the spectral radius of the pull-back operator f^* acting on the Hodge cohomology group $H^{q,q}(X, \mathbb{C})$. It is denoted by $d_q(f)$ or simply by d_q if there is no confusion. We have $d_0 = d_k = 1$ and if $f^n := f \circ \cdots \circ f$ (n times) is the iterate of order n of f, then $d_q(f^n) = d_q^n$.

A theorem by Khovanskii [13], Teissier [18] and Gromov [8] implies that the sequence $q \mapsto \log d_q$ is concave. So, there are integers $0 \leq p \leq p' \leq k$ such that

$$1 = d_0 < \cdots < d_p = \cdots = d_{p'} > \cdots > d_k = 1.$$

An instructive example with $p \neq p'$ is a map f on a product $X = Y \times Z$ of compact Kähler manifolds such that $f(y, z) = (g(y), z)$ for $(y, z) \in Y \times Z$. More interesting examples of maps preserving a fibration were considered in [4].

Most dynamical studies on automorphisms of compact Kähler manifolds are concentrated on the case where the consecutive dynamical degrees are distinct, i.e. $p = p'$. Somehow, this condition insures that the considered dynamical systems have no trivial direction. From now on, we also assume that f satisfies this natural condition. In [6, 7], we constructed for f canonical invariant currents

(Green currents) and ergodic invariant probability measures using the theory of intersection of currents, see also [10].

When the operator f^*, acting on $H^{p,p}(X,\mathbb{C})$, admits only one eigenvalue of maximal modulus, there is only one invariant probability measure obtained as the intersection of a Green (p,p)-current of f and a Green $(k-p,k-p)$-current of f^{-1}. We call it the *equilibrium measure* of f. The above eigenvalue is necessarily equal to d_p and the obtained measure is shown to be mixing, hyperbolic and of maximal entropy. The reader finds in [7] and in Section 4 below some details. Here is our main theorem.

Theorem 1.1. *Let f be a holomorphic automorphism on a compact Kähler manifold (X,ω) and d_q its dynamical degrees. Assume that there is a degree d_p strictly larger than the other ones and that f^*, acting on $H^{p,p}(X,\mathbb{C})$, admits only one eigenvalue of maximal modulus d_p. Then the equilibrium probability measure μ of f is exponentially mixing. More precisely, if δ is a constant such that $\max(d_{p-1},d_{p+1})<\delta<d_p$ and all the eigenvalues of f^*, acting on $H^{p,p}(X,\mathbb{C})$, except d_p, are strictly smaller than δ, then*

$$|\langle\mu,(\varphi\circ f^n)\psi\rangle-\langle\mu,\varphi\rangle\langle\mu,\psi\rangle|\leq A\|\varphi\|_{\mathcal{C}^\beta}\|\psi\|_{\mathcal{C}^{\beta'}}(d_p/\delta)^{-n\beta\beta'/8},$$

for all $\mathcal{C}^\beta$ function φ and all $\mathcal{C}^{\beta'}$ function ψ on X with $0\leq\beta,\beta'\leq 2$. Here, $A=A(\beta,\beta',\delta)$ is a constant independent of φ, ψ and of the integer $n\geq 0$.

Mixing is equivalent to the property that the left hand side of the above inequality converges to 0 when n goes to infinity. In the proof of Theorem 1.1, we use in particular dynamical properties of the map $F:=(f^{-1},f)$ acting on $X\times X$. A Green (k,k)-current of F can be obtained as the limit of $d_p^{-2n}(F^n)^*[\Delta]$, where $[\Delta]$ is the current of integration on the diagonal Δ of $X\times X$. The speed of convergence is the key point in the proof of our result, see Proposition 3.1 below. The idea was already introduced in [3, 5, 7]. However, the use of the pseudoconvexity of $\mathbb{C}^k$ is no longer valid in the compact setting. We will replace it with the use of the Hölder continuity of Green super-potentials.

