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CONGRUENCES OF ALTERNATING MULTIPLE HARMONIC SUMS

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Abstract. In this sequel to [16], we continue to study the congruence properties of the alternating version of multiple harmonic sums. As contrast to the study of multiple harmonic sums where Bernoulli numbers and Bernoulli polynomials play the key roles, in the alternating setting the Euler numbers and the Euler polynomials are also essential.

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1. Introduction

In proving a Van Hamme type congruence the first author was led to consider some congruences involving alternating multiple harmonic sums (AMHS for short) which are defined as follows. Let $d > 0$ and let $\mathbf{s} := (s_1, \dots, s_d) \in (\mathbb{Z}^*)^d$. We define the *alternating multiple harmonic sum* as

$$H(\mathbf{s}; n) := \sum_{1 \le k_1 < k_2 < \cdots < k_d \le n} \prod_{i=1}^{d} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}.$$

By convention we set $H(\mathbf{s}; n) = 0$ any $n < d$. We call $\ell(\mathbf{s}) := d$ and $|\mathbf{s}| := \sum_{i=1}^{d} |s_i|$ its *depth* and *weight*, respectively. We point out that $\ell(\mathbf{s})$ is sometimes called length in the literature. When every s_i is positive we recover the multiple harmonic sums (MHS for short) whose congruence properties are studied in [10, 11, 18, 19]. There is another "non-strict" version of the AMHS defined as follows:

$$S(\mathbf{s}; n) := \sum_{1 \le k_1 \le k_2 \le \cdots \le k_d \le n} \prod_{i=1}^{d} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}.$$

By Inclusion and Exclusion Principle it is easy to see that

$$S(\mathbf{s}; n) = \sum_{\mathbf{r} \preceq \mathbf{s}} H(\mathbf{r}; n), \tag{1}$$

$$H(\mathbf{s}; n) = \sum_{\mathbf{r} \preceq \mathbf{s}} (-1)^{\ell(\mathbf{s}) - \ell(\mathbf{r})} S(\mathbf{r}; n), \tag{2}$$

where $\mathbf{r} \prec \mathbf{s}$ means $\mathbf{r}$ can be obtained from $\mathbf{s}$ by combining some of its parts.

The main goal of this paper is to provide a systematic study of the congruence property of $H(\mathbf{s}; p-1)$ (and $S(\mathbf{s}; p-1)$) for primes $p > |\mathbf{s}| + 2$ by using intimate relations between Bernoulli polynomials, Bernoulli numbers, Euler polynomials, and Euler numbers. Throughout the paper, we often use the abbreviation $S(-) = S(-; p-1)$ and $H(-) = H(-; p-1)$ if no confusion will arise. The following congruences concerning harmonic sums will be crucial for us.

Theorem 1.1. *Let $k \in \mathbb{N}$ and p be a prime. Set $X_p(k) := \frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2(2p-1-k)}$.*

(a) [15, Theorem 5.1] *If $p \ge k + 3$ then*

$$H(k) \equiv \begin{cases} k(k+1)B_{p-2-k}p^2/2(p-2-k) & \pmod{p^3}, \quad \text{if } k \text{ is odd;} \\ -2kX_p(k+1)p & \pmod{p^3}, \quad \text{if } k \text{ is even.} \end{cases} \tag{3}$$

(b) [15, Theorem 5.2] *If* $p \ge k+4$ *then*

$$H(k;(p-1)/2) \equiv \begin{cases} 2(2^k-2)X_p(k) & \pmod{p^2}, \quad \text{if } k>1 \text{ is odd;} \\ -k(2^{k+1}-1)X_p(k+1)p & \pmod{p^3}, \quad \text{if } k \text{ is even;} \\ -2q_p+pq_p^2-\frac{2}{3}p^2q_p^3-\frac{7}{12}p^2B_{p-3} & \pmod{p^3}, \quad \text{if } k=1. \end{cases} \tag{4}$$

Here $q_p=(2^{p-1}-1)/p$ *is the Fermat quotient.*

We now sketch the outline of the paper. We start §2 by recalling some important relations among AMHS such as the stuffle and reversal relations. Then we present some basic properties of Euler polynomials which provide one of the fundamental tools for us in the alternating setting. Then in §2.3 we describe two reduction procedures for $H(\mathbf{s}) \pmod p$ general $\mathbf{s}$, which are used to derive congruences in depth two and depth three cases in §3 and §4, respectively.

Theorem 1.2. *Let* $a,\ell \in \mathbb{N}$, $\mathbf{s}=(s_1,\dots,s_\ell)\in(\mathbb{Z}^*)^\ell$ *and* $\mathbf{s}'=(s_2,\dots,s_\ell)$. *For every prime* $p\ge a+2$ *write* $H(-)=H(-;p-1)$. *Then we have the reduction formulae*

$$\begin{aligned} H(a,\mathbf{s}) \equiv& -\frac{1}{a}H\big((a-1)\oplus s_1,\mathbf{s}'\big)-\frac{1}{2}H(a\oplus s_1,\mathbf{s}') \\ &+\sum_{k=2}^{p-1-a}\binom{p-a}{k}\frac{B_k}{p-a}H\big((k+a-1)\oplus s_1,\mathbf{s}'\big) \pmod p, \\ H(-a,\mathbf{s}) \equiv& \frac{(1-2^{p-a})B_{p-a}}{p-a}\Big(H(\mathbf{s})-H(-s_1,\mathbf{s}')\Big) \\ &-\sum_{k=0}^{p-2-a}\binom{p-1-a}{k}\frac{E_k(0)}{2}H\big((k+a)\oplus(-s_1),\mathbf{s}'\big) \pmod p, \end{aligned}$$

where $s\oplus t=\operatorname{sgn}(st)(|s|+|t|)$ *and* $E_k(0)=2(1-2^{k+1})B_{k+1}/(k+1)$.

In §5 we deal with the homogeneous AMHS of arbitrary depth and provide an explicit formula using the relation between the power sum and elementary symmetric functions and the partition functions. §6 is devoted to a comprehensive study of the weight four AMHS in which identities involving Bernoulli numbers such as those proved in [18] play the leading roles. For example, by writing $H(-)=H(-;p-1)$ we find the following interesting relations (see Proposition 6.1, Proposition 6.2 and Proposition 6.4):

$$\begin{aligned} H(1,-3)\equiv\frac{1}{2}H(-2,2)\equiv&\sum_{k=0}^{p-3}2^kB_kB_{p-3-k} && \pmod p, \\ H(-1,3)\equiv&-\frac{1}{2}q_pB_{p-3} && \pmod p, \\ H(1,-2,-1)\equiv& H(1,-3)-\frac{5}{4}q_pB_{p-3} && \pmod p, \\ H(-1,1,-1,1)\equiv H(1,-1,1,-1)\equiv&-\frac{1}{12}\Big(q_pB_{p-3}+2q_p^4\Big) && \pmod p, \\ H(-1,1,1,-1)\equiv&\frac{1}{12}\Big(6H(1,-3)+7q_pB_{p-3}+2q_p^4\Big) && \pmod p, \end{aligned}$$

for all primes $p\ge 7$. None of the above congruences can be obtained simply by the stuffle and reversal relations.

After studying some special types of AMHS of weight four in §5, we turn to congruence relations involving lower weight AMHS modulo higher powers of primes in the last section. One of the main ideas in these sections is to relate AMHS to sums $U(\mathbf{s};n)$ and $V(\mathbf{s};n)$ defined by (37) and (38), respectively. These sums appeared previously in congruences involving powers of Fermat quotient (see [2, 3, 5, 8]).

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2. Properties of AMHS

2.1. **Stuffle relation.** The most important relation between AMHS is the so called stuffle relation. It is possible to formalize this using words as in [20, §2] or [14, §2.2] which is a generalization of the MHS case (see [10, §2]). Unfortunately, for AMHS we don't have the integral representations which provide another product structure for the alternating multiple zeta values which are the infinite sum version of AMHS.

Fix a positive integer n. Let $\mathfrak{A}$ be the algebra generated by letters y_s for $s \in \mathbb{Z}^*$. Define a multiplication $*$ on $\mathfrak{A}$ by requiring that $*$ distribute over addition, that $\mathbf{1} * w = w * \mathbf{1} = w$ for the empty word $\mathbf{1}$ and any word w, and that, for any two words w_1, w_2 and two letters y_s, y_t $(s, t \in \mathbb{Z}^*)$

$$y_s w_1 * y_t w_2 = y_s(w_1 * y_t w_2) + y_t(y_s w_1 * w_2) + y_{s\oplus t}(w_1 * w_2) \tag{5}$$

where $s \oplus t = \operatorname{sgn}(st)(|s| + |t|)$. Then we get an algebra homomorphism

$$\begin{aligned} H: \quad (\mathfrak{A}, *) &\longrightarrow \{H(\mathbf{s}; n) : \mathbf{s} \in \mathbb{Z}^r, r \in \mathbb{N}\} \\ \mathbf{1} &\longmapsto 1 \\ y_{s_1} \dots y_{s_r} &\longmapsto H(s_1, \dots, s_r; n). \end{aligned}$$

For example,

$$\begin{aligned} H(-2; n)H(-3, 2; n) = H(-2, -3, 2; n) + H(-3, -2, 2; n) + H(-3, 2, -2; n) \\ + H(5, 2; n) + H(-3, -4; n). \end{aligned}$$

There is another kind of relation caused by the reversal of the arguments which we call the *reversal relations*. For any $\mathbf{s} = (s_1, \dots, s_r)$ they have the form

$$\begin{aligned} H(\mathbf{s}; p-1) &\equiv \operatorname{sgn}\Big(\prod_{j=1}^r s_j\Big)(-1)^r H(\overleftarrow{\mathbf{s}}; p-1) \pmod p, \\ S(\mathbf{s}; p-1) &\equiv \operatorname{sgn}\Big(\prod_{j=1}^r s_j\Big)(-1)^r S(\overleftarrow{\mathbf{s}}; p-1) \pmod p, \end{aligned} \tag{6}$$

for any odd prime $p > |\mathbf{s}|$, where $\overleftarrow{\mathbf{s}} = (s_r, \dots, s_1)$.

