STEMS AND SPECTRAL SEQUENCES

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ABSTRACT. We introduce the category $\mathcal{P}stem[n]$ of n-stems, with a functor $\mathcal{P}[n]$ from spaces to $\mathcal{P}stem[n]$. This can be thought of as the n-th order homotopy groups of a space. We show how to associate to each simplicial n-stem $\mathcal{Q}_\bullet$ an $(n+1)$-truncated spectral sequence. Moreover, if $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$ is the Postnikov n-stem of a simplicial space $X_\bullet$, the truncated spectral sequence for $\mathcal{Q}_\bullet$ is the truncation of the usual homotopy spectral sequence of $X_\bullet$. Similar results are also proven for cosimplicial n-stems. They are helpful for computations, since n-stems in low degrees have good algebraic models.

0. Introduction

Many of the spectral sequences of algebraic topology arise as the homotopy spectral sequence of a (co)simplicial space – including the spectral sequence of a double complex, the (stable or unstable) Adams spectral sequence, the Eilenberg-Moore spectral sequence, and so on (see §4.14). Given a simplicial space $X_\bullet$, the E^2-term of its homotopy spectral sequence has the form $E^2_{s,t} = \pi_s\pi_t X_\bullet$, so it may be computed by applying the homotopy group functor dimensionwise to $X_\bullet$.

In this paper we show that the higher terms of this spectral sequence are obtained analogously by applying 'higher homotopy group' functors to $X_\bullet$. These functors are given explicitly in the form of certain *Postnikov stems*, defined in Section 1; the Postnikov 0-stem of a space is equivalent to its homotopy groups.

We then show how the E^r-term of the homotopy spectral sequence of a simplicial space $X_\bullet$ can be described in terms of the $(r-2)$-Postnikov stem of $X_\bullet$, for each $r \geq 2$ (see Theorem 3.14) – and similarly for the homotopy spectral sequence of a cosimplicial space $X^\bullet$ (see Theorem 4.12).

As an application for the present paper, in [BB2] we generalize the first author's result with Mamuka Jibladze (in [BJ]), which shows that the E^3-term of the stable Adams spectral sequence can be identified as a certain secondary derived functor Ext. We do this by showing how to define in general the *higher order derived functors* of a continuous functor $F : \mathcal{C} \to \mathcal{T}_*$, by applying F to a simplicial resolution $W_\bullet$ in $\mathcal{C}$, and taking Postnikov n-stems of $FW_\bullet$.

0.1. **Notation and conventions.** The category of pointed connected topological spaces will be denoted by $\mathcal{T}_*$; that of pointed sets by $\mathcal{S}et_*$; that of groups by $\mathcal{G}p$.

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For any category $\mathcal{C}$, $s\mathcal{C}$ denotes the category of simplicial objects over $\mathcal{C}$, and $c\mathcal{C}$ that of cosimplicial objects over $\mathcal{C}$. However, we abbreviate $s\mathcal{S}et$ to $\mathcal{S}$, $s\mathcal{S}et_*$ to $\mathcal{S}_*$, and $s\mathcal{G}p$ to $\mathcal{G}$. The constant (co)simplicial object on an object $X \in \mathcal{C}$ is written $c(X)_\bullet \in s\mathcal{C}$ (respectively, $c(X)^\bullet \in c\mathcal{C}$). For any small indexing category I, the category of functors $I \to \mathcal{C}$ is denoted by $\mathcal{C}^I$.

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1. Postnikov stems

The Postnikov system of a topological space (or simplicial set) X is the tower of fibrations:

$$\dots \to P^{n+1}X \xrightarrow{p^{n+1}} P^nX \xrightarrow{p^n} P^{n-1}X \dots P^1X \xrightarrow{p^1} P^0X , \tag{1.1}$$

equipped with maps $q^n : X \to P^nX$ (with $p^n \circ q^n = q^{n-1}$), which induce isomorphisms on homotopy groups in degrees $\leq n$. Here P^nX is n-coconnected (that is, $\pi_i P^nX = 0$ for $i > n$) and $\pi_i p^n$ is an isomorphism for $i < n$. The fiber of the map $p^n : P^nX \to P^{n-1}X$ is the Eilenberg-Mac Lane space $K(\pi_nX, n)$, so the fibers are determined up to homotopy by π_*X. Thus a generalization of the homotopy groups of X is provided by the following notion:

1.2. **Definition.** For any $n \geq 0$, a *Postnikov n-stem* in $\mathcal{T}_*$ is a tower:

$$\mathcal{Q} := \left(\dots \to Q_{k+1} \xrightarrow{q_{k+1}} Q_k \xrightarrow{q_k} Q_{k-1} \dots Q_0\right) \tag{1.3}$$

in $\mathcal{T}_*^{(\mathbb{N},\leq)}$, in which Q_k is $(k-1)$-connected and $(n+k)$-coconnected (so that $\pi_i(Q_k) = 0$ for $i < k$ or $i > n+k$) and $\pi_i(q_k)$ is an isomorphism for $k \leq i < n+k$. Here $(\mathbb{N}, \leq)$ is the usual linearly ordered category of the natural numbers. The space Q_k is called the k-th *n-window* of $\mathcal{Q}$.

Such an n-stem is thus a collection of overlapping $(k-1)$-connected $n+k$-types, which may be depicted for $n = 2$ as follows:

$$\begin{array}{cccccccc}
\dots & * & * & * & & & & \\
 & & * & * & * & & & \\
 & & & * & * & * & & \\
 & & & & * & * & * & \dots
\end{array}$$

where each row exhibits the $n+1$ non-trivial homotopy groups (denoted by $*$) of one n-window, and all those in the i-th column (corresponding to π_i) are isomorphic.

We denote by $\mathcal{P}stem[n]$ the full subcategory of Postnikov n-stems in the functor category $\mathcal{T}_*^{(\mathbb{N},\leq)}$ (with model category structure on the latter as in [Hi, 11.6]). Thus the morphisms in $\mathcal{P}stem[n]$ are given by strictly commuting maps of towers, and $f : \mathcal{Q} \to \mathcal{Q}'$ is a weak equivalence (respectively, a fibration) if each $f_k : Q_k \to Q'_k$ is such. This lets us define the homotopy category of Postnikov n-stems, $\operatorname{ho}\mathcal{P}stem[n]$, as a full sub-category of $\operatorname{ho}\mathcal{T}_*^{(\mathbb{N},\leq)}$.

The category $\mathcal{P}stem[n]$ is pointed, has products, and is equipped with canonical functors

$$
\begin{tikzcd}
\mathcal{T}_* \arrow[dr, "\mathcal{P}[n]"] \arrow[drr, bend left, "\mathcal{P}[n-1]"] \arrow[drrrr, bend left, "\mathcal{P}[0]"] & & & & \\
\ldots & \mathcal{P}stem[n] \arrow[r, "\overline{\mathcal{P}}[n-1]"'] & \mathcal{P}stem[n-1] \arrow[r, "\overline{\mathcal{P}}[n-2]"'] & \cdots \arrow[r, "\overline{\mathcal{P}}[0]"'] & \mathcal{P}stem[0]
\end{tikzcd}
\tag{1.4}
$$

which preserve products and weak equivalences.

