Complex contact manifolds and S^1 actions

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Abstract

We prove rigidity and vanishing theorems for several holomorphic Euler characteristics on complex contact manifolds admitting holomorphic circle actions preserving the contact structure. Such vanishings are reminiscent of those of LeBrun and Salamon on Fano contact manifolds but under a symmetry assumption instead of a curvature condition.

1 Introduction

The geometry of complex contact manifolds was first studied by Kobayashi Kobayashi and Boothby Boothby, and more recently by LeBrun LeBrun-contact and Moroianu Moroianu-contact in relation to quaternion-Kähler geometry. Here, we study these manifolds from the point of view of transformation groups.

Inspired by Atiyah and Hirzebruch AH, Hattori proved the vanishing of indices of Dirac operators with coefficients in certain powers of the Spin^c complex line bundle on compact Spin^c manifolds admitting smooth circle actions Hattori. Such vanishings apply to complex contact manifolds since their first Chern class is a multiple of an integral cohomology class.

In this note, we prove the vanishing of several holomorphic Euler characteristics on complex contact manifolds admitting holomorphic circle actions preserving the contact structure. The vector bundles considered in the holomorphic Euler characteristics are tensor products of a suitable exterior power of the contact (distribution) sub-bundle and a power of the canonical line bundle.

These vanishings are reminiscent of those of LeBrun and Salamon for Fano contact manifolds LS. Their vanishings depend on a positive-curvature condition (Fano condition) which, in particular, makes such manifolds projective. Here, we assume the existence of a compatible circle action on a complex contact manifold; in principle, the manifolds may neither fulfill a curvature condition nor be projective.

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The note is organized as follows. In Section 2, we recall the definition and some properties of complex contact manifolds, the rigidity of elliptic operators, state our main theorem (see Theorem 2.1), and describe some properties of the exponents of the action. In Section 3, we carry out index calculations and prove Theorem 2.1. In Section 4 we prove further vanishings under a non-negativity assumption on the exponents of the action.

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2 Preliminaries

2.1 Complex contact manifolds

Let X be a complex manifold and TX its holomorphic tangent bundle.

Definition 2.1 *The complex manifold X is called* contact *if there is a complex-codimension one holomorphic sub-bundle D of TX which is maximally non-integrable, i.e. the tensor*

$$\begin{array}{ccc} D \times D & \longrightarrow & TX/D \\ (v, w) & \mapsto & [v, w] \bmod D \end{array}$$

is non-degenerate for every point of X.

Examples.

- Let V be a compact complex manifold. Then the projectivization of the cotangent bundle $X := \mathbb{P}(T^*V)$ is a complex contact manifold (see [6] for further details). Here a 1-form θ can be defined as follows: $\theta(u) := v(d\pi(u))$, where $u \in T_v(T^*V)$ and $\pi : T^*V \to V$ is the projection onto V. Thus, $D := ker(\theta)$.
- Let M be a quaternion-Kähler manifold. Its twistor space Z is a contact complex manifold (see [9]). In fact, Z is a fiber bundle over M with fiber $\mathbb{C}P^1$, and D is a complex codimension 1 distribution that is transverse to the fibers of $Z \to M$.

Let $L := TX/D$ be the quotient line bundle and $\theta : TX \longrightarrow L$ the tautological projection, so that we have the short exact sequence

$$0 \longrightarrow D \longrightarrow TX \longrightarrow L \longrightarrow 0. \tag{1}$$