2.2. **Euler polynomials.** In the study of congruences of MHS [10, 18, 19] we have seen that Bernoulli numbers play the key roles by virtue of the following identity: ([1, p. 804, 23.1.4-7])

$$\sum_{j=1}^{n-1} j^d = \sum_{r=0}^{d} \binom{d+1}{r} \frac{B_r}{d+1} n^{d+1-r}, \quad \forall n, d \geq 1. \tag{7}$$

In the case of AMHS, however, the Euler polynomials and the Euler numbers are indispensable, too. Recall that the Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Lemma 2.1. *Let $n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\sum_{i=1}^{d-1} (-1)^i i^n = \frac{1}{2}\Big((-1)^{d-1} E_n(d) + E_n(0)\Big) = \sum_{a=0}^{n} \binom{n}{a} F_{n,d,a} d^{n-a}, \tag{8}$$

where

$$F_{n,d,a} = \begin{cases} (-1)^{d-1} E_a(0)/2, & \text{if } a < n; \\ (1 - (-1)^d) E_n(0)/2, & \text{if } a = n > 0; \\ -(1 + (-1)^d)/2, & \text{if } a = n = 0. \end{cases}$$

Moreover, $E_0(0) = 1$ *and for all* $a \in \mathbb{N}$

$$E_a(0) = \frac{2^{a+1}}{a+1}\Big(B_{a+1}\Big(\frac{1}{2}\Big) - B_{a+1}\Big) = \frac{2}{a+1}(1-2^{a+1})B_{a+1}. \tag{9}$$

Proof. Consider the generating function

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{i=1}^{d-1}(-1)^i i^n\right)\frac{t^n}{n!} &= \sum_{i=1}^{d-1}(-1)^i e^{ti} \\ &= \frac{(-e^t)^d - 1}{-e^t - 1} - 1 \\ &= \frac{(-1)^{d-1}e^{dt}+1}{e^t+1} - 1 \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\Big((-1)^{d-1}E_n(d) + E_n(0)\Big)\frac{t^n}{n!} - 1. \end{aligned} \tag{10}$$

Now (8) follows from the notorious equation (see for e.g., [1, p. 805, 23.1.7])

$$E_n(x) = \sum_{a=0}^{n}\binom{n}{a}E_a(0)x^{n-a} \tag{11}$$

for all $n > 0$. Equation (9) is also well-known (see for e.g., item 23.1.20 on p. 805 of loc. cit.). □

Remark 2.2. The classical Euler numbers E_k is defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}.$$

They are related to $E_k(0)$ by the formula (see [1, p. 805, 23.1.7])

$$E_m(0) = \sum_{k=0}^{m}\binom{m}{k}\frac{E_k}{2^k}\left(-\frac{1}{2}\right)^{m-k}.$$

Corollary 2.3. *Let* $a \in \mathbb{Z}_{\geq 0}$ *and* p *be a prime such that* $p \geq a+2$. *Then*

$$H(-a;p-1) \equiv \begin{cases} -\dfrac{2(1-2^{p-a})}{a}B_{p-a} & \pmod{p}, \quad \text{if } a \text{ is odd;} \\ \dfrac{a(1-2^{p-1-a})}{a+1}pB_{p-1-a} & \pmod{p^2}, \quad \text{if } a \text{ is even.} \end{cases} \tag{12}$$

Proof. Taking $d = p$ and $n = p(p-1) - a$ in the Lemma we see that

$$\begin{aligned} H(-a;p-1) \equiv& F_{p(p-1)-a,p,p(p-1)} + p(p(p-1)-a)F_{p(p-1)-a,p,p(p-1)-1-a} \\ \equiv& E_{p(p-1)-a}(0) - \frac{1}{2}paE_{p(p-1)-1-a}(0) \pmod{p^2}, \end{aligned}$$

since all the coefficients in (8) are p-integral by (9) and the property of Bernoulli numbers: B_m is not p-integral if and only if $p-1$ divides $m > 0$. Then the corollary directly follows from (9) and Kummer congruences

$$\begin{aligned} \frac{B_{p(p-1)-a}}{p(p-1)-a} &\equiv \frac{B_{p-1-a}}{p-1-a} \pmod{p}, \\ \frac{B_{p(p-1)-a+1}}{p(p-1)-a+1} &\equiv \frac{B_{p-a}}{p-a} \pmod{p}. \end{aligned}$$

□

Remark 2.4. (a). The corollary can also be obtained from [16, Theorem 2.1] combined with (4). Notice that both terms in [16, Theorem 2.1] contribute nontrivially when a is even since the modulus is p^2. (b). Notice that if a is odd we allow $p = a+2$ to be a prime modulus unlike [4, Lemma 5.1]. If a is even then the corollary is given also by [4, (6.2)].

2.3. **Two reduction formulae.** We now prove two reduction formulae of $H(\mathbf{s})$ for arbitrary composition $\mathbf{s}$, corresponding to the two cases where $\mathbf{s}$ begins with a positive or a negative number.

Theorem 2.5. *Let $a, \ell \in \mathbb{N}$, $\mathbf{s} = (s_1, \dots, s_\ell) \in (\mathbb{Z}^*)^\ell$ and $\mathbf{s}' = (s_2, \dots, s_\ell)$. For any prime $p \geq a+2$ write $H(-) = H(-; p-1)$. Then*

$$H(a, \mathbf{s}) \equiv -\frac{1}{a} H\big((a-1) \oplus s_1, \mathbf{s}'\big) + \sum_{k=1}^{p-1-a} \binom{p-a}{k} \frac{B_k}{p-a} H\big((k+a-1) \oplus s_1, \mathbf{s}'\big) \pmod p. \quad (13)$$

Proof. By definition

$$\begin{aligned} H(a, \mathbf{s}) &\equiv \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} \sum_{j=1}^{j_1-1} j^{p-1-a} \\ &\equiv \sum_{k=0}^{p-1-a} \binom{p-a}{k} \frac{B_k}{p-a} \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} j_1^{p-a-k} \end{aligned}$$

which is exactly the right hand side of (13). □

Theorem 2.6. *Let $a, \ell \in \mathbb{N}$, $\mathbf{s} = (s_1, \dots, s_\ell) \in (\mathbb{Z}^*)^\ell$ and $\mathbf{s}' = (s_2, \dots, s_\ell)$. For any prime $p \geq a+2$ write $H(-) = H(-; p-1)$. Then*

$$\begin{aligned} H(-a, \mathbf{s}) \equiv& \frac{(1-2^{p-a})B_{p-a}}{p-a} \Big(H(\mathbf{s}) - H(-s_1, \mathbf{s}') \Big) \\ &- \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{(1-2^{k+1})B_{k+1}}{k+1} H\big((k+a) \oplus (-s_1), \mathbf{s}'\big) \pmod p. \end{aligned}$$

Proof. By definition and Lemma 2.1

$$\begin{aligned} H(a, \mathbf{s}) \equiv& \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} \sum_{j=1}^{j_1-1} (-1)^j j^{p-1-a} \\ \equiv& \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{E_k(0)}{2} \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|+k+a}} (-1)^{j_1-1} \\ &+ \frac{E_{p-1-a}(0)}{2} \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} \Big(1 - (-1)^{j_1}\Big) \\ \equiv& \frac{E_{p-1-a}(0)}{2} \Big(H(\mathbf{s}) - H(-s_1, \mathbf{s}') \Big) - \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{E_k(0)}{2} H\big((k+a) \oplus (-s_1), \mathbf{s}'\big). \end{aligned}$$

The theorem follows from (9) easily. □

3. AMHS OF DEPTH TWO

In this section we will provide congruence formulae for all depth two AMHS. All but one case are given by very concise values involving Bernoulli numbers or Euler numbers (which are closely related by the identity (9)).

Theorem 3.1. *Let $a, b \in \mathbb{N}$ and a prime $p \geq a+b+2$. Write $S(-) = S(-;p-1)$ and $H(-) = H(-;p-1)$. If $a+b$ is odd then we have*

$$S(a,b) \equiv H(a,b) \equiv \frac{(-1)^b}{a+b}\binom{a+b}{a} B_{p-a-b} \pmod{p}, \quad (14)$$

$$H(-a,-b) \equiv \frac{2^{p-a-b}-1}{a+b}(-1)^b\binom{a+b}{a} B_{p-a-b} \pmod{p}, \quad (15)$$

$$S(-a,-b) \equiv \frac{2^{p-a-b}-1}{a+b}\left(2+(-1)^b\binom{a+b}{a}\right) B_{p-a-b} \pmod{p}, \quad (16)$$

$$H(-a,b) \equiv H(a,-b) \equiv \frac{1-2^{p-a-b}}{a+b} B_{p-a-b} \pmod{p}, \quad (17)$$

$$S(-a,b) \equiv S(a,-b) \equiv \frac{2^{p-a-b}-1}{a+b} B_{p-a-b} \pmod{p}. \quad (18)$$

If $a+b$ is even then we have

$$S(a,b) \equiv H(a,b) \equiv 0 \pmod{p}, \quad (19)$$

$$S(-a,-b) \equiv H(-a,-b) \equiv \frac{2(1-2^{p-a})(1-2^{p-a})}{ab} B_{p-a}B_{p-b} \pmod{p}. \quad (20)$$

Proof. Congruences (14) and (19) follow from [18, Theorem 3.1] (see [10, Theorem 6.1] for a different proof). Congruences (15), (17), and (20) are given by [4, Lemma 6.2] (notice that $(1-2^{p-1})B_{p-1} \equiv q_p \pmod{p}$). Finally, congruences for S version of AMHS are all easy consequence of the H version by the relation $S(\alpha,\beta) = H(\alpha,\beta) + H(\alpha \oplus \beta)$. □

Even though we don't have compact congruence formulae for $H(-a,b)$ and $H(a,-b)$ when $a+b$ is even we can prove two general statements using the two reduction procedures provided by Theorem 2.5 and Theorem 2.6.

Proposition 3.2. *Let $a, b \in \mathbb{N}$ and a prime $p \geq a+b+2$. Write $S(-) = S(-;p-1)$ and $H(-) = H(-;p-1)$. If $a+b$ is even then we have*

$$S(a,-b) \equiv H(a,-b) \equiv -S(-b,a) \equiv -H(-b,a)$$
$$\equiv \sum_{k=0}^{p-a-1} \binom{p-a}{k} \frac{2(1-2^{2p-a-b-k})B_k B_{2p-a-b-k}}{(p-a)(2p-a-b-k)} \pmod{p}. \quad (21)$$

Proof. By Theorem 2.5 we have

$$H(a,-b) \equiv \sum_{k=0}^{p-a-1} \binom{p-a}{k} \frac{B_k}{p-a} \cdot H(-(a+b+k-1)) \pmod{p}.$$

To use Cororllary 2.3 we need to break the sum into two parts, i.e., when $a+b+k < p$ and when $a+b+k \geq p$. In the first case we can replace k by $k+p-1$ and then to get the correct term in (21) we only need to use Fermat's Little Theorem $2^{p+1-a-b-k} \equiv 2^{2p-a-b-k} \pmod{p}$ and Kummer congruence $B_{p+1-a-b-k}/(p+1-a-b-k) \equiv B_{2p-a-b-k}/(2p-a-b-k) \pmod{p}$. This finishes the proof of the proposition. □

The second reduction procedure, Theorem 2.6, provides us another useful result on AMHS of even weight and depth two.

Proposition 3.3. *Let $a, b \in \mathbb{N}$ and a prime $p \geq a+b+2$. Write $S(-) = S(-; p-1)$ and $H(-) = H(-; p-1)$. If $a+b$ is even then we have*

$$\begin{aligned} S(-a,b) &\equiv H(-a,b) \equiv -S(b,-a) \equiv -H(b,-a) \\ &\equiv \sum_{k=1}^{p-2-a-b} \binom{p-1-a}{k} \frac{2(1-2^{k+1})(1-2^{p-a-b-k})B_{k+1}B_{p-a-b-k}}{(k+1)(a+b+k)} \\ &+ \sum_{k=p-1-a-b}^{p-1-a} \binom{p-1-a}{k} \frac{2(1-2^{k+1})(1-2^{2p-1-a-b-k})B_{k+1}B_{2p-1-a-b-k}}{(k+1)(1+a+b+k)} \pmod{p}. \end{aligned}$$

Proof. By Theorem 2.6 we have

$$H(-a,b) \equiv \sum_{k=0}^{p-1-a} \binom{p-1-a}{k} \frac{(1-2^{k+1})B_{k+1}}{k+1} H\big(-(k+a+b)\big) \pmod{p}$$

since $H(b) \equiv 0 \pmod{p}$. The rest follows from Cororllary 2.3 similar to the proof of Proposition 3.2. ☐

The two propositions above will be used in §6 to compute some AMHS congruences explicitly.