Theorem 1.1 still holds under weaker hypothesis: all the eigenvalues of maximal modulus of f^*, acting on $H^{p,p}(X,\mathbb{C})$, are equal to d_p and the spectral radius d_p of this operator is of multiplicity 1, i.e. $\|(f^n)^*\|\sim d_p^n$. The last property can be seen in the Jordan form of the square matrix associated to f^*: the Jordan blocks whose diagonal entries have modulus d_p, are of size 1×1. Very likely, the condition $\|(f^n)^*\|\sim d_p^n$ is necessary because it insures that the cohomology classes associated to $d_p^{-2n}(F^n)^*[\Delta]$ converge exponentially fast. Otherwise, we cannot have a good speed of convergence for the currents $d_p^{-2n}(F^n)^*[\Delta]$. In the considered case, the construction in [7] gives a finite family of invariant probability measures of maximal entropy. They are all exponentially mixing.

Consider now an automorphism f of positive entropy on a compact Kähler surface X. Results by Gromov [9] and Yomdin [20] say that the (topological) entropy of f is equal to $\log d_1$. So, $d_1 > 1$ and the consecutive dynamical degrees of f are distinct. These automorphisms were studied by Cantat in [2]. He showed in particular that all the eigenvalues of f^*, acting on $H^{1,1}(X, \mathbb{C})$, have modulus 1 except two eigenvalues d_1 and $1/d_1$. So, we can apply Theorem 1.1 and deduce the following result.

Corollary 1.2. *Let f be a holomorphic automorphism of positive entropy on a compact Kähler surface X. Then the equilibrium measure of f is exponentially mixing.*

Note that exponential mixing for polynomial automorphisms was proved by Nguyen and the authors in [3, 5]. We refer to Bedford-Kim [1], Keum-Kondo [12], McMullen [14] and Oguiso [15] for interesting examples of automorphisms on compact Kähler manifolds.

Acknowledgement. The first author wishes to express his gratitude to the Max-Planck Institut für Mathematik in Bonn for its hospitality during the preparation of this paper.

2 Super-potentials of currents

Super-potentials were introduced by the authors in order to develop a calculus on positive closed currents. We recall some basic properties and refer to [7] for details.

Let $\mathcal{D}_p$ denote the real space generated by positive closed (p,p)-currents on X. If S is a current in $\mathcal{D}_p$, define the *norm* $\|S\|_*$ of S by

$$\|S\|_* := \min \|S^+\| + \|S^-\|$$

where the minimum is taken over the positive closed currents $S^\pm$ with $S = S^+ - S^-$. Here, $\|S^\pm\|$ denote the *mass* of $S^\pm$ which are defined by

$$\|S^\pm\| := \langle S, \omega^{k-p} \rangle.$$

Observe $\|S^\pm\|$ depend only on the cohomology classes of $S^\pm$ in $H^{p,p}(X, \mathbb{R})$. We say that a subset of $\mathcal{D}_p$ is *$*$-bounded* if it is bounded for the $\|\ \|_*$-norm. Let $\mathcal{D}_p^0$ denote the subspace of currents S in $\mathcal{D}_p$ whose classes $\{S\}$ in $H^{p,p}(X, \mathbb{R})$ are zero.

We consider on $\mathcal{D}_p$ and $\mathcal{D}_p^0$ the following *topology*: a sequence (S_n) in $\mathcal{D}_p$ or $\mathcal{D}_p^0$ converges to a current S if S_n converge to S in the sense of currents and if $\|S_n\|_*$ are bounded by a constant independent of n. Smooth forms are dense in $\mathcal{D}_p$ and $\mathcal{D}_p^0$ for this topology.

For any $0 < l < \infty$, we can associate to $\mathcal{D}_p$ a *norm* $\|\ \|_{\mathcal{C}^{-l}}$ and a *distance* dist_l defined by

$$\|S\|_{\mathcal{C}^{-l}} := \sup_{\|\Phi\|_{\mathcal{C}^l} \leq 1} |\langle S, \Phi \rangle| \quad \text{and} \quad \text{dist}_l(S, S') := \|S - S'\|_{\mathcal{C}^{-l}},$$

where Φ is a smooth test form of bidegree $(k-p, k-p)$ on X. The weak topology on each $*$-bounded subset of $\mathcal{D}_p$ coincides with the topology induced by $\| \|_{\mathcal{C}^{-l}}$. If $0 < l < l' < \infty$ are two constants, then on each $*$-bounded subset of $\mathcal{D}_p$ we have

$$\mathrm{dist}_{l'} \leq \mathrm{dist}_l \leq c_{l,l'}(\mathrm{dist}_{l'})^{l/l'}$$

for some positive constant $c_{l,l'}$.

