Geometric entropy of geodesic currents on free groups

Ilya Kapovich and Tatiana Nagnibeda

Abstract. A geodesic current on a free group F is an F-invariant measure on the set $\partial^2 F$ of pairs of distinct points of the boundary ∂F. The main aim of this paper is to introduce and study the notion of *geometric entropy* $h_T(\mu)$ for a geodesic current μ on a free group F with respect to a point T in the Outer Space $cv(F)$, T thus being an $\mathbb{R}$-tree equipped with a minimal free and discrete isometric action of F. The geometric entropy $h_T(\mu)$ measures the slowest exponential decay rate of the values of μ on cylinder sets in T, with respect to the T-length of the segment defining such a cylinder.

We obtain an explicit formula for $h_{T'}(\mu_T)$, where $T, T' \in cv(F)$ are arbitrary points and where μ_T is the Patterson-Sullivan current corresponding to T, in terms of the volume entropy of T and the extremal distortion of distances in T with respect to distances in T'.

We conclude that for $T \in CV(F)$ (where $CV(F) \subseteq cv(F)$ is the projectivized Outer space consisting of all elements of $cv(F)$ with co-volume 1) and for a Patterson-Sullivan current μ_T corresponding to T, the function $CV(F) \to \mathbb{R}$ mapping T' to $h_{T'}(\mu_T)$, achieves a strict global maximum at $T' = T$.

We also show that for any $T \in cv(F)$ and any geodesic current μ on F, we have $h_T(\mu) \le h(T)$, where $h(T)$ is the volume entropy of T, and the equality is realized when $\mu = \mu_T$. For points $T \in cv(F)$ with simplicial metric (where all edges have length one), we relate the geometric entropy of a current and the measure-theoretic entropy.

1. Introduction

In [**6**] Culler and Vogtmann introduced a free group analogue of the Teichmüller space of a hyperbolic surface now known as Culler-Vogtmann's *Outer space*. The Outer space proved to be a fundamental object in the study of the outer automorphism group of a free group and of individual outer automorphisms.

Let F be a free group of finite rank $k \ge 2$. The *nonprojectivized Outer space* $cv(F)$ consists of all minimal free and discrete isometric actions of F on $\mathbb{R}$-trees. Two trees in $cv(F)$ are considered equal if there exists an F-equivariant isometry between them. Note that for every $T \in cv(F)$ the action of F on T is cocompact.

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There are several topologies on $cv(F)$ that are all known to coincide [**21**]: the equivariant Gromov-Hausdorff convergence topology, the point-wise translation length function convergence topology, and the weak CW-topology (see Section 2 below for more details). There is a natural continuous left action of $Out(F)$ on $cv(F)$ that corresponds to pre-composing an action of F on T with the inverse of an automorphism of F. One often works with the projectivized version $CV(F)$ of $cv(F)$, called the *Outer space*, which consists of all $T \in cv(F)$ such that the quotient graph T/F has volume 1. The space $CV(F)$ is a closed $Out(F)$-invariant subset of $cv(F)$.

A geodesic current is, in the context of negative curvature, a measure-theoretic generalization of the notion of a free homotopy class of a closed curve on a surface and of the notion of a conjugacy class in a group. Let ∂F be the hyperbolic boundary of F and let $\partial^2 F$ be the set of all pairs $(\xi, \zeta) \in \partial F \times \partial F$ such that $\xi \neq \zeta$. There is a natural left translation action of F on ∂F and hence on $\partial^2 F$. A *geodesic current* on F is a positive, finite on compact subsets, Borel measure on $\partial^2 F$ that is F-invariant. (One sometimes also requires currents to be invariant with respect to the "flip" map $\partial^2 F \to \partial^2 F$, $(\xi, \zeta) \mapsto (\zeta, \xi)$, but we do not impose this restriction in this paper). The space $Curr(F)$ of all geodesic currents on F is locally compact and comes equipped with a natural continuous action of $Out(F)$ by linear transformations.

The study of geodesic currents in the context of hyperbolic surfaces was initiated by Bonahon [**1, 2**]. Bonahon extended the notion of a geometric intersection number between two (free homotopy classes of) closed curves on a hyperbolic surface to a symmetric and mapping-class-group invariant notion of an intersection number between two geodesic currents. He also showed that the Liouville embedding of the Teichmüller space into the space of projectivized geodesic currents extends to a topological embedding of Thurston's compactification of the Teichmüller space. The study of geodesic currents also proved useful in the context of free groups (see, for example, [**20, 12, 13, 14, 15, 16, 17, 3, 7**]). Thus in [**12, 13**] Kapovich constructed a canonical Bonahon-type $Out(F)$-invariant continuous "intersection form" $I : cv(F) \times Curr(F) \to \mathbb{R}$. In a recent paper [**16**] Kapovich and Lustig extended this intersection form to the "boundary" of $cv(F)$ and constructed its continuous $Out(F)$-invariant extension $I : \overline{cv} \times Curr(F) \to \mathbb{R}$. Here $\overline{cv}(F)$ is the closure of $cv(F)$ in the equivariant Gromov-Hausdorff (or the length function) topology. It is known that $\overline{cv}(F)$ consists precisely of all the minimal *very small* isometric actions of F on $\mathbb{R}$-trees. The projectivization of $\overline{cv}(F)$ gives the *Thurston compactification* $\overline{CV}(F) = CV(F) \cup \partial CV(F)$ of the Outer space $CV(F)$. Motivated by Bonahon's result, in [**18**] Kapovich and Nagnibeda constructed the *Patterson-Sullivan map* $CV(F) \to \mathbb{P}Curr(F)$ and proved that this map is an $Out(F)$-equivariant continuous embedding (here $\mathbb{P}Curr(F)$ is the space of *projectivized geodesic currents* on F). Since $\mathbb{P}Curr(F)$ is compact, the closure of the image of $CV(F)$ under this map gives a compactification of $CV(F)$. However, unlike in the case of hyperbolic surfaces, this compactification is not the same as Thurston's compactification $\overline{CV}(F)$ of $CV(F)$. Kapovich and Lustig [**15**] proved moreover that there does not exist a continuous $Out(F)$-equivariant map $\partial CV(F) \to \mathbb{P}Curr(F)$.

Let $T \in cv(F)$. Note that T is a proper Gromov-hyperbolic geodesic metric space. Denote by ∂T the hyperbolic boundary of T and by $\partial^2 T$ the set of all pairs $(\xi, \zeta) \in \partial^2 T$ such that $\xi \neq \zeta$. Thus for any $(\xi, \zeta) \in \partial^2 T$ there exists a unique bi-infinite (non-parameterized) oriented geodesic line $[\xi, \zeta] \subseteq T$ in T from ξ to ζ.

We think of $[\xi,\zeta]\subseteq T$ as the image of an isometric embedding from $\mathbb{R}$ to T, with the correct choice of an orientation on $[\xi,\zeta]$.

Since F acts discretely, isometrically and co-compactly on T, the orbit map (for any basepoint in T) defines a quasi-isometry $q_T : F \to T$ (where F is taken with any word metric) and hence a canonical F-equivariant homeomorphism $\partial q_T : F \to \partial T$. In turn, ∂q_T defines an F-equivariant homeomorphism $\partial^2 q_T : \partial^2 F \to \partial^2 T$.

We will use the homeomorphisms ∂q_T and $\partial^2 q_T$ to identify ∂F with ∂T and, similarly, $\partial^2 F$ with $\partial^2 T$. We will often suppress this explicit identification.

The *volume entropy* $h = h(T)$ is defined as

$$h(T) = \lim_{R\to\infty} \frac{\log \#\{g \in F : d_T(x_0, gx_0) \le R\}}{R},$$

where $x_0 \in T$ is a basepoint. It is well known ([**5**]) that the limit always exists and does not depend on the choice of a basepoint $x_0 \in T$. It also coincides with the *critical exponent of the Poincaré series* :

$$\Pi_{x_0}(s) := \sum_{g\in F} e^{-s\ d_T(x_0,gx_0)}\ ,$$

namely, $\Pi_{x_0}(s)$ converges for all $s > h$ and diverges for all $s < h$. Moreover, for every $x_0 \in T$, as $s \to h+$, any weak limit ν of the probability measures

$$\frac{1}{\Pi_{x_0}(s)} \sum_{g\in F} e^{-s\ d(x_0,gx_0)} \mathrm{Dirac}(gx_0).$$

is a measure supported on ∂T. The measure-class of ν is uniquely determined and does not depend on the choice of x_0 or on the choice of a weak limit. Any such ν is called *a Patterson-Sullivan measure* on $\partial T = \partial F$ corresponding to $T \in cv(F)$.

Furman proved in [**9**], in a wider context of hyperbolic groups, that there exists a unique, up to a scalar multiple, F-invariant and flip-invariant nonzero locally finite measure μ_T on $\partial^2 T$ in the measure class of $\nu \times \nu$. Such a measure μ_T is called a *Patterson-Sullivan current* for $T \in cv(F)$. Since μ_T is unique up to a scalar multiple, its projective class $[\mu_T]$ is called *the projective Patterson-Sullivan current* corresponding to $T \in cv(F)$. Moreover, Furman's results imply that for $T \in cv(F)$ the projective Patterson-Sullivan current corresponding to T depends only on the projective class $[T]$ of T, and thus allow to define the Patterson-Sullivan map $CV(F) \to \mathbb{P}Curr(F); [T] \mapsto \mu_T$. We refer the reader to [**18**] for a more detailed discussion.

Let $T \in cv(F)$. Let $x, y \in T, x \neq y$, and $[x, y]$ denote the unique simplicial geodesic between x and y in T. Denote

$$\begin{aligned} Cyl^T_{[x,y]} = Cyl_{[x,y]} := \{(\zeta_1,\zeta_2) \in \partial^2 F : [x,y] \subseteq [\partial q_T(\zeta_1), \partial q_T(\zeta_2)] \\ \text{and the orientations on } [x,y] \text{ and on } [\zeta_1,\zeta_2] \text{ agree}\} \end{aligned}$$

the *two-sided cylinder set corresponding to* $[x, y]$.

For a fixed $T \in cv(F)$, any current $\mu \in Curr(F)$ is uniquely determined by its values on the cylinder sets $Cyl_{[x,y]} \subseteq \partial^2 F$, where $[x, y]$ varies over all nondegenerate geodesic segments in T. Note that since μ is F-invariant, the value $\mu(Cyl_{[x,y]})$ depends only on μ and the path which is the image of $[x, y]$ in the quotient graph T/F. The "weights" $\mu(Cyl_{[x,y]})$ tend to 0 as $d_T(x, y) \to \infty$, and in many interesting cases, as for example in that of Patterson-Sullivan currents, this convergence is exponential. We introduce the notion of *geometric entropy* $h_T(\mu)$ of μ with respect to T to measure the slowest exponential rate of decay of the weights $\mu(Cyl_{[x,y]})$ as $d_T(x,y)$ tends to infinity. More precisely (see Definition 3.1 below):

$$h_T(\mu) := \liminf_{d_T(x,y)\to\infty} \frac{-\log \mu(Cyl_{[x,y]})}{d_T(x,y)}.$$

We first establish some basic properties of geometric entropy in Section 3. In particular $h_T(\mu) = h_T(c\mu)$ for any $c > 0$, $\mu \in Curr(F)$, so that $h_T(\mu)$ depends only on the projective class of μ. We note that for a fixed $\mu \in Curr(F)$ the function $E_\mu : cv(F) \to \mathbb{R}$, $T \mapsto h_T(\mu)$ is continuous (Proposition 3.7). On the other hand, for any $T \in cv(F)$, the function $h_T : Curr(F) \to \mathbb{R}$, $\mu \mapsto h_T(\mu)$ is highly discontinuous. Indeed, there is a dense subset in $Curr(F)$ consisting of so-called "rational" currents (see Definition 5.1 in [**13**]), whose geometric entropy is zero. On the other hand there are many currents with positive geometric entropy.

We obtain an explicit formula for the geometric entropy of a Patterson-Sullivan current μ_T of $T \in cv(F)$ with respect to an arbitrary $T' \in cv(F)$. The geometric entropy of μ_T with respect to T coincides with the volume entropy $h(T)$. We then solve two types of extremal problems regarding maximal values of the geometric entropy with either the tree or the current arguments fixed. Our main results are the following.