4. AMHS of depth three

All the congruences in this section are modulo a prime p. Write $H(-) = H(-; p-1)$. Recall that for depth three MHS modulo p can be determined [18, Theorem 3.5], [10, Theorem 6.2] and [18, (3.13)]. For AMHS, we first observe that for any $\alpha, \beta, \gamma \in \mathbb{Z}$

$$\begin{aligned} H(\alpha,\beta,\gamma) =& H(\alpha)H(\beta)H(\gamma) - H(\gamma)H(\beta,\alpha) - H(\gamma)H(\beta \oplus \alpha) \\ &- H(\gamma,\beta)H(\alpha) - H(\gamma \oplus \beta)H(\alpha) + H(\gamma,\beta,\alpha) \\ &+ H(\gamma \oplus \beta,\alpha) + H(\gamma,\beta \oplus \alpha) + H(\gamma \oplus \beta \oplus \alpha). \end{aligned}$$

This can be easily checked by stuffle relations but the idea is hidden in the general framework set up by Hoffman [9]. Combining with the reversal relations we can obtained the following results without much difficulty. We leave its proof to the interested reader.

Theorem 4.1. *Let p be a prime and $a, b, c \in \mathbb{N}$ such that $p > w$ where $w := a+b+c$. Write $H(-) = H(-; p-1)$. Then we have*

(1). If w is even then

$$2H(a,-b,c) \equiv H(-c-b,a) + H(c,-b-a) \pmod{p}, \tag{22}$$

$$2H(a,b,-c) \equiv -H(-c)H(b,a) + H(-c-b,a) + H(-c,b+a) \pmod{p}, \tag{23}$$

$$2H(-a,-b,-c) \equiv -H(-c)H(-b,-a) - H(-c,-b)H(-a) \tag{24}$$

$$+ H(c+b,-a) + H(-c,a+b) \pmod{p}. \tag{25}$$

(2). If w is odd then

$$2H(a,-b,-c) \equiv H(c+b,a) + H(-c,-b-a) - H(-c)H(-b,a) \pmod{p}, \tag{26}$$

$$2H(-a,b,-c) \equiv -H(-c)H(b,-a) - H(-c,b)H(-a) \tag{27}$$

$$+ H(-c-b,-a) + H(-c,-b-a) \pmod{p}. \tag{28}$$

Because of the reversal relations when the weight is even there remains essentially only one more case to consider in depth three. This is given by the next result which will be used in §6.

Theorem 4.2. *Let a,b,c be positive integers such that $w:=a+b+c$ is even. Then for any prime $p\ge w+3$ we have*

$$\begin{aligned}H(a,-b,-c)\equiv &-\sum_{k=2}^{p-w+1}\binom{p-a}{p-w-k+1}\binom{k+c-1}{c}\frac{(1-2^k)B_kB_{p-w-k+1}}{ak}\\&-\sum_{k=p+1-b-c}^{p-c}\binom{p-a}{2p-w-k}\binom{k+c-1}{c}\frac{(1-2^k)B_kB_{2p-w-k}}{ak}\\&-\frac{(1-2^{p-c})(1-2^{p-a-b})B_{p-a-b}B_{p-c}}{(a+b)c}.\end{aligned}$$

Proof. The proof is essentially a repeated application of Theorem 2.5. But we spell out all the details below because there are some subtle details that we need to attend to.

By (7) and Fermat's Little Theorem we have modulo p

$$\begin{aligned}H(a,-b,-c)&\equiv\sum_{i=1}^{p-1}\frac{(-1)^i}{i^c}\sum_{j=1}^{i-1}(-1)^j j^{p-1-b}\sum_{k=1}^{j-1}k^{p-1-a}\\&\equiv\sum_{i=1}^{p-1}\frac{(-1)^i}{i^c}\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{B_k}{p-a}\sum_{j=1}^{i-1}(-1)^j j^{\rho(k)+p-a-b-k},\end{aligned}$$

where $\rho(k)=0$ if $k<p-a-b$ and $\rho(k)=p-1$ if $k\ge p-a-b$ (to make sure all exponents are positive in the sum of the second line above). By Lemma 2.1

$$\begin{aligned}H(a,-b,-c)&\equiv\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{B_k}{p-a}\sum_{r=0}^{n}\binom{n}{r}\sum_{i=1}^{p-1}(-1)^i i^{n-r-c}F_{n,i,r}\\&\equiv-\sum_{k=0}^{p-1-a}\binom{p-a}{k}\sum_{r=1}^{n-1}\binom{n}{r}\frac{(1-2^{r+1})B_kB_{r+1}}{(p-a)(r+1)}\sum_{i=1}^{p-1}i^{n-c-r}\\&\quad+\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{(1-2^{n+1})B_kB_{n+1}}{(p-a)(n+1)}\sum_{i=1}^{p-1}((-1)^i-1)i^{-c},\end{aligned}$$

where $n=\rho(k)+p-a-b-k$. Here we have used the fact that when $r=0$ we have $F_{n,i,r}=(-1)^{i-1}$ and thus the inner sum is $\sum_{i=1}^{p-1}i^{n-c}\equiv 0 \pmod p$ except when $p-1|n-c$, i.e., except when $n=c$ and $k=p-w$. But then $B_k=0$ since w is even by assumption. Thus

$$\begin{aligned}H(a,-b,-c)\equiv&\sum_{\substack{0\le k<p-a\\ c<n}}\binom{p-a}{k}\binom{n}{n-c}\frac{(1-2^{n-c+1})B_kB_{n-c+1}}{(p-a)(n-c+1)}\\&+H(-c)\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{(1-2^{n+1})B_kB_{n+1}}{(p-a)(n+1)}.\end{aligned}$$

Now if c is even then $H(-c)\equiv 0 \pmod p$. So we may assume c is odd in the last line above. Then $k+n+1$ is always odd so that $B_kB_{n+1}\ne 0$ if and only if $k=1$ and $n=p-a-b-1$. Hence

$$\begin{aligned}H(a,-b,-c)\equiv&\sum_{k=0}^{p-w-1}\binom{p-a}{k}\binom{p-a-b-k}{p-w-k}\frac{(1-2^{p-w-k+1})B_kB_{p-w-k+1}}{(p-a)(p-w-k+1)}\\&+\sum_{k=p-a-b}^{p-1-a}\binom{p-a}{k}\binom{2p-1-a-b-k}{2p-1-w-k}\frac{(1-2^{2p-w-k})B_kB_{2p-w-k}}{(p-a)(2p-w-k)}\\&-\frac{(1-2^{p-c})(1-2^{p-a-b})B_{p-a-b}B_{p-c}}{(a+b)c}.\end{aligned}$$

After substitutions $k\to p-w+1-k$ in the first sum and $k\to 2p-w-k$ in the second sum the theorem follows immediately from Cororllary 2.3, □

Remark 4.3. The condition $p \geq w+3$ in Theorem 4.2 can not be weakened since

$$H(1,-2,-3) \equiv \text{RHS} + 5 \not\equiv \text{RHS} \pmod 7.$$

5. AMHS of arbitrary depth

In this section we provide some general results on AMHS without restrictions on the depth. We first consider the homogeneous case for which the key idea comes from [18, Lemma 2.12] and [10, Theorem 2.3].

Let $p_i = \sum_{j\geq 1} x_j^i$ be the power-sum symmetric functions and $e_i = \sum_{j_1<\cdots<j_i} x_{j_i}\cdots x_{j_i}$ be the elementary symmetric functions of degree i. Let $P(\ell)$ be the set of unordered partitions of ℓ. For $\lambda = (\lambda_1,\dots,\lambda_r) \in P(\ell)$ we set $p_\lambda = \prod_{i=1}^r p_{\lambda_i}$. Recall that the expression of e_l in terms of p_i is given by the following formula (see [13, p.28]):

$$\ell! e_\ell = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ & & & \ddots & 0 \\ p_{\ell-1} & p_{\ell-2} & p_{\ell-3} & \cdots & \ell-1 \\ p_\ell & p_{\ell-1} & p_{\ell-2} & \cdots & p_1 \end{vmatrix} = \sum_{\lambda\in P(\ell)} c_\lambda p_\lambda. \tag{29}$$

Denote by $O(\ell) \subset P(\ell)$ the subset of odd partitions $\lambda = (\lambda_1,\dots,\lambda_r)$ (i.e., λ_i is odd for every part).

Lemma 5.1. *Let $a,\ell \in \mathbb{N}$ and p a prime such that $a\ell < p-1$. Set $H(-) = H(-;p-1)$. For an odd partition $\lambda = (\lambda_1,\dots,\lambda_r) \in O(\ell)$ we put $H_\lambda(-a) = \prod_{i=1}^r H(-\lambda_i a)$. Then*

$$\ell! H\big(\{-a\}^\ell\big) \equiv \sum_{\lambda\in O(\ell)} c_\lambda H_\lambda(-a) \pmod p, \tag{30}$$

where c_λ are given by (29). In particular, if a is even then (30) $\equiv 0 \pmod p$. If $a>1$ is odd then (30) is congruent to a $\mathbb{Q}$-linear combination of $B_{p-\lambda_1 a}\cdots B_{p-\lambda_r a}$ for odd partitions $\lambda = (\lambda_1,\dots,\lambda_r)$. If $a=1$ then (30) is congruent to a $\mathbb{Q}$-linear combination of $q_p^s B_{p-\lambda_{s+1}}\cdots B_{p-\lambda_r}$ for odd partitions $\lambda = (\{1\}^s,\lambda_{s+1},\dots,\lambda_r)$ ($\lambda_i>1$ for all $i>s$), where $q_p = (2^{p-1}-1)/p$ is the Fermat quotient.

Proof. Congruence (30) follows from [18, Lemma 2.12] and [10, Theorem 2.3]. The last part follows from [16, Theorem 2.1] and (4). □

For example, by [16, Theorem 2.1] and (4) we have

$$H(-1) = -H(1) + H(1;\frac{p-1}{2}) \equiv -2q_p + pq_p^2 - \frac{2}{3}p^2q_p^3 - \frac{1}{4}p^2B_{p-3} \pmod{p^3}. \tag{31}$$

Observe that $O(2) = \{(1,1)\}, O(3) = \{(1,1,1),(3)\}, O(4) = \{(1,1,1,1),(1,3)\}$. It is obvious that $c_{(1,\cdots,1)} = 1$, $c_{(3)} = 2$, $c_{(1,3)} = 8$, $c_{(1,1,3)} = 20$, and $c_{(1,1,1,3)} = c_{(3,3)} = 40$, which implies that

$$2H\big(\{-1\}^2\big) \equiv 4q_p^2, \qquad \text{so } H\big(\{-1\}^2\big) \equiv 2q_p^2 \tag{32}$$

$$6H\big(\{-1\}^3\big) \equiv -8q_p^3 + 2H(-3), \qquad \text{so } H\big(\{-1\}^3\big) \equiv -\frac{4}{3}q_p^3 - \frac{1}{6}B_{p-3} \tag{33}$$

$$24H\big(\{-1\}^4\big) \equiv 16q_p^4 + 8H(-1)H(-3), \quad \text{so } H\big(\{-1\}^4\big) \equiv \frac{2}{3}q_p^4 + \frac{1}{3}q_pB_{p-3}, \tag{34}$$

$$5!H\big(\{-1\}^5\big) \equiv -32q_p^5 + 20H(-1)^2H(-3), \quad \text{so } H\big(\{-1\}^5\big) \equiv -\frac{4}{15}q_p^5 - \frac{1}{3}q_p^2B_{p-3} \tag{35}$$

and

$$\begin{aligned} 6!H\big(\{-1\}^6\big) &\equiv 64q_p^6 + 40H(-1)^3H(-3) + 40H(-3)^2, \\ H\big(\{-1\}^6\big) &\equiv \frac{4}{45}q_p^6 + \frac{2}{9}q_p^3B_{p-3} + \frac{1}{72}B_{p-3}^2. \end{aligned} \tag{36}$$

Remark 5.2. Congruences (32) and (33) are not new. See the Remarks on [4, p. 365].

For non-homogeneous $\mathbf{s}$ we don't know too much except for those of very special forms. For example we have the follow easy statement.