1.5. *Remark.* The sequence of functors (1.4) is described by a commuting diagram, in which we may take all maps to be fibrations:

$$
\begin{tikzcd}
 & {} & {} & {} & {} & \\
\cdots \arrow[r] & Q^{n+k+1}_{k+1} \arrow[r, "q^n_{k+1}"] \arrow[d, "p^n_{k+1}"'] \arrow[u, dotted, no head] \arrow[ur] & Q^{n+k}_{k} \arrow[r, "q^n_{k}"] \arrow[d, "p^n_{k}"'] \arrow[u, dotted, no head] \arrow[ur] & Q^{n+k-1}_{k-1} \arrow[r] \arrow[d, "p^n_{k-1}"] \arrow[u, dotted, no head] \arrow[ur] & \cdots \arrow[r] & Q^{n}_{0} \arrow[d, "p^n_0"] \\
\cdots \arrow[r] & Q^{n+k}_{k+1} \arrow[r, "q^{n-1}_{k+1}"] \arrow[d, "p^{n-1}_{k+1}"'] \arrow[ur, "r^{n-1}_{k+1}"] & Q^{n+k-1}_{k} \arrow[r, "q^{n-1}_{k}"] \arrow[d, "p^{n-1}_{k}"'] \arrow[ur, "r^{n-1}_{k}"] & Q^{n+k-2}_{k-1} \arrow[r] \arrow[d, "p^{n-1}_{k-1}"] \arrow[ur] & \cdots \arrow[r] & Q^{n-1}_{0} \arrow[d, "p^n_0"] \\
\cdots \arrow[r] & Q^{n+k-1}_{k+1} \arrow[r, "q^{n-2}_{k+1}"] \arrow[d, dotted, no head] \arrow[ur, "r^{n-2}_{k+1}"] & Q^{n+k-2}_{k} \arrow[r, "q^{n-2}_{k}"] \arrow[d, dotted, no head] \arrow[ur, "r^{n-2}_{k}"] & Q^{n+k-3}_{k-1} \arrow[r] \arrow[d, dotted, no head] \arrow[ur] & \cdots \arrow[r] & Q^{n-2}_{0} \arrow[d, dotted, no head] \\
\cdots \arrow[r] & Q^{k+1}_{k+1} \arrow[r, "q^{0}_{k+1}"] & Q^{k}_{k} \arrow[r, "q^{0}_{k}"] & Q^{k-1}_{k-1} \arrow[r] & \cdots \arrow[r] & Q^{0}_{0}
\end{tikzcd}
\tag{1.6}
$$

Here $\pi_i Q^n_k = 0$ for $i < k$ or $i > n$, and all maps induce isomorphisms in π_i whenever possible. Thus:

(a) The k-th column (from the right) is the Postnikov tower for $Q_k := \lim_n Q^n_k$.
(b) The diagonals are the dual Postnikov system of connected covers for Q^j_0.
(c) The n-th row (from the bottom) is a Postnikov n-stem.
(d) In particular, each space in the 0-stem (the bottom row) is an Eilenberg-Mac Lane space, and the maps q^0_k are nullhomotopic. Thus the homotopy type of the bottom line in $\operatorname{ho}\mathcal{P}stem[0]$ is determined by the collection of homotopy groups $\{\pi_k Q^k_k\}_{k=0}^{\infty}$.

1.7. **Definition.** The motivating example of a Postnikov n-stem is a *realizable* one, associated to a space $X \in \mathcal{T}_*$, and denoted by $\mathcal{P}[n]X$, with $(P[n]X)_k := P^{n+k}X\langle k\rangle$. As usual, $Y\langle k\rangle$ denotes the $(k-1)$-connected cover of a space $Y \in \mathcal{T}_*$. Each fibration $q_k : (P[n]X)_k \to (P[n]X)_{k-1}$ fits into a commuting triangle of fibrations:

$$
\begin{tikzcd}
P^{n+k+1}X\langle k+1\rangle \arrow[rr, "q"] \arrow[dr, "p"] & & P^{n+k}X\langle k\rangle \\
 & P^{n+k}X\langle k+1\rangle \arrow[ur, "r"] &
\end{tikzcd}
\tag{1.8}
$$

in which the maps p and r are the fibration of (1.1) and the covering map, respectively. See [BB1, §10.5] for a natural context in which non-realizable Postnikov n-stems arise.

1.9**. Examples of stems.** The functor $\mathcal{P}[0]_* : \mathcal{T}_* \to \operatorname{ho}\mathcal{P}stem[0]$ induced by $\mathcal{P}[0]$ is equivalent to the homotopy group functor: in fact, the homotopy groups of a space define a functor $\pi_* : \mathcal{T}_* \to \mathcal{K}$ into the product category $\mathcal{K} := \prod_{i=0}^{\infty} \mathcal{K}_i$, where $\mathcal{K}_0 = \mathcal{S}et_*$, $\mathcal{K}_1 = \mathcal{G}p$, and $\mathcal{K}_i = \mathcal{A}b\mathcal{G}p$, for $i \geq 2$. Moreover, there is an equivalence of categories $\vartheta : \mathcal{K} \equiv \operatorname{ho}\mathcal{P}stem[0]$, such that the functor $\mathcal{P}[0]_*$ is equivalent to the composite functor $\vartheta \circ \pi_* : \mathcal{T}_* \to \mathcal{K}$.

Similarly, the functor $\mathcal{T}_* \to \operatorname{ho}\mathcal{P}stem[1]$ induced by $\mathcal{P}[1]$ is equivalent to the secondary homotopy group functor of [BM, §4], in the sense that each secondary homotopy group $\pi_{n,*}X$ completely determines the n-th 1-window of X. Using the results on secondary homotopy groups in [BM], one obtains a homotopy category of algebraic 1-stems which is equivalent to $\operatorname{ho}\mathcal{P}stem[1]$.

A category of algebraic models for 2-stems is only partially known. The homotopy classification of $(k-1)$-conected $(k+2)$-types is described for all k in [Ba]; this theory can be used to classify homotopy types of Postnikov 2-stems.

2. The spectral sequence of a simplicial space

We begin with the construction of the homotopy spectral sequence for a simplicial space (cf. [Q], [BF, Theorem B.5], and [BK1, X,§6]), using the version given by Dwyer, Kan, and Stover in [DKSt2, §8] (see also [Bou2, §2,5], [Bou1], and [DKSt1, §3.6]). For this purpose, we require some explicit constructions for the E^2-model category of simplicial spaces.

2.1. **Definition.** Given a simplicial object $X_\bullet \in s\mathcal{C}$, over a complete pointed category $\mathcal{C}$, for each $n \geq 1$ define its *n-cycles* object to be

$$Z_n X_\bullet \ := \ \{x \in X_n \mid d_i x = * \text{ for } i = 0, \dots, n\} \ .$$

Similarly, the the *n-chains* object for $X_\bullet$ is

$$C_n X_\bullet \ := \ \{x \in X_n \mid d_i x = * \text{ for } i = 1, \dots, n\}$$

Set $Z_0 X_\bullet := X_0$. We denote the map $d_0|_{C_n X_\bullet} \colon C_n X_\bullet \to Z_{n-1} X_\bullet$ by $\mathbf{d}_0^{X_n}$.

2.2. **Notation.** For any non-negatively graded object T_*, we write ΩT_* for the graded object with $(\Omega T_*)_j := T_{j+1}$ for all $j \geq 0$. The notation is motivated by the natural isomorphism of graded groups $\pi_* \Omega X \cong \Omega(\pi_* X)$ for $X \in \mathcal{T}_*$.

