COMMUTATIVE QUOTIENTS OF FINITE W-ALGEBRAS

ALEXANDER PREMET

Abstract. Let e be a nilpotent element in a Chevalley form $\mathfrak{g}_{\mathbb{Z}}$ of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and let $\bar{e} = e \otimes 1$ be the corresponding nilpotent element in the restricted Lie algebra $\mathfrak{g}_{\Bbbk} = \mathfrak{g} \otimes_{\mathbb{Z}} \Bbbk$, where $\Bbbk$ is the algebraic closure of $\mathbb{F}_p$. Assume that $p \gg 0$ and set $\chi := \kappa(\bar{e}, \cdot\,)$, where κ is the Killing form of $\mathfrak{g}_{\Bbbk}$. Let $G_{\Bbbk}$ be the simple, simply connected algebraic $\Bbbk$-group with $\mathfrak{g}_{\Bbbk} = \mathrm{Lie}(G_{\Bbbk})$, write $\mathcal{O}_{\Bbbk}$ for the adjoint $G_{\Bbbk}$-orbit of $\bar{e}$, and denote by $U(\mathfrak{g}, e)$ the finite W-algebra associated to e. In this paper we prove that if $U(\mathfrak{g}, e)$ has a 1-dimensional representation, then the reduced enveloping algebra $U_{\chi}(\mathfrak{g}_{\Bbbk})$ possesses a simple module of dimension $p^{d(\bar{e})}$, where $d(\bar{e})$ is the half-dimension of $\mathcal{O}_{\Bbbk}$. We also show that if e is induced from a nilpotent element e_0 in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ admits 1-dimensional representations, then so does $U(\mathfrak{g}, e)$. This reduces the problem of 1-dimensional representations for finite W-algebras to the case where e is a rigid nilpotent element in a Lie algebra of type F_4, E_6, E_7, E_8. We use Katsylo's results on sections of sheets to determine, in many cases, the Krull dimension of the largest commutative quotient of the algebra $U(\mathfrak{g}, e)$

1. Introduction

1.1. This paper is a continuation of [34]. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Roughly speaking, the main result of [34] states that the primitive ideals of $U(\mathfrak{g})$ having rational infinitesimal characters admit *finite* generalised Gelfand–Graev models. One of the goals of this paper is to remove the unnecessary rationality assumption from the statement of [34, Thm. 1.1] and thus confirm [33, Conjecture 3.2] in full generality; see Theorem 4.2. This was announced in [34, p. 745], and very few changes to the original proof in [34] are actually required.

In the meantime two different proofs of [33, Conjecture 3.2] have appeared in the literature; the first one was found by Losev in [24] and the second one by Ginzburg in [16]. Our proof relies on the method developed in [34], the only difference being that in the present case our base ring is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$ rather than $\mathbb{Q}$. In this setting, we have to produce sufficiently many primes p for which the reduction procedure described in [34] leads to irreducible representations of the p-Lie algebra $\mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$ with p-characters belonging to the modular counterpart of our initial nilpotent orbit; see Section 4.

1.2. Denote by G a simple, simply connected algebraic group over $\mathbb{C}$, let (e, h, f) be an $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and denote by $(\,\cdot\,,\,\cdot\,)$ the G-invariant bilinear form on $\mathfrak{g}$ for which $(e, f) = 1$. Let $\chi \in \mathfrak{g}^*$ by such that $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$ and write $U(\mathfrak{g}, e)$ for the quantisation of the Slodowy slice $e + \mathrm{Ker}\,\mathrm{ad}\, f$ to the adjoint orbit $\mathcal{O} := (\mathrm{Ad}\, G)e$. Recall that $U(\mathfrak{g}, e) = \mathrm{End}_{\mathfrak{g}}\,(Q_{\chi})^{\mathrm{op}}$, where Q_{χ}

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is the generalised Gelfand–Graev $\mathfrak{g}$-module associated with the triple (e, h, f). The module Q_χ is induced from a 1-dimensional module $\mathbb{C}_\chi$ over of a nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ whose dimension equals $\frac{1}{2}\dim \mathcal{O}$. The Lie subalgebra $\mathfrak{m}$ is $(\operatorname{ad} h)$-stable, all eigenvalues of $\operatorname{ad} h$ on $\mathfrak{m}$ are negative, and χ vanishes on $[\mathfrak{m}, \mathfrak{m}]$. The action of $\mathfrak{m}$ on $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is given by $x(1_\chi) = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$; see [31, 13] for more detail. The algebra $U(\mathfrak{g}, e)$ shares many remarkable features with the universal enveloping algebra $U(\mathfrak{g})$ and is often referred to as the enveloping algebra of the Slodowy slice to $\mathcal{O}$. As an example, $U(\mathfrak{g}, e)$ is a deformation of the universal enveloping algebra $U(\mathfrak{z}_\chi)$, where $\mathfrak{z}_\chi$ is the stabiliser of χ in $\mathfrak{g}$; see [33]. It is also known that $U(\mathfrak{g}, e)$ is isomorphic to the Zhu algebra of the vertex W-algebra $W^{\mathrm{aff}}(\mathfrak{g}, e)$. The Zhu algebra of $W^{\mathrm{aff}}(\mathfrak{g}, e)$ is, in turn, isomorphic to the finite W-algebra $W^{\mathrm{fin}}(\mathfrak{g}, e)$ associated with $\mathfrak{g}$ and e; see [9] and [10].

1.3. In [33], the author conjectured that every algebra $U(\mathfrak{g}, e)$ admits a 1-dimensional representation; see [33, Conjecture 3.1(1)]. In [24], Losev proved this conjecture for $\mathfrak{g}$ classical. In this paper, we take another step towards proving [33, Conjecture 3.1(1)]. Recall that $\mathcal{O}$ is said to be *induced* from a nilpotent orbit $\mathcal{O}_0$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, if $\mathcal{O}$ intersects densely with the Zariski closed set $\overline{\mathcal{O}_0} + \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$ with Levi component $\mathfrak{l}$. If $\mathcal{O}$ is not induced, then one says that $\mathcal{O}$ is a *rigid* orbit.

Theorem 1.1. *Suppose the orbit $\mathcal{O}$ is induced from a nilpotent orbit $\mathcal{O}_0$ in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, and let $e_0 \in \mathcal{O}_0$. If the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ admits a 1-dimensional representation, then so does $U(\mathfrak{g}, e)$.*

Theorem 1.1 is proved in Section 3. Combined with [24, Thm. 1.2.3(1)] it reduces proving [33, Conjecture 3.1(1)] to the case of rigid nilpotent orbits in exceptional Lie algebras. We say that $\mathfrak{g}$ is *well-behaved* if for any proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and any nilpotent element $e_0 \in \mathfrak{l}$ the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ admits a 1-dimensional representation. In view of [24, Thm. 1.2.3(1)] the Lie algebras of types A_ℓ, B_ℓ, C_ℓ, D_ℓ, G_2, F_4, E_6 are well-behaved.