Proposition 5.3. *Suppose* $\mathbf{s} = \overleftarrow{\mathbf{s}}$ *is palindromic. If the number of negative components in* $\mathbf{s}$ *and the weight* $|\mathbf{s}|$ *have different parity then*

$$S(\mathbf{s}; p-1) \equiv H(\mathbf{s}; p-1) \equiv 0 \pmod p.$$

Proof. This is obvious by the reversal relations (6) □

In order to state and prove Proposition 5.7, we need to investigate the following two types of sums. Define

$$U(\mathbf{s}; n) := \sum_{1\le k_1<k_2<\cdots<k_\ell\le n} \prod_{i=1}^{\ell} \frac{(-\operatorname{sgn}(s_i)/2+3/2)^{k_i}}{k_i^{|s_i|}}, \tag{37}$$

$$V(\mathbf{s}; n) := \sum_{1\le k_1<k_2<\cdots<k_\ell\le n} \prod_{i=1}^{\ell} \frac{(\operatorname{sgn}(s_i)/4+3/4)^{k_i}}{k_i^{|s_i|}}. \tag{38}$$

For example, $U(-m; n) = \sum_{k=1}^n 2^k/k^m$, $V(-m; n) = \sum_{k=1}^n 1/(2^k k^m)$, and for $\mathbf{s}$ with only positive components $U(\mathbf{s}; n) = V(\mathbf{s}; n) = H(\mathbf{s}; n)$. The sums $U(-m; p-1)$ appeared previously in congruences involving powers of Fermat quotient (see [2, 3, 5, 8]). To compute the congruence involving these sums we need the following preliminary results which will also be needed in §7.

Lemma 5.4. *Assume* $\mathbf{s} = (s_1, \dots, s_\ell) \in (\mathbb{Z}^*)^\ell$. *Let* $\mathbf{e}_j$ *be the standard* j*-th unit vector. Then we have*

$$U(\mathbf{s}; n) \equiv 2^{p\sharp\{j: s_j<0\}}(-1)^{|\mathbf{s}|}\Big(V(\overleftarrow{\mathbf{s}}; n) + p\sum_{j=1}^{\ell} |s_j| V(\overleftarrow{\mathbf{s}\oplus\mathbf{e}_j}; n)\Big) \pmod{p^2} \tag{39}$$

$$V(\mathbf{s}; n) \equiv 2^{-p\sharp\{j: s_j<0\}}(-1)^{|\mathbf{s}|}\Big(U(\overleftarrow{\mathbf{s}}; n) + p\sum_{j=1}^{\ell} |s_j| U(\overleftarrow{\mathbf{s}\oplus\mathbf{e}_j}; n)\Big) \pmod{p^2}. \tag{40}$$

Further,

$$V(-1) \equiv -\frac{1}{2^p}\Big(U(-1) + pU(-2) + p^2U(-3)\Big) \pmod{p^3}. \tag{41}$$

Lemma 5.5. *Let* p *be an odd prime. For all positive integers* d *and* m *we have*

$$\sum_{1\le n_1\le\cdots\le n_d\le m} \frac{(1-x)^{n_1}-1}{n_1\cdots n_d} = \sum_{j=1}^{m} \frac{(-x)^j}{j^d}\binom{m}{j} \tag{42}$$

$$\sum_{1\le n_1\le\cdots\le n_d\le p-1} \frac{(1-x)^{n_1}-1}{n_1\cdots n_d} \equiv \sum_{j=1}^{p-1} \frac{x^j}{j^d}\Big(1 - pH(1;j) + p^2H(1,1;j)\Big) \pmod{p^3}. \tag{43}$$

Proof. Since for all positive integer $j < p$

$$(-1)^j\binom{p-1}{j} \equiv 1 - pH(1;j) + p^2H(1,1;j) \pmod{p^3}, \tag{44}$$

congruence (43) follows easily from (42) which we now prove by mimicking the argument in [12].

Let $\mathbb{Q}^* = \mathbb{Q}\setminus\{0\}$. Define the injective operator

$$\Delta : \mathbb{Q}[x]\setminus\mathbb{Q}^* \longrightarrow \mathbb{Q}[x]\setminus\mathbb{Q}^*$$
$$p(x) \longmapsto xp'(x).$$

It suffices to show that

$$\Delta^d\left(\sum_{1\le n_1\le\cdots\le n_d\le m} \frac{(1-x)^{n_1}-1}{n_1\cdots n_d}\right) = \Delta^d\left(\sum_{j=1}^{m} \frac{(-x)^j}{j^d}\binom{m}{j}\right).$$

Clearly

$$\Delta^d\left(\sum_{j=1}^m \frac{(-x)^j}{j^d}\binom{m}{j}\right)=\sum_{j=1}^m(-x)^j\binom{m}{j}=(1-x)^m-1.$$

On the other hand,

$$\begin{aligned}&\Delta^d\left(\sum_{1\le n_1\le n_2\le\cdots\le n_d\le m}\frac{(1-x)^{n_1}-1}{n_1n_2\cdots n_d}\right)\\ =&\Delta^{d-1}\left(\sum_{1\le n_2\le\cdots\le n_d\le m}\frac{(1-x)^{n_2}-1}{n_2\cdots n_d}\right)\\ &\vdots\\ =&\Delta\left(\sum_{1\le n_d\le m}\frac{(1-x)^{n_d}-1}{n_d}\right)\\ =&-x\sum_{1\le n_d\le m}(1-x)^{n_d-1}\\ =&(1-x)^m-1.\end{aligned}$$

By injectivity of Δ this completes the proof of the lemma. □

Lemma 5.6. *Let d and m be two positive integers. Then*

$$\sum_{1\le n_1\le\cdots\le n_d\le m}\frac{(-1)^{n_d}(1-x)^{n_1}}{n_1\cdots n_d}\binom{m}{n_d}=\sum_{k=1}^m\frac{x^k}{k^d}-\sum_{k=1}^m\frac{1}{k^d}. \tag{45}$$

Proof. The proof, which is completely similar to [12], is left to the interested reader. Note that there is a misprint in [12] where in the definition of $f(x)$ the range of i should be from 1 to j. Namely

$$f(x)=\sum_{k=1}^n\frac{(-1)^{k-1}}{k}\binom{n}{k}\sum_{j=1}^k\frac{1}{j}\sum_{i=1}^j\frac{(1-x)^i}{i}.$$

□

Proposition 5.7. *Let $n\in\mathbb{N}$. For any prime $p>n+2$ write $H(-)=H(-;p-1)$ and similarly for S,U and V. Then*

$$\begin{aligned}H(-1,\{1\}^n)\equiv S(-1,\{1\}^n)&\equiv(-1)^nH(\{1\}^n,-1)\equiv(-1)^nS(\{1\}^n,-1)\\ &\equiv U(-n-1)\equiv(-1)^{n+1}2V(-n-1)\pmod p.\end{aligned}\tag{46}$$

Proof. The congruence $(-1)^nH(\{1\}^n,-1)\equiv U(-n-1)\pmod p$ is the content of [16, Theorem 2.3]. By taking $m=p-1$, $d=n+1$ and $x=2$ in Lemma 5.5 or Lemma 5.6 we get $S(-1,\{1\}^n)\equiv U(-n-1)\pmod p$. The congruence for V follows from (39). The other two congruences follows from the reversal relations (6). □

We now generalize this to the following

Proposition 5.8. *Let m,n be two nonnegative integers and a a positive integer. For any prime $p>a(m+n)+2$ write $H(-)=H(-;p-1)$. Then*

$$H(\{a\}^m,-a,\{a\}^n)\equiv(-1)^{m+n}S(\{a\}^n,-a,\{a\}^m)\pmod p, \tag{47}$$

$$H(\{a\}^m,-a,\{a\}^n)\equiv(-1)^{(m+n+1)(a+1)}S(\{a\}^m,-a,\{a\}^n)\pmod p. \tag{48}$$

Proof. The first congruence (47) follows from (49) in the next Lemma by taking $x=-1$, $k=m+1$ and $d=m+n+1$. The second congruence (48) follows (47) by the reversal relation. □

Let $a, d \in \mathbb{N}$ and a prime $p \geq da+3$. We identify the finite field $\mathbb{F}_p$ of p elements with $\mathbb{Z}/p\mathbb{Z}$. For $1 \leq k \leq d$ define

$$H_{d,k}^{(a)}(x) = \sum_{0<i_1<\cdots<i_d<p} \frac{x^{i_k}}{(i_1\cdots i_d)^a} \in \mathbb{F}_p[x],$$

$$S_{d,k}^{(a)}(x) = \sum_{1\leq i_1\leq\cdots\leq i_d<p} \frac{x^{i_k}}{(i_1\cdots i_d)^a} \in \mathbb{F}_p[x]$$

$$h_{d,k}^{(a)}(x) = \sum_{0<i_1<\cdots<i_d<p} \frac{(-1)^{\sum i_j} x^{i_k}}{(i_1\cdots i_d)^a} \in \mathbb{F}_p[x],$$

$$s_{d,k}^{(a)}(x) = \sum_{1\leq i_1\leq\cdots\leq i_d<p} \frac{(-1)^{\sum i_j} x^{i_k}}{(i_1\cdots i_d)^a} \in \mathbb{F}_p[x],$$

where $\sum i_j = i_1+\cdots+i_d$. For convenience we set $H_{d,0}^{(a)}(x) = H(\{a\}^d) = 0$, $S_{d,d+1}^{(a)}(x) = x^{p-1}S(\{a\}^d) = 0$, $S_{d,0}^{(a)}(x) = xS(\{a\}^d) = 0$ and $H_{d,d+1}^{(a)}(x) = x^pH(\{a\}^d) = 0$ by [18, Theorem 2.13]. Moreover for even a we set $h_{d,0}^{(a)}(x) = H(\{-a\}^d) = 0$, $s_{d,d+1}^{(a)}(x) = (-x)^{p-1}S(\{-a\}^d) = 0$, $s_{d,0}^{(a)}(x) = -xS(\{-a\}^d) = 0$ and $h_{d,d+1}^{(a)}(x) = (-x)^pH(\{-a\}^d) = 0$ by Lemma 5.1 and (12).