2.3. **Definition.** Now assume that $\mathcal{C}$ is a pointed model category of spaces, such as $\mathcal{T}_*$ or $\mathcal{G}$, and $X_\bullet$ is a Reedy fibrant simplicial object over $\mathcal{C}$ – that is, for each $n \geq 1$, the universal face map $\delta_n : X_n \to M_n X_\bullet$ into the n-th matching object of $X_\bullet$ is a fibration (see [Hi, 15.3]). The map $\mathbf{d}_0 = \mathbf{d}_0^{X_n}$ then fits into a fibration sequence in $\mathcal{C}$:

$$\cdots \Omega Z_n X_\bullet \to Z_{n+1} X_\bullet \xrightarrow{j_{n+1}^{X_\bullet}} C_{n+1} X_\bullet \xrightarrow{\mathbf{d}_0^{X_{n+1}}} Z_n X_\bullet \tag{2.4}$$

(see [DKSt2, Prop. 5.7]).

For each $n \geq 0$, the n-th *natural homotopy group* of the simplicial space $X_\bullet$, denoted by $\pi^\natural_n X_\bullet = \pi^\natural_{n,*} X_\bullet$, the cokernel of the map $(\mathbf{d}_0^{X_{n+1}})_\#$ (induced on homotopy groups by $\mathbf{d}_0^{X_{n+1}}$). Note that the cokernel of a maps of groups or pointed sets is generally just a pointed set.

We thus have an exact sequence of graded groups:

$$\pi_* C_{n+1} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+1}})_\#} \pi_* Z_n X_\bullet \xrightarrow{\hat{\vartheta}_n} \pi^\natural_{n,*} X_\bullet \to 0\ . \tag{2.5}$$

Together the groups $(\pi^\natural_{n,k} X_\bullet)_{n,k=0}^\infty$ constitute the *bigraded homotopy groups* of [DKSt2, §5.1].

2.6**. Construction of the spiral sequence.** Applying the functor π_* to the fibration sequence (2.4) yields a long exact sequence, with connecting homomorphism $\partial_\# : \Omega\pi_* Z_n X_\bullet = \pi_* \Omega Z_n X_\bullet \to \pi_* Z_{n+1} X_\bullet$. Note that the inclusion $\iota : C_n X_\bullet \hookrightarrow X_n$ induces an isomorphism $\iota_\star : \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$ for each $n \geq 0$ (see [Bl3, Prop. 2.7]). From (2.5) we see that:

$$\Omega\pi^\natural_n X_\bullet = \Omega\, \mathrm{Coker}\, (\mathbf{d}_0^{X_{n+1}})_\# \cong \mathrm{Im}\ \partial_\# \cong \mathrm{Ker}\, (j_{n+1}^{X_\bullet})_\# \subseteq \pi_* Z_{n+1} X_\bullet\ ,$$

so we obtain a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}\,(j_n)_* \arrow[r, hook] \arrow[d, hook] & B_{n+1}X_\bullet \arrow[r, two heads, "(j_n)_*"] \arrow[d, hook] & B_{n+1}\pi_* X_\bullet \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega\pi^\natural_{n-1} X_\bullet \arrow[r, hook, "\ell_{n-1}"] \arrow[d] \arrow[dr, "s_n"] & \pi_* Z_n X_\bullet \arrow[r, "(j_n^{X_\bullet})_\#"] \arrow[d, "\hat{\vartheta}_n"] & Z_n \pi_* X_\bullet \arrow[r, two heads] \arrow[d, "\vartheta_n"] & \mathrm{Coker}\ h_n \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Ker}\ h_n \arrow[r, hook] \arrow[d] & \pi^\natural_n X_\bullet \arrow[r, "h_n"] \arrow[d] & \pi_n \pi_* X_\bullet \arrow[r, two heads] \arrow[d] & \mathrm{Coker}\ h_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.7}$$

in which $B_{n+1}X_\bullet := \mathrm{Im}\,(\mathbf{d}_0^{X_{n+2}})_\# \subseteq \pi_* Z_n X_\bullet$ and $B_{n+1}\pi_* X_{n+2} := \mathrm{Im}\ \mathbf{d}_0^{\pi_* X_{n+2}}$ are the respective boundary objects. Note that the map $(j_n^{X_\bullet})_\# : \pi_* Z_n X_\bullet \to \pi_* C_n X_\bullet$ induced by the inclusion $j_n^{X_\bullet}$ of (2.4) above in fact factors through $Z_n \pi_* X_\bullet$, as indicated in the middle row of (2.7).

This defines the map of graded groups $h_n : \pi^\natural_n X_\bullet \to \pi_n(\pi_* X_\bullet)$. Note that for $n = 0$ the map $\hat{\iota}_\star$ is an isomorphism, so h_0 is, too. The map $s_n : \Omega\pi^\natural_{n-1} X_\bullet \to \pi^\natural_n X_\bullet$ is the composite of the inclusion $\ell_{n-1} : \mathrm{Ker}\,(j_n^{X_\bullet})_\# \hookrightarrow \pi_* Z_n X_\bullet$ with the quotient map $\hat{\vartheta}_n : \pi_* Z_n X_\bullet \to \pi^\natural_n X_\bullet$ of (2.5), using the natural identification of $\Omega\pi^\natural_n X_\bullet$ with $\mathrm{Ker}\,(j_{n+1}^{X_\bullet})_\#$.

The map $\partial_{n+2} : \pi_{n+2}\pi_* X_\bullet \to \Omega\pi^\natural_n X_\bullet$ is induced by the composite

$$Z_{n+2}\pi_* X_\bullet \subseteq C_{n+2}\pi_* X_\bullet \cong \pi_* C_{n+2} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+2}})_\#} \pi_* Z_{n+1} X_\bullet\ , \tag{2.8}$$

which actually lands in $\operatorname{Ker}(j^{X_\bullet}_{n+1})_\#$ by the exactness of the long exact sequence for the fibration (2.4).

These maps s_n, h_n, and ∂_n fit into a *spiral long exact sequence:*

$$\begin{aligned}\ldots \to \Omega\pi^\natural_{n-1}X_\bullet \xrightarrow{s_n} \pi^\natural_n X_\bullet \xrightarrow{h_n} \pi_n\pi_* X_\bullet \xrightarrow{\partial_n} \Omega\pi^\natural_{n-2}X_\bullet \\ \xrightarrow{s_{n-1}} \pi^\natural_{n-1}X_\bullet \to \ldots \to \pi^\natural_0 X_\bullet \xrightarrow{\cong} \pi_0\pi_* X_\bullet\end{aligned} \tag{2.9}$$

(cf. [DKSt2, 8.1]).

2.10**. The spectral sequence of a simplicial space.** For any simplicial space $X_\bullet \in s\mathcal{T}_*$ (or bisimplicial set), Bousfield and Friedlander showed that there is a first-quadrant spectral sequence of the form

$$E^2_{s,t} = \pi_s\pi_t X_\bullet \;\Rightarrow\; \pi_{s+t}\|X_\bullet\| \;, \tag{2.11}$$

where $\|X_\bullet\| \in \mathcal{T}_*$ is the realization (or the diagonal, in the case of $X_\bullet \in s\mathcal{S}_*$). The spectral sequence is always defined, but $X_\bullet$ must satisfy certain "Kan conditions" to guaranteee *convergence* – see [BF, Theorem B.5].