The projection θ can be thought of as a 1-form with values in the line bundle L, $\theta \in \Gamma(X, \Omega^1(L))$, with $\ker(\theta) = D$. The sub-bundle D must have even rank $2n$ and, therefore, the manifold X has odd complex dimension $2n+1 \geq 3$. Moreover, the non-degeneracy condition implies

$$\theta \wedge (d\theta)^n \in \Gamma(X, \Omega^{2n+1}(L^{n+1}))$$

is nowhere zero. This provides an isomorphism Kobayąshi,LS of the anti-canonical line bundle of X and L^{n+1},

$$\kappa_X^{-1} = \bigwedge^{2n+1} TX \cong L^{n+1}.$$

Since $L = TX/D$, there is a C^∞ isomorphism

$$TX \cong D \oplus L,$$

so that

$$c(X) = c(D) \cdot c(L).$$

There is also the following isomorphism (cf. [9, p. 116])

$$D \cong D^* \otimes L. \tag{2}$$

By means of the splitting principle we can write the Chern classes in terms of formal roots

$$c(D) = (1+y_1)(1+y_2)\cdots(1+y_{2n}),$$

and

$$c(L) = (1+y_{2n+1}),$$

so that

$$c_1(X) = (n+1)y_{2n+1}.$$

2.2 Rigidity of elliptic operators

Let M be a compact manifold and E and F vector bundles over M.

Definition 2.2 *Let $\mathcal{D} : \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator acting on sections of E and F. The index of $\mathcal{D}$ is the virtual vector space* $\mathrm{ind}(\mathcal{D}) = \ker(\mathcal{D}) - \mathrm{coker}(\mathcal{D})$. *If M admits a circle action preserving $\mathcal{D}$, i.e. such that S^1 acts on E and F, and commutes with $\mathcal{D}$,* $\mathrm{ind}(\mathcal{D})$ *admits a Fourier decomposition into complex 1-dimensional irreducible representations of S^1* $\mathrm{ind}(\mathcal{D}) = \sum a_m L^m$, *where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on $\mathbb{C}$ given by $z \mapsto z^m$. The elliptic operator $\mathcal{D}$ is called* rigid *if $a_m = 0$ for all $m \neq 0$, i.e.* $\mathrm{ind}(\mathcal{D})$ *consists only of the trivial representation with multiplicity a_0.*

Remark 2.1 Equivalently, we can take the trace for $z \in S^1$,

$$\mathrm{ind}(\mathcal{D})_z = \mathrm{trace}(z, \sum a_m L^m) = \sum a_m z^m,$$

to get a finite Laurent series in z. Now $\mathcal{D}$ is rigid if and only if $\mathrm{ind}(\mathcal{D})_z$ does not depend on $z \in S^1$.

Example. The deRham complex

$$\mathcal{D} = d + d^*\colon \Omega^{even} \longrightarrow \Omega^{odd}$$

from even-dimensional forms to odd-dimensional ones, where d^* denotes the adjoint of the exterior derivative d, is *rigid* for any circle action on M by isometries since by Hodge theory the kernel and the cokernel of this operator consist of harmonic forms, which by homotopy invariance stay fixed under the circle action.

2.3 Rigidity and vanishing theorem

Now we can state the main theorem,

Theorem 2.1 *Let X be a complex contact manifold, D the contact distribution and $L = TX/D$. Assume X admits a circle action by holomorphic automorphisms preserving the contact structure. Then, the equivariat holomorphic Euler characteristic $\chi(X, \mathcal{O}(\bigwedge^p D^* \otimes L^{-k}))_z$ is rigid, i.e.*

$$\chi(X, \mathcal{O}(\textstyle\bigwedge^p D^* \otimes L^{-k}))_z = \chi(X, \mathcal{O}(\textstyle\bigwedge^p D^* \otimes L^{-k}))$$

for all $z \in S^1$, if

$$\begin{aligned} 0 \le k \le n+1-p, &\qquad \textit{for } 0 \le p \le n, \\ n-p \le k \le 1, &\qquad \textit{for } n+1 \le p \le 2n. \end{aligned}$$

Furthermore

$$\chi(X, \mathcal{O}(\textstyle\bigwedge^p D^* \otimes L^{-k})) = 0$$

if either side of the corresponding inequality is strict.

We postpone the proof of the theorem until Section 3 while we recall other preliminaries.