Given an associative algebra Λ we denote by Λ^{ab} the factor-algebra $\Lambda/\Lambda \cdot [\Lambda, \Lambda]$, where $\Lambda \cdot [\Lambda, \Lambda]$ is the ideal of Λ generated by all commutators $[a, b]$ with $a, b \in \Lambda$. Clearly, Λ^{ab} is the largest commutative quotient of Λ. Since $U(\mathfrak{g}, e)$ is Noetherian, by [31, 4.6], so is the commutative $\mathbb{C}$-algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$. By Hilbert's Nullstellensatz, the maximal spectrum $\mathcal{E} := \operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$ parametrises the 1-dimensional representations of $U(\mathfrak{g}, e)$. Our main goal in Section 3 is to determine the Krull dimension of the algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ under the assumption that $\mathfrak{g}$ is well-behaved. In proving the main results of Section 3 we shall rely on Borho's classification of sheets in semisimple Lie algebras and Katsylo's results on sections of sheets.

Given $x \in \mathfrak{g}$ we denote by G_x the centraliser of x in G. For $d \in \mathbb{N}$, set $\mathfrak{g}^{(d)} := \{x \in \mathfrak{g} \mid \dim G_x = d\}$. The irreducible components of the quasi-affine variety $\mathfrak{g}^{(d)}$ are called *sheets* of $\mathfrak{g}$. The sheets are $(\operatorname{Ad} G)$-stable, locally closed subsets of $\mathfrak{g}$. It is well-known that every sheet contains a unique nilpotent orbit and there is a bijection between the sheets of $\mathfrak{g}$ and the G-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O}_0)$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}_0$ is a rigid nilpotent orbit in $[\mathfrak{l}, \mathfrak{l}]$.

If $\mathfrak{l}$ if a Levi subalgebra of $\mathfrak{g}$, then the centre $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is a toral subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{z}(\mathfrak{l})_{\mathrm{reg}}$ the set of all $z \in \mathfrak{z}(\mathfrak{l})$ for which $\operatorname{ad} z$ acts invertibly $\mathfrak{g}/\mathfrak{l}$. Given a

nilpotent element $e_0 \in [\mathfrak{l},\mathfrak{l}]$ define $\mathcal{D}(\mathfrak{l},e_0) := (\operatorname{Ad} G)\cdot(e_0+\mathfrak{z}(\mathfrak{l})_{\mathrm{reg}})$, a locally closed subset of $\mathfrak{g}$, and call $\mathcal{D}(\mathfrak{l},e_0)$ a *decomposition class* of $\mathfrak{g}$. By [1], every sheet $\mathcal{S}$ of $\mathfrak{g}$ contains a unique open decomposition class. Moreover, if $\mathcal{D}(\mathfrak{l},e_0)$ is such a class, then $\mathcal{O}_0 := (\operatorname{Ad} L)\cdot e_0$ is rigid in $[\mathfrak{l},\mathfrak{l}]$ and the $(\operatorname{Ad} G)$-orbit induced from $\mathcal{O}_0$ is contained in $\mathcal{S}$ (here L is the Levi subgroup of G with $\operatorname{Lie}(L)=\mathfrak{l}$).

The group $C(e) := G_e \cap G_f$ is reductive and its finite quotient $\Gamma(e) := C(e)/C(e)^\circ$ identifies with the component group of G_e. If $\mathcal{S}(e)$ is a sheet containing e, then the set $X := \mathcal{S}(e)\cap(e+\operatorname{Ker}\operatorname{ad} f)$ is $C(e)$-stable and Zariski closed in $\mathfrak{g}$. By [21], the identity component $C(e)^\circ$ acts trivially on X and the component group $\Gamma(e)$ permutes transitively the irreducible components of X. Furthermore, if $\mathcal{D}(\mathfrak{l},e_0)$ is the open decomposition class of $\mathcal{S}(e)$ and Y is any irreducible component of X, then $\dim Y = \dim \mathfrak{z}(\mathfrak{l})$.

For an algebraic variety Z, we denote by $\operatorname{Comp}(Z)$ the set of all irreducible components of Z. Our main result in Section 3 is the following:

Theorem 1.2. *Suppose $\mathfrak{g}$ is well-behaved and $\mathcal{O}$ is not rigid. Let $\mathcal{S}_1,\dots,\mathcal{S}_t$ be the pairwise distinct sheets of $\mathfrak{g}$ containing $e\in\mathcal{O}$. Let $\mathcal{D}(\mathfrak{l}_i,e_i)$ be the open decomposition class of $\mathcal{S}_i$ and $X_i=\mathcal{S}_i\cap(e+\operatorname{Ker}\operatorname{ad} f)$. Then there is a surjection*

$$\tau\colon \operatorname{Comp}(\mathcal{E}) \twoheadrightarrow \operatorname{Comp}(X_1)\sqcup\ldots\sqcup\operatorname{Comp}(X_t)$$

such that $\dim \mathcal{Y} = \dim \mathfrak{z}(\mathfrak{l}_i)$ for every $\mathcal{Y}\in\tau^{-1}(\operatorname{Comp}(X_i))$, where $1\le i\le t$.

It follows from Theorem 1.2 that if $\mathfrak{g}$ is well-behaved and $\mathcal{O}$ is not rigid, then

$$\dim U(\mathfrak{g},e)^{\mathrm{ab}} = \max_{1\le i\le t}\dim \mathfrak{z}(\mathfrak{l}_i).$$

We also show in Section 3 that if $\mathcal{O}$ is rigid and $e\in\mathcal{O}$, then $\mathcal{E}$ is a finite set (possibly empty). In this case we do not require $\mathfrak{g}$ to be well-behaved.

For $\mathfrak{g}=\mathfrak{gl}(N)$, we obtain a much stronger result. Recall that to any partition $\lambda=(p_n\ge p_{n-1}\ge\cdots\ge p_1)$ of N there corresponds a nilpotent element $e_\lambda\in\mathfrak{gl}(N)$ of Jordan type $(p_1,p_2,\dots,p_n)$, and any nilpotent element in $\mathfrak{gl}(N)$ is conjugate to one of the e_λ's. At the end of Section 3 we show that

$$U(\mathfrak{gl}(N),e_\lambda)^{\mathrm{ab}} \;\cong\; \mathbb{C}[X_1,\dots,X_l],\qquad l=p_n.$$

In proving this isomorphism we use Theorem 1.2 and the explicit presentation of finite W-algebras of type A found by Brundan–Kleshchev in [7].

1.4. Our proof of Theorems 1.1 and 1.2 relies on characteristic p methods developed in [34]. We have to generalise several technical results proved in [34]; see Section 2. The algebra $U(\mathfrak{g},e)$ is defined over a suitable localisation $A=\mathbb{Z}[d^{-1}]$ of $\mathbb{Z}$. More precisely, there exists an A-subalgebra $U(\mathfrak{g}_A,e)$ of $U(\mathfrak{g},e)$ free as an A-module and such that $U(\mathfrak{g},e)\cong U(\mathfrak{g}_A,e)\otimes_A\mathbb{C}$. We take a sufficiently large prime p invertible in A, denote by $\Bbbk$ the algebraic closure of $\mathbb{F}_p$, and set $U(\mathfrak{g}_\Bbbk,e)=U(\mathfrak{g}_A,e)\otimes_A\Bbbk$. Here $\mathfrak{g}_\Bbbk=\mathfrak{g}_\mathbb{Z}\otimes_\mathbb{Z}\Bbbk$, where $\mathfrak{g}_\mathbb{Z}$ is a Chevalley $\mathbb{Z}$-form of $\mathfrak{g}$ containing e. We identify e with its image in $\mathfrak{g}_\Bbbk$ and regard $\chi=(e,\cdot\,)$ as a linear function on $\mathfrak{g}_\Bbbk$ (this is possible because the bilinear form $(\cdot\,,\cdot\,)$ is A-valued).