In [DKSt2, §8.4], Dwyer, Kan and Stover showed that (2.11) coincides up to sign, from the E^2-term on, with the spectral sequence associated to the exact couple of (2.4), which we call the *spiral spectral sequence* for $X_\bullet$.

If we assume that each X_n is connected, by taking loops (or applying Kan's functor G, if $X_\bullet \in s\mathcal{S}_*$), we may replace $X_\bullet$ by a bisimplicial group $GX_\bullet \in s\mathcal{G}$, and then (2.11) becomes the spectral sequence of [Q].

3. Simplicial stems and truncated spectral sequences

As noted in §1.9, the E^2-term of any of the above equivalent spectral sequences for a simplicial space $X_\bullet$ is determined explicitly by the simplicial 0-stem of $X_\bullet$.

Our goal is to extend this description to the higher terms of the spectral sequence. For this purpose, fix $n \geq 0$, and consider a simplicial Postnikov n-stem $\mathcal{Q}_\bullet$ (which need not be realizable as $\mathcal{P}[n]X_\bullet$ for some simplicial space $X_\bullet$). This is equivalent to having a collection of simplicial spaces $\mathcal{Q}^{n+k}_\bullet\langle k\rangle$ for each $k \geq 0$, equipped with maps as in (1.3), with $\pi_i\mathcal{Q}^{n+k}_\bullet\langle k\rangle = 0$ for $i < k$ or $i > n+k$.

We assume that $\mathcal{Q}_\bullet$ is *Reedy fibrant* in the sense that for each $k \geq 0$, the simplicial space $\mathcal{Q}^{n+k}_\bullet\langle k\rangle$ is Reedy fibrant. In this case, the "n-stem version" of the spiral long exact sequence is defined as follows: for each $t, i, k \geq 0$, set $\pi^{\natural\,(k,n)}_{t,i}\mathcal{Q}_\bullet := \pi^\natural_{t,i+k}\mathcal{Q}^{n+k}_\bullet\langle k\rangle$ and

$$\pi^{(k,n)}_i\mathcal{Q}_\bullet \;:=\; \pi_{i+k}\mathcal{Q}^{n+k}_\bullet\langle k\rangle \;=\; \begin{cases}\pi_{i+k}\mathcal{Q}_\bullet & \text{if } 0 \leq i \leq n\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

Note that the $(i+k)$-th homotopy group $\pi_{i+k}\mathcal{Q}_\bullet$ of a Postnikov n-stem $\mathcal{Q}_\bullet$ is well-defined, and coincides with $\pi_{i+k}X_\bullet$ for $0 \leq i \leq n$ when $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$.

3.2. **Definition.** The collection of long exact sequences (2.9) for $\mathcal{Q}_\bullet^{n+k}\langle k\rangle$ (indexed by $k\geq 0$):

$$\ldots\Omega\pi^{\natural\,(k,n)}_{t-1,*}\mathcal{Q}_\bullet \xrightarrow{s_t^{(k,n)}} \pi^{\natural\,(k,n)}_{t,*}\mathcal{Q}_\bullet \xrightarrow{h_t^{(k,n)}} \pi_t\pi^{(k,n)}_*\mathcal{Q}_\bullet \xrightarrow{\partial_t^{(k,n)}} \Omega\pi^{\natural\,(k,n)}_{t-2,*}\mathcal{Q}_\bullet\ldots\ , \tag{3.3}$$

together with the maps between adjacent k-windows induced by the map q in (1.6), will be called the *spiral n-system* of $\mathcal{Q}_\bullet$. When $\mathcal{Q}_\bullet=\mathcal{P}[n]X_\bullet$, we will refer to this simply as the spiral n-system of $X_\bullet$.

3.4. *Remark.* Using the exactness of (3.3), definition (3.1) implies that:

$$\pi^{\natural\,(k,n)}_{t,i}\mathcal{Q}_\bullet=\pi^{\natural}_{t,i}\mathcal{Q}_\bullet^{n+k}\langle k\rangle\ =\ 0\quad \text{for } i>n\ , \tag{3.5}$$

by induction on $t\geq 0$. Note, however, that while the groups $\pi_i^{(k,n)}\mathcal{Q}_\bullet$ are explicitly described by (3.1), the dependence of $\pi^{\natural\,(k,n)}_{t,i}\mathcal{Q}_\bullet$ on k and n requires more care.

3.6**. The E^2-term of the spectral sequence.** The spiral 0-system of a simplicial Postnikov 0-stem $\mathcal{Q}_\bullet$ reduces to a series of isomorphisms $h_t:\pi^{\natural\,(k,0)}_{t,*}\mathcal{Q}_\bullet\cong\pi_t\pi^{(k,0)}_*\mathcal{Q}_\bullet$ (for each $k\geq 0$). When $\mathcal{Q}_\bullet=\mathcal{P}[0]X_\bullet$ is the Postnikov 0-stem of a simplicial space $X_\bullet$, this allows us to identify the $E^2_{t,k}$-term of the spiral spectral sequence for $X_\bullet$, which is:

$$\pi_t\pi_kX_\bullet\ =\ \pi_t\pi_kP^{0+k}X_\bullet\langle k\rangle\ =\ \pi_t\pi_k(P[0]X_\bullet)_k\ =\ \pi_t\pi_*^{(k,0)}\mathcal{P}[0]X_\bullet\ =\ \pi_t\pi_*^{(k,0)}\mathcal{Q}_\bullet,$$

with $\pi^{\natural\,(k,0)}_{t,*}\mathcal{Q}_\bullet=\pi^{\natural\,(k,0)}_{t,*}\mathcal{P}[0]X_\bullet$.

The first interesting case is the spiral 1-system, for which we have:

3.7. **Proposition.** *The E^3-term of the spiral spectral sequence for a simplicial space $X_\bullet$ is determined by the spiral 1-system of $X_\bullet$. In fact, $d^2_{t,k}$ may be identified with $\partial_t^{(k,1)}:\pi_t\pi_kX_\bullet\to\Omega\pi^{\natural\,(k,1)}_{t-2,0}X_\bullet$, while $E^3_{t,k}$ is the image of the composite map*

$$\pi^{\natural\,(k,1)}_{t,0}X_\bullet \xrightarrow{h_t^{(k,1)}} \pi_t\pi_kX_\bullet\ \cong\ \pi_t\pi_1^{(k-1,1)}X_\bullet \underset{\cong}{\xleftarrow{h_t^{(k-1,1)}}} \pi^{\natural\,(k-1,1)}_{t,1}X_\bullet \xrightarrow{s_{t+1}^{(k-1,1)}} \pi^{\natural\,(k-1,1)}_{t+1,0}X_\bullet\ . \tag{3.8}$$

Observe that (3.8) involves maps from different windows of the spiral 1-system, implicitly identified using the isomorphisms induced by the map q in (1.6).