Remark 2.2 The bounded and dotted region in the (k, p)-plane in Figure 1, shows the pairs of powers that give rigidity and vanishing theorems for the holomorphic Euler characteristics $\chi(X, \mathcal{O}(\bigwedge^p D^* \otimes L^{-k}))$ when $n = 5$.

2.4 Exponents of the circle action

From now on, we shall assume that the complex contact manifold X admits a circle action by holomorphic automorphisms preserving the contact distribution.

Let N denote a connected component of the S^1-fixed point set M^{S^1}, which is a submanifold and has even real codimension. Let $x \in N$. Since the S^1-action preserves the contact structure, we have S^1-representations on the complex vector spaces T_xX,

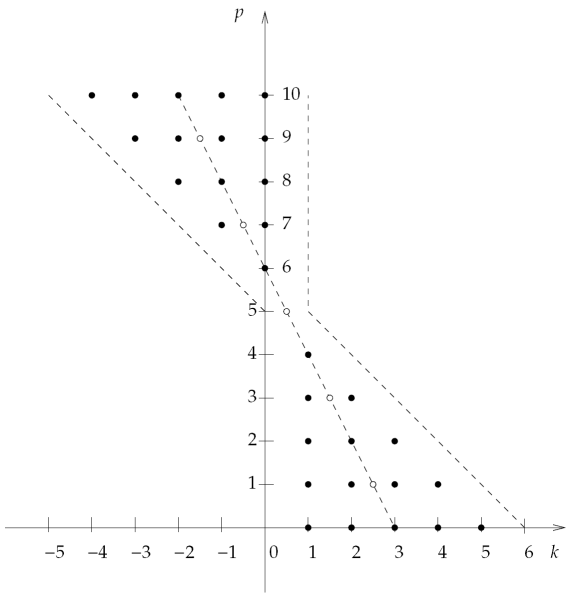

Figure 1: Rigidity and vanishing region for $n = 5$.

D_x and L_x given by the fibers of the bundles TX, D and L at x, as well as an S^1-equivariant exact sequence

$$0 \longrightarrow D_x \longrightarrow T_xX \longrightarrow L_x \longrightarrow 0.$$

First, let us consider T_xX. It decomposes as a finite direct sum of S^1-representations

$$T_xX = \bigoplus_{m\in\mathbb{Z}} V(m). \tag{3}$$

where for each $m \in \mathbb{Z}$, $v \in V(m)$ and $z \in S^1$, z acts on v by multiplication with z^m. Similarly, $D_x \subset T_xX$ will consist of some of these summands. Finally, since $\bigwedge^{2n+1} TX \cong L^{n+1}$,

$$L_x = \bigotimes_m \bigwedge^{\dim V(m)} V(m).$$

The exponents m depend on the connected component N. In order to carry out our computations, we will consider each $V(m)$ to be the sum of appropriately chosen one dimensional representations of S^1 with the same exponent m, and will make no reference to $V(m)$ anymore.

Thus, the holomorphic tangent bundle of X restricted to N splits as a sum of S^1-equivariant line bundles. We can think of such a splitting as follows:

$$TX|_N = \mathcal{L}^{m_1} \oplus \ldots \oplus \mathcal{L}^{m_{2n+1}},$$

where $m_i \in \mathbb{Z}$, $\mathcal{L}^{m_i}$ denotes the line bundle whose fiber is acted on by $z \in S^1$ by multiplication with z^{m_i}. Furthermore, the lines with exponent equal to 0 add up to the tangent bundle of N.