The super-potential of a current S in $\mathcal{D}_p$ is a canonical linear function defined, under some normalization, on the smooth forms in $\mathcal{D}^0_{k-p+1}$. It plays the same role as the potentials of positive closed $(1,1)$-currents which are quasi-p.s.h. functions.

Let $\alpha = (\alpha_1, \ldots, \alpha_h)$ with $h := \dim H^{p,p}(X, \mathbb{R})$ be a fixed family of real smooth closed (p,p)-forms such that the family of classes $\{\alpha\} = (\{\alpha_1\}, \ldots, \{\alpha_h\})$ is a basis of $H^{p,p}(X, \mathbb{R})$. Let R be a current in $\mathcal{D}^0_{k-p+1}$. Since the cohomology class of R is zero, there is a real $(k-p, k-p)$-current U_R such that $dd^c U_R = R$. We call U_R a *potential of* R. Adding to U_R a suitable closed form allows to assume that $\langle U_R, \alpha_i \rangle = 0$ for $i = 1, \ldots, h$ and we say that U_R is *α-normalized*. When R is smooth, we can choose U_R smooth and the *α-normalized super-potential* $\mathcal{U}_S$ of S is defined by

$$\mathcal{U}_S(R) := \langle S, U_R \rangle.$$

The definition does not depend on the choice of U_R.

When the function $\mathcal{U}_S$ extends continuously to $\mathcal{D}^0_{k-p+1}$ for the considered topology, we say that S has a *continuous* super-potential. If S is in $\mathcal{D}^0_p$ then $\mathcal{U}_S$ does not depend on the choice of α; if moreover S is smooth, it has a continuous super-potential and we have the formula

$$\mathcal{U}_S(R) = \mathcal{U}_R(S),$$

where $\mathcal{U}_R$ is the super-potential of R which is also independent of the normalization. We can extend the above equality to the case where S has a continuous super-potential.

We say that $\mathcal{U}_S$ is (l, λ, M)-*Hölder continuous* if it is continuous and if

$$|\mathcal{U}_S(R)| \leq M \|R\|^{\lambda}_{\mathcal{C}^{-l}}$$

for $R \in \mathcal{D}^0_{k-p+1}$ with $\|R\|_* \leq 1$, where $l > 0$, $0 < \lambda \leq 1$ and $M \geq 0$ are constants. If $l' > 0$ is another constant, the above comparison between dist_l and $\mathrm{dist}_{l'}$ implies that when $\mathcal{U}_S$ is (l, λ, M)-Hölder continuous, it is (l', λ', M')-Hölder continuous for some constants λ' and M' which are independent of S.

Here is the main result in this section. It improves Theorem 3.2.6 in [7] and can be seen as a version of the classical exponential estimates for p.s.h. functions.

Proposition 2.1. *Let R be a current in $\mathcal{D}^0_{k-p+1}$ with $\|R\|_* \leq 1$ such that its super-potential $\mathcal{U}_R$ is $(2, \lambda, M)$-Hölder continuous. Then there is a constant $A > 0$ independent of R, λ and M such that the super-potential $\mathcal{U}_S$ of S satisfies*

$$|\mathcal{U}_S(R)| \leq A(1 + \lambda^{-1} \log^+ M),$$

for any current S in $\mathcal{D}_p^0$ with $\|S\|_* \leq 1$, where $\log^+ := \max(0, \log)$.

We will use a family of linear regularizing operators $\mathcal{L}_\theta : \mathcal{D}_p^0 \to \mathcal{D}_p^0$ introduced in [7] with θ in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Let us recall some properties of $\mathcal{L}_\theta$. Fix a constant $c > 0$ large enough which depends only on the geometry of (X, ω).

The operators $\mathcal{L}_\theta$ are continuous for the considered topology on $\mathcal{D}_p^0$ and are bounded for the $\| \ \|_*$-norm, i.e. $\|\mathcal{L}_\theta(S)\|_* \leq c\|S\|_*$ with $c > 0$ independent of θ and S. We have $\mathcal{L}_0(S) = S$ and

$$\mathrm{dist}_2(S, \mathcal{L}_\theta(S)) \leq c\|S\|_*|\theta|.$$

Moreover, $\mathcal{L}_\theta = \mathcal{L}_\infty$ for $|\theta| \geq 1$.