The subalgebra $\mathfrak{m}$ from (1.2) is defined over A and we set $\mathfrak{m}_\Bbbk := \mathfrak{m}_A\otimes_A\Bbbk$, where $\mathfrak{m}_A=\mathfrak{m}\cap\mathfrak{g}_A$ (it can be assumed that $\mathfrak{m}_A$ is a free A-module). By construction, the Lie algebra $\mathfrak{m}_\Bbbk$ possesses a 1-dimensional module on which it acts via χ; we call it $\Bbbk_\chi$.

We then consider the induced $\mathfrak{g}_{\Bbbk}$-module $Q_{\chi,\Bbbk} := U(\mathfrak{g}_{\Bbbk}) \otimes_{U(\mathfrak{m}_{\Bbbk})} \Bbbk_{\chi}$, denote by $\rho_{\Bbbk}$ the corresponding representation of $U(\mathfrak{g}_{\Bbbk})$, and define

$$\widehat{U}(\mathfrak{g}_{\Bbbk}, e) \;:=\; (\mathrm{End}_{\mathfrak{g}_{\Bbbk}} Q_{\chi,\Bbbk})^{\mathrm{op}}.$$

It is easy to see that $U(\mathfrak{g}_{\Bbbk}, e)$ is a subalgebra of $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$. Let $Z_p = Z_p(\mathfrak{g}_{\Bbbk})$ denote the p-centre of $U(\mathfrak{g}_{\Bbbk})$ (it is generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}_{\Bbbk}$, where $x \mapsto x^{[p]}$ is the p-th power map of the restricted Lie algebra $\mathfrak{g}_{\Bbbk}$). Clearly, $\rho_{\Bbbk}(Z_p) \subseteq \widehat{U}(\mathfrak{g}_{\Bbbk}, e)$. Given a subspace V of $\mathfrak{g}_{\Bbbk}$ we write $Z_p(V)$ for the subalgebra of Z_p generated by all $v^p - v^{[p]}$ with $v \in V$. In Section 2 we prove:

Theorem 1.3. *The algebra $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ is generated by $U(\mathfrak{g}_{\Bbbk}, e)$ and $\rho_{\Bbbk}(Z_p)$; moreover, $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ is a free $\rho_{\Bbbk}(Z_p)$-module of rank p^r, where $r = \dim G_e$. There is a subspace $\mathfrak{a}_{\Bbbk}$ of $\mathfrak{g}_{\Bbbk}$ with $\dim\, \mathfrak{a}_{\Bbbk} = \frac{1}{2}\dim\, \mathcal{O}$ such that $\widehat{U}(\mathfrak{g}_{\Bbbk}, e) \cong U(\mathfrak{g}_{\Bbbk}, e) \otimes_{\Bbbk} Z_p(\mathfrak{a}_{\Bbbk})$ as $\Bbbk$-algebras.*

Let $G_{\Bbbk}$ be a simple, simply connected algebraic $\Bbbk$-group with $\mathrm{Lie}(G_{\Bbbk}) = \mathfrak{g}_{\Bbbk}$. Recall that for $\xi \in \mathfrak{g}_{\Bbbk}^*$ the *reduced enveloping algebra* $U_{\xi}(\mathfrak{g}_{\Bbbk})$ is defined as the quotient of $U(\mathfrak{g}_{\Bbbk})$ by its ideal generated by all $x^p - x^{[p]} - \xi(x)^p$ with $x \in \mathfrak{g}_{\Bbbk}$. One of the challenging open problems in the representation theory of $\mathfrak{g}_{\Bbbk}$ is to show that for every $\xi \in \mathfrak{g}_{\Bbbk}^*$ the reduced enveloping algebra $U_{\xi}(\mathfrak{g}_{\Bbbk})$ has a simple module of dimension $p^{(\dim\, \mathcal{O}(\xi))/2}$, where $\mathcal{O}(\xi) = (\mathrm{Ad}^*\, G_{\Bbbk})\xi$. As a consequence of Theorem 1.3 we obtain:

Theorem 1.4. *If the finite W-algebra $U(\mathfrak{g}, e)$ admits a 1-dimensional representation, then for $p \gg 0$ the reduced enveloping algebra $U_{\chi}(\mathfrak{g}_{\Bbbk})$ has a simple module of dimension $p^{(\dim\, \mathcal{O}(\chi))/2}$.*

Together with Theorem 1.1 and [24, Thm. 1.2.3(1)] this yields:

Corollary 1.1. *If $\mathfrak{g}$ is classical and $p \gg 0$, then for any $\xi \in \mathfrak{g}_{\Bbbk}^*$ the reduced enveloping algebra $U_{\xi}(\mathfrak{g}_{\Bbbk})$ has a simple module of dimension $p^{(\dim\, \mathcal{O}(\xi))/2}$.*

It also follows from Theorems 1.1 and 1.4 that if $\mathcal{O}$ is induced from $\mathcal{O}_0 \subset \mathfrak{l}$ and the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ with $e_0 \in \mathcal{O}_0$ has a 1-dimensional representation, then for $p \gg 0$ the reduced enveloping algebra $U_{\chi}(\mathfrak{g}_{\Bbbk})$ has a module of dimension $p^{(\dim\, \mathcal{O}(\chi))/2}$.

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2. Finite W-algebras and their modular analogues

2.1. Let G be a simple, simply connected algebraic group over $\mathbb{C}$, and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Φ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Choose a basis of simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ in Φ, let Φ^+ be the corresponding positive system in Φ, and put $\Phi^- := -\Phi^+$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the corresponding triangular decomposition of $\mathfrak{g}$ and choose a Chevalley basis $\mathcal{B} = \{e_\gamma \mid \gamma \in \Phi\} \cup \{h_\alpha \mid \alpha \in \Pi\}$ in $\mathfrak{g}$. Set $\mathcal{B}^{\pm} := \{e_\alpha \mid \alpha \in \pm\Phi^+\}$. Let $\mathfrak{g}_{\mathbb{Z}}$ and $U_{\mathbb{Z}}$ denote the Chevalley $\mathbb{Z}$-form of $\mathfrak{g}$ and the Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ associated with $\mathcal{B}$. Given a $\mathbb{Z}$-module V and a $\mathbb{Z}$-algebra A, we write $V_A := V \otimes_{\mathbb{Z}} A$.

Take a nonzero nilpotent element $e \in \mathfrak{g}_{\mathbb{Z}}$ and choose $f, h \in \mathfrak{g}_{\mathbb{Q}}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{Q}}$. Denote by $(\,\cdot\,,\,\cdot\,)$ a scalar multiple of the Killing form κ of $\mathfrak{g}$ for which $(e, f) = 1$ and define $\chi \in \mathfrak{g}^*$ by setting $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$ (it follows from the $\mathfrak{sl}_2$-theory that $\kappa(e, f)$ is a positive integer). Given $x \in \mathfrak{g}$ we set $\mathcal{O}(x) := (\operatorname{Ad} G) \cdot x$ and $d(x) := \frac{1}{2} \dim \mathcal{O}(x)$.

Definition 2.1. We call a commutative ring A *admissible* if A is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, $\kappa(e, f) \in A^\times$, and all bad primes of the root system of G and the determinant of the Gram matrix of $(\,\cdot\,,\,\cdot\,)$ relative to a Chevalley basis of $\mathfrak{g}$ are invertible in A.