Theorem A. *(Corollary 5.3 and Theorem 5.5). Let $T \in cv(F)$ and let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current corresponding to T. Let $h(T)$ be the volume entropy of T. Then*

1. *$h_T(\mu_T) = h(T)$;*
2. *for any $T' \in cv(F)$*

$$h_{T'}(\mu_T) = h(T) \inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} = \frac{h(T)}{\sup_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T}}.$$

Here, for $f \in F$ and $T \in cv(F)$, $||f||_T := \inf_{x\in T} d_T(x, fx)$ is the translation length *of f.*

It is known [**24, 12, 13**] that

$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} = \min_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}, \qquad \sup_{f\in F\setminus\{1\}} \frac{||f||_T}{||f||_{T'}} = \max_{f\in F\setminus\{1\}} \frac{||f||_T}{||f||_{T'}}$$

and, moreover, one can algorithmically find $g, f \in F$ realizing the above equalities in a finite subset of $F \setminus \{1\}$ depending only on T. It follows that the geometric entropy as function of T' admits a continuous and strictly positive extension to $\overline{cv}(F)$.

The extremal distortions of the trees T and T' with respect to each other which appear in Theorem A are key ingredients in the recent construction by Francaviglia and Martino [**8**] of asymmetric metrics on the Outer space. Their construction is inspired by Thurston's work on mutual extremal stretching factors (extremal Lipshitz constants) of two points in the Teichmüller space [**22**].

We further use Theorem A to compute extremal values of $h_{T'}(\mu_T)$ as function of $T' \in CV(F)$ and show that this function achieves its strict maximum at T.

Corollary B. *(Corollaries 5.6 and 5.7). Let $T, T' \in CV(F)$ be such that $T \neq T'$. Let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current corresponding to T and let $h(T)$ be the volume entropy of T. Then*

(1) *for any* $T' \in CV(F)$ *such that* $T' \neq T$

$$h_{T'}(\mu_T) < h_T(\mu_T) = h(T);$$

(2) *we have*

$$\inf_{T' \in CV(F)} h_{T'}(\mu_T) = 0.$$

We then proceed to study the geometric entropy as a function of $\mu \in Curr(F)$. Although it is highly discontinuous, we compute its maximal value. Given a current $\mu \in Curr(F)$, we consider a family of measures $\{\mu_x\}_{x\in T}$ on ∂F defined by their values on *one-sided cylinder subsets* of ∂F:

$$Cyl^x_{[x,y]} := \{\xi \in \partial F : \text{ the geodesic ray } [x, \partial_T(\xi)] \text{ in } T \text{ begins with } [x, y]\} \subseteq \partial F,$$

$$\mu_x(Cyl^x_{[x,y]}) := \mu(Cyl_{[x,y]}).$$

If $\mu \in Curr(F)$, $\mu \neq 0$ then there is $x \in T$ such that $\mu_x \neq 0$ (it is enough to take a segment $[x, y]$) such that $\mu(Cyl_{[x,y]}) \neq 0$). Note however, that the action of F on the set of vertices of T is not necessarily transitive and it may happen that $\mu \neq 0$ but for some vertex x of T we have $\mu_x = 0$.

If μ_T is a Patterson-Sullivan current corresponding to T then μ_x is a Patterson-Sullivan measure on ∂F corresponding to T (see [**18**]).

Theorem C. *(Theorem 6.1 and Corollary 6.3.) Let* $T \in cv(F)$ *and let* $h = h(T)$ *be the volume entropy of* T.

(1) *Let* $\mu \in Curr(F)$, $\mu \neq 0$, *and let* $x \in T$ *be such that* $\mu_x \neq 0$. *Then*

$$h_T(\mu) \leq \mathbf{HD}_{\partial T}(\mu_x) \leq h_T(\mu_T) = h(T),$$

where $\mathbf{HD}_{\partial T}(\mu_x)$ *is the Hausdorff dimension of* μ_x *with respect to* ∂T *with the metric* d_x *(definitions are recalled in the beginning of Section 6).*

(2) *For* $T \in cv(F)$ *denote by* $[T]$ *the* projective class *of* T *that is, the set of all* $cT \in cv(F)$ *where* $c > 0$. *If* $T' \in cv(F)$ *is such that* $[T'] \neq [T]$, *then*

$$h_T(\mu_{T'}) < h(T).$$

Part (1) of Theorem C implies that

$$h(T) = h_T(\mu_T) = \max_{\mu\in Curr(F)-\{0\}} h_T(\mu).$$

As was observed in [**11**], if $T_A \in cv(F)$ is the Cayley graph of F with respect to a free basis A and if $T \in cv(F)$ is arbitrary, then

$$\mathbf{HD}_{\partial T}(m_A) = \frac{\log(2k-1)}{\lambda_A(T)}$$

where $k \geq 2$ is the rank of F, where m_A is the "uniform" measure on ∂F corresponding to A, and where $\lambda_A(T)$ is the "generic stretching factor" of T with respect to A. That is, for an element $w_n \in F$ obtained by a simple non-backtracking random walk on T_A of length n, we have $||w_n||_T/||w_n||_A \to \lambda_A(T)$ as $n \to \infty$. Note that in this case $h(T_A) = \log(2k-1)$ and $m_A = (\mu_{T_A})_x$ for x being the vertex of the Cayley graph T_A of F corresponding to $1 \in F$. Also, obviously $\lambda_A(T) \leq \sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_A}$. Thus part (2) of Theorem C agrees with these observations since it says that

$$h_T(\mu_{T_A}) = \frac{\log(2k-1)}{\sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_A}} \leq \mathbf{HD}_{\partial T}(m_A) = \frac{\log(2k-1)}{\lambda_A(T)}.$$

These observations also suggest that if $T_0 \in cv(F)$ is arbitrary (not necessarily corresponding to a free basis) and if $T \in cv(F)$, one can define the "generic stretching factor" of T with respect to T_0 as $\lambda_{T_0}(T) := \frac{h(T_0)}{\mathbf{HD}_{\partial T}(\mu_0)}$ where μ_0 is a Patterson-Sullivan measure on ∂F corresponding to T_0.

Combining Theorem A and Theorem C we obtain:

Corollary D. *Let $T, T' \in cv(F)$.*

(1)
$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} \le \frac{h(T')}{h(T)} \le \sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}.$$

(2) *Suppose that $[T] \neq [T']$. Then*
$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} < \frac{h(T')}{h(T)} < \sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}.$$

Here $[T]$ denotes the projective class of a tree $T \in cv(F)$. Part (2) of Corollary D implies that if $[T] \neq [T']$ and $h(T) = h(T')$ then
$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} < 1 < \sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}.$$
Thus there exist $g, f \in F \setminus \{1\}$ such that $||g||_T < ||g||_{T'}$ and $||f||_T > ||f||_{T'}$. This statement provides an analogue of a theorem of Tad White [**24**] who proved a similar result for $CV(F)$, that is for the situation where points in $cv(F)$ are normalized by co-volume. The above inequality is an analogue of White's result for the situation where we normalize points of $cv(F)$ by volume entropy.

In Theorem 3.9 we also bound the geometric entropy $h_T(\mu)$ by the exponential growth rate of the support of μ (appropriately defined) and observe that currents with support of subexponential growth always have geometric entropy equal to zero.

Suppose that $T \in cv(F)$ is a simplicial tree with all edges of length one, so that we can think of T as $\widetilde{\Gamma}$ for the finite graph $\Gamma = T/F$ with the standard simplicial metric, and without degree-one vertices. There is a natural shift map $\sigma : \Omega(\Gamma) \to \Omega(\Gamma)$ on the space $\Omega(\Gamma)$ of all semi-infinite reduced edge-paths in Γ, corresponding to erasing the first edge of a path. The pair $(\Omega(\Gamma), \sigma)$ is an irreducible subshift of finite type. For every finite reduced edge-path v in Γ there is a natural cylinder set $Cyl_v \subseteq \Omega(\Gamma)$ consisting of all semi-infinite paths $\gamma \in \Omega(\Gamma)$ that have v as an initial segment. We can think of $\Omega(\Gamma)$ as an analogue of the unit tangent bundle for Γ. There is also a natural affine correspondence (see [**13**] for more details) between the space of geodesic currents $Curr(F)$ and the space $\mathcal{M}(\Gamma)$ of finite shift-invariant measures on the space $\Omega(\Gamma)$. For $\mu \in Curr(F)$ the corresponding shift-invariant measure $\widehat{\mu} \in \mathcal{M}(\Gamma)$ is defined by the condition $\widehat{\mu}(Cyl_v) := \mu(Cyl_{[x,y]})$ where v is an arbitrary finite reduced edge path in Γ and where $[x, y]$ is a lift of v to T. In this setting, for any $\mu \in Curr(F)$, we relate the geometric entropy $h_T(\mu)$ and the measure-theoretic entropy of $\widehat{\mu}$ (normalized to be a probability measure). We refer the reader to [**19**] for a more detailed discussion regarding measure-theoretic entropy (also known as metric entropy or Kolmogorov-Sinai entropy) of shift-invariant measures on irreducible subshifts of finite type. In particular, it is known that for such subshifts the measure-theoretic entropy of a shift-invariant probability measure never exceeds the topological entropy of the shift and that the equality is realized by a unique shift-invariant probability measure called the *measure of maximal entropy*. For a shift-invariant probability measure ν on $\Omega(\Gamma)$ we denote its measure-theoretic entropy by $\hbar(\nu)$.

Theorem E. *(Theorem 7.2 and Corollary 7.3.) Let $T \in cv(F)$ be a simplicial tree with simplicial metric and let $\Gamma = T/F$ be the quotient graph (thus all edges in T and Γ have length one, and Γ is a finite connected graph without degree-one vertices). Let $\mu \in Curr(F)$, $\mu \neq 0$, and let $\widehat{\mu} \in \mathcal{M}(\Gamma)$ be the corresponding shift-invariant measure on $\Omega(\Gamma)$, normalized to be a probability measure. Similarly, let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current for T and let $\widehat{\mu}_T \in \mathcal{M}(\Gamma)$ be the corresponding shift-invariant measure on $\Omega(\Gamma)$, normalized to have total mass one. Then we have*

(1) $h_T(\mu) \le \hbar(\widehat{\mu}) \le h_{topol}(\Omega(\Gamma), \sigma) = h(T) = \hbar(\widehat{\mu}_T)$;
(2) *$h_T(\mu) = h(T)$ if and only if there is $c > 0$ such that $c\mu = \mu_T$, that is, $[\mu] = [\mu_T]$ in $\mathbb{P}Curr(F)$.*

(In particular $\widehat{\mu}_T$ is the measure of maximal entropy for $(\Omega(\Gamma), \sigma)$). Note that Theorem E implies part (2) of Corollary C for the case where $T \in cv(F)$ has simplicial metric (all edges have length one).

We believe that a version of Theorem E should hold for an arbitrary $T \in cv(F)$ and not just for a tree with simplicial metric, that is, for the case where $\Gamma = T/F$ is an arbitrary finite metric graph without degree-one vertices. Indeed, in this general case there is still a natural correspondence between $Curr(F)$ and the space of all $\mathbb{R}_{\ge 0}$-invariant measures on the space $\Omega_{\mathbb{R}}(\Gamma)$ of all locally isometric embeddings $\gamma : [0, \infty) \to \Gamma$, where $\gamma(0)$ is not required to be a vertex. (The space $\Omega_{\mathbb{R}}(\Gamma)$ comes equipped with a natural $\mathbb{R}_{\ge 0}$-action). However, in the case of a non-simplicial metric on Γ this $\mathbb{R}_{\ge 0}$ action does not nicely match the "combinatorial" shift σ from Theorem E, which creates unpleasant technical problems with the argument. Proving an analogue of Theorem E for an arbitrary $T \in cv(F)$ requires first developing basic background machinery and formalism for analyzing $\mathbb{R}_{\ge 0}$-invariant measures on $\Omega_{\mathbb{R}}(\Gamma)$ and matching this information with the combinatorial description of geodesic currents used in this paper. We postpone such analysis till a later date.