Lemma 5.9. *For $1 \leq k \leq d$ we have the identity in $\mathbb{F}_p[x]$*

$$H_{d,k}^{(a)}(x) + (-1)^d S_{d,d+1-k}^{(a)}(x) = 0, \tag{49}$$

$$h_{d,k}^{(a)}(x) + (-1)^d s_{d,d+1-k}^{(a)}(x) = 0. \tag{50}$$

Proof. To prove (49) we proceed by induction on d. For $d=1$ it holds since

$$H_{1,1}^{(a)}(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^a} = S_{1,1}^{(a)}(x), \quad H_{1,0}^{(a)}(x) = S_{1,0}^{(a)}(x) = H_{1,2}^{(a)}(x) = S_{1,2}^{(a)}(x) = 0.$$

Now we will follow the idea of the proof of [16, Theorem 2.3]. For $1 \leq k \leq d$ we have that

$$\begin{aligned}\left(x\frac{d}{dx}\right)^a H_{d,k}^{(a)}(x) &= \sum_{0<i_1<\cdots<i_{k-1}<i_{k+1}<\cdots<i_d<p} \frac{1}{(i_1\cdots i_{k-1}i_{k+1}\cdots i_d)^a} \sum_{i_k=i_{k-1}+1}^{i_{k+1}-1} x^{i_k} \\ &= \sum_{0<i_1<\cdots<i_{k-1}<i_{k+1}<\cdots<i_d<p} \frac{1}{(i_1\cdots i_{k-1}i_{k+1}\cdots i_d)^a} \cdot \frac{x^{i_{k+1}} - x^{i_{k-1}+1}}{x-1} \\ &= \frac{1}{x-1} H_{d-1,k}^{(a)}(x) - \frac{x}{x-1} H_{d-1,k-1}^{(a)}(x),\end{aligned}$$

that is

$$(x-1)\left(x\frac{d}{dx}\right)^a H_{d,k}^{(a)}(x) = H_{d-1,k}^{(a)}(x) - xH_{d-1,k-1}^{(a)}(x).$$

In a similar way we have that

$$(x-1)\left(x\frac{d}{dx}\right)^a S_{d,k}^{(a)}(x) = xS_{d-1,k}^{(a)}(x) - S_{d-1,k-1}^{(a)}(x).$$

Hence

$$\begin{aligned}&(x-1)\left(x\frac{d}{dx}\right)^a \left(H_{d,k}^{(a)}(x) + (-1)^d S_{d,d+1-k}^{(a)}(x)\right) \\ =&H_{d-1,k}^{(a)}(x) - xH_{d-1,k-1}^{(a)}(x) + (-1)^d xS_{d-1,d+1-k}^{(a)}(x) - (-1)^d S_{d-1,d-k}^{(a)}(x) \\ =&H_{d-1,k}^{(a)}(x) + (-1)^{d-1} S_{d-1,d-k}^{(a)}(x) - x\left(H_{d-1,k-1}^{(a)}(x) + (-1)^{d-1} S_{d-1,d-(k-1)}^{(a)}(x)\right) = 0.\end{aligned}$$

Thus $\left(x\frac{d}{dx}\right)^{a-1}\left(H_{d,k}^{(a)}(x) + (-1)^d S_{d,d+1-k}^{(a)}(x)\right) = c$ for some constant $c \in \mathbb{F}_p$ since this polynomial has degree less than p. By letting $x=0$ we see that $c=0$. Repeating this process a times yields (49).

The congruence (50) can be proved in a similar way. In particular, we can show easily that

$$-(x+1)\left(x\frac{d}{dx}\right)^a h^{(a)}_{d,k}(x) = h^{(a)}_{d-1,k}(-x) + xh^{(a)}_{d-1,k-1}(-x),$$
$$(x+1)\left(x\frac{d}{dx}\right)^a s^{(a)}_{d,k}(x) = xs^{(a)}_{d-1,k}(-x) + s^{(a)}_{d-1,k-1}(-x).$$

So by induction we get

$$(x+1)\left(x\frac{d}{dx}\right)^a \left(H^{(a)}_{d,k}(x) + (-1)^d S^{(a)}_{d,d+1-k}(x)\right) = 0,$$

which quickly leads to (50). This concludes the proof of the lemma. □

Similar to Proposition 5.8 we also have the following

Proposition 5.10. *Let m, n be two nonnegative integers. Let a be a positive even integer. For any prime $p > a(m+n)+2$ write $H(-) = H(-; p-1)$. Then*

$$H(\{-a\}^m, a, \{-a\}^n) \equiv S(\{-a\}^m, a, \{-a\}^n) \pmod p, \tag{51}$$
$$H(\{-a\}^m, a, \{-a\}^n) \equiv (-1)^{(m+n)} S(\{-a\}^n, a, \{-a\}^m) \pmod p. \tag{52}$$

Proof. The second congruence (51) follows from (50) by taking $x = -1$, $k = m+1$ and $d = m+n+1$. The first congruence (52) follows (51) by the reversal relation since a is even. □

6. AMHS of weight four

In [16] the first author studied the congruence properties of AMHS of weight less than four. In this section, applying the results obtained in the previous sections we can analyze the weight four AMHS in some detail. First we treat some special congruences which can not be obtained by just using the stuffle relations and the reversal relations. Let $p \geq 7$ be a prime and set $A = A_p, \cdots, K = K_p$ as follows:

$$A := \sum_{k=2}^{p-3} B_k B_{p-3-k}, \quad B := \sum_{k=2}^{p-3} 2^k B_k B_{p-3-k}, \quad C := \sum_{k=2}^{p-3} 2^{p-3-k} B_k B_{p-3-k},$$
$$D := \sum_{k=2}^{p-3} \frac{B_k B_{p-3-k}}{k}, \quad E := \sum_{k=2}^{p-3} \frac{2^k B_k B_{p-3-k}}{k}, \quad F := \sum_{k=2}^{p-3} \frac{2^{p-3-k} B_k B_{p-3-k}}{k},$$
$$G := \sum_{k=2}^{p-3} k B_k B_{p-3-k}, \quad J := \sum_{k=2}^{p-3} 2^k k B_k B_{p-3-k}, \quad K := \sum_{k=2}^{p-3} 2^{p-3-k} k B_k B_{p-3-k}.$$

Then by [18, Cororllary 3.6] and simple computation

$$A \equiv -B_{p-3}, \quad G \equiv 0, \quad C \equiv B - \frac{3}{4}A, \quad K \equiv -3B - J + 3A \pmod p. \tag{53}$$

Proposition 6.1. *For all prime $p \geq 7$ write $H(-) = H(-; p-1)$. Then we have*

$$H(1,-3) \equiv \frac{1}{2} H(-2,2) \equiv B - A \equiv \sum_{k=0}^{p-3} 2^k B_k B_{p-3-k} \pmod p, \tag{54}$$
$$H(-1,3) \equiv -\frac{1}{2} q_p B_{p-3} \pmod p. \tag{55}$$

Proof. We take congruence modulo p throughout this proof. By Proposition 3.3 we have

$$H(-3,1) \equiv -\frac{1}{3}\sum_{k=1}^{p-6}(k+2)(k+3)(1-2^{k+1})(1-2^{p-4-k})\frac{B_{k+1}B_{p-4-k}}{k+4} - \frac{1}{2}q_p B_{p-3}$$
$$\equiv \frac{1}{3}\sum_{k=2}^{p-3}(k+1)(k+2)(1-2^k)(1-2^{p-3-k})\frac{B_k B_{p-3-k}}{k} - \frac{1}{2}q_p B_{p-3}.$$

by the substitution $k \to p-4-k$. Similarly we can get

$$H(-2,2) \equiv -\sum_{k=2}^{p-3}(k+2)(1-2^k)(1-2^{p-3-k})\frac{B_kB_{p-3-k}}{k} + \frac{3}{2}q_pB_{p-3}$$

$$H(-1,3) \equiv 2\sum_{k=2}^{p-3}(1-2^k)(1-2^{p-3-k})\frac{B_kB_{p-3-k}}{k} - 2q_pB_{p-3}$$

Using (53) we reduce the above to

$$3H(-3,1) \equiv 3A - 3B + \frac{5}{2}D - 2E - 2F - \frac{3}{2}q_pB_{p-3} \tag{56}$$

$$H(-2,2) \equiv 2B - 2A - \frac{5}{2}D + 2E + 2F + \frac{3}{2}q_pB_{p-3} \tag{57}$$

$$H(-1,3) \equiv \qquad + \frac{5}{2}D - 2E - 2F - 2q_pB_{p-3} \tag{58}$$

On the other hand, by Proposition 3.2 we get

$$\begin{aligned}
H(1,-3) \equiv & \frac{2}{p-1}\sum_{k=0}^{p-2}\binom{p-1}{k}\frac{1-2^{2p-4-k}}{2p-4-k}B_kB_{2p-4-k} \\
\equiv & -2\sum_{k=0}^{p-5}\frac{1-2^{p-3-k}}{p-3-k}B_kB_{p-3-k} + 2q_pB_{p-3} \\
\equiv & -2\sum_{k=2}^{p-3}\frac{1-2^k}{k}B_kB_{p-3-k} + 2q_pB_{p-3} \\
\equiv & 2E - 2D + 2q_pB_{p-3}.
\end{aligned}$$

by the substitution $k \to p-3-k$. Thus by the reversal relation

$$H(-3,1) \equiv -H(1,-3) \equiv 2D - 2E - 2q_pB_{p-3}. \tag{59}$$

Similarly we find

$$H(-2,2) \equiv -H(2,-2) \equiv B - A + 2E - 2D + 2q_pB_{p-3}, \tag{60}$$

$$H(-1,3) \equiv -H(3,-1) \equiv \frac{1}{3}\Big(-J + 3A - 3B + 2D - 2E\Big). \tag{61}$$

Then by adding (56), (57), (59) and (60) altogether we have

$$4H(-3,1) + 2H(-2,2) \equiv 0$$

which implies the first congruence in (54). Now adding (59) and (60) yields

$$-H(-3,1) \equiv H(-3,1) + H(-2,2) \equiv B - A \tag{62}$$

which is the second congruence in (54). Plugging this into (56) we see that

$$\frac{5}{2}D - 2E - 2F - \frac{3}{2}q_pB_{p-3} \equiv 0$$

which combined with (58) produces (55). This finishes the proof of the proposition. □

Proposition 6.2. *For all prime $p \ge 7$ write $H(-) = H(-;p-1)$. Then we have*

$$H(1,-1,-2) \equiv \frac{1}{2}H(1,-3) + \frac{1}{2}J \pmod p, \tag{63}$$

$$H(1,-2,-1) \equiv H(1,-3) - \frac{5}{4}q_pB_{p-3} \pmod p, \tag{64}$$

$$H(2,-1,-1) \equiv -H(1,-3) - \frac{1}{2}J + \frac{3}{4}q_pB_{p-3} \pmod p. \tag{65}$$

Proof. By Theorem 4.2, Proposition 6.1 and (53) we get

$$
\begin{aligned}
2H(1,-1,-2) &\equiv -\sum_{k=2}^{p-3}(k+1)(1-2^k)B_kB_{p-3-k} \equiv B-A+J-G = H(1,-3)+J \qquad (66)\\
H(1,-2,-1) &\equiv -\sum_{k=2}^{p-3}(1-2^k)B_kB_{p-3-k} - \frac{5}{4}q_pB_{p-3} \equiv H(1,-3) - \frac{5}{4}q_pB_{p-3}\\
H(2,-1,-1) &\equiv \sum_{k=2}^{p-3}\frac{(k+2)(1-2^k)B_kB_{p-3-k}}{-2} + \frac{3}{4}q_pB_{p-3} \equiv -H(1,-3) - \frac{1}{2}J + \frac{3}{4}q_pB_{p-3},
\end{aligned}
$$

as claimed. ☐

By (23) and (62) it is readily seen that

$$2H(1,1,-2) \equiv H(-3,1) + H(-2,2) \equiv -H(-3,1).$$

In fact, by the stuffle and the reversal relations we can find congruences of all weight four AMHS of depth up to three. By the reversal relations we only need to list about half of the values.

Proposition 6.3. *For all prime $p \geq 7$ write $H(-) = H(-;p-1)$ and set $H_{\bar{2}11} := H(-2,1,1)$. Then $H_{\bar{2}11} = (A-B)/2$ and*

$$
\begin{aligned}
&H(4) \equiv H(-4) \equiv H(2,2) \equiv H(-2,-2) \equiv H(1,3) \equiv H(1,-2,1) \equiv H(-1,-2,-1) \equiv 0,\\
&H(1,-3) \equiv -2H_{\bar{2}11}, \quad H(2,-2) \equiv 4H_{\bar{2}11}, \quad H(1,-1,2) \equiv 3H_{\bar{2}11},\\
&H(-1,-3) \equiv \tfrac{1}{2}q_pB_{p-3}, \qquad H(3,-1) \equiv \tfrac{1}{2}q_pB_{p-3},\\
&H(1,-1,-2) \equiv -H_{\bar{2}11} + \tfrac{1}{2}J, \qquad H(-2,-1,-1) \equiv 2H_{\bar{2}11} - q_pB_{p-3},\\
&H(-1,2,1) \equiv -H_{\bar{2}11} + \tfrac{5}{4}q_pB_{p-3}, \qquad H(-1,1,2) \equiv 2H_{\bar{2}11} - \tfrac{3}{4}q_pB_{p-3},\\
&H(-2,1,-1) \equiv 3H_{\bar{2}11} - \tfrac{1}{2}J + \tfrac{3}{4}q_pB_{p-3}, \qquad H(1,-2,-1) \equiv -2H_{\bar{2}11} - \tfrac{5}{4}q_pB_{p-3},\\
&H(-1,2,-1) \equiv -4H_{\bar{2}11} + J - \tfrac{5}{2}q_pB_{p-3}, \qquad H(2,-1,-1) \equiv 2H_{\bar{2}11} - \tfrac{1}{2}J + \tfrac{3}{4}q_pB_{p-3}.
\end{aligned}
$$

We now turn to the depth four cases.

