A CASSON-LIN TYPE INVARIANT FOR LINKS

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Abstract. In 1992, Xiao-Song Lin constructed an invariant $h(K)$ of knots $K \subset S^3$ via a signed count of conjugacy classes of irreducible $SU(2)$ representations of $\pi_1(S^3 - K)$ with trace-free meridians. Lin showed that $h(K)$ equals one half times the knot signature of K. Using methods similar to Lin's, we construct an invariant $h(L)$ of two-component links $L \subset S^3$. Our invariant is a signed count of conjugacy classes of projective $SU(2)$ representations of $\pi_1(S^3 - L)$ with a fixed 2-cocycle and corresponding non-trivial w_2. We show that $h(L)$ is, up to a sign, the linking number of L.

1. Introduction

One of the characteristic features of the fundamental group of a closed 3-manifold is that its representation variety in a compact Lie group tends to be finite, in a properly understood sense. This has been a guiding principle for defining invariants of 3-manifolds ever since Casson defined his λ-invariant for integral homology 3-spheres via a signed count of the $SU(2)$ representations of the fundamental group, where signs were determined using Heegaard splittings.

Among numerous generalizations of Casson's construction, we will single out the invariant of knots in S^3 defined by Xiao-Song Lin [9] via a signed count of $SU(2)$ representations of the fundamental group of the knot exterior. The latter is a 3-manifold with non-empty boundary so the above finiteness principle only applies after one imposes a proper boundary condition. Lin's choice of the boundary condition, namely, that all of the knot

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meridians are represented by trace-free $SU(2)$ matrices, resulted in an invariant $h(K)$ of knots $K \subset S^3$. Lin further showed that $h(K)$ equals half the knot signature of K.

The signs in Lin's construction were determined using braid representations for knots. Austin (unpublished) and Heusener and Kroll [7] extended this construction by letting the meridians of the knot be represented by $SU(2)$ matrices with a fixed trace which need not be zero. Their construction gives, for each choice of the trace, a knot invariant which equals one half times the equivariant knot signature.

In this paper, we extend Lin's construction to two-component links L in S^3. In essence, we replace the count of $SU(2)$ representations with a count of *projective* $SU(2)$ representations of $\pi_1(S^3 - L)$, in the sense of [11], with a fixed 2-cocycle representing a non-trivial element in the second group cohomology of $\pi_1(S^3 - L)$. The resulting signed count is denoted by $h(L)$. The two main results of this paper are then as follows.

Theorem 1. *For any two-component link $L \subset S^3$, the integer $h(L)$ is a well defined invariant of L.*

Theorem 2. *For any two-component link $L = \ell_1 \cup \ell_2$ in S^3, one has*

$$h(L) \;=\; \pm\,\mathrm{lk}\,(\ell_1, \ell_2).$$

It is worth mentioning that our choice of the 2-cocycle imposes Lin's trace-free condition on us. This is in contrast to Lin's construction, where the choice of boundary condition seemed somewhat arbitrary. This also means one should not expect to extend our construction to $SU(2)$ representations with non-zero trace boundary condition.

Shortly after Casson introduced his invariant for homology 3-spheres, Taubes [12] gave a gauge theoretic description of it in terms of a signed count of flat $SU(2)$ connections. After Lin's work, but before Heusener and Kroll, a gauge theoretic interpretation of the Lin invariant was given by Herald [6]. He used this interpretation to define an extension of the Lin invariant, now known as the Herald–Lin invariant, to knots in arbitrary homology spheres, with arbitrary fixed-trace (possibly non-zero) boundary condition.

Another attractive feature of the gauge theoretic approach is that it can be used to produce ramified versions of the above invariants. Floer [4] introduced the instanton homology theory whose Euler characteristic is twice the Casson invariant. We expect that our invariant will have a similar interpretation, perhaps along the lines of the knot instanton homology theory of Kronheimer and Mrowka [8], which in turn is a variant of the orbifold Floer homology of Collin and Steer [3]. We hope to discuss this elsewhere, together with possible extensions to links in homology spheres and to links of more than two components.

2. Braids and representations

Let F_n be a free group of rank $n \geq 2$, with a fixed generating set $x_1, \ldots, x_n$. We will follow the conventions of [10] and define the n-string braid group $\mathcal{B}_n$ to be the subgroup of $\operatorname{Aut}(F_n)$ generated by the automorphisms $\sigma_1, \ldots, \sigma_{n-1}$, where the action of σ_i is given by

$$\begin{aligned} \sigma_i : \quad x_i &\mapsto x_{i+1} \\ x_{i+1} &\mapsto (x_{i+1})^{-1}\, x_i\, x_{i+1} \\ x_j &\mapsto x_j, \quad j \neq i, i+1. \end{aligned}$$

The natural homomorphism $\mathcal{B}_n \to S_n$ onto the symmetric group on n letters, $\sigma \mapsto \bar{\sigma}$, maps each generator σ_i to the transposition $\bar{\sigma}_i = (i, i+1)$. A useful observation is that, for any $\sigma \in \operatorname{Aut}(F_n)$, one has

$$\sigma(x_i) = w\, x_{\bar{\sigma}^{-1}(i)}\, w^{-1} \tag{1}$$

for some word $w \in F_n$. One can also observe that σ preserves the product $x_1 \cdots x_n$, that is,

$$\sigma(x_1 \cdots x_n) = x_1 \cdots x_n \tag{2}$$

2.1. $SU(2)$ **representations.** Consider the Lie group $SU(2)$ of unitary two-by-two matrices with determinant one, i.e. complex matrices

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

such that $u\bar{u} + v\bar{v} = 1$. We will often identify $SU(2)$ with the group $Sp(1)$ of unit quaternions via

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \quad \mapsto \quad u + vj \, \in \mathbb{H}.$$

Let $R_n = \operatorname{Hom}(F_n, SU(2))$ be the space of $SU(2)$ representations of F_n, and identify it with $SU(2)^n$ by sending a representation $\alpha : F_n \to SU(2)$ to the vector $(\alpha(x_1), \ldots, \alpha(x_n))$ of $SU(2)$ matrices. The above representation $\mathcal{B}_n \to \operatorname{Aut}(F_n)$ then gives rise to the representation

$$\rho : \; \mathcal{B}_n \longrightarrow \operatorname{Diff}(R_n) \tag{3}$$

via $\rho(\sigma)(\alpha) = \alpha \circ \sigma^{-1}$. We will abbreviate $\rho(\sigma)$ to σ. We will also denote $X = (X_1, \ldots, X_n) \in R_n$ and write $\sigma(X) = (\sigma(X)_1, \ldots, \sigma(X)_n)$.

Example. For any $(X_1, \ldots, X_n) \in R_n$, we have $\sigma_1(X_1, X_2, X_3, \ldots, X_n) = (X_1 X_2 X_1^{-1}, X_1, X_3, \ldots, X_n)$.