Let us note however, in view of the above discussion, that one can directly define an analogue of the notion of *geometric entropy for any shift-invariant* (or even not necessarily invariant) *measure on an irreducible subshift of finite type*, and study this notion in its own right. Thus let A be a finite alphabet, let A^ω be the full shift (the set of all semi-infinite words in A), let $\sigma : A^\omega \to A^\omega$ be the standard shift map and let $\Omega \subseteq A^\omega$ be a subshift of finite type. For any finite word v that occurs as an initial segment of some element of Ω we define Cyl_v to be the set of all $\gamma \in \Omega$ that begin with v. If μ is a finite positive Borel measure on Ω, we can define its *geometric entropy* as:

$$h_{geom}(\mu) = \liminf_{|v| \to \infty} \frac{-\log \mu(Cyl_v)}{|v|},$$

where $|v|$ is the ordinary combinatorial length of the word v. Similar to the results above, one can show that $h_{geom}(\mu) \le h_{topol}(\Omega)$ and that if μ is a shift invariant probability measure then $h_{geom}(\mu) \le \hbar(\mu) \le h_{topol}(\Omega)$. Also, similarly to Theorem C, one can show that $h_{geom}(\mu) \le \mathbf{HD}_d(\mu)$ with respect to restriction d to Ω of the standard metric from A^ω. In this paper we concentrate on the setting of geodesic currents since it is more invariant and allows us to avoid choosing Ω, (which would correspond to fixing a particular simplicial tree $T \in cv(F)$).

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2. The Culler-Vogtmann Outer Space

The Culler-Vogtmann outer space, introduced by Culler and Vogtmann in a seminal paper [**6**], is a free group analogue of the Teichmüller space of a closed surface of negative Euler characteristic. We refer the reader to the original paper [**6**] and to a survey paper [**23**] for a detailed discussion of the basic facts listed in this section and for the further references.

Definition 2.1 (Non-projectivized Outer Space)**.** Let F be a finitely generated free group of rank $k \geq 2$.

The *non-projectivized outer space* $cv(F)$ consists of all minimal free and discrete isometric actions of F on $\mathbb{R}$-trees. Two such trees are considered equal in $cv(F)$ if there exists an F-equivariant isometry between them. The space $cv(F)$ is endowed with the equivariant Gromov-Hausdorff convergence topology.

It turns out that every $T \in cv(F)$ is uniquely determined by its *translation length function* $\ell_T : F \to \mathbb{R}$, where for every $g \in F$

$$\ell_T(g) = ||g||_T = \min_{x \in T} d_T(x, gx)$$

is the *translation length* of g. Note that $\ell_T(g) = \ell_T(hgh^{-1})$ for every $g, h \in F$. Thus ℓ_T can be thought of as a function on the set of conjugacy classes in F. The space $cv(F)$ comes equipped with a natural left $Out(F)$-action by homeomorphisms. At the length function level, if $\phi \in Out(F)$, $T \in cv(F)$ and $g \in F$ we have

$$\ell_{\phi T}([g]) = \ell_T(\phi^{-1}[g]).$$

It is known that the equivariant Gromov-Hausdorff topology on $cv(F)$ coincides with the pointwise convergence topology at the level of length functions. Thus for $T_n, T \in cv(F)$ we have $\lim_{n\to\infty} T_n = T$ if and only if for every $g \in F$ we have $\lim_{n\to\infty} \ell_{T_n}(g) = \ell_T(g)$.

Points of $cv(F)$ have a more explicit combinatorial description as "marked metric graph structures" on F:

Definition 2.2 (Metric graph structure)**.** Let Γ be a finite connected graph without degree-one and degree-two vertices. A *metric graph structure* $\mathcal{L}$ on Γ is a function $\mathcal{L} : E\Gamma \to \mathbb{R}$ such that for every $e \in E\Gamma$ we have

$$\mathcal{L}(e) = \mathcal{L}(e^{-1}) > 0.$$

More generally, define a *semi-metric graph structure* $\mathcal{L}$ on Γ to be a function $\mathcal{L} : E\Gamma \to \mathbb{R}$ such that for every $e \in E\Gamma$ we have

$$\mathcal{L}(e) = \mathcal{L}(e^{-1}) \geq 0.$$

A semi-metric graph structure $\mathcal{L}$ on Γ is *nondegenerate* if there exists a subforest Z in Γ such that $\mathcal{L}(e) > 0$ for every $e \in E\Gamma - EZ$.

For a semi-metric graph structure $\mathcal{L}$ on Γ define the *volume* of $\mathcal{L}$ as

$$vol(\mathcal{L}) = \sum_{e \in E^+\Gamma} \mathcal{L}(e)$$

where $E\Gamma = E^+\Gamma \sqcup E^-\Gamma$ is any orientation on Γ.

If $\mathcal{L}$ is a semi-metric graph structure on Γ and $v = e_1, \dots, e_n$ is an edge-path in Γ, we denote

$$\mathcal{L}(v) = \sum_{i=1}^{n} \mathcal{L}(e_i)$$

and call $\mathcal{L}(v)$ the *$\mathcal{L}$-length* of v.

Convention 2.3. For a (finite or infinite) graph Δ, denote by $V\Delta$ the set of all vertices of Δ, and denote by $E\Delta$ the set of all oriented edges of Δ. Combinatorially, we use Serre's convention regarding graphs. Namely, Δ comes equipped with three functions: $o : E\Delta \to V\Delta$; $t : E\Delta \to V\Delta$; and ${}^{-1} : E\Delta \to E\Delta$, such that $e^{-1} \neq e$, $(e^{-1})^{-1} = e$ for every $e \in E\Delta$, $o(e) = t(e^{-1})$, and $t(e) = o(e^{-1})$ for every $e \in E\Delta$. The edge e^{-1} is called the *inverse* of e. An *orientation* on Δ is a partition $E\Delta = E^+\Delta \sqcup E^-\Delta$, where for every $e \in E\Delta$ one of the edges e, e^{-1} belongs to $E^+\Delta$ and the other edge belongs to $E^-\Delta$.

An *edge-path* γ in Δ is a sequence of oriented edges which connects a vertex $o(\gamma)$ (origin) with a vertex $t(\gamma)$ (terminus). A path is called *reduced* if it does not contain a back-tracking, that is a path of the form ee^{-1}, where $e \in E\Delta$.

If $\gamma = e_1, \dots, e_n$ is an edge-path in Δ, where $e_i \in E\Delta$, we call n the *simplicial length* of γ and denote it by $|\gamma|$.

We denote by $\mathcal{P}(\Delta)$ the set of all finite reduced edge-paths in Δ. For a vertex $x \in V\Delta$, we denote by $\mathcal{P}_x(\Delta)$ the collection of all $\gamma \in \mathcal{P}(\Delta)$ that begin with x.

Definition 2.4 (Marking or Simplicial chart)**.** Let Γ be a finite connected graph without degree-one and degree-two vertices such that $\pi_1(\Gamma) \cong F$. Let $\alpha : F \to \pi_1(\Gamma, p)$ be an isomorphism, where p is a vertex of Γ. We call such α a *simplicial chart* or a *marking* for F.

Definition 2.5 (Marked metric graph structure)**.** Let F be a free group of finite rank $k \geq 2$. A *marked (semi-)metric graph structure* on F is a pair $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$ is a simplicial chart for F and $\mathcal{L}$ is a (semi-)metric structure on Γ. (A marked semi-metric graph structure $(\alpha, \mathcal{L})$ is *non-degenerate* if $\mathcal{L}$ is nondegenerate.)

Convention 2.6. Let $(\alpha, \mathcal{L})$ be a marked metric graph structure on F. Then $(\alpha, \mathcal{L})$ defines a point $T \in cv(F)$ as follows. Topologically, let $T = \widetilde{\Gamma}$, with an action of F on T via α. We lift the metric structure $\mathcal{L}$ from Γ to T by giving every edge in T the same length as that of its projection in Γ. This makes T into an $\mathbb{R}$-tree equipped with a minimal free and discrete isometric action of F. Thus $T \in cv(F)$ and in this situation we will sometimes use the notation $T = (\alpha, \mathcal{L}) \in cv(F)$. Note that $T/F = \Gamma$. Moreover, it is not hard to see that every point of $cv(F)$ arises in this fashion and that $CV(F)$ is exactly the set of all those $T = (\alpha, \mathcal{L}) \in cv(F)$ where $(\alpha, \mathcal{L})$ is a marked metric graph structure on F with $vol(\mathcal{L}) = 1$.

Note also that any nondegenerate marked semi-metric graph structure on F also defines a point in $cv(F)$ by first contracting the edges of $\mathcal{L}$-length 0, and then proceeding as above.

Definition 2.7 (Elementary charts)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F. Fix an orientation $E\Gamma = E^+\Gamma \cup E^-\Gamma$ on Γ and let $E^+\Gamma = \{e_1, \dots, e_m\}$, where $m = \#E^+\Gamma$.

Let $V_\alpha \subseteq cv(F)$ be the set of all $T = (\alpha, \mathcal{L})$ where $\mathcal{L}$ is a nondegenerate semi-metric structure on Γ. Let U_α be the set of all $T = (\alpha, \mathcal{L})$ where $\mathcal{L}$ is a metric structure on Γ. Thus $U_\alpha \subseteq V_\alpha$. We call V_α the *elementary chart* corresponding to α and we call U_α the *elementary open chart* corresponding to α.

There is a natural map $\lambda_\alpha : V_\alpha \to \mathbb{R}^m$ defined as $\lambda_\alpha(\alpha, \mathcal{L}) = (\mathcal{L}(e_1), \dots, \mathcal{L}(e_m))$. It is known that $\lambda_\alpha : V_\alpha \to \mathbb{R}^m$ is injective and is a homeomorphism onto its image. In particular, $\lambda_\alpha(U_\alpha)$ is the positive open cone in $\mathbb{R}^m$, that is, $\lambda_\alpha(U_\alpha)$ consists of all points in $\mathbb{R}^m$ all of whose coordinates are positive. Therefore U_α is homeomorphic to an open cone in $\mathbb{R}^m$.

The space $cv(F)$ is the union of open cones U_α taken over all simplicial charts α on F. Moreover, every point $T \in cv(F)$ belongs to only finitely many of the elementary charts V_α. It is also known that the standard topology on $cv(F)$ coincides with the weakest topology for which all the maps $\lambda_\alpha^{-1} : \lambda_\alpha(V_\alpha) \to V_\alpha$ are continuous.

Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart, let $T = \widetilde{\Gamma}$ and let $j : T \to \Gamma$ denote the covering. It is easy to see that for $\mu_n, \mu \in Curr(F)$ we have $\lim_{n\to\infty} \mu_n = \mu$ if and only if $\lim_{n\to\infty} \mu_n(Cyl_\gamma) = \mu(Cyl_\gamma)$ for every $\gamma \in \mathcal{P}(T)$. Moreover, for $\mu, \mu' \in Curr(F)$ we have $\mu = \mu'$ if and only if $\mu(Cyl_\gamma) = \mu'(Cyl_\gamma)$ for every $\gamma \in \mathcal{P}(T)$.

Note that for any $f \in F$ and $\gamma \in \mathcal{P}(T)$ we have $fCyl_\gamma = Cyl_{f\gamma}$. Since geodesic currents are, by definition, F-invariant, for a geodesic current μ and for $\gamma \in \mathcal{P}(T)$ the value $\mu(Cyl_\gamma)$ only depends on the label $j(\gamma) \in \mathcal{P}(\Gamma)$ of γ.

Notation 2.8. For this reason, for any reduced edge-path $v \in \mathcal{P}(\Gamma)$, we denote by $\langle v, \mu\rangle_\alpha$ the value $\mu(Cyl_\gamma)$ where $\gamma \in \mathcal{P}(T)$ is any reduced edge-path with label v.

We finish this Section with the following basic lemma, needed later, which shows that for $T, T' \in cv(F)$ extremal distortions of the translation length functions for T and T' give the optimal stretching constants for F-equivariant quasi-isometries between T and T'.

Lemma 2.9. *Let $T, T' \in cv(F)$ and let $\phi : T \to T'$ be an F-equivariant quasi-isometry. Then the following hold:*

(1)
$$\lambda_1 := \inf_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} = \liminf_{d_T(x,y)\to\infty} \frac{d_{T'}(\phi(x),\phi(y))}{d_T(x,y)}.$$
(2)
$$\lambda_2 := \sup_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} = \limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(\phi(x),\phi(y))}{d_T(x,y)}.$$
(3) *There is $C > 0$ such that for any $x, y \in T$ we have*
$$\lambda_1 d_T(x,y) - C \le d_{T'}(\phi(x),\phi(y)) \le \lambda_2 d_T(x,y) + C.$$

Proof. Let $x_0 \in T$ be a base-point and let $x_0' = \phi(x_0) \in T'$. Thus $\phi(gx_0) = gx_0'$ for all $g \in F$.