It is clear from the definition that every admissible ring is a Noetherian domain. Given a finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}$ we denote by $\pi(A)$ the set of all primes $p \in \mathbb{N}$ such that $A/\mathfrak{P} \cong \mathbb{F}_p$ for some maximal ideal $\mathfrak{P}$ of A.

Let $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. Then $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, by the $\mathfrak{sl}_2$-theory, and all subspaces $\mathfrak{g}(i)$ are defined over $\mathbb{Q}$. Also, $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$. We define a skew-symmetric bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{g}(-1)$ by setting $\langle x, y\rangle := (e, [x, y])$ for all $x, y \in \mathfrak{g}(-2)$. This skew-symmetric bilinear form is nondegenerate, hence there exists a basis $B = \{z'_1, \dots, z'_s, z_1, \dots, z_s\}$ of $\mathfrak{g}(-1)$ contained in $\mathfrak{g}_{\mathbb{Q}}$ and such that

$$\langle z'_i, z_j\rangle = \delta_{ij}, \qquad \langle z_i, z_j\rangle \,=\, \langle z'_i, z'_j\rangle = 0 \qquad\qquad (1 \le i, j \le s).$$

As explained in [34, 4.1], after enlarging A if need be, one can assume that $\mathfrak{g}_A = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_A(i)$, that each $\mathfrak{g}_A(i) := \mathfrak{g}_A \cap \mathfrak{g}(i)$ is a freely generated over A by a basis of the vector space $\mathfrak{g}(i)$, and that B is a free basis of the A-module $\mathfrak{g}_A(-1)$.

Put $\mathfrak{m} := \mathfrak{g}(-1)^0 \oplus \sum_{i \le -2} \mathfrak{g}(i)$ where $\mathfrak{g}(-1)^0$ denotes the $\mathbb{C}$-span of $z'_1, \dots, z'_s$. Then $\mathfrak{m}$ is a nilpotent Lie subalgebra of dimension $d(e)$ in $\mathfrak{g}$ and χ vanishes on the derived subalgebra of $\mathfrak{m}$; see [31] for more detail. It follows from our assumptions on A that $\mathfrak{m}_A = \mathfrak{g}_A \cap \mathfrak{m}$ is a free A-module and a direct summand of $\mathfrak{g}_A$. More precisely, $\mathfrak{m}_A = \mathfrak{g}_A(-1)^0 \oplus \sum_{i \le -2} \mathfrak{g}_A(i)$, where $\mathfrak{g}_A(-1)^0 = \mathfrak{g}_A \cap \mathfrak{g}(-1) = Az'_1 \oplus \cdots \oplus Az'_s$.

Enlarging A further we may assume that $e, f \in \mathfrak{g}_A$ and that $[e, \mathfrak{g}_A(i)]$ and $[f, \mathfrak{g}_A(i)]$ are direct summands of $\mathfrak{g}_A(i{+}2)$ and $\mathfrak{g}_A(i{-}2)$, respectively. Then $\mathfrak{g}_A(i{+}2) = [e, \mathfrak{g}_A(i)]$ for all $i \ge 0$; see [34, 4.1].

Write $\mathfrak{g}_e$ for the centraliser of e in $\mathfrak{g}$. Similar to [31, 4.2 and 4.3] we choose a basis $x_1, \dots, x_r, x_{r+1}, \dots, x_m$ of the free A-module $\mathfrak{p}_A := \bigoplus_{i \ge 0} \mathfrak{g}_A(i)$ such that

(a) $x_i \in \mathfrak{g}_A(n_i)$ for some $n_i \in \mathbb{Z}_+$;
(b) $x_1, \dots, x_r$ is a free basis of the A-module $\mathfrak{g}_A \cap \mathfrak{g}_e$;
(c) $x_{r+1}, \dots, x_m \in [f, \mathfrak{g}_A]$.

2.2. Let Q_χ be the generalised Gelfand-Graev $\mathfrak{g}$-module associated to e. Recall that $Q_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$, where $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is a 1-dimensional $\mathfrak{m}$-module such that $x \cdot 1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$. Given $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s$ we let $x^{\mathbf{a}} z^{\mathbf{b}}$ denote the monomial $x_1^{a_1} \cdots x_m^{a_m} z_1^{b_1} \cdots z_s^{b_s}$ in $U(\mathfrak{g})$. Set $Q_{\chi,A} := U(\mathfrak{g}_A) \otimes_{U(\mathfrak{m}_A)} A_\chi$, where $A_\chi = A1_\chi$. Note that $Q_{\chi,A}$ is a $\mathfrak{g}_A$-stable A-lattice in Q_χ with $\{x^{\mathbf{i}} z^{\mathbf{j}} \otimes 1_\chi, \mid (\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s\}$ as a free basis; see [34] for more detail. Given $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^m \times Z_+^s$ we set

$$|(\mathbf{a}, \mathbf{b})|_e \,:=\, \sum_{i=1}^{m} a_i(n_i + 2) + \sum_{i=1}^{s} b_i.$$

According to [31, Thm. 4.6], the algebra $U(\mathfrak{g},e) := (\mathrm{End}_{\mathfrak{g}}\, Q_\chi)^{\mathrm{op}}$ is generated over $\mathbb{C}$ by endomorphisms $\Theta_1,\dots,\Theta_r$ such that

$$\Theta_k(1_\chi) \;=\; \Big(x_k + \sum_{0<|(\mathbf{i},\mathbf{j})|_e\le n_k+2} \lambda^k_{\mathbf{i},\mathbf{j}}\, x^{\mathbf{i}} z^{\mathbf{j}}\Big)\otimes 1_\chi, \qquad 1\le k\le r, \tag{1}$$

where $\lambda^k_{\mathbf{i},\mathbf{j}}\in\mathbb{Q}$ and $\lambda^k_{\mathbf{i},\mathbf{j}}=0$ if either $|(\mathbf{i},\,\mathbf{j})|_e = n_k+2$ and $|\mathbf{i}|+|\mathbf{j}|=1$ or $\mathbf{i}\neq\mathbf{0}$, $\mathbf{j}=\mathbf{0}$, and $i_l=0$ for $l>r$. Moreover, the monomials $\Theta_1^{i_1}\cdots\Theta_r^{i_r}$ with $(i_1,\dots,i_r)\in\mathbb{Z}_+^r$ form a PBW basis of the vector space $U(\mathfrak{g},e)$.