2.2. **Extension to the wreath product** $\mathbb{Z}_2 \wr \mathcal{B}_n$**.** The wreath product $\mathbb{Z}_2 \wr \mathcal{B}_n$ is the semidirect product of $\mathcal{B}_n$ with $(\mathbb{Z}_2)^n$, where $\mathcal{B}_n$ acts on $(\mathbb{Z}_2)^n$ by permuting the coordinates, $\sigma(\varepsilon_1, \ldots, \varepsilon_n) = (\varepsilon_{\bar{\sigma}(1)}, \ldots, \varepsilon_{\bar{\sigma}(n)})$. Thus the elements of $\mathbb{Z}_2 \wr \mathcal{B}_n$ are the pairs $(\varepsilon, \sigma) \in (\mathbb{Z}_2)^n \times \mathcal{B}_n$, with the group multiplication law

$$(\varepsilon, \sigma) \cdot (\varepsilon', \sigma') \;=\; (\varepsilon\sigma(\varepsilon'), \, \sigma\sigma').$$

The representation (3) can be extended to a representation

$$\rho : \; \mathbb{Z}_2 \wr \mathcal{B}_n \longrightarrow \operatorname{Diff}(R_n) \tag{4}$$

by defining

$$\rho(\varepsilon, \sigma)(X) = \varepsilon \cdot \sigma(X) = (\varepsilon_1 \sigma(X)_1, \ldots, \varepsilon_n \sigma(X)_n),$$

where ε_i are viewed as elements of the center $\mathbb{Z}_2 = \{\pm 1\}$ of $SU(2)$. That (4) is a representation follows by a direct calculation after one observes that, because of (1),

$$\sigma(X)_i = AX_{\bar\sigma(i)}A^{-1} \quad \text{for some } A \in SU(2). \tag{5}$$

Again, we will abuse notations and write simply $\varepsilon\sigma$ for both (ε, σ) and $\rho(\varepsilon, \sigma)$.

Example. For any $(X_1, \dots, X_n) \in R_n$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{Z}_2)^n$, we have $(\varepsilon\sigma_1)(X_1, X_2, X_3, \dots, X_n) = (\varepsilon_1\, X_1X_2X_1^{-1}, \varepsilon_2\, X_1, \varepsilon_3\, X_3, \dots, \varepsilon_n\, X_n)$. Also $\sigma_1(\varepsilon X) = \sigma_1(\varepsilon)\sigma_1(X) = (\varepsilon_2 X_1X_2X_1^{-1}, \varepsilon_1 X_1, \varepsilon_3 X_3, \dots, \varepsilon_n X_n)$.

2.3. **Braids and link groups.** The closure $\hat\sigma$ of a braid $\sigma \in \mathcal{B}_n$ is a link in S^3 with link group

$$\pi_1(S^3 - \hat\sigma) = \langle\, x_1, \dots, x_n \mid x_i = \sigma(x_i),\ i = 1, \dots, n \,\rangle,$$

where each x_i represents a meridian of $\hat\sigma$. One can easily see that the fixed points of the diffeomorphism $\sigma : R_n \to R_n$ are representations $\pi_1(S^3 - \hat\sigma) \to SU(2)$. This paper grew out of the observation that a fixed point $\alpha = (\alpha(x_1), \dots, \alpha(x_n))$ of the diffeomorphism $\varepsilon\sigma : R_n \to R_n$ gives rise to a representation $\operatorname{ad}\alpha : \pi_1(S^3 - \hat\sigma) \to SO(3)$ by composing with the adjoint representation $\operatorname{ad} : SU(2) \to SO(3)$. Depending on ε, the representation $\operatorname{ad}\alpha$ may or may not lift to an $SU(2)$ representation, the obstruction being the second Stiefel–Whitney class $w_2(\operatorname{ad}\alpha) \in H^2(\pi_1(S^3 - \hat\sigma); \mathbb{Z}_2)$.

3. Definition of $h(\varepsilon\sigma)$

Every link in S^3 is the closure $\hat\sigma$ of a braid σ; see Alexander [1]. Let σ be a braid whose closure $\hat\sigma$ has two components. We will associate with it, for a carefully chosen ε, an integer $h(\varepsilon\sigma)$. We will prove in Section 4 that h is an invariant of the link $\hat\sigma$.

3.1. **Choice of ε.** The number of components of the link $\hat{\sigma}$ is exactly the number of cycles in the permutation $\bar{\sigma}$. We will be interested in two component links, that is, the closures of braids σ with

$$\bar{\sigma} = (i_1 \dots i_m)(i_{m+1} \dots i_n) \quad \text{for some } \ 1 \le m \le n-1. \tag{6}$$

Given such a braid σ, choose a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{Z}_2)^n$ such that

$$\varepsilon_{i_1} \cdots \varepsilon_{i_m} = \varepsilon_{i_{m+1}} \cdots \varepsilon_{i_n} = -1. \tag{7}$$

This choice of ε is dictated by the following two considerations. First, we wish to preserve condition (2) in the form

$$(\varepsilon\sigma)(X)_1 \cdots (\varepsilon\sigma)(X)_n = X_1 \cdots X_n, \tag{8}$$

and second, we want the fixed points α of the diffeomorphism $\varepsilon\sigma : R_n \to R_n$ to have non-zero $w_2\,(\operatorname{ad}\alpha)$.

Lemma 3.1. *Let α be a fixed point of $\varepsilon\sigma : R_n \to R_n$ with ε as in (7) then $w_2\,(\operatorname{ad}\alpha) \neq 0$.*

Proof. The class $w_2\,(\operatorname{ad}\alpha)$ is the obstruction to lifting $\operatorname{ad}\alpha$ to an $SU(2)$ representation. Extend α arbitrarily to a function $\alpha : \pi_1(S^3 - \hat{\sigma}) \to SU(2)$ lifting $\operatorname{ad}\alpha$ then $w_2\,(\operatorname{ad}\alpha)$ will vanish if and only if there is a function $\eta : \pi_1(S^3 - \hat{\sigma}) \to \mathbb{Z}_2 = \{\pm 1\}$ such that $\eta \cdot \alpha$ is a representation. Suppose that such a function exists, and denote $\eta(x_i) = \eta_i = \pm 1$. Also, assume without loss of generality that $\bar{\sigma} = (1 \dots m)(m+1 \dots n)$. It follows from (5) that in order to satisfy the relations $X_i = (\varepsilon\sigma)(X)_i$ we must have $\eta_1 = \varepsilon_1\eta_2 = \varepsilon_1\varepsilon_2\eta_3 = \dots = \varepsilon_1 \cdots \varepsilon_m\eta_1 = -\eta_1$, a contradiction with $\eta_1 = \pm 1$. $\square$

The above result concerning $w_2\,(\operatorname{ad}\alpha)$ can be sharpened using the following algebraic topology lemma.