Proposition 6.4. *Let $p \geq 7$ be a prime and write $H(-) = H(-;p-1)$. Then we have*

$$
\begin{aligned}
H(1,-1,-1,1) &\equiv -\frac{1}{2}\Big(H(1,-3) + J + q_p^4\Big) \pmod p, \qquad (67)\\
H(-1,-1,1,1) \equiv H(1,1,-1,-1) &\equiv \frac{1}{24}\Big(6J + 7q_pB_{p-3} + 8q_p^4\Big) \pmod p, \qquad (68)\\
H(-1,1,-1,1) \equiv H(1,-1,1,-1) &\equiv -\frac{1}{12}\Big(q_pB_{p-3} + 2q_p^4\Big) \pmod p, \qquad (69)\\
H(-1,1,1,-1) &\equiv \frac{1}{12}\Big(6H(1,-3) + 7q_pB_{p-3} + 2q_p^4\Big) \pmod p. \qquad (70)
\end{aligned}
$$

Proof. By Theorem 2.5

$$
\begin{aligned}
H(1,-1,-1,1) &\equiv -H(-1,-1,1) - \frac{1}{2}H(-2,-1,1) - \sum_{k=2}^{p-3}B_kH(-(k+1),-1,1),\\
H(1,1,-1,-1) &\equiv -H(1,-1,-1) - \frac{1}{2}H(2,-1,-1) - \sum_{k=2}^{p-3}B_kH(k+1,-1,-1).
\end{aligned}
$$

Using reversal relations and (26) we see that

$$\begin{aligned} H(1,-1,-1,1) \equiv &- H(-1,-1,1) - \frac{1}{2}H(-2,-1,1) \\ &+ \frac{1}{2}\sum_{k=2}^{p-3} B_k\Big(H(k+2,1) + H(-(k+1),-2) - H(-(k+1))H(-1,1)\Big), \quad (71) \\ H(1,1,-1,-1) \equiv &- H(1,-1,-1) - \frac{1}{2}H(2,-1,-1) \\ &- \frac{1}{2}\sum_{k=2}^{p-3} B_k\Big(H(2,k+1) + H(-1,-(k+2)) - H(-1)H(-1,k+1)\Big). \quad (72) \end{aligned}$$

Note that by Theorem 2.5

$$\begin{aligned} -H(-1,1) \equiv &H(1,-1) \equiv -\sum_{k=0}^{p-3}(-1)^k B_k H(-(k+1)), \\ H(1,1,-1) \equiv &- H(1,-1) - \frac{1}{2}H(2,-1) + \sum_{k=2}^{p-3} B_k H(-1,k+1). \end{aligned}$$

We may use (14), and (15) to simplify (71) and (72) further. For all $k = 2, \dots, p-5$ we have

$$H(k+2,1) \equiv - B_{p-3-k}, \qquad H(-(k+1),-2) \equiv \frac{1}{2}(2^{p-3-k}-1)(k+2)B_{p-3-k}, \quad (73)$$

$$H(2,k+1) \equiv -\frac{(k+2)B_{p-3-k}}{2}, \qquad H(-1,-(k+2)) \equiv (2^{p-3-k}-1)B_{p-3-k}. \quad (74)$$

However, one has to be very careful in applying these formulae because the formulae might fail when $k = p-3$. We need to compute these separately as follows:

$$\begin{aligned} H(p-1,1) =&\sum_{i=1}^{p-1}\frac{1}{i}\sum_{j=1}^{i-1}\frac{1}{j^{p-1}} \equiv \sum_{i=1}^{p-1}\frac{i-1}{i} \equiv -1 \equiv -B_0, \\ H(-(p-2),-2) =&\sum_{i=1}^{p-1}\frac{(-1)^i}{i^2}\sum_{j=1}^{i-1}\frac{(-1)^j}{j^{p-2}} \equiv \sum_{i=1}^{p-1}\frac{(-1)^i}{i^2}\left((-1)^i\frac{1-2i}{4} - \frac{1}{4}\right) \equiv 0, \\ H(2,p-2) \equiv &H(p-2,2) = -\sum_{i=1}^{p-1}\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j^{p-2}} \equiv -\sum_{i=1}^{p-1}\frac{i-1}{2i} \equiv \frac{1}{2} \equiv 0, \\ H(-1,-(p-1)) \equiv &- H(-(p-1),-1) = -\sum_{i=1}^{p-1}\frac{(-1)^i}{i}\sum_{j=1}^{i-1}\frac{(-1)^j}{j^{p-1}} \equiv \sum_{i=1}^{p-1}\frac{1+(-1)^i}{2i} \equiv \frac{1}{2}H(-1) \equiv -q_p \end{aligned}$$

by (31). We see that only $H(-1,-(p-1))$ fails the formula in (74) and therefore we get

$$\begin{aligned} H(1,-1,-1,1) \equiv &- H(-1,-1,1) - \frac{1}{2}H(-2,-1,1) + \frac{1}{2}H(-1)H(-1,1) - \frac{1}{2}H(-1,1)^2 \\ &+ \frac{1}{2}\sum_{k=2}^{p-3} B_k\left(-B_{p-3-k} + \frac{1}{2}(2^{p-3-k}-1)(k+2)B_{p-3-k}\right) \end{aligned}$$

and

$$\begin{aligned} H(1,1,-1,-1) \equiv &H(-1,-1,1) - \frac{1}{2}H(2,-1,-1) + \frac{1}{2}H(-1)\Big(H(1,1,-1) + H(1,-1) + \frac{1}{2}H(2,-1)\Big) \\ &- \frac{1}{2}\sum_{k=2}^{p-3} B_k\left(-\frac{(k+2)B_{p-3-k}}{2} + (2^{p-3-k}-1)B_{p-3-k}\right) + \frac{1}{2}q_p B_{p-3}. \end{aligned}$$

Now by [16, Cororllary 2.4, Cororllary 2.5] we know $H(-1,1)\equiv -q_p^2$, $H(-1,-1,1)\equiv q_p^3+\frac{7}{8}B_{p-3}$, and $H(1,1,-1)\equiv -\frac{1}{3}q_p^3-\frac{7}{24}B_{p-3}$. Together with (31) these yield

$$H(1,-1,-1,1)\equiv -\frac{1}{2}B_{p-3}-\frac{1}{2}H(-2,-1,1)-\frac{1}{2}q_p^4-\frac{1}{2}A+\frac{1}{4}\sum_{k=0}^{p-3}(2^{p-3-k}-1)(k+2)B_kB_{p-3-k},$$

$$H(1,1,-1,-1)\equiv B_{p-3}-\frac{1}{2}H(2,-1,-1)+\frac{1}{3}q_p^4+\frac{2}{3}q_pB_{p-3}+A+\frac{1}{4}G-\frac{1}{2}\sum_{k=0}^{p-3}2^{p-3-k}B_kB_{p-3-k}.$$

It now follows from (53) and the substitution $k\to p-3-k$ that

$$H(1,-1,-1,1)\equiv -\frac{1}{2}H(-2,-1,1)-\frac{1}{2}q_p^4+\frac{1}{4}\sum_{k=0}^{p-3}(1-2^k)(k+1)B_kB_{p-3-k},$$

$$H(1,1,-1,-1)\equiv -\frac{1}{2}H(2,-1,-1)-\frac{1}{2}H(1,-3)+\frac{2}{3}q_pB_{p-3-k}+\frac{1}{3}q_p^4.$$

Hence (67) and (68) quickly follow from (66) and (65).

Finally, (69) follows from stuffle relations applied to $H(-1)H(1,-1,1)$ and then (70) from stuffle relations applied to $H(-1)H(1,1,-1)$. This finishes the proof of the proposition. □

For other depth four and weight four AMHS we have the following relations derived from the stuffle relations and the congruences obtained above:

$$\begin{aligned}
H(1,1,-1,1)\equiv{}&2H_{\bar{2}11}+3H(-1,1,1,1)+\frac{1}{2}q_pB_{p-3},\\
H(-1,-1,1,-1)\equiv{}&6H_{\bar{2}11}+3H(1,-1,-1,-1)-4q_pB_{p-3}-2q_p^4,\\
H(-1,-1,-1,-1)\equiv{}&\frac{1}{3}q_pB_{p-3}+\frac{2}{3}q_p^4.
\end{aligned}$$

On the other hand, we can only deduce from Theorem 2.5 and Theorem 2.6 that

$$\begin{aligned}
H(1,1,1,-1)\equiv{}&-H(1,1,-1)-\frac{1}{2}H(2,1,-1)-\sum_{k=2}^{p-3}B_kH(k+1,1,-1),\\
H(-1,-1,-1,1)\equiv{}&-q\Big(H(-1,-1,1)-H(1,-1,1)\Big)-\frac{1}{2}H(2,-1,1)\\
&+\sum_{k=2}^{p-3}(1-2^k)B_kH(k+1,-1,1),
\end{aligned}$$

where, by the reduction theorems again,

$$\begin{aligned}
H(k+1,1,-1)\equiv{}&-\frac{1}{k+1}H(k+1,-1)-\frac{1}{2}H(k+2,-1)\\
&+\sum_{j=2}^{p-k-2}\binom{p-k-1}{j}\frac{B_j}{p-k-1}H(j+k+1,-1),\\
H(k+1,-1,1)\equiv{}&-\frac{1}{k+1}H(-(k+1),1)-\frac{1}{2}H(-(k+2),1)\\
&+\sum_{j=2}^{p-k-2}\binom{p-k-1}{j}\frac{B_j}{p-k-1}H(-(j+k+1),1).
\end{aligned}$$

Observe that the indices k and j in the above sums can be both taken to be even numbers. Thus by Proposition 3.2 and Proposition 3.3

$$\begin{aligned}
H(j+k+1,-1)\equiv{}&\sum_{i=0}^{p-j-k-2}\binom{p-j-k-1}{i}\frac{2(1-2^{2p-i-j-k-2})B_iB_{2p-i-j-k-2}}{(p-j-k-1)(2p-i-j-k-2)},\\
\equiv{}&\sum_{i=0}^{p-j-k-2}\binom{p-j-k-1}{i}\frac{2(1-2^{p-i-j-k-1})B_iB_{p-i-j-k-1}}{(j+k+1)(i+j+k+1)},
\end{aligned}$$

$$
\begin{aligned}
H(-(j+k+1),1) \equiv & \sum_{i=1}^{p-j-k-4} \binom{p-2-j-k}{i} \frac{2(1-2^{i+1})(1-2^{p-i-j-k-2})B_{i+1}B_{p-i-j-k-2}}{(i+1)(i+j+k+2)} \\
& + \frac{2(1-2^{p-j-k-1})(1-2^{p-1})B_{p-j-k-1}B_{p-1}}{p-j-k-1} \\
\equiv & \sum_{i=2}^{p-j-k-3} \binom{p-2-j-k}{i} \frac{2(1-2^{i})(1-2^{p-i-j-k-1})B_{i}B_{p-i-j-k-1}}{i(i+j+k+1)} \\
& - q_p \frac{2(1-2^{p-j-k-1})B_{p-j-k-1}}{j+k+1}.
\end{aligned}
$$