Let $x \in N$. Since $\bigwedge^{2n+1} TX = L^{n+1}$, L_x has exponent

$$h = \frac{1}{n+1} \left(\sum_{i=1}^{2n+1} m_i \right). \tag{4}$$

Since the exponents of D_x are $m_1, \ldots, m_{2n}$, the exponents of D_x^* must be $-m_1, \ldots, -m_{2n}$. By (2), the exponents of $D_x^* = D_x \otimes L_x^{-1}$ are

$$\begin{aligned} m_1 - h &= -m_{\sigma(1)} \\ &\vdots \\ m_{2n} - h &= -m_{\sigma(2n)}, \end{aligned}$$

where σ is a permutation of $\{1, 2, \ldots, 2n\}$, depending on the connected component N. The relevant fact here is

$$m_i + m_{\sigma(i)} = h. \tag{5}$$

Furthermore,

$$\bigwedge^{2n} D^* = \bigwedge^{2n} D \otimes L^{2n}$$

which implies

$$2nh = 2(m_1 + m_2 + \ldots + m_{2n}),$$

i.e.

$$nh = m_1 + m_2 + \ldots + m_{2n}. \tag{6}$$

Combining (4) with (6)

$$h = m_{2n+1}. \tag{7}$$

3 Proof of Theorem 2.1

We will use the notation set up in the previous section. Let us now consider the Hilbert polynomials in two variables given by the following holomorphic Euler characteristics

$$\chi(X, \mathcal{O}(\bigwedge^p D^* \otimes L^{-k})),$$

where $p, k \in \mathbb{Z}$, and $0 \le p \le 2n$. By the Atiyah-Singer index theorem, they can be computed by the following

$$\begin{aligned} &\langle \mathrm{ch}(\bigwedge^p D^* \otimes L^{-k}) \, \mathrm{Td}(X), [X] \rangle = \\ &= \langle \mathrm{ch}(\bigwedge^p D^*) \, e^{-ky_{2n+1}} \, e^{(n+1)y_{2n+1}/2} \, \widehat{A}(X), [X] \rangle \\ &= \langle \mathrm{ch}(\bigwedge^p D^*) \, e^{(-2k+n+1)y_{2n+1}/2} \prod_{i=1}^{2n+1} \frac{y_i/2}{\sinh(y_i/2)}, [X] \rangle \\ &= \left\langle \left(\sum_{1 \le i_1 < \ldots < i_p \le 2n} e^{-y_{i_1} - \ldots - y_{i_p}} \right) e^{(-2k+n+1)y_{2n+1}/2} \prod_{i=1}^{2n+1} \frac{y_i/2}{\sinh(y_i/2)}, [X] \right\rangle, \end{aligned} \tag{8}$$

where ch, Td and $\widehat{A}$ denote the Chern character, the Todd genus and the $\widehat{A}$-genus respectively.

Since the manifold X admits a holomorphic S^1 action preserving the contact structure, we can apply the Atiyah-Singer fixed point theorem AŞ3 to obtain the equivariant version of the index for $z \in S^1$ (cf. [5, p. 67])

$$
\begin{aligned}
&\chi(X, \mathcal{O}(\textstyle\bigwedge^p D^* \otimes L^{-k}))_z = \\
&= \sum_{N \in X^{S^1}} \left\langle \left(\sum_{1 \le i_1 < \ldots < i_p \le 2n} z^{m_{i_1}+\ldots+m_{i_p}} e^{-y_{i_1}-\ldots-y_{i_p}} \right) \times \right. \\
&\quad \left. \times\, z^{(2k-n-1)h/2} e^{(-2k+n+1)y_{2n+1}/2} \left(\prod_{m_i=0} \frac{y_i}{2} \right) \left(\prod_{i=1}^{2n+1} \frac{1}{z^{-m_i/2} e^{y_i/2} - z^{m_i/2} e^{-y_i/2}} \right), [N] \right\rangle \\
&= \sum_{N \in X^{S^1}} \left\langle \left(\sum_{1 \le i_1 < \ldots < i_p \le 2n} e^{-y_{i_1}-\ldots-y_{i_p}} \cdot e^{(-2k+n+1)y_{2n+1}/2} \left(\prod_{m_i=0} \frac{y_i}{2} \right) \times \right. \right. \\
&\quad \left. \left. \times\; z^{m_{i_1}+\ldots+m_{i_p}+kh} \left(\prod_{i=1}^{2n+1} \frac{1}{e^{y_i/2} - z^{m_i} e^{-y_i/2}} \right) \right), [N] \right\rangle, \qquad (9)
\end{aligned}
$$

where we have substituted

$$
\begin{array}{ccc}
M & \text{by} & N, \\
e^{\pm y_i} & \text{by} & z^{\mp m_i} e^{\pm y_i}.
\end{array}
$$

in the formula (8) for $\chi(X, \mathcal{O}(\bigwedge^p D^* \otimes L^{-k}))$ in order to obtain the formula (9).