Let us show Part (2). For any $g \in F \setminus \{1\}$ we have
$$\lim_{n\to\infty} \frac{d_T(x_0, g^n x_0)}{n} = ||g||_T, \quad \lim_{n\to\infty} \frac{d_{T'}(x_0', g^n x_0')}{n} = ||g||_{T'}$$
and hence
$$\lim_{n\to\infty} \frac{d_{T'}(x_0', g^n x_0')}{d_T(x_0, g^n x_0)} = \frac{||g||_{T'}}{||g||_T}.$$
Then $\frac{||g||_{T'}}{||g||_T} \le \limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(\phi(x),\phi(y))}{d_T(x,y)}$ and so

$$\sup_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} \le \limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(\phi(x),\phi(y))}{d_T(x,y)}.$$

Suppose now that $d_T(p_n,q_n)\to\infty$ and

$$\lim_{n\to\infty} \frac{d_{T'}(\phi(p_n),\phi(q_n))}{d_T(p_n,q_n)} = \limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(\phi(x),\phi(y))}{d_T(x,y)}.$$

There is a constant $M = M(T') \ge 1$ such that there are some $g_n, h_n \in F$ with $d_T(p_n, g_n x_0), d_T(q_n, h_n x_0) \le M$ and such that the geodesic $[g_n x_0, h_n x_0]$ projects to a closed cyclically reduced path in T/F. That is, the points $g_n x_0, h_n x_0$ belong to the axis of the element $h_n g_n^{-1}$ and $d_T(g_n x_0, h_n x_0) = ||h_n g_n^{-1}||_T$. Translating $[p_n, q_n]$ by g_n^{-1} we may assume that $g_n = 1$ for every $n \ge 1$. Thus $d(x_0, h_n x_0) = ||h_n||_T$ and $d_T(p_n, x_0), d_T(q_n, h_n x_0) \le M$. Hence $|d_T(p_n,q_n) - ||h_n||_T| \le 2M$. Note that x_0 and $h_n x_0$ belong to the axis of h_n in T.

Denote $p_n' = \phi(p_n)$, $q_n' = \phi(q_n)$. Since ϕ is an F-equivariant quasi-isometry, the ϕ-image of the axis of h_n in T is an h_n-invariant quasigeodesic in T' which is at a bounded Hausdorff distance from an h_n-invariant geodesic in T', that is, from the axis of h_n in T'. Hence there is some constant $C \ge 1$ such that $|d_{T'}(p_n', q_n') - ||h_n||_{T'}| \le C$.

Therefore

$$\lim_{n\to\infty} \frac{d_{T'}(\phi(p_n),\phi(q_n))}{d_T(p_n,q_n)} \le \limsup_{n\to\infty} \frac{||h_n||_T + M}{||h_n||_{T'} - C} \le \sup_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T}.$$

Hence

$$\sup_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} = \limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(\phi(x),\phi(y))}{d_T(x,y)},$$

as required.

Part (1) is established using a similar argument to part (2) and we omit the details.

For part (3), let $x, y \in T$ be arbitrary and let $x' = \phi(x)$, $y' = \phi(y)$. As in the proof of (2), there exist elements $g, h \in F$ such that $d_T(x, gx_0), d_T(y, hx_0) \le M$ and such that the geodesic $[gx_0, hx_0]$ projects to a closed cyclically reduced path in T/F. By exactly the same argument as above we deduce

$$d_{T'}(x',y') \le ||h||_{T'} + C \le \lambda_2 ||h||_T + C \le$$
$$\lambda_2(d_T(x,y) + 2M) + C = \lambda_2 d_T(x,y) + (2M\lambda + C),$$

as required. The proof that $d_{T'}(x',y') \ge \lambda_1 d_T(x,y) - (2M\lambda + C)$, is similar, and we omit the details.

□

3. Geometric entropy of a geodesic current

Definition 3.1 (Geometric entropy of a current)**.** Let $\mu \in Curr(F)$ and let $T \in cv(F)$. Define the *geometric entropy of* μ *with respect to* T as

$$h_T(\mu) := \liminf_{\substack{d_T(x,y)\to\infty \\ x,y\in T}} \frac{-\log \mu(Cyl_{[x,y]})}{d_T(x,y)}.$$

If $\mu(Cyl_{[x,y]}) = 0$, we interpret $\log 0 = -\infty$. Thus for $\mu = 0$ and any $T \in cv(F)$ we have $h_T(\mu) = \infty$. For any $\mu \in Curr(F)$, $\mu \neq 0$ and any $T \in cv(F)$ we have $0 \le h_T(\mu) < \infty$. Informally, $h_T(\mu)$ measures the slowest exponential decay rate (with respect to $d_T(x,y)$) of the "weights" $\mu(Cyl_{[x,y]})$ as $d_T(x,y) \to \infty$. Thus, taking into account that T is a discrete $\mathbb{R}$-tree, we see that if $h_T(\mu) > s > 0$ then there exists $C > 0$ such that for every $x, y \in T$ with $d_T(x,y) \ge 1$ we have

$$\mu(Cyl_{[x,y]}) \le C \ \exp(-s\, d_T(x,y)).$$

The following is a more combinatorial interpretation of the geometric entropy that follows immediately from unraveling the definitions (we use notation 2.8).

Proposition 3.2. *Let $\mu \in Curr(F)$, $\mu \neq 0$ and let $T \in cv(F)$ be determined by the pair $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$ is an isomorphism and $\mathcal{L}$ is a metric graph structure on Γ. Then*

(1)
$$h_T(\mu) := \liminf_{\substack{|v|\to\infty \\ v\in\mathcal{P}(\Gamma)}} \frac{-\log\langle v,\mu\rangle_\alpha}{\mathcal{L}(v)} = \liminf_{\substack{\mathcal{L}(v)\to\infty \\ v\in\mathcal{P}(\Gamma)}} \frac{-\log\langle v,\mu\rangle_\alpha}{\mathcal{L}(v)};$$

(2) *if $h_T(\mu) > s \ge 0$ then there exists $C > 0$ such that, for every $v \in \mathcal{P}(\Gamma)$*
$$\langle v,\mu\rangle_\alpha \le C \ \exp(-s\mathcal{L}(v)).$$

We already noted that if $\mu = 0$, we get $h_T(\mu) = \infty$ for any $T \in cv(F)$. On the other hand, there is a family of so-called rational currents $\{\eta_g\}_{g\in F\setminus\{1\}}$, such that for any $T \in cv(F)$, $h_T(\eta_g) = 0$.

The current η_g is defined as follows: for every Borel subset $S \subseteq \partial F^2$, $\eta_g(S)$ is equal to the number of F-translates of the bi-infinite geodesic $(g^{-\infty}, g^{\infty})$ that belong to S.

The following statement is an immediate corollary of the definition of geometric entropy:

Proposition 3.3. *Let $\mu \in Curr(F), \mu \neq 0$ and let $T \in cv(F)$.*

(1) *For any $c > 0$ we have $h_T(\mu) = h_T(c\mu)$.*
(2) *For any $c > 0$ we have $h_{cT}(\mu) = \frac{1}{c} h_T(\mu)$.*

Thus $h_T(\mu)$ depends only on the projective class $[\mu] \in \mathbb{P}Curr(F)$ of μ.

Lemma 3.4. *Let $T, T' \in cv(F)$. Let $\phi : T \to T'$ be an F-equivariant quasi-isometry. There exists an integer $M = M(\phi) \ge 1$ with the following property.*

Let $x_1, x_2 \in T$ and let $x_1' = \phi(x_1), x_2' = \phi(x_2)$. Let $y_1, y_2 \in [x_1', x_2']$ be such that $d_{T'}(x_1', y_1) = d_{T'}(x_2', y_2) = M$. Then

$$\phi(Cyl_{[x_1,x_2]}) \subseteq Cyl_{[y_1,y_2]}.$$

Proof. This statement is a straightforward consequence of the "Bounded Cancellation Lemma" [**4**] for quasi-isometries between Gromov-hyperbolic spaces. Indeed, let $[x,y] \subseteq T$ be a geodesic segment in T and let γ be a geodesic ray in T with initial point y, such that the path $[x,y]\gamma$ is a geodesic (that is, there is no cancellation between $[x,y]$ and γ. There is a unique geodesic ray γ' in T' starting at $\phi(y)$ such that γ' is at a finite Hausdorff distance from $\phi(\gamma)$. The "Bounded Cancellation Lemma" implies that the cancellation between $[\phi(x), \phi(y)]$ and γ' is bounded by some constant $M \ge 1$ depending only on the quasi-isometry ϕ. It is not hard to check that this constant M satisfies the requirements of the proposition and we leave the details to the reader. □

Proposition 3.5. *Let $T, T' \in cv(F)$. Let $\phi : T \to T'$ be an F-equivariant quasi-isometry. There exists an integer $M_1 = M_1(\phi) \geq 1$ with the following property.*

Let $x_1, x_2 \in T$ and let $x_1' = \phi(x_1), x_2' = \phi(x_2)$.

Then there exist points $p_1, p_2, q_1, q_2 \in T'$ such that $[p_1, p_2] = [x_1', x_2'] \cap [q_1, q_2]$, such that $d(q_i, x_i') \leq M_1$ and such that

$$Cyl_{[q_1,q_2]} \subseteq \phi(Cyl_{[x_1,x_2]}).$$

Proof. Let $\psi : T' \to T$ be an F-equivariant quasi-isometry which is a quasi-inverse of ϕ. Let $M = M(\psi) > 0$ be the constant provided by Lemma 3.4 for ψ.

Choose $\xi, \zeta \in \partial T$ such that the bi-infinite geodesic $[x_1, x_2] \subseteq [\xi, \zeta]$. Then $x_1' = \phi(x_1)$ and $x_2' = \phi(x_2)$ are at distance at most $C_1 = C_1(\phi)$ from $[\phi(\xi), \phi(\zeta)]$. Thus $[\phi(\xi), \phi(\zeta)] \cap [x_1', x_2'] = [p_1, p_2]$, where $d(x_i', p_i) \leq C_1$.

Recall that ψ is a quasi-isometry that is a quasi-inverse of ϕ, so that $\psi(\phi(\xi)) = \xi$ and $\psi(\phi(\zeta)) = \zeta$. Note also that $[x_1, x_2] \subseteq [\xi, \zeta] = [\psi(\phi(\xi)), \psi(\phi(\zeta))]$ and that $[p_1, p_2] \subseteq [\phi(\xi), \phi(\zeta)]$. Therefore there is some $C_2 = C_2(\phi) > 0$ such that if $q_1 \in [\phi(\xi), p_1]$, $q_2 \in [p_2, \phi(\zeta)]$ are such that $d(q_i, p_i) \geq C_2$ then $[x_1, x_2] \subseteq [\psi(q_1), \psi(q_2)]$ and $d(x_i, \psi(q_i)) \geq M = M(\psi)$.

Choose $q_1 \in [\phi(\xi), p_1]$, $q_2 \in [p_2, \phi(\zeta)]$ such that $d(q_i, p_i) = C_2$. Thus $[x_1, x_2] \subseteq [\psi(q_1), \psi(q_2)]$ and $d(x_i, \psi(q_i)) \geq M$. Note that $d(x_i, \psi(q_i)) \leq C_3 = C_3(\psi)$ since ψ is a quasi-isometry and $d(x_i', q_i) \leq C_2 + C_1$.