Lemma 3.2. *Let $\widehat{\sigma}$ be a link of two components. If $\widehat{\sigma}$ is non-split then $H^2(\pi_1(S^3 - \hat{\sigma}); \mathbb{Z}_2) = \mathbb{Z}_2$. Otherwise, $H^2(\pi_1(S^3 - \hat{\sigma}); \mathbb{Z}_2) = 0$.*

Proof. If $\widehat{\sigma}$ is non-split then $S^3 - \widehat{\sigma}$ is a $K(\pi, 1)$ by the Sphere Theorem, hence $H^2(\pi_1(S^3 - \widehat{\sigma}); \mathbb{Z}_2) = H^2(S^3 - \widehat{\sigma}; \mathbb{Z}_2) = \mathbb{Z}_2$. If $\widehat{\sigma}$ is split then $K(\pi_1(S^3 - \widehat{\sigma}), 1)$

has the homotopy type of a one-point union of two circles and the result again follows. □

Corollary 3.3. *Let $\widehat{\sigma}$ be a split link of two components, and let ε be chosen as in (7). Then the diffeomorphism $\varepsilon\sigma : R_n \to R_n$ has no fixed points.*

3.2. **The zero-trace condition.** A naive way to define $h(\varepsilon\sigma)$ would be as the intersection number of the graph of $\varepsilon\sigma : R_n \to R_n$ with the diagonal in the product $R_n \times R_n$. One can observe though that, in addition to this intersection not being transversal, its points $(X, X) = (X_1, \dots, X_n, X_1, \dots, X_n)$ have the property that tr $X_1 = \ldots =$ tr $X_n = 0$. This can be seen as follows.

Assume without loss of generality that $\bar{\sigma} = (1 \dots m)(m + 1 \dots n)$. Then the relations $X = \varepsilon\sigma(X)$ together with (5) imply that

$$\begin{aligned} X_1 = \varepsilon_1 \sigma(X)_1 &= \varepsilon_1 A_1 \cdot X_{\bar{\sigma}(1)} \cdot A_1^{-1} = \varepsilon_1 A_1 X_2 A_1^{-1} \\ &= \varepsilon_1 A_1 \cdot \varepsilon_2\, \sigma(X)_2 \cdot A_1^{-1} = \varepsilon_1 \varepsilon_2\, A_1 A_2 \cdot X_{\bar{\sigma}(2)} \cdot A_2^{-1} A_1^{-1} = \ldots \\ &= \varepsilon_1 \cdots \varepsilon_m\, (A_1 \cdots A_m) \cdot X_1 \cdot (A_1 \dots A_m)^{-1}. \end{aligned}$$

Since trace is conjugation invariant and $\varepsilon_1 \dots \varepsilon_m = -1$, we conclude that tr $X_1 = \ldots =$ tr $X_m = 0$. Similarly, tr $X_{m+1} = \ldots =$ tr $X_n = 0$.

Therefore, in our definition we will restrict ourselves to the subset of R_n consisting of $X = (X_1, \dots, X_n)$ with tr $X_1 = \ldots =$ tr $X_n = 0$. The non-transversality problem will be addressed below by factoring out the conjugation symmetry and lowering the dimension of the ambient manifold.

3.3. **The definition.** The subset of $SU(2)$ consisting of the matrices with zero trace is a conjugacy class in $SU(2)$ diffeomorphic to S^2. Define

$$Q_n = \{(X_1, \dots, X_n) \in R_n \mid \text{ tr } X_i = 0\,\} \subset R_n,$$

so that Q_n is a manifold diffeomorphic to $(S^2)^n$. Also define

$$H_n = \{(X_1, \dots, X_n, Y_1, \dots, Y_n) \in Q_n \times Q_n \mid X_1 \cdots X_n = Y_1 \cdots Y_n\,\}.$$

This is no longer a manifold due to the presence of *reducibles*. We call a point $(X_1, \dots, X_n, Y_1, \dots, Y_n) \in Q_n \times Q_n$ reducible if all X_i and Y_j commute with each other or, equivalently, if there is a matrix $A \in SU(2)$ such that

AX_iA^{-1} and AY_iA^{-1} are all diagonal matrices, $i = 1, \dots, n$. The subset $S_n \subset Q_n \times Q_n$ of reducibles is closed.

Lemma 3.4. *$H_n^* = H_n - S_n$ is an open manifold of dimension $4n - 3$.*

Proof. Let us consider the open manifold $(Q_n \times Q_n)^* = Q_n \times Q_n - S_n$ of dimension $4n$ and the map $f : (Q_n \times Q_n)^* \to SU(2)$ given by

$$f(X_1, \dots, X_n, Y_1, \dots, Y_n) = X_1 \cdots X_n Y_n^{-1} \cdots Y_1^{-1}. \tag{9}$$

According to Lemma 1.5 of [9], this map has $1 \in SU(2)$ as a regular value. Since $H_n^* = f^{-1}(1)$, the result follows. □

Because of (5) and the fact that multiplication by $-1 \in SU(2)$ preserves the zero trace condition, the representation (4) gives rise to a representation

$$\rho : \ \mathbb{Z}_2 \wr \mathcal{B}_n \longrightarrow \mathrm{Diff}(Q_n). \tag{10}$$

Given $\varepsilon\sigma \in \mathbb{Z}_2 \wr \mathcal{B}_n$ such that (6) and (7) are satisfied, consider two submanifolds of $Q_n \times Q_n$: one is the graph of $\varepsilon\sigma : Q_n \to Q_n$,

$$\Gamma_{\varepsilon\sigma} = \{ \, (X, \varepsilon\sigma(X)) \mid X \in Q_n \, \},$$

and the other the diagonal,

$$\Delta_n = \{ \, (X, X) \mid X \in Q_n \, \}.$$

Note that both $\Gamma_{\varepsilon\sigma}$ and Δ_n are subsets of $H_n :$ this is obvious for Δ_n, and follows from equation (8) for $\Gamma_{\varepsilon\sigma}$.