Remark 3.1 We wish to control the behaviour of (9) at 0 and ∞ thought of as a rational function in z. Let us consider the function of $z \in \mathbb{C}$

$$F(z) = \frac{z^l}{e^x - z^m e^{-x}}$$

where x is an unknown and $l, m \in \mathbb{Z}$. First, let us assume $m > 0$. Thus, if $l > 0$ then

$$\lim_{z \to 0} F(z) = \lim_{z \to 0} \frac{z^l}{e^x - z^m e^{-x}} = 0, \qquad (10)$$

and if $l - m < 0$ then

$$\lim_{z \to \infty} F(z) = \lim_{z \to \infty} \frac{z^{l-m}}{z^{-m} e^x - e^{-x}} = 0. \qquad (11)$$

This means that for $0 < l < m$, $F(z)$ has zeroes at 0 and at ∞. If the inequalities are non-strict $0 \le l \le m$, then the limits are bounded. Similarly for $m \le 0$, we get that for $0 \ge l \ge m$, $F(z)$ has zeroes at 0 and at ∞, and if the inequalities are not strict then the limits of $F(z)$ are bounded.

By Remark 3.1, we will have control over the behaviour of each factor of each summand in $\chi(X, \mathcal{O}(\bigwedge^p D^* \otimes L^k))_z$ at 0 and ∞ if

$$\left|m_{i_1} + \ldots + m_{i_p} + kh\right| \leq \sum_{i=1}^{2n+1} |m_i|$$

for every p-tuple $1 \leq i_1 < \ldots < i_p \leq 2n$.

We will consider three cases which show the general pattern, where n will be as large as needed to illustrate the procedure.

- Case $p = 0$. In this case, we need to determine bounds for k such that

$$|kh| \leq \sum_{i=1}^{2n+1} |m_i|.$$

  - When $k = 0$ there is nothing to do.
  - When $k = 1$ and there exists i such that $\sigma(i) \neq i$, by (5)

$$|h| \leq |m_i + m_{\sigma(i)}| \leq |m_i| + |m_{\sigma(i)}| \leq \sum_{i=1}^{2n+1} |m_i|.$$

    If there is no such i, $m_1 = m_2 = h/2$ so that

$$|h| = |m_1 + m_2| \leq |m_1| + |m_2| \leq \sum_{i=1}^{2n+1} |m_i|.$$

  - When $k = 2$ and there exist $i \neq j$ such that $\sigma(i) \neq i$ and $\sigma(j) \neq j$, by (5)

$$\begin{aligned} |2h| &\leq |m_i + m_{\sigma(i)} + m_j + m_{\sigma(j)}| \\ &\leq |m_i| + |m_{\sigma(i)}| + |m_j| + |m_{\sigma(j)}| \\ &\leq \sum_{i=1}^{2n+1} |m_i|. \end{aligned}$$

    If there is only one i such that $\sigma(i) \neq i$, there must be $j, k \neq i$ with $m_j = m_k = h/2$ so that

$$|2h| \leq |m_i + m_{\sigma(i)} + m_j + m_k| \leq |m_i| + |m_{\sigma(i)}| + |m_j| + |m_k| \leq \sum_{i=1}^{2n+1} |m_i|.$$

    If there is no such i, $m_1 = m_2 = m_3 = m_4 = h/2$ so that

$$|2h| \leq |m_1 + m_2 + m_3 + m_4| \leq |m_1| + |m_2| + |m_3| + |m_4| \leq \sum_{i=1}^{2n+1} |m_i|.$$

We continue like this until $k = n+1$, where the last h is replaced with $h = m_{2n+1}$ so that

$$0 \leq k \leq n+1.$$

- Case $p=1$. For the sake of simplicity, let us determine bounds for k such that

$$|m_1 + kh| \leq \sum_{i=1}^{2n+1} |m_i|.$$

The argument will be analogous for all other m_i.