Proof. Because $n=1$ throughout, we abbreviate $\pi^{\natural\,(k,1)}_{t,i}\mathcal{Q}_\bullet$ to $\pi^{\natural\,(k)}_{t,i}\mathcal{Q}_\bullet$, and $\pi_i^{(k,1)}\mathcal{Q}_\bullet$ to $\pi_i^{(k)}\mathcal{Q}_\bullet$, observing that $\pi_i^{(k)}\mathcal{Q}_\bullet$ is simply $\pi_{i+k}X_\bullet$ for $i=0,1$, and zero otherwise, since $\mathcal{Q}_\bullet=\mathcal{P}[1]X_\bullet$. Thus the spiral 1-system (3.3) is non-trivial for each $t\geq 1$ in (internal) degrees $i=0,1$ only, and we can write it in two rows:

$$\begin{array}{cccccccccccc}
0 & \longrightarrow & \pi^{\natural\,(k)}_{t,1}\mathcal{Q}_\bullet & \xrightarrow{\cong} & \pi_t\pi_1^{(k)}\mathcal{Q}_\bullet & \longrightarrow & 0 & \longrightarrow & \pi^{\natural\,(k)}_{t-1,1}\mathcal{Q}_\bullet & \xrightarrow{\cong} & \pi_{t-1}\pi_1^{(k)}\mathcal{Q}_\bullet \\
\Omega\pi^{\natural\,(k)}_{t-1,0}\mathcal{Q}_\bullet & \xrightarrow{s_t} & \pi^{\natural\,(k)}_{t,0}\mathcal{Q}_\bullet & \xrightarrow{h_t} & \pi_t\pi_0^{(k)}\mathcal{Q}_\bullet & \xrightarrow{\partial_t} & \Omega\pi^{\natural\,(k)}_{t-2,0}\mathcal{Q}_\bullet & \xrightarrow{s_{t-1}} & \pi^{\natural\,(k)}_{t-1,0}\mathcal{Q}_\bullet & \xrightarrow{h_{t-1}} & \pi_{t-1}\pi_0^{(k)}\mathcal{Q}_\bullet
\end{array}$$

Since $\mathcal{Q}_\bullet:=\mathcal{P}[1]X_\bullet$ is the simplicial Postnikov 1-stem of $X_\bullet$, we actually have a collection of two-row long exact sequences, one for each k-window of $\mathcal{P}[1]X_\bullet$.

For each such k-window $\mathcal{P}_k[1]X_\bullet$, we can use the top row to identify

$$\Omega\pi^{\natural\,(k)}_{t,0}\mathcal{Q}_\bullet \;=\; \Omega\pi^{\natural}_{t,0}\mathcal{P}_k[1]X_\bullet \;=\; \pi^{\natural}_{t,1}\mathcal{P}_k[1]X_\bullet \;=\; \pi^{\natural\,(k)}_{t,1}\mathcal{Q}_\bullet$$

with $\pi_t\pi^{(k)}_1\mathcal{Q}_\bullet = \pi_t\pi^{(1)}_t\mathcal{P}_k[1]X_\bullet = \pi_t\pi_{k+1}X_\bullet$, so the bottom row reduces to:

(3.9)
$$\begin{tikzcd}
\pi_{t-1}\pi_{k+1}X_\bullet \arrow[rr, "s^{(k,1)}_t"] \arrow[dr] & & \pi^{\natural\,(k)}_{t,0}\mathcal{Q}_\bullet \arrow[rr, "h^{(k,1)}_t"] \arrow[dr] & & \pi_t\pi_k X_\bullet \arrow[rr, "\partial^{(k,1)}_t"] \arrow[dr] & & \pi_{t-2}\pi_{k+1}X_\bullet \\
& \mathrm{Im}\,(s^{(k,1)}_t) \arrow[ur] & & \begin{array}{c}\mathrm{Im}\,(h^{(k,1)}_t)\\ =\\ \mathrm{Ker}\,(\partial^{(k,1)}_t)\end{array} \arrow[ur] & & \mathrm{Im}\,(\partial^{(k,1)}_t) \arrow[ur] &
\end{tikzcd}$$

Note that the following part of the E^1-term of the exact couple for the fibration sequence $C_{n+1}P^1\Omega^iX_\bullet \to Z_nP^1\Omega^iX_\bullet$, (as in (2.4)):

$$\begin{tikzcd}
\pi_1 Z_{t-1}P^1\Omega^kX_\bullet \arrow[r, "(j_{t-1})_\#"] \arrow[ddd, "\partial_*"] & \pi_1 C_{t-1}P^1\Omega^kX_\bullet \arrow[r, "(d_0^{t-1})_\#"] & \pi_1 Z_{t-2}P^1\Omega^kX_\bullet \arrow[r, "(j_{t-2})_\#"] \arrow[d] \arrow[dr] \arrow[ddd, bend right=60, "\partial_*"'] & \pi_1 C_{t-2}X_\bullet \arrow[r] & \cdots \\
& & \Omega\pi^{\natural\,(k)}_{t-2,0}X_\bullet = \pi^{\natural\,(k)}_{t-2,1}X_\bullet \arrow[dr, "h^{(k+1,1)}_{t-2,1}", "\cong"'] \arrow[dd, hook] & Z_{t-2}\pi_1P^1\Omega^kX_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_{t-2}"] & \\
& & & \pi_{t-2}\pi_{k+1}X_\bullet & \\
\pi_0 Z_tP^1\Omega^kX_\bullet \arrow[r, "(j_t)_\#"] \arrow[d, "\hat{\vartheta}_t"] \arrow[dr] & \pi_0 C_tP^1\Omega^kX_\bullet \arrow[r, "(d_0^t)_\#"] & \pi_0 Z_{t-1}P^1\Omega^kX_\bullet \arrow[r, "(j_{t-1})_\#"] \arrow[d, "\hat{\vartheta}_{t-1}"] \arrow[dr] & \pi_0 C_{t-1}P^1\Omega^kX_\bullet \arrow[r] & \cdots \\
\pi^{\natural\,(k)}_{t,0}X_\bullet \arrow[dr, "h^{(k,1)}_t"] & Z_t\pi_kX_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_t"] & \pi^{\natural\,(k)}_{t-1,0}X_\bullet \arrow[dr, "h^{(k,1)}_{t-1}"] & Z_{t-1}\pi_kX_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_{t-1}"] & \\
& \pi_t\pi_kX_\bullet \arrow[uuuur, "\partial^{(k,1)}_{t,0}"] & & \pi_{t-1}\pi_kX_\bullet &
\end{tikzcd}$$

is naturally isomorphic to the exact couple for $C_{n+1}\Omega^kX_\bullet \to Z_n\Omega^kX_\bullet$, since C_{n+1} and Z_n are limits, so they commute with P^1, and then $\pi_1P^1Z_{t-1}\Omega^kX_\bullet \cong \pi_1Z_{t-1}\Omega^kX_\bullet$, and so on. This does not imply, of course, that $\pi^{\natural\,(k)}_{t,1}X_\bullet \cong \pi^{\natural}_{t,k+1}X_\bullet$.