Proposition 3.5. *The intersection $\Gamma_{\varepsilon\sigma} \cap \Delta_n \subset H_n$ consists of irreducible representations.*

Proof. Assume without loss of generality that $\bar\sigma = (1 \dots m)(m+1 \dots n)$, and suppose that $(X, X) = (X_1, \dots, X_n, X_1, \dots, X_n) \in \Gamma_{\varepsilon\sigma} \cap \Delta_n$ is reducible. Then all of the X_i commute with each other and, in particular, $\sigma(X) = (X_{\bar\sigma(1)}, \dots, X_{\bar\sigma(n)})$. The equality $X = \varepsilon\sigma(X)$ then implies that $X_1 = \varepsilon_1 X_{\bar\sigma(1)} = \varepsilon_1 X_2 = \varepsilon_1\varepsilon_2 X_{\bar\sigma(2)} = \ldots = \varepsilon_1 \cdots \varepsilon_m X_1 = -X_1$, a contradiction with $X_1 \in SU(2)$. □

Let $\Gamma^*_{\varepsilon\sigma} = \Gamma_{\varepsilon\sigma} \cap H^*_n$ and $\Delta^*_n = \Delta_n \cap H^*_n$ be the irreducible parts of $\Gamma_{\varepsilon\sigma}$ and Δ_n, respectively. They are both open submanifolds of H^*_n of dimension $2n$.

Corollary 3.6. *The intersection $\Delta^*_n \cap \Gamma^*_{\varepsilon\sigma} \subset H^*_n$ is compact.*

Proof. Proposition 3.5 implies that $\Delta^*_n \cap \Gamma^*_{\varepsilon\sigma} = \Delta_n \cap \Gamma_{\varepsilon\sigma}$, and the latter intersection is obviously compact as it is the intersection of two compact subsets of H_n. □

The group $SO(3) = SU(2)/\{\pm 1\}$ acts freely by conjugation on H^*_n, Δ^*_n, and $\Gamma^*_{\varepsilon\sigma}$. Denote the resulting quotient manifolds by

$$\widehat{H}_n = H^*_n/SO(3), \quad \widehat{\Delta}_n = \Delta^*_n/SO(3), \quad \text{and} \quad \widehat{\Gamma}_{\varepsilon\sigma} = \Gamma^*_{\varepsilon\sigma}/SO(3).$$

The dimension of $\widehat{H}_n$ is $4n-6$, and $\widehat{\Delta}_n$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ are submanifolds, each of dimension $2n-3$. Since the intersection $\widehat{\Delta}_n \cap \widehat{\Gamma}_{\varepsilon\sigma}$ is compact, one can isotope $\widehat{\Gamma}_{\varepsilon\sigma}$ into a submanifold $\widetilde{\Gamma}_{\varepsilon\sigma}$ using an isotopy with compact support so that $\widehat{\Delta}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma}$ consists of finitely many points. Define

$$h(\varepsilon\sigma) = \#_{\widehat{H}_n} (\widehat{\Delta}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})$$

as the algebraic intersection number, where the orientations of $\widehat{H}_n$, $\widehat{\Delta}_n$, and $\widetilde{\Gamma}_{\varepsilon\sigma}$ are described in the following subsection. It is obvious that $h(\varepsilon\sigma)$ does not depend on the perturbation of $\widehat{\Gamma}_{\varepsilon\sigma}$ so we will simply write

$$h(\varepsilon\sigma) = \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle_{\widehat{H}_n}.$$

3.4. **Orientations.** Orient the copy of $S^2 \subset SU(2)$ cut out by the trace zero condition arbitrarily, and endow $Q_n = (S^2)^n$ and $Q_n \times Q_n$ with product orientations. The diagonal Δ_n and the graph $\Gamma_{\varepsilon\sigma}$ are naturally diffeomorphic to Q_n via projection onto the first factor, and they are given the induced orientations. Note that if we reverse the orientation of S^2, then the orientation of Q_n is reversed if n is odd. Hence the orientations of both Δ_n and $\Gamma_{\varepsilon\sigma}$ are reversed if n is odd, while the orientation of $Q_n \times Q_n = (S^2)^{2n}$ is preserved regardless of the parity of n.

Orient $SU(2)$ by the standard basis $\{i, j, k\}$ in its Lie algebra $\mathfrak{su}(2)$, and orient $H^*_n = f^{-1}(1)$ by applying the base–fiber rule to the map (9). The adjoint action of $SO(3)$ on $S^2 \subset SU(2)$ is orientation preserving, hence the $SO(3)$ quotients $\widehat{H}_n$, $\widehat{\Delta}_n$, and $\widehat{\Gamma}_{\varepsilon\sigma}$ are orientable. We orient them using the base–fiber rule. The discussion in the previous paragraph shows that reversing orientation on S^2 may reverse the orientations of $\widehat{\Delta}_n$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ but that it does not affect the intersection number $\langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle_{\widehat{H}_n}$.

4. The link invariant h

In this section, we will prove Theorem 1. This will be accomplished by proving that $h(\varepsilon\sigma)$ is independent first of ε and then of σ.

4.1. **Independence of ε.** We will first show that, for a fixed σ whose closure $\hat{\sigma}$ is a link of two components, $h(\varepsilon\sigma)$ is independent of the choice of ε as long as ε satisfies (7).

Proposition 4.1. *Let ε and ε' be such that (7) is satisfied. Then $h(\varepsilon\sigma) = h(\varepsilon'\sigma)$.*

Proof. Assume without loss of generality that $\bar{\sigma} = (1\dots m)(m+1\dots n)$ and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_n)$. Define $\delta = (\delta_1, \dots, \delta_n)$ as the vector in $(\mathbb{Z}_2)^n$ with coordinates

$$\delta_1 = 1 \quad \text{and} \quad \delta_{k+1} \;=\; \delta_k\, \varepsilon_k\, \varepsilon'_k \quad \text{for} \quad k = 1, \dots, n-1,$$

and define the involution $\tau : Q_n \to Q_n$ by the formula

$$\tau(X) = \delta X = (\delta_1 X_1, \delta_2 X_2, \dots, \delta_n X_n).$$

Recall that $Q_n = (S^2)^n$ so that τ is a diffeomorphism which restricts to each of the factors S^2 as either the identity or the antipodal map. In particular, τ need not be orientation preserving.

The map $\tau\times\tau : Q_n\times Q_n \to Q_n\times Q_n$ obviously preserves the irreducibility condition and commutes with the $SO(3)$ action. It gives rise to an orientation preserving automorphism of $\widehat{H}_n$ which will again be called $\tau \times \tau$. It is clear that $(\tau \times \tau)(\widehat{\Delta}_n) = \widehat{\Delta}_n$. It is also true that $(\tau \times \tau)(\widehat{\Gamma}_{\varepsilon\sigma}) = \widehat{\Gamma}_{\varepsilon'\sigma}$, which can be seen as follows. Given a pair $(\delta X, \delta\varepsilon\sigma(X))$ whose conjugacy class belongs to $(\tau\times\tau)(\widehat{\Gamma}_{\varepsilon\sigma})$, write it as

$$(\delta X, \delta\varepsilon\sigma(X)) = (\delta X, \delta\varepsilon\sigma(\delta\delta X)) = (\delta X, \delta\varepsilon\sigma(\delta)\,\sigma(\delta X))$$

using the multiplication law in the group $\mathbb{Z}_2 \wr \mathcal{B}_n$. The conjugacy class of this pair belongs to $\Gamma_{\varepsilon'\sigma}$ if and only if $\delta\varepsilon\sigma(\delta) = \varepsilon'$. That this condition holds can be verified directly from the definition of δ.