Thus $[x_1, x_2] \subseteq [\psi(q_1), \psi(q_2)]$ and $d(x_i, \psi(q_i)) \geq M = M(\psi)$. Then by Lemma 3.4, applied to ψ, we have

$$\psi(Cyl_{[q_1,q_2]}) \subseteq Cyl_{[x_1,x_2]}.$$

Applying ϕ, we obtain

$$Cyl_{[q_1,q_2]} \subseteq \phi(Cyl_{[x_1,x_2]}).$$

Note that by construction $[p_1, p_2] = [x_1', x_2'] \cap [q_1, q_2]$. Moreover, $d(q_i, x_i') \leq C_1 + C_2$. Thus all the requirements of the proposition are satisfied, which completes the proof. $\square$

Corollary 3.6. *Let $\mu \in Curr(F)$ and let $T, T' \in cv(F)$. Then*

$$h_T(\mu) > 0 \iff h_{T'}(\mu) > 0.$$

Proof. Suppose that $h_{T'}(\mu) > 0$. Hence there exists $s > 0$, $C > 0$ such that for every $x, y \in T'$, $x \neq y$, we have

$$\mu(Cyl_{[x,y]}) \leq C \exp(-s\, d_{T'}(x, y)).$$

Let $\phi : T \to T'$ be a F-equivariant (λ, λ)-quasi-isometry, where $\lambda \geq 1$ (i.e. a quasi-isometry with both the multiplicative and the additive constants equal to

λ). Let $M, C > 0$ be provided by Lemma 3.4. Let $x_1, x_2 \in T$ be such that $N := d_T(x_1, x_2) > 20\lambda^2 M$. Let $x_i' = \phi(x_i) \in T'$, $i = 1, 2$. Thus $d_{T'}(x_1', x_2') \geq N/\lambda - \lambda \geq N/2\lambda$. Let $y_1, y_2 \in [x_1', x_2']$ be such that $d_{T'}(x_1', y_1) = d_{X'}(x_2', y_2) = M$. Thus

$$d_{T'}(y_1, y_2) \geq \frac{N}{2\lambda} - 2M \geq \frac{N}{3\lambda} = \frac{d_T(x_1, x_2)}{3\lambda}.$$

By Lemma 3.4 we have:

$$\phi(Cyl_{[x_1,x_2]}) \subseteq Cyl_{[y_1,y_2]}$$

Recall that under the identifications $\partial q_T : \partial T \to \partial F$ and $\partial q_{T'} : \partial T' \to \partial F$, for any $S \subseteq \partial T$, $\partial q_T(S) = \partial q_{T'}(\phi(S)) \subseteq \partial F$. Therefore

$$\mu(Cyl_{[x_1,x_2]}) \le \mu(Cyl_{[y_1,y_2]}) \le C\exp(-s d_{T'}(y_1,y_2)) \le$$
$$\le C\exp(-s\frac{d_T(x_1,x_2)}{3\lambda}).$$

This implies that $h_T(\mu) \ge \frac{s}{3\lambda} > 0$, as required. □

Corollary 3.6 implies that the following notion is well-defined. We say that a current $\mu \in Curr(F)$ has *exponential decay* or *decays exponentially fast* if for some (equivalently, for any) $T \in cv(F)$ we have $h_T(\mu) > 0$. Similarly, we say that μ has *subexponential decay* if for some (equivalently, for any) $T \in cv(F)$ we have $h_T(\mu) = 0$.

Recall that for a graph Γ we denote by $\mathcal{P}(\Gamma)$ the set of all nontrivial reduced finite edge-paths in Γ.

Proposition 3.7. *Let $\mu \in Curr(F), \mu \neq 0$. Then the function*

$$E_\mu : cv(F) \to \mathbb{R}, \quad T \mapsto h_T(\mu)$$

is continuous.

Proof. It suffices to check that for every simplicial chart $\alpha : F \to \pi_1(\Gamma, p)$ on F the restriction of E_μ to the elementary chart $V_\alpha \subseteq cv(F)$ is continuous.

Let $E\Gamma = E^+\Gamma \sqcup E^-\Gamma$ be an orientation on Γ, let $m = \#E^+\Gamma$ and let $E^+\Gamma = \{e_1, \dots, e_m\}$. Recall that V_α consists of all points of the form $(\alpha, \mathcal{L})$, where $\mathcal{L}$ is a nondegenerate semi-metric structure on Γ.

Let $T = (\alpha, \mathcal{L}) \in U_\alpha$. Let $\epsilon > 0$ be arbitrary. Choose $\epsilon_1 > 0$ such that $\epsilon_1 h_T(\mu) < \epsilon$. Then there exists a neighborhood Ω of T in V_α with the following property. For any $v \in \mathcal{P}(\Gamma)$ and for any $T' = (\alpha, \mathcal{L}') \in \Omega$ we have

$$(1-\epsilon_1)\mathcal{L}'(v) \le \mathcal{L}(v) \le (1+\epsilon_1)\mathcal{L}'(v).$$

Therefore

$$\frac{-\log\langle v,\mu\rangle_\alpha}{\mathcal{L}(v)}(1-\epsilon_1) \le \frac{-\log\langle v,\mu\rangle_\alpha}{\mathcal{L}'(v)} \le \frac{-\log\langle v,\mu\rangle_\alpha}{\mathcal{L}(v)}(1+\epsilon_1).$$

It follows that

$$h_T(\mu)(1-\epsilon_1) \le h_{T'}(\mu) \le h_T(\mu)(1+\epsilon_1)$$

and hence

$$|E_\mu(T) - E_\mu(T')| = |h_{T'}(\mu) - h_T(\mu)| \le h_\mu(T)\epsilon_1 \le \epsilon.$$

Thus E_μ is continuous at the point T and, since $T \in V_\alpha$ was arbitrary, E_μ is continuous on V_α, as required. □

Notation 3.8. Let $\alpha : F \to \pi_1(\Gamma)$ be a simplicial chart and let $\mu \in Curr(F)$ be a geodesic current. Define

$$supp_\alpha(\mu) := \{v \in \mathcal{P}(\Gamma) : \langle v,\mu\rangle_\alpha > 0\}.$$

We refer to $supp_\alpha(\mu)$ as the *support of μ with respect to α*.

Let $\mathcal{L}$ be a metric graph structure on Γ and let $T_{(\alpha,\mathcal{L})} \in cv(F)$ correspond to the universal cover of Γ with the edge-length lifted from $\mathcal{L}$. Define

$$\rho(T_{(\alpha,\mathcal{L})}, \mu) := \liminf_{R\to\infty} \frac{\log\beta(R)}{R}$$

where $\beta(R) = \#\{v \in supp_\alpha(\mu) : \mathcal{L}(v) \le R\}$. Thus $\rho(T_{(\alpha,\mathcal{L})})$ measures the exponential growth rate of the support $supp_\alpha(\mu)$ with respect to the metric structure $\mathcal{L}$.

Let now $\mu \in Curr(F)$, $\mu \neq 0$ and let $T \in cv(F)$ be arbitrary. There exists an (essentially unique) marked metric graph structure $(\alpha, \mathcal{L})$ such that $T = T_{(\alpha,\mathcal{L})}$. Put $\rho(T, \mu) := \rho(T_{(\alpha,\mathcal{L})})$.

It is not hard to show that if $\rho(T_0, \mu) = 0$ for some $T_0 \in cv(F)$ then $\rho(T, \mu) = 0$ for every $T \in cv(F)$.

Theorem 3.9. *Let $T \in cv(F)$ and $\mu \in Curr(F)$, $\mu \neq 0$. Then*

$$h_T(\mu) \le \rho(T, \mu)$$

Proof. We may assume that $T = T_{(\alpha,\mathcal{L})}$ for some marked metric graph structure $(\alpha : F \to \pi_1(\Gamma), \mathcal{L})$ on F. If $h_T(\mu) = 0$, there is nothing to prove. Suppose now that $h_T(\mu) > 0$. Let $s > 0$ be arbitrary such that $s < h_T(\mu)$. Thus there exists $n_0 \ge 1$ such that for any $p, q \in T$ with $d_T(p, q) \ge n_0$ we have $\mu(Cyl_{[p,q]}) \le \exp(-s d_T(p, q))$.

Since $\mu \neq 0$, there exists an edge e of Γ such that $\langle e, \mu\rangle_\alpha > 0$, that is $e \in supp_\alpha(\mu)$. Let $[x, y] \subseteq T$ be a lift of e to T. Thus $\mu(Cyl_{[x,y]}) = \langle e, \mu\rangle_\alpha > 0$. For any integer $n \ge n_0$ let $[x, z_1], \dots [x, z_m]$ be all geodesic segments of length n in T that contain $[x, y]$ as an initial segment and such that $\mu(Cyl_{[x,p_i]}) > 0$, where $p_i = z_i$ if z_i is a vertex of T, whereas if z_i is an interior point of an edge, p_i is the endpoint of that edge further away from x than z_i. Then $\mu(Cyl_{[x,p_i]}) \le \exp(-sn)$. Moreover, the cylinders $Cyl_{[x,p_1]}, \dots, Cyl_{[x,p_m]}$ are disjoint and $\mu(Cyl_{[x,y]}) = \sum_{i=1}^m \mu(Cyl_{[x,p_i]})$. Let $v_i \in \mathcal{P}(\Gamma)$ be the path in Γ which labels the segment $[x, p_i]$ in T. Note that by construction $n \le \mathcal{L}(v_i) \le n + c$, where $c > 0$ is the length of the longest edge in $(\Gamma, \mathcal{L})$. Thus, again, by construction, $m \le \beta(n + c)$. We have

$$0 < \langle e, \mu\rangle_\alpha = \sum_{i=1}^m \langle v_i, \mu\rangle_\alpha \le m \exp(-sn) \le \beta(n + c) \exp(-sn).$$

It follows that $\rho(T, \mu) \ge s$ since if $\rho(T, \mu) < s$ then $\beta(n + c) \exp(-sn) \to 0$ as $n \to \infty$, contradicting the fact that $\langle e, \mu\rangle_\alpha > 0$. Thus for every $0 < s < h_T(\mu)$ we have $\rho(T, \mu) \ge s$. Therefore $\rho(T, \mu) \ge h_T(\mu)$, as required. □

Remark 3.10. Suppose $T = T_{(\alpha,\mathcal{L})}$ where for every edge e of Γ we have $\mathcal{L}(e) = 1$. Let $\Omega(\Gamma, \mu)$ be the set of all semi-infinite reduced edge paths $\gamma = e_1, \dots, e_n, \dots$ in Γ such that for every finite subpath v of γ we have $v \in supp_\alpha(\mu)$. We can think of $\Omega(\Gamma, \mu)$ as a subset of the set A^ω of all semi-infinite words in the alphabet A consisting of all oriented edges of Γ. The subset $\Omega(\Gamma, \mu) \subseteq A^\omega$ is clearly invariant under the shift map $\sigma : A^\omega \to A^\omega$ which erases the first letter of a semi-infinite word from A^ω. Thus $\Omega(\Gamma, \mu)$ is a subshift (not necessarily of finite type) of the full shift (A^ω, σ). It is easy to see from the definitions that in this case $\rho(T_{(\alpha,\mathcal{L})})$ is equal to the topological entropy $h_{topol}(\Omega(\Gamma, \mu))$ of the subshift $\Omega(\Gamma, \mu)$ of the set A^ω. In this situation Theorem 3.9 says that $h_T(\mu) \le h_{topol}(\Omega(\Gamma, \mu))$.

As we already observed, it is not hard to show that if $\rho(T_0, \mu) = 0$ for some $T_0 \in cv(F)$ then $\rho(T, \mu) = 0$ for every $T \in cv(F)$. In this case Theorem 3.9 implies $h_T(\mu) = 0$ for every $T \in cv(F)$.

4. Tame currents

Definition 4.1 (Tame current with respect to a tree)**.** Let $\mu \in Curr(F)$ and $T \in cv(F)$. We say that μ is *tame with respect to* T or T*-tame* if for every $M \geq 1$ there is $C = C(M) \geq 1$ such that whenever $a_1, b_1, a_2, b_2 \in T$ satisfy $d(a_1, a_2) \leq M, d(b_1, b_2) \leq M$, $a_1 \neq b_1$, $a_2 \neq b_2$ then

$$\frac{1}{C}\mu(Cyl_{[a_2,b_2]}) \leq \mu(Cyl_{[a_1,b_1]}) \leq C\mu(Cyl_{[a_2,b_2]}).$$

We call $C = C(M)$ the T*-tameness constant* corresponding to M.

Lemma 4.2. *Let $T \in cv(F)$ and $\mu \in Curr(F)$. Suppose that there is some $N \geq 1$ such that for every $M \geq N$ there exists $D = D(M) \geq 1$ such that whenever $[x, y] \subseteq [a, b]$ where $x \neq y$ and $d_T(a, x) = d_T(y, b) = M$ then*

$$\mu(Cyl_{[x,y]}) \leq D\mu(Cyl_{[a,b]}).$$

Then μ is T-tame.