Let $p \geq 1$ be a constant and let $q \geq 1$ such that when $p < k+1$, $1/q = 1/p - 1 + k/(k+1)$ and $q = \infty$ when $p \geq k+1$. We always have $\|\mathcal{L}_\infty(S)\|_{L^1} \leq c\|S\|_*$. If S is an L^p form, $p \geq 1$, then $\mathcal{L}_\infty(S)$ is an L^q form satisfying

$$\|\mathcal{L}_\infty(S)\|_{L^q} \leq c\|S\|_{L^p}.$$

Here, $c > 0$ is a constant large enough. Recall also that the function $u_S(\theta) := \mathcal{U}_{\mathcal{L}_\theta(S)}(R)$ is continuous and is constant out of the unit disc. It satisfies

$$\|dd^c u_S(\theta)\| \leq c\|S\|_*\|R\|_*.$$

The last properties hold for R smooth and extend by continuity to currents R with a continuous super-potential.

Proof of Proposition 2.1. For R and S as in the proposition, we have $\|S\|_* \leq 1$ and $\|R\|_* \leq 1$. Multiplying S by a constant allows to assume that $\|S\|_* \leq c^{-k-3}$. Define $S_0 := S$ and $S_{i+1} := \mathcal{L}_\infty(S_i)$ for $0 \leq i \leq k+1$. Define also $u_i(\theta) := \mathcal{U}_{\mathcal{L}_\theta(S_i)}(R)$ and $m_i := u_i(0) = u_{i-1}(\infty)$. Using inductively the above estimates, we get $\|S_i\|_* \leq 1/c$, $\|dd^c u_i\| \leq 1$ and $\|S_{k+2}\|_{L^\infty} \leq 1$. The last inequality implies that $|m_{k+2}|$ is bounded by a constant independent of S, R. Indeed, R always admits a potential U_R of bounded L^1-norm and we have $m_{k+2} = \langle S_{k+2}, U_R \rangle$.

We need to show that $|m_0| \leq A(1 + \lambda^{-1}\log^+ M)$ for some constant $A > 0$. For this purpose, we can assume that $M > 1$ and it is enough to check that $|m_i - m_{i+1}| \leq A(1 + \lambda^{-1}\log M)$ for some constant $A > 0$. We have $m_i - m_{i+1} = v_i(0)$ where $v_i := u_i - m_{i+1}$. The above properties of u_i imply that v_i are continuous, vanish outside the unit disc and satisfy $\|dd^c v_i\| \leq 1$. The classical exponential estimates for subharmonic functions imply that $\|e^{|v_i|}\|_{L^1(\mathbb{P}^1)} \leq c$ for some universal constant $c > 0$, see [7, Lemma 2.2.4] and [11, Th. 4.4.5]. We then deduce that there is a θ satisfying $|\theta| \leq M^{-1/\lambda}$ and $|v_i(\theta)| \leq (A-1) + A\lambda^{-1}\log M$ for a fixed constant A large enough. Finally, using the Hölder continuity of $\mathcal{U}_R$, we get

$$\begin{aligned} |v_i(0) - v_i(\theta)| &= |\mathcal{U}_{S_i}(R) - \mathcal{U}_{\mathcal{L}_\theta(S_i)}(R)| = |\mathcal{U}_R(S_i) - \mathcal{U}_R(\mathcal{L}_\theta(S_i))| \\ &\leq M\,\mathrm{dist}_2(S_i, \mathcal{L}_\theta(S_i))^\lambda \leq M|\theta|^\lambda \leq 1. \end{aligned}$$

Therefore, $|v_i(0)| \leq A(1 + \lambda^{-1}\log M)$. This completes the proof. □

3 Convergence towards Green currents

Let f, d_q and δ be as in Theorem 1.1. Fix a constant $\delta_0 < \delta$, close enough to δ, so that δ_0 satisfies also the same properties as δ. We recall some known facts and refer to [7] for details. By Poincaré duality, the dynamical degree d_q of f is equal to the degree $d_{k-q}(f^{-1})$ of f^{-1}. Since the mass of a positive closed current can be computed cohomologically, if S is in $\mathcal{D}_q$ and R is in $\mathcal{D}_{k-p+1}$, we have $\|(f^n)^*(S)\|_* \leq cd_p^n\|S\|_*$ and $\|(f^n)_*(R)\|_* \leq c\delta_0^n\|R\|_*$ for some constant $c > 0$ independent of S, R and n.