Recall that the orientations of $\widehat{\Delta}_n$, $\widehat{\Gamma}_{\varepsilon\sigma}$, and $\widehat{\Gamma}_{\varepsilon'\sigma}$ are induced by the orientation of Q_n. Therefore, the maps $\tau\times\tau : \widehat{\Delta}_n \to \widehat{\Delta}_n$ and $\tau\times\tau : \widehat{\Gamma}_{\varepsilon\sigma} \to \widehat{\Gamma}_{\varepsilon'\sigma}$ are either both orientation preserving or both orientation reversing depending on whether $\tau : Q_n \to Q_n$ preserves or reverses orientation. Hence we have

$$h(\varepsilon\sigma) = \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle_{\widehat{H}_n} = \langle\, (\tau\times\tau)(\widehat{\Delta}_n), (\tau\times\tau)(\widehat{\Gamma}_{\varepsilon\sigma})\,\rangle_{(\tau\times\tau)(\widehat{H}_n)}$$
$$= \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon'\sigma}\,\rangle_{\widehat{H}_n} = h(\varepsilon'\sigma).$$

□

From now on, we will drop ε from the notation and simply write $h(\sigma)$ for $h(\varepsilon\sigma)$ assuming that a choice of ε satisfying (7) has been made.

4.2. **Independence of σ.** In this section, we will show that $h(\sigma)$ only depends on the link $\hat{\sigma}$, not on a particular choice of braid σ, by verifying that h is preserved under Markov moves. We will follow the proof of [9, Theorem 1.8] which goes through with little change once the right ε are chosen.

Recall that two braids $\alpha\in\mathcal{B}_n$ and $\beta\in\mathcal{B}_m$ have isotopic closures $\hat{\alpha}$ and $\hat{\beta}$ if and only if one braid can be obtained from the other by a finite sequence of Markov moves; see for instance [2]. A type 1 Markov move replaces $\sigma\in\mathcal{B}_n$ by $\xi^{-1}\sigma\xi\in\mathcal{B}_n$ for any $\xi\in\mathcal{B}_n$. A type 2 Markov move means replacing $\sigma\in\mathcal{B}_n$ by $\sigma_n^{\pm1}\sigma\in\mathcal{B}_{n+1}$, or the inverse of this operation.

Proposition 4.2. *The invariant $h(\sigma)$ is preserved by type 1 Markov moves.*

Proof. Let ξ, $\sigma\in\mathcal{B}_n$ and assume as usual that $\bar{\sigma} = (1\dots m)(m+1\dots n)$. Then

$$\overline{\xi^{-1}\sigma\xi} = (\bar{\xi}(1)\dots\bar{\xi}(m))(\bar{\xi}(m+1)\dots\bar{\xi}(n))$$

has the same cycle structure as $\bar{\sigma}$. To compute $h(\xi^{-1}\sigma\xi)$, we will make a choice of $\varepsilon \in (\mathbb{Z}_2)^n$ which satisfies condition (7) with respect to the braid $\xi^{-1}\sigma\xi$, that is, $\varepsilon_{\bar{\xi}(1)} \cdots \varepsilon_{\bar{\xi}(m)} = \varepsilon_{\bar{\xi}(m+1)} \cdots \varepsilon_{\bar{\xi}(n)} = -1$.

The braid ξ gives rise to the map $\xi : Q_n \to Q_n$. It acts by permutation and conjugation on the S^2 factors in Q_n hence it is orientation preserving (we use the fact that S^2 is even dimensional). It induces an orientation preserving map $\xi \times \xi : Q_n \times Q_n \to Q_n \times Q_n$, which preserves the irreducibility condition and commutes with the $SO(3)$ action. Equation (2) then ensures that we have a well defined orientation preserving automorphism $\xi \times \xi : \widehat{H}_n \to \widehat{H}_n$.

That this automorphism preserves the diagonal, $(\xi \times \xi)(\widehat{\Delta}_n) = \widehat{\Delta}_n$, is obvious. Concerning the graphs, let $(X, \varepsilon\xi^{-1}\sigma\xi(X)) \in \widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi}$ then

$$
\begin{aligned}
&(\xi \times \xi)(X, \varepsilon\xi^{-1}\sigma\xi(X)) \\
&\qquad\qquad = (\xi(X), \xi(\varepsilon\xi^{-1}\sigma\xi(X))) = (\xi(X), \xi(\varepsilon)\sigma(\xi(X))) \in \widehat{\Gamma}_{\xi(\varepsilon)\sigma}.
\end{aligned}
$$

Therefore, $(\xi \times \xi)(\widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi}) = \widehat{\Gamma}_{\xi(\varepsilon)\sigma}$. Since $\xi : Q_n \to Q_n$ is orientation preserving, the above identifications of the diagonals and graphs via $\xi \times \xi$ are also orientation preserving.

Observe that $\xi(\varepsilon)_i = \varepsilon_{\bar{\xi}(i)}$ hence $\xi(\varepsilon)$ satisfies (7) with respect to σ and thus can be used to compute $h(\sigma)$. The following calculation now completes the argument:

$$
\begin{aligned}
h(\xi^{-1}\sigma\xi) = \langle \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi} \rangle_{\widehat{H}_n} &= \langle (\xi \times \xi)(\widehat{\Delta}_n), (\xi \times \xi)(\widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi}) \rangle_{(\xi\times\xi)(\widehat{H}_n)} \\
&= \langle \widehat{\Delta}_n, \widehat{\Gamma}_{\xi(\varepsilon)\sigma} \rangle_{\widehat{H}_n} = h(\sigma).
\end{aligned}
$$

□

Proposition 4.3. *The invariant $h(\sigma)$ is preserved by type 2 Markov moves.*

Proof. Given $\sigma \in \mathcal{B}_n$ and ε satisfying (7), change σ to $\sigma_n\sigma \in \mathcal{B}_{n+1}$ and let $\varepsilon' = \sigma_n(\varepsilon, 1)$. If $X = (X_1, \ldots, X_n)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ then

$$
\begin{aligned}
(\sigma_n\sigma)(X, X_{n+1}) &= \sigma_n(\sigma(X), X_{n+1}) \\
&= (\sigma(X)_1, \ldots, \sigma(X)_{n-1}, \, \sigma(X)_n X_{n+1} \sigma(X)_n^{-1}, \, \sigma(X)_n))
\end{aligned}
$$

and $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_{n-1}, 1, \varepsilon_n)$. In particular, ε' satisfies (7) with respect to $\sigma_n\sigma$. Consider the embedding $g : Q_n \times Q_n \to Q_{n+1} \times Q_{n+1}$ given by

$$g(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = (X_1, \ldots, X_n, Y_n, Y_1, \ldots, Y_n, Y_n)$$