Proof. Suppose that for every $M \geq N$ there is $D = D(M) \geq 1$ as in Lemma 4.2.

We need to prove that μ is T-tame. It is easy to see that it suffices to verify the conditions of Definition 4.1 for all sufficiently large M.

Let $M \geq N$ be arbitrary. Suppose now that $[x, y] \subseteq [a, b]$ with $d_T(a, x) = d_T(y, b) \leq M$ and $x \neq y$. Choose a geodesic segment $[a', b']$ in T such that $[a, b] \subseteq [a', b']$ and such that $d_T(a', x) = d_T(b', y) = M$. Then $Cyl_{[a',b']} \subseteq Cyl_{[a,b]} \subseteq Cyl_{[x,y]}$. Hence by assumption on $D = D(M)$ we have

$$\mu(Cyl_{[a,b]}) \leq \mu(Cyl_{[x,y]}) \leq D\mu(Cyl_{[a',b']}) \leq D\mu(Cyl_{[a,b]})$$

so that

$$\mu(Cyl_{[a,b]}) \leq \mu(Cyl_{[x,y]}) \leq D\mu(Cyl_{[a,b]}). \tag{†}$$

Thus (†) holds whenever $[x, y] \subseteq [a, b]$ with $d_T(a, x) = d_T(y, b) \leq M$.

Suppose now that $M \geq N$ and $a_1, b_1, a_2, b_2 \in T$ satisfy $d(a_1, a_2) \leq M, d(b_1, b_2) \leq M$. We may assume that $d(a_1, b_1) \geq 3M$ since otherwise the requirements of Definition 4.1 are easily satisfied. (Indeed, if $d(a_1, b_1) \leq 3M$, then the points a_1, a_2, b_1, b_2 lie in an F-translated of a fixed closed ball of radius $6M$, which is a finite subtree of T). Then $[a_1, b_1] \cap [a_2, b_2]$ is a non-degenerate geodesic segment. Put $[x, y] = [a_1, b_1] \cap [a_2, b_2]$.

Then

$$d(x, a_1), d(x, a_2), d(y, b_1), d(y, b_2) \leq M$$

and $[x, y] \subseteq [a_1, b_1]$, $[x, y] \subseteq [a_2, b_2]$. Then by (†) we have

$$\mu(Cyl_{[a_1,b_1]}) \leq \mu(Cyl_{[x,y]}) \leq D\mu(Cyl_{[a_2,b_2]})$$

and

$$\mu(Cyl_{[a_2,b_2]}) \leq \mu(Cyl_{[x,y]}) \leq D\mu(Cyl_{[a_1,b_1]}).$$

Therefore μ is T-tame with as required. □

Proposition 4.3. *Let $\mu \in Curr(F)$ and let $T, T' \in cv(F)$. Then μ is tame with respect to T if and only if μ is tame with respect to T'.*

Proposition 4.3 implies that the following notion is well-defined and does not depend on the choice of $T \in cv(F)$:

Definition 4.4 (Tame current)**.** Let $\mu \in Curr(F)$. We say that μ is *tame* if for some (equivalently, for any) $T \in cv(F)$ the current μ is T-tame.

Proof. Suppose that μ is tame with respect to T'.

Let $x \in T$ and $x' \in T'$ be arbitrary vertices. Let $\phi : T \to T'$ be an F-equivariant (λ, λ)-quasi-isometry such that $\phi(x) = x'$. Let $M = M(\phi) \geq 1$ be the constant provided by Lemma 3.4.

We need to prove that μ is tame with respect to T. By Lemma 4.2 it suffices to show that the conditions of Lemma 4.2 hold for μ.

Let $M_0 \geq 1$ be sufficiently big (to be specified later) and suppose $s, t, a, b \in T$ are such that $[s, t] \subseteq [a, b]$ with $d_T(s, a) = d_T(t, b) = M_0$.

Let $s', t' \in [\phi(a), \phi(b)]$ be such that

$$d_{T'}(\phi(s), s') = d_{T'}(\phi(s), [\phi(a), \phi(b)]) \quad \text{and}$$
$$d_{T'}(\phi(t), t') = d_{T'}(\phi(t), [\phi(a), \phi(b)]).$$

Since ϕ is a quasi-isometry and T, T' are Gromov-hyperbolic, we have

$$d_{T'}(\phi(s), s'), d_{T'}(\phi(t), t') \leq C_1,$$

where $C_1 = C_1(\phi) > 0$ is some constant. Note that $[\phi(a), \phi(b)] \cap [\phi(s), \phi(t)] = [s', t']$. We may assume that M_0 was chosen big enough so that

$$d_{T'}(\phi(a), s'), d_{T'}(\phi(b), t') \geq M_1 = M_1(\psi)$$

where $M_1 = M_1(\psi)$ is the constant provided by Proposition 3.5.

Proposition 3.5 implies that there exist $p_1, p_2, q_1, q_2 \in T'$ such that $[p_1, p_2] = [q_1, q_2] \cap [\phi(a), \phi(b)]$ and such that

$$Cyl_{[q_1,q_2]} \subseteq \phi(Cyl_{[a,b]})$$

and such that $d_{T'}(q_1, \phi(a)), d_{T'}(q_2, \phi(b)) \leq M_1$. Thus $d_{T'}(\phi(a), s'), d_{T'}(\phi(b), t') \geq M_1 = M_1(\psi)$ implies that $[s', t'] \subseteq [p_1, p_2]$.

Let $s'', t'' \in [s', t'] \subseteq [\phi(a), \phi(b)]$ be such that $d_{T'}(s', s'') = d_{T'}(t', t'') = M$. Thus $s'', t'' \in [\phi(s), \phi(t)]$ and

$$M \leq d_{T'}(\phi(s), s'') \leq M + C_1, \quad M \leq d_{T'}(\phi(t), t'') \leq M + C_1.$$

Then by Lemma 3.4

$$\phi(Cyl_{[s,t]}) \subseteq Cyl_{[s'',t'']}$$

and hence

$$\mu(Cyl_{[s,t]}) \leq \mu(Cyl_{[s'',t'']}).$$

Note that since ϕ is a (λ, λ)-quasi-isometry and $d_T(a, s) = M_0$, $d_T(b, t) = M_0$ then

$$d_{T'}(\phi(a), \phi(s)), d_{T'}(\phi(b), \phi(t)) \leq \lambda M_0 + \lambda.$$

Since $d_{T'}(\phi(s), s'), d_{T'}(\phi(t), t') \leq C_1$, it follows that

$$d_{T'}(\phi(a), s'), d_{T'}(\phi(b), t') \leq \lambda M_0 + \lambda + C_1.$$

Since $d_{T'}(s', s'') = d_{T'}(t', t'') = M$, we have

$$d(\phi(a), s''), d(\phi(b), t'') \leq M + \lambda M_0 + \lambda + C_1.$$

Since $p_1 \in [\phi(a), s'']$, $p_2 \in [t'', \phi(b)]$ and $d_{T'}(q_1, \phi(a)), d_{T'}(q_2, \phi(b)) \leq M_1$, we get

$$d_{T'}(s'', q_1), d_{T'}(t'', q_2) \leq M_1 + M + \lambda M_0 + \lambda + C_1.$$

Put $M_2 = M_1 + M + \lambda M_0 + \lambda + C_1$. Since μ is T'-tame,

$$\mu(Cyl_{[s'',t'']}) \le C_2 \mu(Cyl_{[q_1,q_2]}),$$

where $C_2 = C(M_2)$ is the T'-tameness constant for μ corresponding to M_2.

Recall that $Cyl_{[q_1,q_2]} \subseteq \phi(Cyl_{[a,b]})$ and therefore

$$\mu(Cyl_{[q_1,q_2]}) \le \mu(Cyl_{[a,b]}).$$

Thus we have

$$\mu(Cyl_{[s,t]}) \le \mu(Cyl_{[s'',t'']}) \le C_2\mu(Cyl_{[q_1,q_2]}) \le \\ \le C_2 \mu(Cyl_{[a,b]}).$$

Hence by Lemma 4.2 μ is T-tame, as required.

□

5. The geometric entropy function on the Outer Space

Theorem 5.1. *Let $T \in cv(F)$ and let $\mu \in Curr(F)$ be a tame current. Then for any $T' \in cv(F)$ we have:*

$$h_{T'}(\mu) \ge h_T(\mu) \inf_{f \in F\setminus\{1\}} \frac{||f||_T}{||f||_{T'}}.$$

Proof. Put $h = h_T(\mu)$.

Let $x \in T$ and $x' \in T'$ be arbitrary vertices. Let $\phi : T' \to T$ be an F-equivariant quasi-isometry such that $\phi(x') = x$. Thus $\phi(gx') = gx$ for every $g \in F$. Let $M \ge 1$ be provided by Lemma 3.4.

Let $a_n, b_n \in T'$ be such that $\lim_{n\to\infty} d_{T'}(a_n, b_n) = \infty$ and such that

$$h_{T'}(\mu) = \lim_{n\to\infty} \frac{-\log \mu(Cyl_{[a_n,b_n]})}{d_{T'}(a_n, b_n)}.$$

Note that there exists some constant $M' = M'(T') \ge 1$ (it can be taken equal to $vol(\mathcal{L}'))$, such that for any finite reduced edge-path v in T'/F there exists a cyclically reduced closed edge-path $\widehat{v}$ in T'/F containing v or contained in v (as a subpath) and such that $|\mathcal{L}'(v) - \mathcal{L}'(\widehat{v})| \le M'$.

Then there exists $h_n, g_n \in F$ such that $d_{T'}(a_n, h_n x') \le M'$, $d_{T'}(b_n, g_n x') \le M'$ and such that $[h_n x', g_n x']$ projects to a closed cyclically reduced path in T'/F.

After translating $[a_n, b_n]$ by h_n^{-1}, we may assume that $h_n = 1$. Thus $[x', g_n x']$ is contained in the axis of g_n and $d_{T'}(x', g_n x') = ||g_n||_{T'}$. Note that $\lim\limits_{n\to\infty} d_{T'}(a_n, b_n) = \infty$ implies $\lim\limits_{n\to\infty} ||g_n||_{T'} = \infty$.

Since μ is tame, there exists a constant $C_1 \ge 1$ such that

$$\mu(Cyl_{[a_n,b_n]}) \le C_1 \mu(Cyl_{[x',g_n x']}).$$

Therefore

$$\frac{-\log \mu(Cyl_{[a_n,b_n]})}{d_{T'}(a_n, b_n)} \ge \frac{-\log \mu(Cyl_{[x',g_n x']}) - \log C_1}{d_{T'}(x', g_n x') + 2M'}.$$

and

$$h_{T'}(\mu) \ge \liminf_{n\to\infty} \frac{-\log \mu(Cyl_{[x',g_n x']})}{d_{T'}(x', g_n x')}.$$

Note that $\phi(x') = x$ and $\phi(g_n x') = g_n x$. Moreover, since $d_{T'}(x', g_n x') = ||g_n||_{T'}$ and ϕ is a quasi-isometry, there is a constant $C_2 > 0$ independent of n such that for every $n \geq 1$

$$|d_T(x, g_n x) - ||g_n||_T| \leq C_2.$$

Also, we have $\lim_{n\to\infty} ||g_n||_T = \infty$. Let $[y_n, z_n] \subseteq [x, g_n x]$ be such that $d_T(x, y_n) = d_T(g_n x, z_n) = M$, where M is a constant provided by Lemma 3.4, with

$$\phi\left(Cyl_{[x', g_n x']}\right) \subseteq Cyl_{[y_n, z_n]}$$

and hence

$$\mu(Cyl_{[x', g_n x']}) \leq \mu(Cyl_{[y_n, z_n]}) \leq C_3 \mu(Cyl_{[x, g_n x]}),$$

where the constant $C_3 \geq 1$ exists and the last inequality holds since μ is tame.