By Perron-Frobenius theorem, the eigenspace H associated to the eigenvalue d_p of f^* acting on $H^{p,p}(X,\mathbb{R})$ is a real line. Therefore, $d_p^{-n}(f^n)^*$ converge to a linear operator $L_\infty : H^{p,p}(X,\mathbb{R}) \to H$. Under the hypothese of Theorem 1.1, it is easy to deduce that on $H^{p,p}(X,\mathbb{R})$

$$\|d_p^{-n}(f^n)^* - L_\infty\| \leq c(d/\delta_0)^{-n}$$

for some constant $c > 0$. A *Green (p,p)-current* T_+ of f is a non-zero positive closed (p,p)-current invariant under $d_p^{-1}f^*$, i.e. $f^*(T_+) = d_pT_+$. Its cohomology class $\{T_+\}$ generates the real line H. Moreover, it is known [7] that T_+ is the unique positive closed current in $\{T_+\}$. So, if S is a current in $\mathcal{D}_p$, then $d_p^{-n}(f^n)^*(S)$ converge to a multiple of T_+. Here is the main result of this section.

Proposition 3.1. *Let f, d_q, δ be as in Theorem 1.1 and S a current in $\mathcal{D}_p$. Let r be the constant such that $d_p^{-n}(f^n)^*(S)$ converge to rT_+. Let R be a current in $\mathcal{D}^0_{k-p+1}$ with $\|R\|_* \leq 1$ whose super-potential $\mathcal{U}_R$ is $(2,\lambda,1)$-Hölder continuous. Let $\mathcal{U}_+$, $\mathcal{U}_n$ be the α-normalized super-potentials of T_+ and of $d_p^{-n}(f^n)^*(S)$. Then*

$$|\mathcal{U}_n(R) - r\mathcal{U}_+(R)| \leq A(d/\delta)^{-n}$$

where $A > 0$ is a constant independent of R and of n.

We first prove the following lemma.

Lemma 3.2. *Let R be a current in $\mathcal{D}^0_{k-p+1}$ whose super-potential $\mathcal{U}_R$ is $(2,\lambda,M)$-Hölder continuous. Then, there is a constant $A_0 \geq 1$ independent of R, λ, M such that the super-potential $\mathcal{U}_{f_*(R)}$ of $f_*(R)$ is $(2,\lambda,A_0M)$-Hölder continuous.*

Proof. Let T be a current in $\mathcal{D}^0_p$ such that $\|T\|_* \leq 1$. We have seen that $\|f^*(T)\|_* \leq c$ for some constant $c \geq 1$ independent of T. Define $T' := c^{-1}f^*(T)$. If T is smooth and U_T is a smooth potential of T, then $f^*(U_T)$ is a smooth potential of $f^*(T)$ and we have

$$\mathcal{U}_{f_*(R)}(T) = \langle f_*(R), U_T\rangle = \langle R, f^*(U_T)\rangle = \mathcal{U}_R(f^*(T)).$$

Since $\mathcal{U}_R$ is continuous and smooth forms are dense in $\mathcal{D}^0_p$, we deduce that $\mathcal{U}_{f_*(R)}$ is continuous and $\mathcal{U}_{f_*(R)}(T) = \mathcal{U}_R(f^*(T))$ for every T in $\mathcal{D}^0_p$. Therefore,

$$|\mathcal{U}_{f_*(R)}(T)| = c|\mathcal{U}_R(T')| \leq cM\|T'\|^\lambda_{\mathcal{C}^{-2}}.$$

Now, it is enough to show that $\|f^*(T)\|_{\mathcal{C}^{-2}} \leq c'\|T\|_{\mathcal{C}^{-2}}$ for some constant $c' > 0$. Consider test $(k-p, k-p)$-forms Φ such that $\|\Phi\|_{\mathcal{C}^2} \leq 1$. Since f^{-1} is smooth, there is a constant $c' > 0$ such that $\|f_*(\Phi)\|_{\mathcal{C}^2} \leq c'$. It follows that

$$\|f^*(T)\|_{\mathcal{C}^{-2}} = \sup_\Phi |\langle f^*(T), \Phi\rangle| = \sup_\Phi |\langle T, f_*(\Phi)\rangle| \leq c'\|T\|_{\mathcal{C}^{-2}}.$$