We therefore see from (2.7) and (2.8) that the differential $d^2_{t,k} : E^2_{t,k} \to E^2_{t-2,k+1}$ may be identified with:

(3.10)
$$\pi_t\pi_kX_\bullet \cong \pi_t\pi^{(k,1)}_0X_\bullet \xrightarrow{\partial^{(k,1)}_{t,0}} \Omega\pi^{\natural\,(k)}_{t-2,0}X_\bullet = \pi^{\natural\,(k)}_{t-2,1}X_\bullet \overset{h_t}{\cong} \pi_{t-2}\pi^{(k,1)}_1X_\bullet \cong \pi_{t-2}\pi_{k+1}X_\bullet$$

Now by definition, $E^3_{t,k}$ fits into a commutative diagram:

$$
\begin{tikzcd}
E^2_{t+2,k-1} \arrow[r, "d^2_{t+2,k-1}"] \arrow[d, two heads, "r"'] & E^2_{t,k} \arrow[r, two heads, "q"] & \mathrm{Coker}\,(d^2_{t+2,k-1}) \\
\mathrm{Im}\,(d^2_{t+2,k-1}) \arrow[r, hook, "\ell"] & \mathrm{Ker}\,(d^2_{t,k}) \arrow[r, two heads, "s"] \arrow[u, hook, "j"'] & E^3_{t,k} \arrow[u, hook, "\kappa"']
\end{tikzcd}
\tag{3.11}
$$

with exact rows, ℓ j and κ monic, and thus $E^3_{t,k} \cong \mathrm{Im}\,(q \circ j)$.

From the exactness of (3.3) (together with (3.9)) we see that $\mathrm{Coker}\,(d^2_{t+2,k-1}) = \mathrm{Coker}\,(\partial^{(k-1,1)}_{t+2}) = \mathrm{Im}\,(s^{(k-1,1)}_{t+1})$ and $\mathrm{Ker}\,(d^2_{t,k}) = \mathrm{Ker}\,(\partial^{(k,1)}_{t}) = \mathrm{Im}\,(h^{(k,1)}_{t})$, so $E^3_{t,k} = \mathrm{Im}\,(q \circ j)$ is indeed the image of the map in (3.8). ∎

3.12. **Definition.** An *r-truncated spectral sequence* is one defined up to and including the E^r-term, together with the differential $d^n : E^r_{t,i} \to E^r_{t-r-1,t+r}$, but without requiring that $d^r \circ d^r = 0$ (so the E^{r+1}-term is defined in terms of the r-truncated spectral sequence only if $d^r d^r = 0$).

The main example is the n-truncation of an (ordinary) spectral sequence (such as that of a simplicial space). In this case we do have $d^r \circ d^r = 0$, of course.

3.13. **Corollary.** *Any Reedy fibrant simplicial Postnikov 1-stem has a well-defined 2-truncated spiral spectral sequence. Moreover, if $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$ for some simplicial space $X_\bullet$, this 2-truncated spectral sequence coincides with the 2-truncation of the Bousfield-Friedlander spectral sequence for $X_\bullet$.*

In general, we have a less explicit description of the higher terms in the spiral spectral sequence:

3.14. **Theorem.** *For each $r \geq 0$, the E^{r+2}-term of the spiral spectral sequence for a simplicial space $X_\bullet$ is determined by the spiral r-system of $X_\bullet$. Moreover, for any $\alpha \in E^{r+1}_{t,i}$, we have $d^{r+1}_{t,i}(\alpha) = \beta \in E^{r+1}_{t-r-1,i+r}$ if and only if α and β have representatives $\bar{a} \in \pi_t\pi_i X_\bullet$ and $\bar{b} \in \pi_{t-r-1}\pi_{i+r}X_\bullet$, respectively, such that:*

$$
(s^{(i,r)}_{t-2,1}) \circ (s^{(i,r)}_{t-3,2}) \circ \cdots \circ (s^{(i,r)}_{t-r,r-1}) \circ (h^{(i,r)}_{t-r-1,r})^{-1}(\bar{b}) \;=\; \partial^{(i,r)}_{t,0}(\bar{a})
\tag{3.15}
$$

Proof. We naturally identify $\pi^{\natural\,(i,r)}_{t,k} X_\bullet$ with $\pi^{\natural\,(i,r-s)}_{t,k+s} X_\bullet$ for $k \geq s$, and similarly for the maps in (3.3), so the spiral $(r-1)$-system embeds in the spiral r-system (with an index shift).

Again we write out the E^1-term of the spiral exact couple:

$$
\begin{tikzcd}
 & \pi_r C_{t-r} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-r})_\#"] & \pi_r Z_{t-r-1} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-r-1})_\#"] \arrow[d, "\hat{\vartheta}_{t-r-1}"] \arrow[dr, "(j^{X_\bullet}_{t-r-1})_\#"] & \pi_r C_{t-r-1} P^r \Omega^i X_\bullet & \\
 & & \Omega\pi^{\natural\,(i,r)}_{t-r-1,r-1} X_\bullet = \pi^{\natural\,(i,r)}_{t-r-1,r} X_\bullet \arrow[dr, "h^{(i,r)}_{t-r-1,r}", "\cong"'] \arrow[dd, hook, "\ell_{t-r-1,r}"] & Z_{t-r-1}\pi_{i+r} X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_{t-r-1}"] & \\
 & & & \pi_{t-r-1}\pi_{i+r} X_\bullet & \\
 & \pi_{r-1} C_{t-r+1} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-r+1})_\#"] & \pi_{r-1} Z_{t-r} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-r})_\#"] & \pi_{r-1} C_{t-r} P^r \Omega^i X_\bullet & \\
 & \vdots & \vdots & \vdots & \\
 & \pi_2 C_{t-2} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-2})_\#"] & \pi_2 Z_{t-3} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-3})_\#"] \arrow[d, "\hat{\vartheta}_{t-3}"] \arrow[dr, "(j^{X_\bullet}_{t-3})_\#"] & \pi_2 C_{t-3} P^r \Omega^i X_\bullet & \\
 & & \Omega\pi^{\natural\,(i,r)}_{t-3,1} X_\bullet = \pi^{\natural\,(i,r)}_{t-3,2} X_\bullet \arrow[dr, "h^{(i,r)}_{t-3,2}"] \arrow[dd, hook, "\ell_{t-3,2}"] \arrow[ddd, bend right=70, "s^{(i,r)}_{t-3,1}"'] & Z_{t-3}\pi_{i+2} X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_{t-3}"] & \\
 & & & \pi_{t-3}\pi_{i+2} X_\bullet & \\
 & \pi_1 C_{t-1} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-1})_\#"] & \pi_1 Z_{t-2} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-2})_\#"] \arrow[d, "\hat{\vartheta}_{t-2}"] \arrow[dr, "(j^{X_\bullet}_{t-2})_\#"] & \pi_1 C_{t-2} P^r \Omega^i X_\bullet & \\
 & & \Omega\pi^{\natural\,(i,r)}_{t-2,0} X_\bullet = \pi^{\natural\,(i,r)}_{t-2,1} X_\bullet \arrow[dr, "h^{(i,r)}_{t-2,1}"] \arrow[dd, hook, "\ell_{t-2,1}"] & Z_{t-2}\pi_{i+1} X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_{t-2}"] & \\
 & & & \pi_{t-2}\pi_{i+1} X_\bullet & \\
\pi_0 Z_t P^r \Omega^i X_\bullet \arrow[r, "(j_t)_\#"] \arrow[d, "\hat{\vartheta}_t"] \arrow[dr, "(j^{X_\bullet}_t)_\#"] & \pi_0 C_t P^r \Omega^i X_\bullet \arrow[r, "(d_0^t)_\#"] & \pi_0 Z_{t-1} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-1})_\#"] & \pi_0 C_{t-1} P^r \Omega^i X_\bullet \arrow[r] & \ldots \\
\pi^{\natural\,(i)}_{t,0} X_\bullet \arrow[dr, "h^{(i,r)}_t"] & Z_t \pi_i X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, "\vartheta_t"'] & & & \\
 & \pi_t \pi_i X_\bullet \arrow[uuuur, "\partial^{(i,r)}_{t,0}"] & & &
\end{tikzcd}
$$