Thus

$$\frac{-\log \mu(Cyl_{[x', g_n x']})}{d_{T'}(x', g_n x')} \geq \frac{-\log \mu(Cyl_{[x, g_n x]}) - \log C_2}{d_{T'}(x', g_n x')} =$$
$$\frac{-\log \mu(Cyl_{[x, g_n x]}) - \log C_2}{d_T(x, g_n x)} \cdot \frac{d_T(x, g_n x)}{d_{T'}(x', g_n x')} =$$
$$\frac{-\log \mu(Cyl_{[x, g_n x]}) - \log C_2}{d_T(x, g_n x)} \cdot \frac{d_T(x, g_n x)}{||g_n||_{T'}} \geq$$
$$\frac{-\log \mu(Cyl_{[x, g_n x]}) - \log C_2}{d_T(x, g_n x)} \cdot \frac{||g_n||_T - C_3}{||g_n||_{T'}}$$

Therefore

$$h_{T'}(\mu) \geq \liminf_{n\to\infty} \frac{-\log \mu(Cyl_{[x', g_n x']})}{d_{T'}(x', g_n x')} \geq$$
$$\liminf_{n\to\infty} \frac{-\log \mu(Cyl_{[x, g_n x]}) - \log C_2}{d_T(x, g_n x)} \cdot \frac{||g_n||_T - C_3}{||g_n||_{T'}} \geq$$
$$\geq h_T(\mu) \liminf_{n\to\infty} \frac{||g_n||_T - C_3}{||g_n||_{T'}} = h_T(\mu) \liminf_{n\to\infty} \frac{||g_n||_T}{||g_n||_{T'}} \geq$$
$$\geq h_T(\mu) \inf_{g \in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}},$$

as required. □

The following statement is an immediate corollary of the explicit formula for the Patterson-Sullivan current in the Outer Space context obtained by Kapovich and Nagnibeda ([**18**], see Proposition 5.3):

Proposition 5.2. *Let $T \in cv(F)$ and let $h = h(T)$ be the critical exponent of T. Let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current corresponding to T. Then there exist constants $C_1 > C_2 > 0$ such that for any distinct vertices x and y of T we have*

$$C_2 \exp(-h\, d_T(x, y)) \leq \mu_T(Cyl_{[x,y]}) \leq C_1 \exp(-h\, d_T(x, y)).$$

Together with the definitions of geometric entropy and of tameness, Proposition 5.2 immediately implies:

Corollary 5.3. *Let $T \in cv(F)$ and let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current corresponding to T. Let $h = h(T)$ be the critical exponent of T.*

Then μ_T is tame and $h_T(\mu_T) = h(T)$.

Proposition 5.4. *Let $T \in cv(F)$ and let $h = h(T)$ be the critical exponent of T. Let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current corresponding to T.*

Let $T' \in cv(F)$. Then for any $g \in F \setminus \{1\}$ we have

$$h_{T'}(\mu_T) \le h \frac{||g||_T}{||g||_{T'}}$$

and therefore

$$h_{T'}(\mu_T) \le h \inf_{f \in F \setminus \{1\}} \frac{||f||_T}{||f||_{T'}}.$$

Proof. By Proposition 5.2 there exists $C > 0$ such that for any distinct vertices $x, y \in T$ we have

$$\mu(Cyl_{[x,y]}) \ge C \exp(-h d_T(x,y)).$$

Note that for any $x \in T, x' \in T'$ we have

$$||g||_T = \lim_{n\to\infty} \frac{d_T(x, g^n x)}{n}, \quad ||g||_{T'} = \lim_{n\to\infty} \frac{d_{T'}(x', g^n x')}{n}$$

and hence

$$\lim_{n\to\infty} \frac{d_T(x, g^n x)}{d_{T'}(x', g^n x')} = \frac{||g||_T}{||g||_{T'}}.$$

Let $x \in T$ and $x' \in T'$ be arbitrary vertices. Let $\phi : T \to T'$ be an F-equivariant quasi-isometry such that $\phi(x) = x'$. Thus $\phi(g^n x) = g^n x'$ for every $n \in \mathbb{Z}$. Let $M \ge 1$ be provided by Lemma 3.4. For $n \to \infty$ let $[y_n, z_n] \subseteq [x', g^n x']$ be such that $d(x', y_n) = d(z_n, g^n x') = M$. Then by Lemma 3.4 we have

$$\phi(Cyl_{[x,g^n x]}) \subseteq Cyl_{[y_n,z_n]}.$$

Hence

$$\mu(Cyl_{[y_n,z_n]}) \ge \mu(Cyl_{[x,g^n x]}) \ge C \exp(-h d_T(x, g^n x))$$

and so

$$\frac{-\log\left(\mu(Cyl_{[y_n,z_n]})\right)}{d_{T'}(y_n, z_n)} \le \frac{h d_T(x, g^n x) - \log C}{d_{T'}(y_n, z_n)} \le \frac{h d_T(x, g^n x) - \log C}{d_{T'}(x', g^n x') - 2M}.$$

Hence

$$h_{T'}(\mu) \le \liminf_{n\to\infty} \frac{h d_T(x, g^n x) - \log C}{d_{T'}(x', g^n x') - 2M} = h \frac{||g||_T}{||g||_{T'}},$$

as required. $\square$

Since Patterson-Sullivan currents are tame, Proposition 5.4 and Theorem 5.1 imply

Theorem 5.5. *Let $T \in cv(F)$ and let $h = h(T)$ be the critical exponent of T. Let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current for T. Let $T' \in cv(F)$. Then*

$$h_{T'}(\mu_T) = h(T) \inf_{f \in F \setminus \{1\}} \frac{||f||_T}{||f||_{T'}}.$$

Corollary 5.6. *Let $T \in CV(F)$ and let $h = h(T)$ be the critical exponent of T. Let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current for T. Let $T' \in CV(F)$ be such that $T' \ne T$. Then*

$$h_{T'}(\mu_T) < h_T(\mu_T) = h(T).$$

Thus

$$h(T) = h_T(\mu_T) = \max_{T' \in CV(T)} h_{T'}(\mu_T)$$

and this maximum is strict.

PROOF. A result of Tad White [**24**] implies that, when $T' \neq T$, there exists a nontrivial $g \in F$ such that $||g||_T < ||g||_{T'}$. Therefore by Theorem 5.5

$$h_{T'}(\mu_T) = h(T)\frac{||g||_T}{||g||_{T'}} < h(T).$$

□

Corollary 5.7. *Let $T \in CV(F)$ and let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current for T. Then*

$$\inf_{T' \in CV(F)} h_{T'}(\mu_T) = 0.$$

PROOF. Recall that F is a free group of rank $k \geq 2$. Let A be a free basis of F and let T_A be the Cayley graph of F with respect to A, where every edge has length $1/k$. Thus $T_A \in CV(F)$. Let $a \in A$. There exists a sequence $\phi_n \in Out(F)$ such that $\lim_{n\to\infty} ||\phi_n a||_A = \infty$ and hence $\lim_{n\to\infty} ||\phi_n a||_{T_A} = \frac{1}{k}\lim_{n\to\infty} ||\phi_n a||_A = \infty$. Put $T_n = \phi_n^{-1} T_A$. Thus $T_n \in CV(F)$ and

$$||a||_{T_n} = ||a||_{\phi_n^{-1} T_A} = ||\phi_n a||_{T_A} \longrightarrow_{n\to\infty} \infty.$$

Therefore by Theorem 5.5 we have

$$h_{T_n}(\mu) \leq h(T)\frac{||a||_T}{||a||_{T_n}} \longrightarrow_{n\to\infty} 0.$$

Hence

$$\inf_{T' \in CV(F)} h_{T'}(\mu) = 0,$$

as required.

□

6. The maximal geometric entropy problem for a fixed tree

Recall that, as observed in Introduction, the function $h_T(\cdot) : Curr(F) \to \mathbb{R}, \mu \mapsto h_T(\mu)$ is not continuous. Nevertheless, it turns out that it is possible to find the maximal value of $h_T(\cdot)$ on $Curr(F) - \{0\}$.

Recall that if (X, d) is a metric space and ν is a positive measure on X, then the *Hausdorff dimension* $\mathbf{HD}_X(\nu)$ *of* ν *with respect to* X is defined as

$$\mathbf{HD}_X(\nu) := \inf\{\mathbf{HD}(S) : S \subseteq X \text{ such that } \nu(X - S) = 0\}.$$

Thus $\mathbf{HD}_X(\nu)$ is the smallest Hausdorff dimension of a subset of X of full ν-measure. Note that this obviously implies that $\mathbf{HD}_X(\nu) \leq \mathbf{HD}(X)$.

Let $T \in cv(F)$. Recall that if $x \in T$, $\xi, \zeta \in \partial T$, we denote by $(\xi|\zeta)_x$ the distance $d_T(x, y)$ where $[x, \xi] \cap [x, \zeta] = [x, y]$. Let $x \in T$ be a base-point. The boundary ∂T is metrized by setting $d_x(\xi, \zeta) = \exp(-(\xi|\zeta)_x)$ for $\xi, \zeta \in \partial T$. It is well-known (see [**5**]) that $\mathbf{HD}(\partial T, d_x) = h(T)$.

Recall that, as explained in the Introduction, given a current $\mu \in Curr(F)$ and $T \in cv(F)$, we introduce a family of measures $\{\mu_x\}_{x\in T}$ on ∂F defined by their values on all the one-sided cylinder subsets of ∂F:

$$Cyl^x_{[x,y]} := \{\xi \in \partial F : \text{ the geodesic ray } [x, \partial_T(\xi)] \text{ in } T \text{ begins with } [x, y]\} \subseteq \partial F,$$

$$\mu_x(Cyl^x_{[x,y]}) := \mu(Cyl_{[x,y]}).$$

It is not hard to see that if $\mu \neq 0$ then there exists $x \in T$ such that $\mu_x \neq 0$.

Theorem 6.1. *Let $\mu \in Curr(F)$, $\mu \neq 0$, let $T \in cv(F)$, and let $x \in T$ be such that $\mu_x \neq 0$. Then*

$$h_T(\mu) \le \mathbf{HD}_{\partial T}(\mu_x) \le h(T).$$

PROOF. As observed by Kaimanovich in [**10**] (see formula (1.3.3)), the following formula holds for the Hausdorff dimension of a measure ν on ∂T endowed with the metric d_x as above.

$$(\ddagger) \qquad \mathbf{HD}_{\partial T}(\nu) = \text{ess sup}_{\xi\in\partial T} \liminf_{k\to\infty} \frac{-\log \nu\left(B_x(\xi,k)\right)}{k}$$

Here $B_x(\xi,k)$ is the set of all $\zeta \in \partial T$ such that $(\xi|\zeta)_x \ge k$, that is $B_x(\xi,k) = Cyl^x_{[x,y_k]}$ where $[x,y_k]$ is the initial segment of $[x,\xi]$ of length k. The essential supremum in ($\ddagger$) is taken with respect to ν.