The monomial $\Theta_1^{i_1}\cdots\Theta_r^{i_r}$ is said to have *Kazhdan degree* $\sum_{i=1}^r a_i(n_i+2)$. For $k\in\mathbb{Z}_+$ we let $U(\mathfrak{g},e)_k$ denote the $\mathbb{C}$-span of all monomials $\Theta_1^{i_1}\cdots\Theta_r^{i_r}$ of Kazhdan degree $\le k$. The union $\bigcup_{k\ge 0} U(\mathfrak{g},e)_k$ is an increasing algebra filtration of $U(\mathfrak{g},e)$, called the *Kazhdan filtration*; see [31]. The corresponding graded algebra $\mathrm{gr}\,U(\mathfrak{g},e)$ is a polynomial algebra in $\mathrm{gr}\,\Theta_1,\dots,\mathrm{gr}\,\Theta_r$. It is immediate from [31, Thm. 4.6] that there exist polynomials $F_{ij}\in\mathbb{Q}[X_1,\dots,X_r]$, where $1\le i<j\le r$, such that

$$[\Theta_i,\Theta_j] \;=\; F_{ij}(\Theta_1,\dots,\Theta_r) \qquad (1\le i<j\le r). \tag{2}$$

Moreover, if $[x_i,x_j] \;=\; \sum_{k=1}^r \alpha_{ij}^k\, x_k$ in $\mathfrak{g}_e$, then

$$F_{ij}(\Theta_1,\dots,\Theta_r) \;\equiv\; \sum_{k=1}^r \alpha_{ij}^k\Theta_k + q_{ij}(\Theta_1,\dots,\Theta_r) \quad \big(\mathrm{mod}\, U(\mathfrak{g},e)_{n_i+n_j}\big),$$

where the initial form of $q_{ij}\in\mathbb{Q}[X_1,\dots,X_r]$ has total degree ≥ 2 whenever $q_{ij}\neq 0$. By [34, Lemma 4.1], the algebra $U(\mathfrak{g},e)$ is generated by $\Theta_1,\dots,\Theta_r$ subject to the relations (2). As in [34], we assume that our admissible ring A contains all $\lambda^k_{\mathbf{i},\mathbf{j}}$ in (1) and all coefficients of the F_{ij}'s in (2).

2.3. Let N_χ denote the ideal of codimension one in $U(\mathfrak{m})$ generated by all $x-\chi(x)$ with $x\in\mathfrak{m}$. Then $Q_\chi\cong U(\mathfrak{g})N_\chi$ as $\mathfrak{g}$-modules. By construction, the left ideal $\mathcal{I}_\chi := U(\mathfrak{g})N_\chi$ of $U(\mathfrak{g})$ is a $\big(U(\mathfrak{g}),U(\mathfrak{m})\big)$-bimodule. The fixed point space $(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}$ carries a natural algebra structure given by $(x+\mathcal{I}_\chi)\cdot(y+\mathcal{I}_\chi)=xy+\mathcal{I}_\chi$ for all $x,y\in U(\mathfrak{g})$. Moreover, $U(\mathfrak{g})/\mathcal{I}_\chi\cong Q_\chi$ as $\mathfrak{g}$-modules via the $\mathfrak{g}$-module map sending $1+\mathcal{I}_\chi$ to 1_χ, and $(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}\cong U(\mathfrak{g},e)$ as algebras. Any element of $U(\mathfrak{g},e)$ is uniquely determined by its effect on the generator $1_\chi\in Q_\chi$ and the canonical isomorphism between $(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}$ and $U(\mathfrak{g},e)$ is given by $u\mapsto u(1_\chi)$ for all $u\in(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}$. It is clear that this isomorphism is defined over A. In what follows we shall identify Q_χ with $U(\mathfrak{g})/\mathcal{I}_\chi$ and $U(\mathfrak{g},e)$ with $(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}$.

Let $U(\mathfrak{g})=\bigcup_{j\in\mathbb{Z}}\mathsf{K}_jU(\mathfrak{g})$ be the Kazhdan filtration of $U(\mathfrak{g})$; see [13, 4.2]. Recall that $\mathsf{K}_jU(\mathfrak{g})$ is the $\mathbb{C}$-span of all products $x_1\cdots x_t$ with $x_i\in\mathfrak{g}(n_i)$ and $\sum_{i=1}^t(n_i+2)\le j$ (the identity element is in $\mathsf{K}_0U(\mathfrak{g})$ by convention). The Kazhdan filtration on Q_χ is defined by $\mathsf{K}_jQ_\chi := \pi(\mathsf{K}_jU(\mathfrak{g}))$, where $\pi\colon U(\mathfrak{g})\twoheadrightarrow U(\mathfrak{g})/\mathcal{I}_\chi$ is the canonical homomorphism; see [13, 4.3]. It turns Q_χ into a filtered $U(\mathfrak{g})$-module. As explained in [13] the Kazhdan grading of $\mathrm{gr}\,Q_\chi$ has no negative components. The Kazhdan filtration of $U(\mathfrak{g},e)$ defined in (2.2) is nothing but the filtration of $U(\mathfrak{g},e)=(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}$ induced from the Kazhdan filtration of Q_χ through the embedding $(U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathrm{ad}\,\mathfrak{m}}\hookrightarrow Q_\chi$; see [13] for more detail.

Let $U(\mathfrak{g}_A,e)$ denote the A-span of all monomials $\Theta_1^{i_1}\cdots\Theta_r^{i_r}$ with $(i_1,\dots,i_r)\in\mathbb{Z}_+^r$. Our assumptions on A guarantee that $U(\mathfrak{g}_A,e)$ is an A-subalgebra of $U(\mathfrak{g},e)$ contained

in $(\mathrm{End}_{\mathfrak{g}_A} Q_{\chi,A})^{\mathrm{op}}$. It is immediate from the above discussion that $Q_{\chi,A}$ identifies with the $\mathfrak{g}_A$-module $U(\mathfrak{g}_A)/U(\mathfrak{g}_A)N_{\chi,A}$, where $N_{\chi,A}$ stands for the A-subalgebra of $U(\mathfrak{m}_A)$ generated by all $x-\chi(x)$ with $x\in\mathfrak{m}_A$. Hence $U(\mathfrak{g}_A,e)$ embeds into the A-algebra $\big(U(\mathfrak{g}_A)/U(\mathfrak{g}_A)N_{\chi,A}\big)^{\mathrm{ad}\ \mathfrak{m}_A}$. Since $Q_{\chi,A}$ is a free A-module with basis $\{x^{\mathbf{i}}z^{\mathbf{j}}\otimes 1_\chi,\mid(\mathbf{i},\mathbf{j})\in\mathbb{Z}_+^m\times\mathbb{Z}_+^s\}$, easy induction on Kazhdan degree (based on [31, Lemma 4.5] and the formula displayed in [31, p. 27]) shows that

$$U(\mathfrak{g}_A,e)\ =\ (\mathrm{End}_{\mathfrak{g}_A} Q_{\chi,A})^{\mathrm{op}}\ \cong\ \big(U(\mathfrak{g}_A)/U(\mathfrak{g}_A)N_{\chi,A}\big)^{\mathrm{ad}\ \mathfrak{m}_A}. \tag{3}$$

Repeating verbatim Skryabin's argument in [31, p. 53] one also observes that $Q_{\chi,A}$ is free as a right $U(\mathfrak{g}_A,e)$-module.

2.4. We now pick $p\in\pi(A)$ and denote by $\Bbbk$ the algebraic closure of $\mathbb{F}_p$. Since the form $(\,\cdot\,,\,\cdot\,)$ is A-valued on $\mathfrak{g}_A$, it induces a symmetric bilinear form on the Lie algebra $\mathfrak{g}_\Bbbk\cong\mathfrak{g}_A\otimes_A\Bbbk$. We use the same symbol to denote this bilinear form on $\mathfrak{g}_\Bbbk$. Let $G_\Bbbk$ be the simple, simply connected algebraic $\Bbbk$-group with hyperalgebra $U_\Bbbk=U_\mathbb{Z}\otimes_Z\Bbbk$. Note that $\mathfrak{g}_\Bbbk=\mathrm{Lie}(G_\Bbbk)$ and the form $(\,\cdot\,,\,\cdot\,)$ is $(\mathrm{Ad}\ G_\Bbbk)$-invariant and nondegenerate. For $x\in\mathfrak{g}_A$ we set $\bar{x}:=x\otimes 1$, an element of $\mathfrak{g}_\Bbbk$. To ease notation we identify e,f with the nilpotent elements $\bar{e},\bar{f}\in\mathfrak{g}_\Bbbk$ and χ with the linear function $(e,\,\cdot\,)$ on $\mathfrak{g}_\Bbbk$ (this will cause no confusion).