  – When $k=0$ there is nothing to do.

  – When $k=1$ and there exists $i \neq 1$ such that $\sigma(i) \neq i$, by (5)

$$|m_1 + h| \leq |m_1 + m_i + m_{\sigma(i)}| \leq |m_1| + |m_i| + |m_{\sigma(i)}| \leq \sum_{i=1}^{2n+1} |m_i|.$$

  If there is no such i, $m_2 = m_3 = h/2$ so that

$$|m_1 + h| \leq |m_1 + m_2 + m_3| \leq |m_1| + |m_2| + |m_3| \leq \sum_{i=1}^{2n+1} |m_i|.$$

  – When $k=2$ and there exist $i \neq j$ different from 1 and such that $\sigma(i) \neq i$ and $\sigma(j) \neq j$, by (5)

$$\begin{aligned} |m_1 + 2h| &\leq |m_1 + m_i + m_{\sigma(i)} + m_j + m_{\sigma(j)}| \\ &\leq |m_1| + |m_i| + |m_{\sigma(i)}| + |m_j| + |m_{\sigma(j)}| \\ &\leq \sum_{i=1}^{2n+1} |m_i|. \end{aligned}$$

  If there is only one i such that $\sigma(i) \neq i$, there must be $j, k \neq i, 1$ with $m_j = m_k = h/2$ so that

$$\begin{aligned} |m_1 + 2h| &\leq |m_1 + m_i + m_{\sigma(i)} + m_j + m_k| \\ &\leq |m_1| + |m_i| + |m_{\sigma(i)}| + |m_j| + |m_k| \\ &\leq \sum_{i=1}^{2n+1} |m_i|. \end{aligned}$$

  If there is no such $i \neq 1$, $m_2 = m_3 = m_4 = m_5 = h/2$ so that

$$\begin{aligned} |m_1 + 2h| &\leq |m_1 + m_2 + m_3 + m_4 + m_5| \\ &\leq |m_1| + |m_2| + |m_3| + |m_4| + |m_5| \\ &\leq \sum_{i=1}^{2n+1} |m_i|. \end{aligned}$$

We continue like this until $k = n$, where the last h is replaced with $h = m_{2n+1}$ so that

$$0 \leq k \leq n.$$

- Case $p = n + 1$. Just as before, let us determine bounds for k such that

$$|m_1 + \ldots + m_n + m_{n+1} + kh| \leq \sum_{i=1}^{2n+1} |m_i|.$$

The argument will be analogous for all other $(n+1)$-tuples.

  - When $k = 0$ there is nothing to do.
  - When $k = 1$, in the worst case scenario $\sigma(i) \neq 1, 2, \ldots, n$ for $i = 1, 2, \ldots, n$, so that by (7) we can only substitute one $h = m_{2n+1}$.

$$\begin{aligned} |m_1 + \ldots + m_n + m_{n+1} + h| &\leq |m_1 + \ldots + m_n + m_{n+1} + m_{2n+1}| \\ &\leq |m_1| + \ldots + |m_n| + |m_{n+1}| + |m_{2n+1}| \\ &\leq \sum_{i=1}^{2n+1} |m_i|. \end{aligned}$$

  - This time it could also happen that, for instance, $\sigma(1) = n+1$, and one could subtract one $h = m_1 + m_{\sigma(1)}$.

$$\begin{aligned} |m_1 + \ldots + m_n + m_{n+1} - h| &\leq |m_2 + \ldots + m_n| \\ &\leq |m_2| + \ldots + |m_n| \\ &\leq \sum_{i=1}^{2n+1} |m_i|. \end{aligned}$$

Thus

$$-1 \leq k \leq 1.$$

In this fashion, we can obtain all the inequalities stated in the theorem.