Applied to μ_x, formula ($\ddagger$) yields:

$$\begin{aligned} h_T(\mu) &= \liminf_{d_T(y,z)\to\infty} \frac{-\log\mu(Cyl_{[y,z]})}{d_T(y,z)} \le \\ &\le \liminf_{\substack{z\in T\\ d_T(x,z)\to\infty}} \frac{-\log\mu(Cyl_{[x,z]})}{d_T(x,z)} \le \\ &\le \text{ess sup}_{\xi\in\partial T} \liminf_{\substack{z\in T\\ d_T(x,z)\to\infty}} \frac{-\log\mu_x(Cyl^x_{[x,z]})}{d_T(x,z)} = \\ &= \mathbf{HD}_{\partial T}(\mu_x) \le \mathbf{HD}(\partial T) = h(T). \end{aligned}$$

□

Proposition 6.2. *Let $T, T' \in cv(F)$ be such that $h := h(T) = h(T')$. Let μ_T be a Patterson-Sullivan current corresponding to T and suppose that $h_{T'}(\mu_T) = h$. Then T and T' are in the same projective class.*

PROOF. Let $\mu_{T'} \in Curr(F)$ be a Patterson-Sullivan current corresponding to T'. We will first show that μ_T is absolutely continuous with respect to $\mu_{T'}$. Let $\phi : T \to T'$ be an F-equivariant (λ,λ)-quasi-isometry, where $\lambda \ge 1$. By Proposition 5.2 there is a constant $C \ge 1$ such that for any $x, y \in T$ with $d_T(x,y) \ge 1$

$$\frac{1}{C}\exp(-hd_T(x,y)) \le \mu_T Cyl_{[x,y]} \le C\exp(-hd_T(x,y)).$$

Let $x', y' \in \phi(T')$ be arbitrary such that $d_{T'}(x',y') \ge \lambda^2 + \lambda$. Let $x, y \in T$ be such that $x' = \phi(x)$, $y' = \phi(y)$. Since μ_T is tame, Lemma 3.4 and Proposition 3.5 imply that there is some constant $C_1 \ge 1$ such that

$$\frac{1}{C_1}\mu_T(Cyl_{[x',y']}) \le \mu_T(Cyl_{[x,y]}) \le C_1\mu_T(Cyl_{[x',y']}).$$

Thus

$$\mu_T(Cyl_{[x',y']}) \ge \frac{1}{C_1}\mu_T(Cyl_{[x,y]}) \ge \frac{1}{C_1 C}\exp(-hd_T(x,y)).$$

On the other hand, since by assumption $h_{T'}(\mu_T) = h$, it follows that for any $\epsilon > 0$ there exists $C_2 = C_2(\epsilon) \ge 1$ such that

$$\mu_T(Cyl_{[x',y']}) \le C_2\exp(-(h-\epsilon)d_{T'}(x',y')).$$

Thus
$$\frac{1}{C_1C}\exp(-hd_T(x,y)) \le C_2\exp(-(h-\epsilon)d_{T'}(x',y')).$$
Hence
$$hd_T(x,y) \ge (h-\epsilon)d_{T'}(x',y') - \log(C_2C_1C).$$
and so
$$\frac{h}{h-\epsilon} \ge \limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(x',y')}{d_T(x,y)}.$$
Since $\epsilon > 0$ was arbitrary, it follows that
$$\limsup_{d_T(x,y)\to\infty} \frac{d_{T'}(x',y')}{d_T(x,y)} \le 1.$$
Therefore
$$\sup_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} \le 1.$$
By Lemma 2.9 this implies that there is a constant $C_3 \ge 1$ such that
$$d_T(x',y') \le d_T(x,y) + C_3.$$
Then
$$\mu_T(Cyl_{[x',y']}) \le \frac{1}{C_1C}\exp(-hd_T(x,y)) \le$$
$$\frac{1}{C_1C}\exp(-h(d_{T'}(x',y') - C_3)) = \frac{\exp(hC_3)}{C_1C}\exp(-hd_{T'}(x',y')) \le$$
$$\frac{C'\exp(hC_3)}{C_1C}\mu_{T'}(Cyl_{[x',y']}).$$
The above inequality holds for any $x', y' \in \phi(T)$ with $d_{T'}(x',y') \ge \lambda^2 + \lambda$. Since ϕ is a quasi-isometry and μ_T is tame, it follows that there exists a constant $C' \ge 1$ such that for any $x', y' \in T'$ with $d_{T'}(x',y') \ge 1$ we have
$$\mu_T(Cyl_{[x',y']}) \le C'\mu_{T'}(Cyl_{[x',y']}).$$
Hence μ_T is absolutely continuous with respect to $\mu_{T'}$. A result of Furman [**9**] now implies that the translation length functions $||.||_T$ and $||.||_{T'}$ are scalar multiples of each other, as required. □

Corollary 6.3. *Let $T_1, T_2 \in cv(F)$ be such two elements that do not lie in the same projective class. Let μ_{T_2} be a Patterson-Sullivan current for T_2. Then $h_{T_1}(\mu_{T_2}) < h(T_1)$.*

Proof. After replacing T_2 by a scalar multiple of T_2 we may assume that $h(T_1) = h(T_2)$. Note that the projective Patterson-Sullivan current depends only on the projective class of an element of $cv(F)$, so that this replacement does not change μ_{T_2}. Now the statement of the corollary follows immediately from Theorem 6.1 and Proposition 6.2. □

Corollary 6.4. *Let $T, T' \in cv(F)$. Then*

(1)
$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} \le \frac{h(T')}{h(T)} \le \sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}.$$

(2) *Suppose that T and T' are not in the same projective class. Then*

$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} < \frac{h(T')}{h(T)} < \sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}.$$

Proof.

(1) Let μ_T be a Patterson-Sullivan current corresponding to T. By Theorem 5.5 and Theorem 6.1 we have

$$h(T) \inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} = h_{T'}(\mu_T) \leq h(T')$$

and hence

$$\inf_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} \leq \frac{h(T')}{h(T)}.$$

A symmetric argument shows that

$$\inf_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} \leq \frac{h(T)}{h(T')}.$$

Clearly,

$$\inf_{g\in F\setminus\{1\}} \frac{||g||_{T'}}{||g||_T} = \frac{1}{\sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}}}$$

and hence

$$\sup_{g\in F\setminus\{1\}} \frac{||g||_T}{||g||_{T'}} \geq \frac{h(T')}{h(T)},$$

as required.

The proof of part (2) is exactly the same, but Corollary 6.3 implies that all the inequalities involved are now strict. □

7. Relation to measure-theoretic entropy

In this section we relate the geometric entropy of a current $\mu \in Curr(F)$ with respect to $T = \widetilde{\Gamma}$, where Γ is a finite graph with simplicial metric (every edge has length one), to the measure-theoretic entropy of the corresponding shift-invariant measure $\widehat{\mu}$ on the (appropriately defined) geodesic flow space of the graph Γ.

Definition 7.1. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart. Let $T = \widetilde{\Gamma}$. We endow both Γ and T with simplicial metrics by giving each edge length one. Thus $T \in cv(F)$. Define $\Omega(\Gamma)$ to be the set of all semi-infinite reduced edge-paths

$$\gamma = e_1, e_2, \dots, e_n, \dots$$

in Γ. Note that $\Omega(\Gamma)$ is naturally identified with the disjoint union of $\#V\Gamma$ copies of ∂T, corresponding to the $\#V\Gamma$ different possibilities of the initial vertex of $\gamma \in \Omega(\Gamma)$. We topologize $\Omega(\Gamma)$ accordingly. Thus, topologically, $\Omega(\Gamma)$ is a disjoint union of $\#V\Gamma$ copies of the Cantor set.

There is a natural shift transformation $\sigma : \Omega(\Gamma) \to \Omega(\Gamma)$ defined as

$$\sigma(e_1, e_2, e_3, \dots) = e_2, e_3, \dots$$

for every $\gamma = e_1, e_2, e_3, \dots \in \Omega(\Gamma)$. Then σ is a continuous transformation and the pair $(\Omega(\Gamma), \sigma)$ is easily seen to be an irreducible subshift of finite type, where the alphabet is the set of oriented edges of Γ.

The space $\Omega(\Gamma)$ has a natural set of cylinder sets which generate its Borel sigma-algebra: for every nontrivial reduced edge-path v in Γ let Cyl_v be the set of all $\gamma \in \Omega(\Gamma)$ that have v as an initial segment. It is easy to see that there is a natural affine isomorphism between the space of geodesic currents $Curr(F)$ and the space $\mathcal{M}(\Gamma)$ of all positive σ-invariant Borel measures on $\Omega(\Gamma)$ of finite total mass. For a current $\mu \in Curr(F)$ the corresponding measure $\widehat{\mu} \in \mathcal{M}(\Gamma)$ is defined by the following condition. For every nontrivial reduced edge-path v in Γ, let $[x, y] \subseteq T$ be any lift of v to T. Then

$$\widehat{\mu}(Cyl_v) = \mu(Cyl_{[x,y]}).$$

Suppose now that $\mu \in Curr(F)$ is normalized so that the corresponding measure $\widehat{\mu} \in \Omega(\Gamma)$ is a probability measure (recall that multiplication by a positive scalar does not change the geometric entropy). For the probability measure $\widehat{\mu}$ one can consider its classical *measure-theoretic* entropy $\hbar(\widehat{\mu})$ with respect to the shift σ, also known as the *Kolmogorov-Sinai entropy* or *metric entropy*, see e.g. [**19**].

Theorem 7.2. *Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart. Let $T = \widetilde{\Gamma}$ and endow both Γ and T with simplicial metric. Let $(\Omega(\Gamma), \sigma)$ be as in Definition 7.1. Let $\mu \in Curr(F)$ be such that the corresponding measure $\widehat{\mu} \in \mathcal{M}(\Gamma)$ is a probability measure. Then*

$$h_T(\mu) \le \hbar(\widehat{\mu}) \le h_{topol}(\sigma) = h(T).$$

Proof. The fact that $h_{topol}(\sigma) = h(T)$ is a straightforward exercise which easily follows from the definitions of both quantities. The fact that $\hbar(\widehat{\mu}) \le h_{topol}(\sigma)$ is also completely general for subshifts of finite type (see e.g. Ch 6. of [**19**]). If $h_T(\mu) = 0$ then the inequality $h_T(\mu) \le \hbar(\widehat{\mu})$ is obvious. Suppose now that $h_T(\mu) > 0$.

Note that

$$\Omega(\Gamma) = \sqcup_{e \in E\Gamma} Cyl_e$$

is a generating partition with respect to σ. Therefore the measure-theoretic entropy of $\widehat{\mu}$ can be computed using this partition and is easily seen to be

$$\hbar(\widehat{\mu}) = \lim_{n \to \infty} \sum_{|v|=n} -\frac{\widehat{\mu}(Cyl_v) \log \widehat{\mu}(Cyl_v)}{n}.$$

Suppose now that $0 < s < h_T(\mu)$. Then there exists $n_0 \ge 1$ such that for any geodesic edge-path $[x, y]$ in T of length $n \ge n_0$ we have

$$\mu(Cyl_{[x,y]}) \le \exp(-sn).$$

Hence for any reduced edge-path v in Γ of length $n \ge n_0$ we have $\widehat{\mu}(Cyl_v) \le \exp(-sn)$ and $-\log \widehat{\mu}(Cyl_v) \ge sn$. Therefore from the above formula for $\hbar(\widehat{\mu})$ we get

$$\hbar(\widehat{\mu}) \ge \lim_{n \to \infty} \sum_{|v|=n} \frac{\widehat{\mu}(Cyl_v) sn}{n} = s,$$

since for every $n \ge 1$ we have $\Omega(\Gamma) = \sqcup_{|v|=n} Cyl_v$ and so $\sum_{|v|=n} \widehat{\mu}(Cyl_v) = 1$. Thus we see that for every $0 < s < h_T(\mu)$ we have $\hbar(\widehat{\mu}) \ge s$. Therefore $\hbar(\widehat{\mu}) \ge h_T(\mu)$ as claimed.

$\square$

It is also well-known (again see, for example, Ch. 6 of [**19**]) that for irreducible subshifts of finite type such as $(\Omega(\Gamma), \sigma)$ in Theorem 7.2, there exists a unique probability measure $\mu_\Omega \in \mathcal{M}(\Gamma)$ such that $\hbar(\mu_\Gamma) = h_{topol}(\sigma) = h(T)$. The measure μ_Ω is known as the *measure of maximal entropy*. We already know that for the Patterson-Sullivan current μ_T we have $h_T(\mu_T) = h(T)$. Thus, if μ_T is normalized so that the corresponding measure $\widehat{\mu}_T \in \mathcal{M}(\Gamma)$ is a probability measure, then by Theorem 7.2 we have $h_T(\mu_T) = \hbar(\widehat{\mu}_T) = h(T)$. Hence $\widehat{\mu}_T$ is the unique measure of maximal geometric entropy μ_Ω.

Corollary 7.3. *Let α, Γ and T be as in Theorem 7.2. Let $\mu_T \in Curr(F)$ be the Patterson-Sullivan current for T normalized so that the corresponding measure $\widehat{\mu}_T \in \mathcal{M}(\Gamma)$ is a probability measure. Then*

(1) *$\hbar(\widehat{\mu}_T) = h(T)$, so that $\widehat{\mu}_T = \mu_\Omega$, the unique probability measure of maximal entropy;*
(2) *for $\mu \in Curr(F)$ we have $h_T(\mu) = h(T)$ if and only if μ is proportional to μ_T.*

Theorem 7.2 and Corollary 7.3 yield Theorem E from the Introduction.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/
E-mail address: kapovich@math.uiuc.edu

Section de mathématiques, Université de Genève, 2-4, rue du Lièvre, c.p. 64, 1211 Genève, Switzerland
http://www.unige.ch/∼tatiana/
E-mail address: tatiana.smirnova-nagnibeda@unige.ch