The Lie algebra $\mathfrak{g}_\Bbbk=\mathrm{Lie}(G_\Bbbk)$ carries a natural $[p]$-mapping $x\mapsto x^{[p]}$ equivariant under the adjoint action of $G_\Bbbk$. For every $x\in\mathfrak{g}_\Bbbk$ the element $x^p-x^{[p]}$ of the universal enveloping algebra $U(\mathfrak{g}_\Bbbk)$. The subalgebra of $U(\mathfrak{g}_\Bbbk)$ generated by all $x^p-x^{[p]}\in U(\mathfrak{g}_\Bbbk)$ is called the *p-centre* of $U(\mathfrak{g}_\Bbbk)$ and denoted $Z_p(\mathfrak{g}_\Bbbk)$ or Z_p for short. It is immediate from the PBW theorem that Z_p is isomorphic to a polynomial algebra in $\dim\mathfrak{g}$ variables and $U(\mathfrak{g}_\Bbbk)$ is a free Z_p-module of rank $p^{\dim\mathfrak{g}}$. For every maximal ideal J of Z_p there is a unique linear function $\eta=\eta_J\in\mathfrak{g}_\Bbbk^*$ such that

$$J=\ \langle x^p-x^{[p]}-\eta(x)^p1\mid\ x\in\mathfrak{g}_\Bbbk\rangle.$$

Since the Frobenius map of $\Bbbk$ is bijective, this enables us to identify the maximal spectrum $\mathrm{Specm}\,Z_p$ with $\mathfrak{g}_\Bbbk^*$.

Given $\xi\in\mathfrak{g}_\Bbbk^*$ we denote by I_ξ the two-sided ideal of $U(\mathfrak{g}_\Bbbk)$ generated by all $x^p-x^{[p]}-\xi(x)^p1$ with $x\in\mathfrak{g}_\Bbbk$, and set $U_\xi(\mathfrak{g}_\Bbbk):=U(\mathfrak{g}_k)/I_\xi$. The algebra $U_\xi(\mathfrak{g}_\Bbbk)$ is called the *reduced enveloping algebra* of $\mathfrak{g}_\Bbbk$ associated to ξ. The preceding remarks imply that $\dim_\Bbbk U_\xi(\mathfrak{g}_\Bbbk)=p^{\dim\mathfrak{g}}$ and $I_\xi\cap Z_p=J_\xi$, the maximal ideal of Z_p associated with ξ. Every irreducible $\mathfrak{g}_\Bbbk$-module is a module over $U_\xi(\mathfrak{g}_\Bbbk)$ for a unique $\xi=\xi_V\in\mathfrak{g}_\Bbbk^*$. The linear function ξ_V is called the *p-character* of V; see [30] for more detail. By [30], any irreducible $U_\xi(\mathfrak{g}_\Bbbk)$-module has dimension divisible by $p^{(\dim\mathfrak{g}-\dim\mathfrak{z}_\xi)/2}$, where $\mathfrak{z}_\xi=\{x\in\mathfrak{g}_\Bbbk\mid\xi([x,\mathfrak{g}_\Bbbk])=0\}$ is the stabiliser of ξ in $\mathfrak{g}_\Bbbk$.

2.5. For $i\in\mathbb{Z}$, set $\mathfrak{g}_\Bbbk(i):=\mathfrak{g}_A(i)\otimes_A\Bbbk$ and put $\mathfrak{m}_\Bbbk:=\mathfrak{m}_A\otimes_A\Bbbk$. Due to our assumptions on A the elements $\bar{x}_1,\dots,\bar{x}_r$ form a basis of the centraliser $(\mathfrak{g}_\Bbbk)_e$ of e in $\mathfrak{g}_\Bbbk$ and that $\mathfrak{m}_\Bbbk$ is a nilpotent subalgebra of dimension $d(e)$ in $\mathfrak{g}_\Bbbk$. Set $Q_{\chi,\Bbbk}:=U(\mathfrak{g}_\Bbbk)\otimes_{U(\mathfrak{m}_\Bbbk)}\Bbbk_\chi$, where $\Bbbk_\chi=A_\chi\otimes_A\Bbbk=\Bbbk 1_\chi$. Clearly, $\Bbbk 1_\chi$ is a 1-dimensional $\mathfrak{m}_\Bbbk$-module with the property that $x(1_\chi)=\chi(x)1_\chi$ for all $x\in\mathfrak{m}_\Bbbk$. Define

$$\widehat{U}(\mathfrak{g}_\Bbbk,e)\ :=\ (\mathrm{End}_{\mathfrak{g}_\Bbbk} Q_{\chi,\Bbbk})^{\mathrm{op}}.$$

It follows from our discussion in (2.2) and (2.3) that $Q_{\chi,\Bbbk} \cong Q_{\chi,A} \otimes_A \Bbbk$ as modules over $\mathfrak{g}_\Bbbk$ and $Q_{\chi,\Bbbk}$ is a free right module over the k-algebra

$$U(\mathfrak{g}_\Bbbk, e) := U(\mathfrak{g}_A, e) \otimes_A \Bbbk.$$

Thus we may identify $U(\mathfrak{g}_\Bbbk, e)$ with a subalgebra of $\widehat{U}(\mathfrak{g}_\Bbbk, e)$. Note that the algebra $U(\mathfrak{g}_\Bbbk, e)$ has $\Bbbk$-basis consisting of all monomials $\bar{\Theta}_1^{i_1} \cdots \bar{\Theta}_r^{i_r}$ with $(i_1, \dots, i_r) \in \mathbb{Z}_+^r$, where $\bar{\Theta}_i := \Theta_i \otimes 1 \in U(\mathfrak{g}_A, e) \otimes_A \Bbbk$. Given a polynomial $g \in A[X_1, \dots, X_n]$ we let ${}^p g$ denote the image of g in the polynomial algebra $\Bbbk[X_1, \dots, X_n] = A[X_1, \dots, X_n] \otimes_A \Bbbk$. Since all polynomials F_{ij} are in $A[X_1, \dots, X_r]$, it follows from the relations (2) that

$$[\bar{\Theta}_i, \bar{\Theta}_j] = {}^p F_{ij}(\bar{\Theta}_1, \dots, \bar{\Theta}_r) \qquad (1 \le i < j \le r). \tag{4}$$

Lemma 2.1. *The algebra $U(\mathfrak{g}_\Bbbk, e)$ is generated by the elements $\bar{\Theta}_1, \dots, \bar{\Theta}_r$ subject to the relations (4).*