One can easily see that $g(H_n) \subset H_{n+1}$ and that g commutes with the conjugation, thus giving rise to an embedding $\hat{g} : \widehat{H}_n \to \widehat{H}_{n+1}$. A straightforward calculation using the above formulas for $\sigma_n\sigma$ and ε' then shows that

$$\hat{g}\,(\widehat{\Delta}_n) \subset \widehat{\Delta}_{n+1}, \quad \hat{g}\,(\widehat{\Gamma}_{\varepsilon\sigma}) \subset \widehat{\Gamma}_{\varepsilon'\sigma_n\sigma}\,, \quad \text{and} \quad \hat{g}\,(\widehat{\Delta}_n \cap \widehat{\Gamma}_{\varepsilon\sigma}) = \widehat{\Delta}_{n+1} \cap \widehat{\Gamma}_{\varepsilon'\sigma_n\sigma}.$$

Now, one can achieve all the necessary transversalities and match the orientations in exactly the same way as in the second half of the proof of [9, Theorem 1.8]. This shows that $h(\sigma_n\sigma) = h(\sigma)$. The proof of the equality $h(\sigma_n^{-1}\sigma) = h(\sigma)$ is similar. □

5. The invariant $h(\sigma)$ as the linking number

In this section we will prove Theorem 2, that is, show that for any link $\hat{\sigma} = \ell_1 \cup \ell_2$ of two components, one has

$$h(\sigma) \;=\; \pm\,\mathrm{lk}\,(\ell_1, \ell_2).$$

Our strategy will be to show that the invariant $h(\sigma)$ and the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$ change according to the same rule as we change a crossing between two strands from two different components of $\hat{\sigma} = \ell_1 \cup \ell_2$ (the link $\hat{\sigma}$ will need to be oriented for that, although a particular choice of orientation will not matter). After changing finitely many such crossings, we will arrive at a split link, for which both the invariant $h(\sigma)$ and the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$ vanish; see Corollary 3.3. The change of crossing as above obviously changes the linking number by ± 1. To calculate the effect of the crossing change on $h(\sigma)$, we will follow [9] and reduce the problem to a calculation in the pillowcase $\widehat{H}_2$.

5.1. **The pillowcase.** We begin with a geometric description of $\widehat{H}_2$ as a pillowcase, compare with [9, Lemma 1.2]. Remember that

$$H_2 = \{(X_1, X_2, Y_1, Y_2) \in Q_2 \times Q_2 \mid X_1 X_2 = Y_1 Y_2\,\}.$$

We will use the identification of $SU(2)$ with $Sp(1)$ when convenient. Since X_2 is trace free, we may assume that $X_2 = i$ after conjugation. Conjugating by $e^{i\varphi}$ will not change X_2 but, for an appropriate choice of φ, will make X_1 into

$$X_1 = \begin{pmatrix} ir & u \\ -u & -ir \end{pmatrix},$$

where both r and u are real, and u is also non-negative. Since $r^2 + u^2 = 1$ we can write $r = \cos\theta$ and $u = \sin\theta$ for a unique θ such that $0 \le \theta \le \pi$. In the quaternionic language, $X_1 = i\,e^{-k\theta}$ with $0 \le \theta \le \pi$. Similarly, the condition $\operatorname{tr}(Y_2) = \operatorname{tr}(Y_1^{-1} X_1 X_2) = 0$ implies that $Y_1 = i\,e^{-k\psi}$, this time with $-\pi \le \psi \le \pi$. To summarize,

$$X_1 = ie^{-k\theta}, \;\; X_2 = i, \;\; Y_1 = ie^{-k\psi}, \;\; Y_2 = ie^{-k(\psi-\theta)}.$$

Thus $\widehat{H}_2$ is parameterized by the rectangle $[0,\pi]\times[-\pi,\pi]$, with proper identifications along the edges and with the reducibles removed. The reducibles occur when both θ and ψ are multiples of π, hence $\widehat{H}_2$ a 2-sphere with the points $A = (0,0)$, $B = (\pi,0)$, $A' = (0,\pi)$, and $B' = (\pi,\pi)$ removed; see Figure 1. According to [9], the orientation on the front sheet of $\widehat{H}_2$ coincides with the standard orientation on the (θ,ψ) plane.

Example. Let $\sigma = \sigma_1^2$ so that $\widehat{\sigma} = \ell_1 \cup \ell_2$ is the Hopf link with $\operatorname{lk}(\ell_1,\ell_2) = \pm 1$. To calculate $h(\sigma)$, we let $\varepsilon = (-1,-1)$, the only available choice satisfying (7), and consider the submanifolds $\widehat{\Delta}_2$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ of $\widehat{H}_2$. We have, in quaternionic notations, $\widehat{\Delta}_2 = \{(ie^{-k\theta}, i, ie^{-k\theta}, i)\}$, which is the diagonal $\psi = \theta$ in the pillowcase. A straightforward calculation shows that $\widehat{\Gamma}_{\varepsilon\sigma} \subset \widehat{H}_2$ is given by $\psi = 3\theta - \pi$. As can be seen in Figure 1, the intersection $\widehat{\Delta}_2 \cap \widehat{\Gamma}_{\varepsilon\sigma}$ consists of one point coming with a sign. Hence $h(\sigma_1^2) = \pm 1$, which is consistent with the fact that $\operatorname{lk}(\ell_1,\ell_2) = \pm 1$.

Example. Let $\sigma = \sigma_1^{2n}$ then arguing as above one can show that $\widehat{\Gamma}_{\varepsilon\sigma} \subset \widehat{H}_2$ is given by $\psi = (2n+1)\theta - \pi$. In this case there are n intersection points all of which come with the same sign. This shows that $h(\sigma_1^{2n}) = \pm n$, which is again consistent with the fact that $\operatorname{lk}(\ell_1,\ell_2) = \pm n$.