This completes the proof. □

Proof of Proposition 3.1. We have seen that $\|d_p^{-n}(f^n)^* - L_\infty\| \lesssim (d/\delta_0)^{-n}$ on $H^{p,p}(X, \mathbb{R})$. So, the computation in [7, Lemma 4.2.3] shows that if $\mathcal{U}_S$ is continuous, $|\mathcal{U}_n(R) - r\mathcal{U}_+(R)| \lesssim (d/\delta)^{-n}$. Therefore, subtracting from S a smooth closed (p,p)-form allows to assume that $\{S\} = 0$ and hence $r = 0$.

Define $R_n := c^{-1}\delta_0^{-n}(f^n)_*(R)$ where $c \geq 1$ is a fixed constant large enough. We have $\|R_n\|_* \leq 1$. Lemma 3.2 implies by induction that $\mathcal{U}_{R_n}$ is $(2, \lambda, A_0^n)$-Hölder continuous. As in the proof of this lemma, we obtain $\mathcal{U}_n(R) = c(d_p/\delta_0)^{-n}\mathcal{U}_S(R_n)$. Finally, we deduce from Proposition 2.1 that

$$|\mathcal{U}_n(R)| = c(d_p/\delta_0)^{-n}|\mathcal{U}_S(R_n)| \lesssim n(d_p/\delta_0)^{-n}.$$

The result follows. □

4 Exponential mixing

In this section, we prove Theorem 1.1. Theory of interpolation between the Banach spaces $\mathcal{C}^0$ and $\mathcal{C}^2$ [19] implies that it is enough to consider the case $\beta = \beta' = 2$, see [3, 5] for details. Assume now that φ and ψ are $\mathcal{C}^2$ functions such that $\|\varphi\|_{\mathcal{C}^2} \leq 1$ and $\|\psi\|_{\mathcal{C}^2} \leq 1$. Subtracting from ψ a constant allows to assume also that $\langle\mu, \psi\rangle = 0$. We have to show that

$$|\langle\mu, (\varphi \circ f^n)\psi\rangle| \lesssim (d/\delta)^{-n/2}.$$

We only need to consider the case where n is even. Indeed, if n is odd, we can replace φ with $\varphi \circ f$ and deduce the result from the first case. So, it is enough to check that

$$|\langle\mu, (\varphi \circ f^{2n})\psi\rangle| \lesssim (d/\delta)^{-n}.$$

We will apply Proposition 3.1 to the automorphism F of $X \times X$ defined by $F(x, y) := (f^{-1}(x), f(y))$. By Künneth formula [17, Th. 11.38], there is a canonical isomorphism

$$H^{q,q}(X \times X, \mathbb{C}) = \bigoplus_{s+r=q} H^{s,r}(X, \mathbb{C}) \otimes H^{r,s}(X, \mathbb{C}).$$

It is not difficult to see that F^* preserves the above decomposition. So, the dynamical degree of order k of F is equal to d_p^2. It was shown in [4] that the spectral radius of f^* on $H^{r,s}(X,\mathbb{C})$, which is also the spectral radius of f_* on $H^{k-r,k-s}(X,\mathbb{C})$, is smaller or equal to $\sqrt{d_r d_s}$. Therefore, the dynamical degrees and the eigenvalues of F^* on $H^{k,k}(X\times X,\mathbb{R})$, except d_p^2, are strictly smaller than $d_p\delta_0$. So, we can apply Proposition 3.1 to F.

Let $[\Delta]$ denote the positive closed (k,k)-current associated to the diagonal Δ of $X\times X$. Recall μ is the wedge-product $T_+\wedge T_-$ of a Green (p,p)-current T_+ associated to f and a Green $(k-p,k-p)$-current T_- associated to f^{-1}. We have $f^*(T_+)=d_pT_+$ and $f_*(T_-)=d_pT_-$. Hence, $F_*(T_+\otimes T_-)=d_p^2T_+\otimes T_-$. We deduce from the uniqueness of Green currents that any Green (k,k)-current of F^{-1} is a multiple of $T_+\otimes T_-$. In particular, it has a Hölder continuous super-potential.