Proof. We argue as in the proof of [34, Lemma 4.1]. Let $\mathbf{I}$ be the two-sided ideal of the free associative algebra $\Bbbk\langle X_1, \dots, X_r\rangle$ generated by all $[X_i, X_j] - {}^p F_{ij}(X_1, \dots, X_r)$ with $1 \le i < j \le r$. Let $\bar{X}_i$ denote the image of X_i of in the factor-algebra $\mathcal{U} := \Bbbk\langle X_1, \dots, X_r\rangle/\mathbf{I}$. There is a natural algebra epimorphism $\psi\colon \mathcal{U} \twoheadrightarrow U(\mathfrak{g}_\Bbbk, e)$ sending $\bar{X}_i$ to $\bar{\Theta}_i$ for all i. For $k \in \mathbb{Z}_+$ let $\mathcal{U}_k$ denote the $\Bbbk$-span of all products $\bar{X}_{j_1} \cdots \bar{X}_{j_m}$ with $\sum_{t=1}^m (n_{j_t} + 2) \le k$ and let $\mathcal{U}'$ be the $\Bbbk$-span of all monomials $\bar{X}_1^{i_1} \cdots \bar{X}_r^{i_r}$ with $(i_1, \dots, i_r) \in \mathbb{Z}_+^r$. Double induction on k and m (upward on k and downward on m) based on the relations (4) shows that $\mathcal{U}' = \mathcal{U}$. Since the monomials $\bar{\Theta}_1^{i_1} \cdots \bar{\Theta}_r^{i_r}$ with $(i_1, \dots, i_r) \in \mathbb{Z}_+^r$ are linearly independent over $\Bbbk$, we obtain $\mathcal{U} \cong U(\mathfrak{g}_\Bbbk, e)$, as required. $\square$

Given an associative algebra Λ we set $\Lambda^{\mathrm{ab}} := \Lambda/\Lambda \cdot [\Lambda, \Lambda]$, where $\Lambda \cdot [\Lambda, \Lambda]$ is the (two-sided) ideal of Λ generated by all commutators $[a, b] = ab - ba$ with $a, b \in \Lambda$. It is immediate from [34, Lemma 4.1] that $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to the quotient of the polynomial algebra $\mathbb{C}[X_1, \dots, X_r]$ by its ideal generated by all polynomials F_{ij} with $1 \le i < j \le r$). Given a subfield K of $\mathbb{C}$ containing A we denote by $\mathcal{E}(K)$ the set of all common zeros of the polynomials F_{ij} in the affine space $\mathbb{A}_K^r$. Clearly, the A-defined Zariski closed set $\mathcal{E}(\mathbb{C})$ parametrises the 1-dimensional representations of the algebra $U(\mathfrak{g}, e)$. Let $\mathcal{E}(\Bbbk)$ denote the set of all common zeros of the polynomials ${}^p F_{ij}$ in $\mathbb{A}_\Bbbk^r$. By Lemma 2.1, the set $\mathcal{E}(\Bbbk)$ parametrises the 1-dimensional representations of the algebra $U(\mathfrak{g}_\Bbbk, e)$. This has the following consequence:

Corollary 2.1. *If the algebras $U(\mathfrak{g}_\Bbbk, e)$, where $\Bbbk = \overline{\mathbb{F}}_p$, afford 1-dimensional representations for infinitely many $p \in \pi(A)$, then the finite W-algebra $U(\mathfrak{g}, e)$ has a 1-dimensional representation.*

Proof. Suppose for a contradiction $U(\mathfrak{g}, e)$ has no 1-dimensional representations. Then $\mathcal{E}(\overline{\mathbb{Q}}) = \varnothing$, where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Since $\overline{\mathbb{Q}}$ is algebraically closed, there exists a finite Galois extension K of $\mathbb{Q}$ and polynomials $g_{ij} \in K[X_1, \dots, X_r]$ such that $\sum_{i,j} g_{ij} F_{ij} = 1$. Let $\mathcal{O}_K$ denote the ring of algebraic integers of K. Rescaling the coefficients of the g_{ij}'s if necessary, we can find $h_{ij} \in \mathcal{O}_K[X_1, \dots, X_r]$ such that $\sum_{i,j} h_{ij} F_{ij} = \tilde{n}$ for some positive integer $\tilde{n}$. For each $p \in \pi(A)$ choose $\mathfrak{P} \in \operatorname{Spec} \mathcal{O}_K$ with $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$. Since $\mathcal{O}_K$ is a Dedekind ring, $\mathcal{O}/\mathfrak{P} \cong \mathbb{F}_q$ for some p-power q. Let

$$\varphi\colon \mathcal{O}[X_1, \dots, X_r] \longrightarrow (\mathcal{O}_K/\mathfrak{P})[X_1, \dots, X_r] \hookrightarrow \Bbbk[X_1, \dots, X_r]$$

denote the homomorphism of polynomial algebras induced by inclusion $\mathbb{F}_q \hookrightarrow \Bbbk$. Note that $\varphi(F_{ij}) = {}^pF_{ij}$ and $\varphi(\widetilde{n})$ is just the residue of $\tilde{n}$ modulo p. As $\tilde{n}$ has finitely many prime divisors, we derive that the ideal of $\Bbbk[X_1, \dots, X_r]$ generated by the ${}^pF_{ij}$'s coincides with $\Bbbk[X_1, \dots, X_r]$ for almost all $p \in \pi(A)$. As $\mathcal{E}(\Bbbk) = \varnothing$ for all such p, this implies that the algebra $U(\mathfrak{g}_\Bbbk, e)$ has no 1-dimensional representations for almost all $p \in \pi(A)$. Since this contradicts our assumption, the corollary follows. $\square$

2.6. Let $\mathfrak{g}_A^*$ be the A-module dual to $\mathfrak{g}_A$, so that $\mathfrak{g}^* = \mathfrak{g}_A^* \otimes_A \mathbb{C}$ and $\mathfrak{g}_\Bbbk^* = \mathfrak{g}_A^* \otimes_A \Bbbk$. Let $\mathfrak{m}_A^\perp$ denote the set of all linear functions on $\mathfrak{g}_A$ vanishing on $\mathfrak{m}_A$, a free A-submodule and a direct summand of $\mathfrak{g}_A^*$ (by our assumptions on A). Note that $\mathfrak{m}_A^\perp \otimes_A \mathbb{C}$ and $\mathfrak{m}_A \otimes_A \Bbbk$ identify naturally with with the annihilators $\mathfrak{m}^\perp := \{f \in \mathfrak{g} \mid f(\mathfrak{m}) = 0\}$ and $\mathfrak{m}_\Bbbk^\perp := \{f \in \mathfrak{g}_\Bbbk^* \mid f(\mathfrak{m}_\Bbbk) = 0\}$, respectively.

For $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$ we set $Q_\chi^\eta := Q_{\chi,\Bbbk}/I_\eta Q_{\chi,\Bbbk}$, where I_η is the ideal of $U(\mathfrak{g}_\Bbbk)$ generated by all $x^p - x^{[p]} - \eta(x)^p 1$ with $x \in \mathfrak{g}_\Bbbk$. Evidently, Q_χ^η is a $\mathfrak{g}_\Bbbk$-module with p-character χ. Note that $Q_\chi^\chi = Q_\chi^{[p]}$ in the notation of [34, 4.3]. Each $\mathfrak{g}_\Bbbk$-endomorphism $\Theta_i \otimes 1$ of $Q_{\chi,\Bbbk} = Q_{\chi,A} \otimes_A \Bbbk$ preserves the submodule $I_\eta Q_{\chi,\Bbbk}$, hence induces a $\mathfrak{g}_\Bbbk$-endomorphism of Q_χ^η. To ease notation we call this endomorphism θ_i. Let $U_\eta(\mathfrak{g}_\Bbbk, e)$ denote the algebra $\bigl(\mathrm{End}_{\mathfrak{g}_\Bbbk}\, Q_\chi^\eta\bigr)^{\mathrm{op}}$. Since the restriction of η to $\mathfrak{m}_\Bbbk$ coincides with that of χ, the ideal of $U(\mathfrak{m}_\Bbbk)$ generated by all $x - \eta(x)$ with $x \in \mathfrak{m}_\Bbbk$ equals $N_{\chi,\Bbbk} = N_{\chi,A} \otimes_A \Bbbk$ and $\Bbbk_\chi = \Bbbk_\eta$ as $\mathfrak{m}_\Bbbk$-modules.