On the one hand, the right hand side in (9) can be considered as a meromorphic function with possibly a finite number of poles on the unit circle and at the origin. On the other hand, since $\chi(X, \mathcal{O}(\bigwedge^p D^* \otimes L^{-k}))_z$ is an index, it is also a finite Laurent polynomial in z and can be regarded as a meromorphic funtion of the form $\sum_j a_j z^j$, $a_j \in \mathbb{Z}$, for finitely many $j \in \mathbb{Z}$. Taking the limits at 0 and ∞ of both sides tells us that $a_j = 0$ for $j \neq 0$ if the inequalities of Theorem 2.1 are fulfilled. Furthermore, if one side of the corresponding inequality is strict, $a_0 = 0$ as well. $\square$

4 Special circle actions

If the complex contact manifold admits a circle action whose exponents $\{m_i\}$ are all non-negative, then one can prove further rigidity and vanishing results, such as the following.

Proposition 4.1 *Let X be a complex contact manifold, D the contact distribution and $L = TX/D$. Assume X admits a circle action by holomorphic automorphisms preserving the contact structure, whose exponents $\{m_i\}$ are all non-negative at any S^1-fixed point component. Then, the equivariat holomorphic Euler characteristic $\chi(X, \mathcal{O}(S^pD^* \otimes L^{-k}))_z$ is rigid, i.e.*

$$\chi(X, \mathcal{O}(S^pD^* \otimes L^{-k}))_z = \chi(X, \mathcal{O}(S^pD^* \otimes L^{-k})),$$

$z \in S^1$, if $0 \leq k \leq n+1-p$, for $0 \leq p \leq n$, where S^pD^* denotes the p-th symmetric tensor power of the bundle D^*. Furthermore

$$\chi(X, \mathcal{O}(S^pD^* \otimes L^{-k})) = 0$$

if one side of the corresponding inequality is strict.

Proof. For the sake of simplicity we will consider the case $p = 2$.
Recall that

$$m_{2n+1} = \frac{1}{n+1}\left(\sum_{i=1}^{2n+1} m_i\right) = h.$$

Since all the exponents are non-negative, the relevant inequalities now take the form

$$0 \leq m_i + m_j + km_{2n+1} \leq (n+1)m_{2n+1},$$

for $1 \leq i \leq j \leq 2n$.

- If $\sigma(i) = i$ and $\sigma(j) = j$, the inequality becomes

  $$0 \leq (k+1)m_{2n+1} \leq (n+1)m_{2n+1}.$$

  so that $-1 \leq k \leq n$.
- If $\sigma(i) = i$ and $\sigma(j) \neq j$, by (5) $m_i = m_{2n+1}/2$, $m_j < m_{2n+1}$ and

  $$0 \leq m_i + m_j + km_{2n+1} < m_{2n+1} + (k+1/2)m_{2n+1} \leq (n+1)m_{2n+1}.$$

  which requires $0 \leq k < n - 1/2$.
- If $\sigma(i) \neq i$ and $\sigma(j) \neq j$, by (5) $m_i < m_{2n+1}$, $m_j < m_{2n+1}$ and

  $$0 \leq m_i + m_j + km_{2n+1} < 2m_{2n+1} + km_{2n+1} \leq (n+1)m_{2n+1},$$

  which requires $0 \leq k \leq n-1$.

Therefore k must satisfy

$$0 \leq k \leq n-1.$$

Similarly for other values of p. □

Remark 4.1 The homogeneous complex contact manifold

$$Z = \frac{SO(8)}{SO(4) \times U(2)}$$

does not admit a holomorphic action with non-negative exponents such as the one in Proposition 4.1, since by he Bott-Borel-Weil theorem

$$\chi(Z, \mathcal{O}(S^2 D^* \otimes L^{-1})) = 1.$$

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