Consequently, both $H(1,1,1,-1)$ and $H(-1,-1,-1,1)$ can be written as a triple sum with most of the terms involving products $B_iB_jB_kB_{p-i-j-k-2}$. It is very likely that modulo p we cannot reduce $H(1,1,1,-1)$ and $H(-1,-1,-1,1)$ to a linear combination of AMHS of depths up to three. At least in theory one possible way to check this hypothesis is to find six infinite sets of primes $S_1 = \{p_1^{(k)} : k \geq 1\}, \dots, S_6 = \{p_6^{(k)} : k \geq 1\}$ for each of the following six elements:

$$
\begin{aligned}
&b_1(p) = J_p, \quad b_2(p) = H(1,-3), \quad b_3(p) = H(1,-1,-1,-1), \\
&b_4(p) = q_p^4, \quad b_5(p) = q_pB_{p-3}, \quad b_6(p) = H(-1,1,1,1),
\end{aligned}
$$

such that for each choice $(p_1^{(k)},\dots,p_6^{(k)})$ we always have $b_j(p_i^{(k)}) \equiv 0 \pmod{p_i^{(k)}}$ for all $i \neq j$ and $b_j(p_j^{(k)}) \not\equiv 0 \pmod{p_j^{(k)}}$ for all $j = 1,\dots,6$. In practice this is extremely difficult to carry out. For example, if $b_4(p) \equiv 0 \pmod p$ then the prime p is called a Wieferich prime. The only known Wieferich primes are 1093 and 3511 and if any other Wieferich primes exist, they must be greater than 6.7×10^{15} according to [6]. It turns out that

$$
\begin{aligned}
[J_p, H(1,-3), H(1,-1,-1,-1), H(-1,1,1,1)] &\equiv [1023, 529, 670, 952] \pmod{1093}, \\
[J_p, H(1,-3), H(1,-1,-1,-1), H(-1,1,1,1)] &\equiv [1618, 2160, 1620, 540] \pmod{3511}.
\end{aligned}
$$

In order to understand the general mod p structure of AMHS we need to consider some infinite algebras similar to the adeles (see [21]).

7. Congruence modulo prime powers

In this last section we shall study the congruence properties of AMHS of small weights modulo higher powers of primes p. We first need some results concerning the sums $U(\mathbf{s}; p-1)$ defined by (37).

Proposition 7.1. *Let p be a prime ≥ 7. Let A and B be defined as in §6. Write $U(-) = U(-; p-1)$. Then we have*

$$U(-1) \equiv -2q_p - \frac{7}{12}p^2B_{p-3} \pmod{p^3}, \tag{75}$$

$$U(-2) \equiv -q_p^2 + \frac{2}{3}pq_p^3 + \frac{7}{6}pB_{p-3} \pmod{p^2}, \tag{76}$$

$$U(-1,1) \equiv q_p^2 - \frac{2}{3}pq_p^3 - \frac{1}{12}pB_{p-3} \pmod{p^2}, \tag{77}$$

$$U(1,-1) \equiv -\frac{13}{12}pB_{p-3} \pmod{p^2}, \tag{78}$$

$$U(-3) \equiv -\frac{1}{3}q_p^3 - \frac{7}{24}B_{p-3} \pmod{p}, \tag{79}$$

$$U(-2,1) \equiv \frac{1}{3}q_p^3 - \frac{23}{24}B_{p-3} \pmod{p}, \tag{80}$$

$$U(1,-2) \equiv \frac{5}{4}B_{p-3} \pmod{p}, \tag{81}$$

$$U(2,-1) \equiv -\frac{3}{4}B_{p-3} \pmod{p}, \tag{82}$$

$$U(-1,2) \equiv \frac{1}{3}q_p^3 + \frac{25}{24}B_{p-3} \pmod p, \quad (83)$$

$$U(1,1,-1) \equiv -\frac{1}{2}B_{p-3} \pmod p, \quad (84)$$

$$U(1,-1,1) \equiv \frac{1}{2}B_{p-3} \pmod p, \quad (85)$$

$$U(-1,1,1) \equiv -\frac{1}{3}q_p^3 - \frac{7}{24}B_{p-3} \pmod p, \quad (86)$$

$$U(-4) \equiv H(-1,1,1,1) \equiv -H(1,1,1,-1) \pmod p, \quad (87)$$

$$U(1,-3) \equiv A - B + \frac{5}{4}q_pB_{p-3} \pmod p, \quad (88)$$

$$U(-3,1) \equiv H(1,1,1,-1) + B - A - \frac{5}{4}q_pB_{p-3} \pmod p. \quad (89)$$

Proof. Throughout the proof we write $H(-) = H(-;p-1)$ and similarly for U and V. We will prove the congruences in the following order: (79), (75), (80), (81), (76), (77), (78), (84), (82), (83), and (85) to (89).

First, (79) follows from [5, (4)] and $H(-3) \equiv -\frac{1}{2}B_{p-3}$ (take $a = 3$ in (12)). Taking $d = 1$ and $m = p-1$ in (42) we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(1-x)^k-1}{k} =& \sum_{j=1}^{p-1}\frac{(-x)^j}{j}\binom{p-1}{j} = \sum_{j=1}^{p-1}\frac{(-x)^j}{j}\left(\binom{p}{j}-\binom{p-1}{j-1}\right)\\ =& -p\sum_{j=1}^{p-1}\frac{x^j}{j^2}(-1)^{j-1}\binom{p-1}{j-1} - \frac{1}{p}\sum_{j=1}^{p-1}(-x)^j\binom{p}{j}\\ =& -p\sum_{j=1}^{p-1}\frac{x^j}{j^2}(-1)^{j-1}\binom{p-1}{j-1} + \frac{(x-1)^p-x^p+1}{p}. \quad (90)\end{aligned}$$

Since

$$(-1)^{j-1}\binom{p-1}{j-1} \equiv 1 - pH(1;j-1) \pmod{p^2}$$

letting $x = -1$ in (90) we get

$$U(-1) - H(1) \equiv -pH(-2) + p^2H(1,-2) - 2q_p \pmod{p^3}. \quad (91)$$

So it is not hard to see that (75) can be obtained from the following: $H(1,-2) \equiv \frac{1}{4}B_{p-3} \pmod p$ by [16, Cororllary 2.4], $H(-2) \equiv \frac{1}{2}pB_{p-3} \pmod{p^2}$ by (12), and the well-known fact $H(1) \equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}$ (see, for e.g., (3)). Letting $x = 1/2$ in (90) we have that

$$V(-1) - H(1) \equiv -pV(-2) + p^2V(1,-2) + \frac{q_p}{2^{p-1}} \pmod{p^3}.$$

Multiplied by -2^p this yields by Lemma 5.4

$$U(-1) + 2^pH(1) \equiv p^2U(-3) + p^2U(-2,1) - 2q_p \pmod{p^3}. \quad (92)$$

Thus (80) follows from (75), (79) and $H(1) \equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}$. Further, by the stuffle relation

$$U(1,-2) \equiv H(1)U(-2) - U(-2,1) - U(-3) \equiv \frac{5}{4}B_{p-3} \pmod p$$

we get (81). Now letting $x = 2$ in (90) we see that

$$H(-1) - H(1) \equiv -pU(-2) + p^2U(1,-2) - 2q_p \pmod{p^3}. \quad (93)$$

Hence (76) follows from (81) and (31). Then taking $d = 2$ and $x = -1$ in (43) we get

$$\begin{aligned}U(-1,1) + U(-2) \equiv& H(1,1) + H(2) + H(-2) - pH(1,-2) - pH(-3)\\ \equiv& \frac{13}{12}pB_{p-3} \pmod{p^2}. \quad (94)\end{aligned}$$

Thus $U(1,-1)=U(-1)H(1)-U(-1,1)-U(-2)\equiv -\frac{13}{12}pB_{p-3}\pmod{p^2}$ which is (77). Then (78) follows from the stuffle relation $U(-1,1)=H(1)U(-1)-U(1,-1)-U(-2)$. Moreover, taking $d=1$ and $x=2$ in (43) we get

$$H(-1)-H(1)\equiv U(-1)-p\Big(U(-2)+U(1,-1)\Big)+p^2\Big(U(1,-2)+U(1,1,-1)\Big). \pmod{p^3}$$

This implies (84) because of (75), (76), (77) and (81).

Next, taking $d=3$ and $x=-1$ in (43) we get

$$\begin{aligned} U(-1,1,1)+U(-1,2)+U(-2,1)\equiv& H(-3)+S(1,1,1)-U(-3)\\ \equiv&\frac{1}{3}q_p^3-\frac{5}{24}B_{p-3}\pmod{p} \end{aligned}\tag{95}$$

since $S(1,1,1)=H(1,1,1)+H(1)H(2)\equiv 0\pmod{p}$. Taking $d=3$ and $x=1/2$ in (43) we get

$$V(-1,1,1)+V(-2,1)+V(-1,2)\equiv 0 \pmod{p}$$

which implies by Lemma 5.4

$$U(1,1,-1)+U(1,-2)+U(2,-1)\equiv 0 \pmod{p}. \tag{96}$$

Thus (82) follows from (81) and (84) immediately. Then (83) follows easily from the stuffle relation of $H(2)U(-1)$. Also, (85) and (86) follow from the stuffle relations:

$$\begin{aligned} H(1)U(1,-1)=&U(2,-1)+U(1,-2)+2U(1,1,-1)+U(1,-1,1)\equiv 0 \pmod{p},\\ H(1)U(-1,1)=&U(-2,1)+U(-1,2)+2U(-1,1,1)+U(1,-1,1)\equiv 0 \pmod{p}. \end{aligned}$$

We now turn to the last three congruences of weight four. By [16, Theorem 2.3] we know (87) holds. Hence taking $d=4$, $m=p-1$ and $x=2$ in Lemma 5.6 and using (44) we get

$$\begin{aligned} S(-1,1,1)+p\Big(U(-4)+H(-1,2,1)+H(-2,1,1)+H(-3,1)-H(1)S(-1,1,1)\Big)&\\ \equiv U(-3)-H(1) \pmod{p^2}.& \end{aligned}\tag{97}$$

On the other hand, taking $d=2$ and $x=2$ in (43) we get

$$S(-1,1,1)-S(1,1,1)\equiv U(-3)-pU(-4)-pU(1,-3) \pmod{p^2}. \tag{98}$$

Comparing (97) and (98), using $S(1,1,1)=H(1,1,1)+H(1)H(2)\equiv 0\pmod{p^2}$ by [18, Theorem 2.13], and $H(-1,2,1)+H(-2,1,1)\equiv\frac{5}{4}q_pB_{p-3}\pmod{p}$ by Proposition 6.3, we can deduce (88). Finally, (89) follows from (87) and the stuffle relation $U(-3,1)=H(1)U(-3)-U(1,-3)-U(-4)$. This finishes the proof of the proposition. □

Lemma 7.2. *Let $a,b\in\mathbb{Z}^*$ and write $H(-)=H(-;p-1)$. Then*

$$H(a,b)\equiv(-1)^{a+b}\operatorname{sgn}(ab)\Big(H(b,a)+p|b|H(\operatorname{sgn}(b)+b,a)+p|a|H(b,\operatorname{sgn}(a)+a)\Big) \pmod{p^2}.$$