Recall that $\|\varphi\|_{\mathcal{C}^2}\leq 1$ and $\|\psi\|_{\mathcal{C}^2}\leq 1$. Define $\Phi(x,y):=\varphi(x)\psi(y)$. Since the $\mathcal{C}^2$-norm of this function is bounded, $dd^c\Phi$ is a current in $\mathcal{D}^0_2(X\times X)$ with bounded $\|\ \|_*$-norm. If $\mathcal{U}$ is its super-potential and T is a current in $\mathcal{D}^0_{2k}(X\times X)$, then $\mathcal{U}(T)=\langle\Phi,T\rangle$. Clearly, $\mathcal{U}$ is $(2,1,M)$-Hölder continuous for some constant $M>0$ independent of φ,ψ. By Proposition 3.4.2 in [7], the wedge-product of currents with Hölder continuous super-potentials has also a Hölder continuous super-potential. We deduce from the proof of that proposition and the comparison between the distances dist_l that $R:=(T_+\otimes T_-)\wedge dd^c\Phi$ is a current in $\mathcal{D}^0_{k+1}(X\times X)$ with a $(2,\lambda,M')$-Hölder continuous super-potential for some constants λ,M' independent of φ,ψ. This current R has also a bounded $\|\ \|_*$-norm. Multiplying φ by a constant allows us to assume that $\|R\|_*\leq 1$ and $M'=1$.

Proposition 3.1 applied to F, $[\Delta]$ instead of f, S yields

$$|\mathcal{U}_{d_p^{-2n}(F^n)^*[\Delta]}(R)-m|\lesssim (d_p/\delta)^{-n}\quad\text{where}\quad m:=\lim_{n\to\infty}\mathcal{U}_{d_p^{-2n}(F^n)^*[\Delta]}(R).$$

On the other hand, since $F_*(T_+\otimes T_-)=d_p^2T_+\otimes T_-$ and since Green currents are well approximated by smooth forms, the following calculus holds (see [7])

$$\begin{aligned}\mathcal{U}_{d_p^{-2n}(F^n)^*[\Delta]}(R) &= \langle d_p^{-2n}(F^n)^*[\Delta],\Phi(T_+\otimes T_-)\rangle\\ &= \langle[\Delta],d_p^{-2n}(\Phi\circ F^{-n})(F^n)_*(T_+\otimes T_-)\rangle\\ &= \langle[\Delta],(\Phi\circ F^{-n})T_+\otimes T_-\rangle\\ &= \langle(T_+\otimes T_-)\wedge[\Delta],\Phi\circ F^{-n}\rangle.\end{aligned}$$

The same arguments and the fact that $\mu=T_+\wedge T_-$ is invariant yield

$$\mathcal{U}_{d_p^{-2n}(F^n)^*[\Delta]}(R)=\langle T_+\wedge T_-,(\varphi\circ f^n)(\psi\circ f^{-n})\rangle=\langle\mu,(\varphi\circ f^{2n})\psi\rangle.$$

We deduce from the mixing of μ that the last integral tends to 0 since $\langle\mu,\psi\rangle=0$. Therefore, we have $m=0$. This together with the above estimate on the super-potential of $d_p^{-2n}(F^n)^*[\Delta]$ implies that

$$|\langle\mu,(\varphi\circ f^{2n})\psi\rangle|\lesssim(d_p/\delta)^{-n},$$

and completes the proof of Theorem 1.1.

Remark 4.1. Let $\delta_+ \geq d_{p-1}$ (resp. $\delta_- \geq d_{p+1}$) denote the smallest number such that the eigenvalues of f^* acting on $H^{p,p}(X, \mathbb{C})$, except d_p, are of modulus smaller than or equal to δ_+ (resp. δ_-). Theorem 1.1 still holds for any δ such that

$$\frac{2 \log \delta_+ \log \delta_-}{\log \delta_+ + \log \delta_-} < \log \delta < \log d_p.$$

Indeed, there are positive integers l, m such that

$$\max\left(\delta_+^l, \delta_-^m\right) < \delta^{\frac{l+m}{2}}$$

and it is enough to follow the proof of Theorem 1.1 where we replace F with the automorphism (f^{-l}, f^m). The details are left to the reader.

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