The differential $d^{r+1}_{t,i} : E^{r+1}_{t,i} \to E^{r+1}_{t-r-1,i+r}$ may then be described as a "relation" (cf. [BK3, §3.1]) in the usual way:

Given a class $\alpha \in E^{r+1}_{t,i}$, choose a representative for it $a \in E^1_{t,i} = \pi_0 C_t P^r \Omega^i X_\bullet$. Since it is a cycle for $d^1_{t,i} = (j_{t-1})_\# \circ (d^t_0)_\#$, it lies in $Z_t \pi_i X_\bullet$ and thus represents an element $\bar{a} \in \pi_t \pi_i X_\bullet = E^2_{t,i}$. From the exactness of the middle row of (2.7) we see that $(d^t_0)_\#(a) \in \mathrm{Ker}\,((j_{t-1})_\#) = \Omega \pi^{\natural\,(i,r)}_{t-2,0} X_\bullet$, and in fact $(d^t_0)_\#(a)$ represents $\partial^{(i,r)}_{t,0}(\bar{a})$. Since $\hat{\vartheta}_{t-2}$ is surjective, we can choose $e_{t-2} \in \pi_1 Z_{t-2} P^r \Omega^i X_\bullet$ mapping to $(d^t_0)_\#(a)$. Because $d^2_{t,i}(\bar{a}) = h^{(i,r)}_{t-2,1} \circ \partial^{(i,r)}_{t,0}(\bar{a})$, as in the proof of Proposition 3.7 (though $h^{(i,r)}_{t-2,1}$ need no longer be an isomorphism!), we see that it is represented by $(j_{t-2})_*(e_{t-2})$. If $r = 1$, we are done. Otherwise, we know that $d^2_{t,i}(\bar{a}) = 0$, so we can choose e_{t-2} so that $(j_{t-2})_*(e_{t-2}) = 0$, using exactness of the third column of of (2.7). Again this implies that $e_{t-2} \in \mathrm{Ker}\,((j_{t-2})_\#) = \Omega \pi^{\natural\,(i,r)}_{t-3,1} X_\bullet$, and $d^3_{t,i}(\langle a \rangle)$ is represented by $h^{(i,r)}_{t-3,2}(e_{t-2})$. Moreover, we see from (2.7) that $s^{(i,r)}_{t-3,1}(e_{t-2}) = \partial^{(i,r)}_{t,0}(\bar{a})$, using the identification $\Omega \pi^{\natural\,(i,r)}_{t-2,0} X_\bullet = \pi^{\natural\,(i,r)}_{t-2,1} X_\bullet$.

Choosing a lift to $e_{t-3} \in \pi_2 Z_{t-3} P^r \Omega^i X_\bullet$, we may assume that $(j_{t-3})_*(e_{t-3}) = 0$, so $e_{t-3} \in \Omega \pi^{\natural\,(i,r)}_{t-4,2} X_\bullet$ and $s^{(i,r)}_{t-4,2}(e_{t-3}) = e_{t-2}$. Continuing in this way, we finally reach $e_{t-r-1} \in \Omega \pi^{\natural\,(i,r)}_{t-r-1,r-1} X_\bullet$ with $s^{(i,r)}_{t-r-2,r}(e_{t-r-1}) = e_{t-r}$, and so on, and see that $d^{r+1}_{t,i}(\langle a \rangle)$ is represented by $h^{(i,r)}_{t-r-1,r}(e_{t-r-1})$. Since (as in the proof of Proposition 3.7) $h^{(i,r)}_{t-r-1,r}$ is an isomorphism, we deduce that $d^{r+1}_{t,i}(\alpha)$ is as in (3.15). □

3.16. *Remark.* From the exactness of (3.3) we have $\mathrm{Im}\,(\partial^{(i,r)}_{t,0}) = \mathrm{Ker}\,(s^{(i,r)}_{t-1,0})$, so the image of $d^{r+1}_{t,i}$ as described in (3.15) is $\mathrm{Ker}\,(\sigma^{r+1}_{t,i})$, where $\sigma^{r+1}_{t,i} := (s^{(i,r)}_{t-1,0}) \circ (s^{(i,r)}_{t-2,1}) \circ (s^{(i,r)}_{t-3,2}) \circ \cdots \circ (s^{(i,r)}_{t-r,r-1})$. Therefore, $E^{r+1}_{t+r-1,i+r}$ embeds naturally in $\mathrm{Im}\,(\sigma^{r+1}_{t,i})$.

3.17. **Corollary.** *Every Reedy fibrant simplicial Postnikov r-stem has a well-defined $(r+1)$-truncated spiral spectral sequence. If $\mathcal{Q}_\bullet = \mathcal{P}[r]X_\bullet$ for some simplicial space $X_\bullet$, this truncated spectral sequence coincides with the $(r+1)$-truncation of the Bousfield-Friedlander spectral sequence for $X_\bullet$.*

Thus the bigraded homomorophism

$$d^{r+1} \circ d^{r+1} : E^r_{t,i} \to E^{r+1}_{t-2r-2,i+2r} \qquad (t \geq 2r+2, i \geq 0)$$

serves as the first obstruction to the realizablity of the simplicial Postnikov r-stem $\mathcal{Q}_\bullet$ by a simplicial space $X_\bullet$.

4. A cosimplicial version

There are actually four variants of the above spectral sequence which we might consider, for a simplicial or cosimplicial object over simplicial or cosimplicial sets. The case of bicosimplicial sets is in principle strictly dual to that of bisimplicial sets, but because the category of cosimplicial *sets* has no (known) useful model category structure, we must restrict to bicosimplicial abelian groups − or equivalently, ordinary double complexes. Thus the main new case of interest is that of cosimplicial simplicial sets, or *cosimplicial spaces.*

4.1**. The spectral sequence of a cosimplicial space.** If $X^\bullet \in c\mathcal{S}_*$ is a fibrant cosimplicial pointed space with total space $\operatorname{Tot} X^\bullet$, there are various constructions for the homotopy spectral sequence of $X^\bullet$:

(a) Using the tower of fibrations for $(\operatorname{Tot}_n X^\bullet)_{n=0}^\infty$ (cf. [BK1, X,§6]).
(b) Using "relations" on the normalized cochains $N^n\pi_t X^\bullet := \pi_t X^n \cap \operatorname{Ker}(s^0) \cap \ldots \cap \operatorname{Ker}(s^{n-1})$ (cf. [BK3, §7]).
(c) Using a cofibration sequence dualizing (2.4) (cf. [R, §3]).

Bousfield and Kan showed that the result is essentially unique (see [BK3]). Since the main ingredient needed for to define the spiral exact couple is the diagram (2.7), we use the first approach:

4.2. **Definition.** For any Reedy fibrant cosimplicial pointed space $X^\bullet \in c\mathcal{S}_*$, consider the fibration sequence

$$F_n X^\bullet \xrightarrow{j_n} \operatorname{Tot}_n X^\bullet \xrightarrow{p_n} \operatorname{Tot}_{n-1} X^\bullet \ , \tag{4.3}$$

where $\operatorname{Tot}_n X^\bullet := \operatorname{map}_{c\mathcal{S}_*}(\operatorname{sk}_n \mathbf{\Delta}, X^\bullet)$ and the fibration p_n is induced by the inclusion of cosimplicial spaces $\operatorname{sk}_{n-1}\mathbf{\Delta} \hookrightarrow \operatorname{sk}_n \mathbf{\Delta}$.