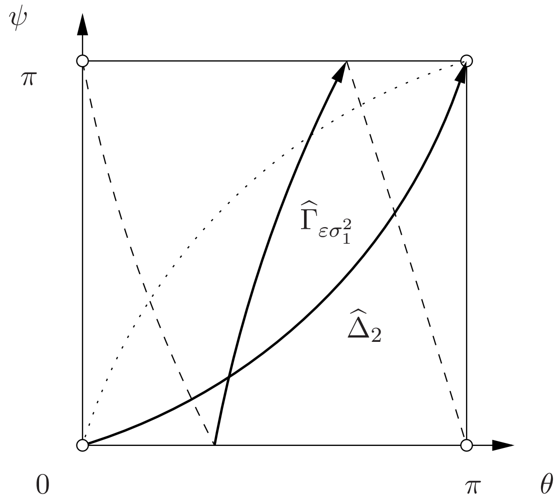

FIGURE 1. The pillowcase

5.2. **The difference cycle.** Given a two component link $\hat{\sigma}$, fix an orientation on it. A particular choice of orientation will not matter because we are only interested in identifying $h(\sigma)$ with the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$ up to sign. We wish to change one of the crossings between the two components of $\hat{\sigma}$. Using a sequence of first Markov moves, we may assume that the first two strands of σ belong to two different components of $\hat{\sigma}$, and that the crossing change occurs between these two strands. Furthermore, we may assume that the crossing change makes σ into $\sigma_1^{\pm 2}\sigma$, where the sign depends on the type of the crossing we change. Note that the braid $\sigma_1^{\pm 2}\sigma$ has the same permutation type as σ; in particular, its closure is a link of two components. In fact, if we let $\sigma' = \sigma_1^{-2}\sigma$ then

$$h(\sigma_1^{-2}\sigma) - h(\sigma) = h(\sigma') - h(\sigma_1^2\sigma') = -\,(h(\sigma_1^2\sigma') - h(\sigma')),$$

hence we only need to understand the difference $h(\sigma_1^2\sigma) - h(\sigma)$. Let us fix $\varepsilon = (-1, -1, 1, \dots, 1)$. Since σ_1^2 and ε commute, we have

$$\begin{aligned} h(\sigma_1^2\sigma) - h(\sigma) &= \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma_1^2\sigma}\,\rangle - \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle \\ &= \langle\, \widehat{\Gamma}_{\sigma_1^{-2}}, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle - \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle = \langle\, \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle, \end{aligned}$$

where all intersection numbers are taken in $\widehat{H}_n$. This leads us to consider the *difference cycle* $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$ which is carried by $\widehat{H}_n$. The next step in our argument will be to reduce the analysis of the above intersection to an intersection theory in the pillowcase $\widehat{H}_2$.

5.3. **The pillowcase reduction.** Let us consider the subset $V_n \subset H_n$ consisting of all $(X_1, \dots, X_n, Y_1, \dots, Y_n) \in H_n$ such that $X_k = Y_k$ for $k = 3, \dots, n$. Equivalently, V_n consists of all $(X_1, \dots, X_n, Y_1, \dots, Y_n) \in Q_n \times Q_n$ such that $(X_1, X_2, Y_1, Y_2) \in H_2$ and $X_k = Y_k$ for all $k = 3, \dots, n$. Therefore, V_n can be identified as

$$V_n = H_2 \times \Delta_{n-2} \;\subset\; (Q_2 \times Q_2) \times (Q_{n-2} \times Q_{n-2}).$$

Lemma 5.1. *The quotient* $\widehat{V}_n = (H_2^* \times \Delta_{n-2})/SO(3)$ *is a submanifold of* $\widehat{H}_n$ *of dimension* $2n-2$.

Proof. Since H_2^* and Δ_{n-2} are smooth manifolds of dimensions five and $2n-4$, respectively, and their product contains no reducibles, the statement follows. □

Lemma 5.2. *The difference cycle* $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$ *is carried by* $\widehat{V}_n$.

Proof. Observe that neither $\widehat{\Gamma}_{\sigma_1^{-2}}$ nor $\widehat{\Delta}_n$ are subsets of $\widehat{V}_n$. However, their portions that do not fit in $\widehat{V}_n$,

$$\widehat{\Gamma}_{\sigma_1^{-2}} - (\widehat{\Gamma}_{\sigma_1^{-2}} \cap \widehat{V}_n) \quad \text{and} \quad \widehat{\Delta}_n - (\widehat{\Delta}_n \cap \widehat{V}_n),$$

are exactly the same. Namely, they consist of the equivalence classes of $2n$–tuples $(X_1, \dots, X_n, X_1, \dots, X_n)$ such that X_1 commutes with X_2. These cancel in the difference cycle $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$, thus making it belong to $\widehat{V}_n$. □

One can isotope $\widehat{\Gamma}_{\varepsilon\sigma}$ into $\widetilde{\Gamma}_{\varepsilon\sigma}$ using an isotopy with compact support so that $\widetilde{\Gamma}_{\varepsilon\sigma}$ is transverse to $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$. The latter means precisely that $\widetilde{\Gamma}_{\varepsilon\sigma}$ stays away from $(S_2 \times \Delta_{n-2})/SO(3)$ and is transverse to both $\widehat{\Gamma}_{\sigma_1^{-2}}$ and $\widehat{\Delta}_n$; a precise argument can be found in [7, page 491]. We further extend this isotopy to make $\widetilde{\Gamma}_{\varepsilon\sigma}$ transverse to $\widehat{V}_n$ so that their intersection is a naturally oriented 1-dimensional submanifold of $\widehat{H}_n$.

The natural projection $p : V_n \to H_2$ induces a map $\hat{p} : \widehat{V}_n \to \widehat{H}_2$. Use a further small compactly supported isotopy of $\widetilde{\Gamma}_{\varepsilon\sigma}$, if necessary, to make $\hat{p}\,(\widehat{V}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})$ into a 1-submanifold of $\widehat{H}_2$. The proofs of Lemmas 2.2 and 2.3 in Lin [9] then go through with little change to give us the following identity

$$\langle\, \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n,\ \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle_{\widehat{H}_n} = \langle\, \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_2,\ \hat{p}\,(\widehat{V}_n \cap \tilde{\Gamma}_{\varepsilon\sigma})\,\rangle_{\widehat{H}_2}.$$

5.4. **Computation in the pillowcase.** We begin by studying the behavior of $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ near the corners of $\widehat{H}_2$.

Proposition 5.3. *There is a neighborhood around A' in the pillowcase $\widehat{H}_2$ inside which $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ is a curve approaching A'.*

Proof. Let us consider the submanifold

$$\Delta'_n = \{(X_1, X_2, X_3, \dots, X_n; Y_1, Y_2, X_3, \dots, X_n)\} \ \subset\ Q_n \times Q_n$$

and observe that $V_n \cap \Gamma_{\varepsilon\sigma} = \Delta'_n \cap \Gamma_{\varepsilon\sigma}$. We will show that the intersection of Δ'_n with $\Gamma_{\varepsilon\sigma}$ is transversal at $(1, \varepsilon\, 1) = (i, \dots, i; -i, -i, i, \dots, i)$. This will imply that $\Delta'_n \cap \Gamma_{\varepsilon\sigma}$ a submanifold of dimension four in a neighborhood of $(1, \varepsilon\, 1)$ and, after factoring out the $SO(3)$ symmetry, that $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ is a curve approaching $A' = p\,(1, \varepsilon\, 1)$.