Proof. By definition

$$\begin{aligned} H(a,b)=&\sum_{1\le m<n<p}\frac{\operatorname{sgn}(a)^m\operatorname{sgn}(b)^n}{m^{|a|}n^{|b|}}=\sum_{1\le n<m<p}\frac{\operatorname{sgn}(a)^{p-m}\operatorname{sgn}(b)^{p-n}}{(p-m)^{|a|}(p-n)^{|b|}}\\ =&(-1)^{a+b}\operatorname{sgn}(ab)\sum_{1\le n<m<p}\frac{\operatorname{sgn}(b)^n\operatorname{sgn}(a)^m}{n^{|b|}m^{|a|}}\frac{1}{(1-p/n)^{|b|}(1-p/m)^{|a|}}. \end{aligned}$$

The lemma follows easily. □

Proposition 7.3. *Let A and B be defined as in* §6. *For all prime $p\ge 7$ write $H(-)=H(-;p-1)$ and set (see Theorem* 1.1*(b))*

$$X=X_p(3):=\frac{B_{p-3}}{p-3}-\frac{B_{2p-4}}{4p-8}.$$

Then we have

$$\begin{aligned} H(-1,-1) \equiv& 2q_p^2 + p\left(2X - 2q_p^3\right) + p^2\left(\frac{11}{6}q_p^4 + \frac{1}{2}q_pB_{p-3}\right) && \pmod{p^3} && (99)\\ \equiv& 2q_p^2 - 2pq_p^3 - \frac{1}{3}pB_{p-3} && \pmod{p^2}, && (100)\\ H(1,-1) \equiv& q_p^2 - pq_p^3 - \frac{13}{24}pB_{p-3} && \pmod{p^2}, && (101)\\ H(-1,1) \equiv& -q_p^2 + pq_p^3 + \frac{1}{24}pB_{p-3} && \pmod{p^2}, && (102)\\ H(-3) \equiv& 3X && \pmod{p^2}, && (103)\\ H(-2,1) \equiv& H(1,-2) \equiv -\frac{3}{2}X && \pmod{p^2}, && (104)\\ H(2,-1) \equiv& -\frac{3}{2}X - \frac{7}{6}pq_pB_{p-3} + p(B-A) && \pmod{p^2}, && (105)\\ H(-1,2) \equiv& -\frac{3}{2}X - \frac{1}{6}pq_pB_{p-3} + p(A-B) && \pmod{p^2}. && (106) \end{aligned}$$

Proof. First, (99) follows readily from the shuffle relation $H(-1)^2 = 2H(-1,-1)+H(2)$, the congruence (31), and $H(2) \equiv -4pX \pmod{p^3}$ by (3). This clearly implies (100) since by Kummer congruence $X \equiv -B_{p-3}/6 \pmod p$.

Next, taking $d = 1$, and $x = -1$ in (43) we get

$$U(-1) - H(1) \equiv H(-1) - p\Big(H(-2) + H(1,-1)\Big) \pmod{p^2}.$$

Combining this with (75) and using (31) we get

$$H(1,-1) \equiv q_p^2 - \frac{2}{3}pq_p^3 - \frac{1}{4}pB_{p-3} + pH(1,1,-1) \equiv q_p^2 - pq_p^3 - \frac{13}{24}pB_{p-3} \pmod{p^2}$$

which is (101). Then (102) is deduced from the stuffle relation

$$H(-1,1) \equiv H(-1)H(1) - H(1,-1) - H(-2) \equiv -q_p^2 + pq_p^3 + \frac{1}{24}pB_{p-3} \pmod{p^2}.$$

Turning to weight three we get by [16, Theorem 2.1]

$$H(-3) \equiv \frac{1}{4}H(3;(p-1)/2) \pmod{p^2}.$$

Hence (103) follows from (4). Now by Lemma 7.2 we have the reversal relation

$$H(-2,1) \equiv H(1,-2) + pH(2,-2) + 2pH(1,-3) \equiv H(1,-2) \pmod{p^2} \qquad (107)$$

by Proposition 6.1. On the other hand, by stuffle relation

$$H(-2,1) + H(1,-2) = H(1)H(-2) - H(-3) \equiv -3X \pmod{p^2}.$$

Together with (107) this clearly yields (104).

Finally, by stuffle relation and (103)

$$H(2,-1) + H(-1,2) = H(2)H(-1) - H(-3) \equiv -\frac{4}{3}pq_pB_{p-3} - 3X \pmod{p^2}. \qquad (108)$$

By Lemma 7.2 we have the reversal relation

$$H(2,-1) \equiv H(-1,2) + pH(-2,2) + 2pH(-1,3) \equiv H(1,-2) + 2p(B-A) - pq_pB_{p-3} \pmod{p^2}$$

by Proposition 6.1. Combining with (108) this implies (105) and (106).

We have finished the proof of the proposition. □

Corollary 7.4. *For every prime $p \geq 7$ we have*

$$U(-1;p-1) \equiv -2q_p + \frac{7}{2}Xp^2 + \frac{1}{2}p^3H(-3,1;p-1) \pmod{p^4}. \qquad (109)$$

Proof. Write $U(-) = U(-;p-1)$ and $H(-) = H(-;p-1)$. Taking $x = -1$ in (90) we get

$$U(-1) - H(1) \equiv -pH(-2) + p^2H(1,-2) - p^3H(1,1,-2) - 2q_p \pmod{p^4}.$$

Thus the corollary follows from (104), Proposition 6.3, the following congruences

$$\begin{aligned} H(1) \equiv& 2p^2X && \pmod{p^4} \quad \text{(by [15, Remark 5.1])},\\ H(2) \equiv& -4pX && \pmod{p^3} \quad \text{(by (3))},\\ H(2,(p-1)/2) \equiv& -14pX && \pmod{p^3} \quad \text{(by (4))}, \end{aligned}$$

and $H(-2) = \frac{1}{2}H(2,(p-1)/2) - H(2)$. □

Proposition 7.5. *For all prime $p \ge 7$ write $H(-) = H(-;p-1)$, $h_{31} := H(3,1;(p-1)/2)$, and set X as above. Then*

$$H(-1,-2) \equiv \frac{9}{2}X - \frac{5}{6}pq_pB_{p-3} + \frac{1}{4}ph_{31} \pmod{p^2}, \tag{110}$$

$$H(-2,-1) \equiv -\frac{9}{2}X - \frac{1}{6}pq_pB_{p-3} - \frac{1}{4}ph_{31} \pmod{p^2}. \tag{111}$$

Proof. By expanding $H(1,2;p-1)$ we get

$$H(1,2;(p-1)/2) - H(2,1;(p-1)/2) \tag{112}$$

$$\equiv H(1,2;p-1) - H(1;(p-1)/2)\Big(H(2;(p-1)/2) + 2pH(3;(p-1)/2)\Big) \tag{113}$$

$$+ pH(2,2;(p-1)/2) + 2ph_{31} \pmod{p^2} \tag{114}$$

$$\equiv -6X - \frac{10}{3}pq_pB_{p-3} + 2ph_{31} \pmod{p^2} \tag{115}$$

since $2H(2,2;(p-1)/2) = H(2;(p-1)/2)^2 - H(4;(p-1)/2) \equiv 0 \pmod{p}$ and

$$H(1,2) \equiv -H(2,1) \equiv -6X \pmod{p^2} \tag{116}$$

by [17, Theorem 2.3]. Hence

$$\begin{aligned} H(-1,-2) \equiv& \frac{1}{4}\Big(H(1,2;(p-1)/2) - H(2,1;(p-1)/2) - ph_{31}\Big) - H(1,2) \pmod{p^2}\\ \equiv& \frac{9}{2}X - \frac{5}{6}pq_pB_{p-3} + \frac{1}{4}ph_{31} \pmod{p^2} \end{aligned}$$

which is (110). Finally, (111) follows from

$$H(-1,-2) + H(-2,-1) = H(-1)H(-2) - H(3) \equiv -pq_pB_{p-3} \pmod{p^2}.$$

This completes the proof of the proposition. □

Finally, we consider the depth three cases.

Proposition 7.6. *Let A and B be defined as in §6. For all prime $p \ge 7$ set X as above, write $H(-) = H(-;p-1)$ and $h_{31} := H(3,1;(p-1)/2)$ as above. Then we have*

$$H(-1,-1,-1) \equiv -\frac{4}{3}q_p^3 + X + p\Big(2q_p^4 + \frac{2}{3}q_pB_{p-3}\Big) \pmod{p^2}, \tag{117}$$

$$H(-1,1,-1) \equiv \frac{p}{2}\Big(q_pB_{p-3} + B - A\Big) \pmod{p^2}, \tag{118}$$

$$H(1,-1,-1) \equiv -q_p^3 + \frac{21}{4}X + p\Big(\frac{3}{2}q_p^4 + \frac{3}{8}q_pB_{p-3} + \frac{A}{4} - \frac{B}{4} + \frac{1}{8}h_{31}\Big) \pmod{p^2}, \tag{119}$$

$$H(-1,-1,1) \equiv q_p^3 - \frac{21}{4}X + p\Big(-\frac{3}{2}q_p^4 + \frac{1}{8}q_pB_{p-3} + \frac{A}{4} - \frac{B}{4} - \frac{1}{8}h_{31}\Big) \pmod{p^2}, \tag{120}$$

$$H(1,-1,1) \equiv -2H(1,1,-1) + 3X + p\Big(\frac{7}{6}q_pB_{p-3} + A - B\Big), \pmod{p^2}, \tag{121}$$

$$H(-1,1,1) \equiv H(1,1,-1) - p\Big(\frac{1}{2}q_pB_{p-3} + A - B\Big) \pmod{p^2}. \tag{122}$$

Proof. First, (117) follows from the stuffle relation

$$3H(-1,-1,-1) = H(-1)H(-1,-1) - H(-1,2) - H(2,-1).$$

By reversal relation (similar to the proof of Lemma 7.2) we can show easily that

$$H(-1,1,-1) \equiv -H(-1,1,-1) - p\Big(H(-2,1,-1) + H(-1,2,-1) + H(-1,1,-2)\Big) \pmod{p^2}.$$

Notice that $H(-1,1,-2) \equiv H(-2,1,-1) \pmod{p}$. Hence (118) follows from Proposition 6.3. It then implies (119) and (120) by the two stuffle relations:

$$\begin{aligned} 2H(1,-1,-1) =& H(-1)H(1,-1) - H(-1,1,-1) - H(-2,-1) - H(1,2), \\ 2H(-1,-1,1) =& H(-1)H(-1,1) - H(-1,1,-1) - H(-1,-2) - H(2,1), \end{aligned}$$

Proposition 7.3, Proposition 7.5 and (116). Similarly, the last two congruences follow immediately from the stuffle relations:

$$\begin{aligned} H(1,-1,1) =& H(1)H(1,-1) - 2H(1,1,-1) - H(2,-1) - H(1,-2), \\ 2H(-1,1,1) =& H(1)H(-1,1) - H(1,-1,1) - H(-2,1) - H(-1,2), \end{aligned}$$

and Proposition 7.3. This completes the proof of the proposition. □

Remark 7.7. Currently, we are not able to express $H(1,1,-1) \pmod{p^2}$ explicitly. In fact, by (97) it is not hard to find that

$$H(1,1,-1) \equiv U(-3) - p\Big(U(-4) + \frac{7}{12}q_p B_{p-3} + A - B\Big) \pmod{p^2},$$

hence it is equivalent to determining $U(-3) \pmod{p^2}$.

In conclusion, we remark that to study weight w AMHS modulo p^2 we need information of weight $w+1$ AMHS modulo p, and if we change modulus to p^3 then we would need to know AMHS of weight $w+2$ modulo p. Hence, it is possible that results in [2] might provide some help in determining the congruence of weight three AMHS modulo prime squares.

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