The cokernel of $(j_n)_\# : \pi_* F_n X^\bullet \hookrightarrow \pi_* \operatorname{Tot}_n X^\bullet$ is called the n-th *natural (graded) cohomotopy group* of $X^\bullet$, and denoted by $\pi^n_{\natural *} X^\bullet$.

4.4. *Remark.* We may identify $F_n X^\bullet$ with the looped normalized cochain object $\Omega^n N^n X^\bullet$, where

$$N^n X^\bullet \ := \ X^n \cap \operatorname{Ker}(s^0) \cap \ldots \cap \operatorname{Ker}(s^{n-1}) \ , \tag{4.5}$$

and $\pi_* N^n X^\bullet$ with $N^n \pi_* X^\bullet$ (see [BK1, X, Proposition 6.3]).

Moreover, the composite

$$\pi_{*+1}\Omega^n N^n X^\bullet \cong \pi_{*+1} F_n X^\bullet \xrightarrow{(j_n)_\#} \pi_{*+1} \operatorname{Tot}_n X^\bullet \xrightarrow{\partial_n} \pi_* F_{n+1} X^\bullet \cong \pi_* \Omega^{n+1} N^{n+1} X^\bullet$$

(where ∂_n is the connecting homomotphism for the (4.3)), may then be identified with the differential

$$\delta^n := \sum_{i=0}^{n} (-1)^i d^i \ : N^n \pi_* X^\bullet \to N^{n+1} \pi_* X^\bullet \ , \tag{4.6}$$

for the normalized cochain complex $N^* \pi_* X^\bullet$, so that

$$\operatorname{Ker}(\delta^n) / \operatorname{Coker}(\delta^{n+1}) \cong \pi^n \pi_* X^\bullet \tag{4.7}$$

(cf. [BK1, X, §7.2]).

4.8. **Proposition.** *For any pointed cosimplicial space* $X^\bullet$ *there is a natural* spiral *long exact sequence:*

$$\begin{aligned} \ldots \to \ \Omega\pi^{n-1}_{\natural *} X^\bullet \ \xrightarrow{s^n} \ \pi^n_{\natural *} X^\bullet \ \xrightarrow{h^n} \ \pi^n \pi_* X^\bullet \ \xrightarrow{\partial^n} \ \Omega \pi^{n-2}_{\natural *} X^\bullet \\ \xrightarrow{s^{n-1}} \ \pi^{n-1}_{\natural *} X^\bullet \ \to \ \ldots \ \to \ \pi^0_{\natural *} X^\bullet \ \xrightarrow{\cong} \ \pi^0 \pi_* X^\bullet \end{aligned} \tag{4.9}$$

Proof. By choosing a fibrant replacement in the model category of cosimplicial simplicial sets defined in [BK1, X, §5], if necessary, we may assume that $X^\bullet$ is Reedy fibrant. We then obtain a commutative diagram as in (2.7) with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{Ker}(j_n)_* \arrow[r, hook] \arrow[d, hook] & B^{n+1}X^\bullet \arrow[r, two heads, "(j_n)_*"] \arrow[d, hook] & B^{n+1}\pi_*X^\bullet \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega\pi^{n-1}_{\natural *}X^\bullet \arrow[r, hook, "\ell_{n-1}"] \arrow[d, two heads] \arrow[dr, "s_n"] & \pi_* \operatorname{Tot}_n X^\bullet \arrow[r, "(j_n^{X^\bullet})_\#"] \arrow[d, "\hat{\vartheta}_n"] & Z^n\pi_*X^\bullet \arrow[r, two heads] \arrow[d, "\vartheta_n"] & \operatorname{Coker}\ h^n \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \operatorname{Ker}\ h^n \arrow[r, hook] \arrow[d] & \pi^n_{\natural *}X^\bullet \arrow[r, "h_n"] \arrow[d] & \pi^n\pi_*X^\bullet \arrow[r, two heads] \arrow[d] & \operatorname{Coker}\ h^n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
\tag{4.10}
$$

in which $B^{n+1}X^\bullet := \operatorname{Im}(j_{n+1})_\# \subseteq \pi_* \operatorname{Tot}_n X^\bullet$ and $B^{n+1}\pi_*X^\bullet := \operatorname{Im}(\delta^{n+1}) = \operatorname{Im}(\partial_{n+1} \circ (j_{n+1})_\#)$ are the respective coboundary objects.

The construction of the maps h^n, s^n, and ∂^n, and the proof of the exactness of (4.9), are then precisely as in §2.6. □

4.11. **Definition.** The *spiral n-system* of a pointed cosimplicial space $X^\bullet \in c\mathcal{S}_*$ is defined to be the collection of long exact sequences (4.9) for the Postnikov n-stem functor $\mathcal{P}[n]$ applied to $X^\bullet$, one for each k-window of $\mathcal{P}[n]X^\bullet$.

As in Definition 3.2, this may actually be defined for a cosimplicial Postnikov n-stem $\mathcal{P}^\bullet$, not necessarily realizable as $\mathcal{P}^\bullet = \mathcal{P}[n]X^\bullet$.

By construction, the homotopy spectral sequence of a (fibrant) cosimplicial space $X^\bullet$, obtained as in (4.1), is associated to the spiral exact couple (4.9). The proofs of Proposition 3.7 and Theorem 3.14 use only the description of the spiral exact couple for $X_\bullet$ derived from (4.10), so by using (4.10) instead we can prove their analogues in the cosimplicial case, and show:

4.12. **Theorem.** *The E_{r+2}-term of the homotopy spectral sequence for a cosimplicial space $X^\bullet$ is determined by the spiral r-system of $X^\bullet$.*

An analogue of Corollary 3.17 also holds, as well as:

4.13. **Proposition.** *The differential $d_2^{t,i} : E_2^{t,i} \to E_2^{t+2,i+1}$ may be identified with $\partial^t_{(i,1)} : \pi^t\pi_iX^\bullet \to \Omega\pi^{t+2,0}_{\natural\,(i)}X^\bullet$.*

4.14. **Examples.** As noted in the introduction, many commonly used spectral sequences arise as the spiral spectral sequence of an appropriate (co)simplicial space, so Theorems 3.14 and 4.12 allow us to extract their E^r- or E_r-terms from the appropriate spiral systems. For instance:

(a) Segal's homology spectral sequence (cf. [Se]), the van Kampen spectral sequence (cf. [St]), and the Hurewicz spectral sequence (cf. [Bl1]) are constructed using bisimplicial sets.
(b) The unstable Adams spectral sequences of [BCKQRS, BK2] and [BCM, §4], Rector's version of the Eilenberg-Moore spectral sequence (cf. [R]), and Anderson's generalization of the latter (cf. [An]) are all associated to cosimplicial spaces.
(c) The usual construction of the stable Adams spectral sequence for $\pi_*^s X \otimes \mathbb{Z}/p$ (cf. [Ad, §3]) uses a tower of (co)fibrations, rather than a cosimplicial space, but when X is finite dimensional, it agrees in a range with the unstable version for $\Sigma^N X$, so Theorem 4.12 applies stably, too.

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