Note that $\dim \Delta'_n = 2n + 4$ hence the dimension of $T_{(1,\varepsilon\, 1)}(\Delta'_n \cap \Gamma_{\varepsilon\sigma}) = T_{(1,\varepsilon\, 1)}\Delta'_n \cap T_{(1,\varepsilon\, 1)}\Gamma_{\varepsilon\sigma}$ is at least four. Therefore, checking the transversality amounts to showing that this dimension is exactly four. Write

$$T_{(1,\varepsilon 1)}(\Delta'_n) = \{(u_1, \dots u_n; v_1, v_2, u_3, \dots, u_n)\} \subset T_{(1,\varepsilon 1)}(Q_n \times Q_n)$$

and

$$T_{(1,\varepsilon 1)}(\Gamma_{\varepsilon\sigma}) = \{(u_1, \dots, u_n; d_1(\varepsilon\sigma)(u_1, \dots, u_n)\} \subset T_{(1,\varepsilon 1)}(Q_n \times Q_n).$$

Then $T_{(1,\varepsilon 1)}(\Delta'_n) \cap T_{(1,\varepsilon\, 1)}(\Gamma_{\varepsilon\sigma})$ consists of the vectors $(u_1, \dots, u_n) \in T_1\, Q_n = T_i\, S^2 \oplus \dots \oplus T_i\, S^2$ that solve the matrix equation

$$\begin{bmatrix} d_1(\sigma) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ u_3 \\ \vdots \\ u_n \end{bmatrix}; \tag{11}$$

since $\varepsilon = (-1, -1, 1, \ldots, 1)$, we can safely replace $[d_1(\varepsilon\sigma)]$ by $[d_1(\sigma)]$. It is shown in [10] that $[d_1(\sigma)]$ is the Burau matrix of σ with parameter equal to -1. It is a real matrix acting on $T_1 Q_n = \mathbb{C}^n$, hence all we need to show is that the space of $(u_1, \ldots, u_n) \in \mathbb{R}^n$ solving (11) has real dimension two. Let us write

$$\begin{bmatrix} d_1(\sigma) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is a 2×2 matrix and D is an $(n-2) \times (n-2)$ matrix. Equation (11) is equivalent to

$$\begin{bmatrix} C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 - D \end{bmatrix} \begin{bmatrix} u_3 \\ \vdots \\ u_n \end{bmatrix},$$

so the proposition will follow as soon as we show that $1 - D$ is invertible. The invertibility of $1 - D$ is a consequence of the following two lemmas. □

Lemma 5.4. *Let $\sigma \in \mathcal{B}_n$ then the Burau matrix of σ with parameter -1 and the permutation matrix of $\bar{\sigma}^{-1}$ are the same modulo 2.*

Proof. According to [2], the Burau matrix of σ with parameter t is the matrix

$$\left.\frac{\partial \sigma(x_i)}{\partial x_j}\right|_{x_i = t}$$

where x_i are generators of the free group and ∂ is the derivative in the Fox free differential calculus; see [5]. Applying the Fox calculus we obtain:

$$\frac{\partial\sigma(x_i)}{\partial x_j} = \frac{\partial(wx_{\bar\sigma^{-1}(i)}w^{-1})}{\partial x_j} = \frac{\partial w}{\partial x_j} + w\left(\frac{\partial(x_{\bar\sigma^{-1}(i)}w^{-1})}{\partial x_j}\right) = \frac{\partial w}{\partial x_j} +$$
$$w\left(\frac{\partial x_{\bar\sigma^{-1}(i)}}{\partial x_j} + x_{\bar\sigma^{-1}(i)}\frac{\partial w^{-1}}{\partial x_j}\right) = \frac{\partial w}{\partial x_j} + w\frac{\partial x_{\bar\sigma^{-1}(i)}}{\partial x_j} - wx_{\bar\sigma^{-1}(i)}w^{-1}\frac{\partial w}{\partial x_j},$$

where w is a word in the x_i. After evaluating at $t = -1$ and reducing modulo 2, the above becomes simply $\partial x_{\bar\sigma^{-1}(i)}/\partial x_j$, which is the permutation matrix of $\bar\sigma^{-1}$. □

Lemma 5.5. *Let $\sigma \in \mathcal{B}_n$ be such that $\widehat{\sigma}$ is a two component link. Then $1 - D$ is invertible.*

Proof. Our assumption in this section has been that $\bar\sigma = (1, \cdots)(2, \cdots)$. We may further assume that

$$\bar\sigma = (1, 3, 4, \dots, k)\,(2, k+1, k+2, \dots, n)$$

by applying a sequence of first Markov moves fixing the first two strands of σ. The matrix $D \pmod 2$ is obtained by crossing out the first two rows and first two columns in the permutation matrix of $\bar\sigma$; see Lemma 5.4. This description implies that $D \pmod 2$ is upper diagonal, and hence so is $(1-D) \pmod 2$. The diagonal elements of the latter matrix are all equal to one, therefore, $\det(1 - D) = 1 \pmod 2$ so $1 - D$ is invertible. □

Remark 5.6. The orientation of the component of $\hat p\ (\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ limiting to A' can be read off its description near A' given in the proof of Proposition 5.3. In particular, this orientation is independent of the choice of σ.

Proposition 5.7. *There are neighborhoods around A and B' in the pillowcase $\widehat{H}_2$ which are disjoint from $\hat p\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$.*

Proof. The arguments for A and B' are essentially the same so we will only give the proof for A. Assuming the contrary we have a curve in $\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma}$

limiting to a reducible representation in $V_n \cap \Gamma_{\varepsilon\sigma}$. After conjugating if necessary, this representation must have the form

$$(i, i, e^{i\varphi_3}, \dots, e^{i\varphi_n}, i, i, e^{i\varphi_3}, \dots, e^{i\varphi_n}).$$

Using the fact that $\varepsilon = (-1, -1, 1, \dots, 1)$ and arguing as in the proof of Proposition 3.5, we arrive at the contradiction $i = -i$. □

5.5. **Proof of Theorem 2.** According to Proposition 5.3, near A', the 1-submanifold $\hat{p}\,(\widehat{V}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})$ is a curve approaching A'. According to Proposition 5.7, the other end of this curve must approach B. Therefore,

$$h(\sigma_1^2\sigma) - h(\sigma) \;=\; \langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_2,\, \hat{p}\,(\widehat{V}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})\rangle_{\widehat{H}_2}$$

is the same as the intersection number of an arc going from A' to B with the difference cycle $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_2$. This number is either 1 or -1 but it is the same for all σ; see Remark 5.6. This is sufficient to prove that $h(\sigma)$ is the linking number up to an overall sign.

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