In what follows we require a slight generalisation of [34, Prop. 4.1].

Lemma 2.2. *The following are true:*

(i) $Q_\chi^\eta \cong U_\eta(\mathfrak{g}_\Bbbk) \otimes_{U_\eta(\mathfrak{m}_\Bbbk)} \Bbbk_\chi$ *as* $\mathfrak{g}_\Bbbk$*-modules;*

(ii) $U_\eta(\mathfrak{g}_\Bbbk, e) \cong \bigl(U_\eta(\mathfrak{g}_\Bbbk)/U_\eta(\mathfrak{g}_\Bbbk) N_{\chi,\Bbbk}\bigr)^{\mathrm{ad}\,\mathfrak{m}_\Bbbk}$*;*

(iii) Q_χ^η *is a projective generator for* $U_\eta(\mathfrak{g}_\Bbbk)$ *and* $U_\eta(\mathfrak{g}_\Bbbk) \cong \mathrm{Mat}_{p^{d(e)}}\bigl(U_\eta(\mathfrak{g}_\Bbbk, e)\bigr)$*;*

(iv) *the monomials* $\theta_1^{i_1} \cdots \theta_r^{i_r}$ *with* $0 \le i_k \le p-1$ *form a* $\Bbbk$*-basis of* $U_\eta(\mathfrak{g}_\Bbbk, e)$*.*

Proof. Let $\bar{1}_\chi$ be the image of $1_\chi \in Q_{\chi,\Bbbk}$ in Q_χ^η. By the universality property of induced modules the is a surjection $\tilde{\alpha}\colon Q_{\chi,\Bbbk} = U(\mathfrak{g}_\Bbbk) \otimes_{U(\mathfrak{m}_\Bbbk)} \Bbbk_\chi \twoheadrightarrow U_\eta(\mathfrak{g}_\Bbbk) \otimes_{U_\eta(\mathfrak{m}_\Bbbk)} \Bbbk_\chi$. As $I_\eta Q_{\chi,\Bbbk} \subseteq \mathrm{Ker}\,\tilde{\alpha}$, it gives rise to an epimorphism $\alpha\colon Q_\chi^\eta \twoheadrightarrow U_\eta(\mathfrak{g}_\Bbbk) \otimes_{U_\eta(\mathfrak{m}_\Bbbk)} \Bbbk_\chi$. On the other hand, Q_χ^η is generated by its 1-dimensional $U_\eta(\mathfrak{m}_\Bbbk)$-submodule $\Bbbk\bar{1}_\chi = \Bbbk_\chi$. The universality property of induced $U_\eta(\mathfrak{g}_\Bbbk)$-modules now shows that there is a surjection $\alpha'\colon U_\eta(\mathfrak{g}_\Bbbk) \otimes_{U_\eta(\mathfrak{m}_\Bbbk)} \Bbbk_\chi \twoheadrightarrow Q_\chi^\eta$. But then α is an isomorphism by dimension reasons, proving (i). Part (ii) is an immediate consequence of part (i); see [31, p. 10] for more detail.

Suppose $\mathfrak{m}_\Bbbk \cap \mathfrak{z}_\eta$ contains a nonzero element, say y, and write $y = \sum_{i \le -1} y_i$ with $y_i \in \mathfrak{g}_\Bbbk(i)$. Let $d \in \mathbb{Z}$ be such that $y_d \ne 0$ and $y_i = 0$ for $i > d$. Since $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$, we can write $\eta = (e + a\,,\,\cdot\,)$ for some $a \in \sum_{i \le 1} \mathfrak{g}_\Bbbk(i)$. As $\mathfrak{z}_\eta = (\mathfrak{g}_\Bbbk)_{e+a}$ and $\mathfrak{z}_\chi = (\mathfrak{g}_\Bbbk)_e$, our choice of d forces $y_d \in \mathfrak{m}_\Bbbk \cap \mathfrak{z}_\chi$. Since $(\mathfrak{g}_\Bbbk)_e \subset \sum_{i \ge 0} \mathfrak{g}_\Bbbk(i)$, this is impossible. So $\mathfrak{m}_\Bbbk \cap \mathfrak{z}_\eta = 0$, implying that $\mathfrak{m}_\Bbbk$ is an η-admissible subalgebra of dimension $d(e)$ in $\mathfrak{g}_\Bbbk$; see [31, 2.3 and 2.6]. Part (iii) now follows from [31, Thm. 2.3].

By (i) and (ii), the Kazhdan filtration of the module $Q_{\chi,\Bbbk}$ indices that on the algebra $U_\eta(\mathfrak{g}_\Bbbk, e) = (Q_{\chi,\Bbbk}/I_\eta Q_{\chi,\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_\Bbbk}$. Repeating verbatim the argument from the proof of [31, Thm. 3.4(i)] one obtains that the monomials $\theta_1^{i_1} \cdots \theta_r^{i_r}$ with $0 \le i_k \le p-1$

are linearly independent in $U_\eta(\mathfrak{g}_\Bbbk, e)$. Since $\dim U_\eta(\mathfrak{g}_\Bbbk, e) = p^r$ by part (iii), these monomials form a basis of $U_\eta(\mathfrak{g}_\Bbbk, e)$. □

2.7. Recall from (2.1) the A-basis $\{x_1, \dots, x_r, x_{r+1}, \dots, x_m\}$ of $\mathfrak{p}_A$. Set

$$X_i = \begin{cases} z_i & \text{if } 1 \le i \le s, \\ x_{r-s+i} & \text{if } s+1 \le i \le m-r+s. \end{cases}$$

For $\mathbf{a} \in \mathbb{Z}_+^{d(e)}$, put $X^{\mathbf{a}} := X_1^{a_1} \cdots X_{d(e)}^{a_{d(e)}}$ and $\bar{X}^{\mathbf{a}} := \bar{X}_1^{a_1} \cdots \bar{X}_{d(e)}^{a_{d(e)}}$, elements of $U(\mathfrak{g}_A)$ and $U(\mathfrak{g}_\Bbbk)$, respectively. By [34, Lemma 4.2(i)], the monomials $X^{\mathbf{a}} \otimes 1_\chi$ with $\mathbf{a} \in \mathbb{Z}^{d(e)}$ form a free basis of the right $U(\mathfrak{g}_A, e)$-module $Q_{\chi,A}$.

Lemma 2.3. *Let $\bar{1}_\chi$ be the image of $1_\chi \in Q_{\chi,\Bbbk}$ in Q_χ^η. For every $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$ the right $U_\eta(\mathfrak{g}_\Bbbk, e)$-module Q_χ^η is free with basis $\big\{\bar{X}^{\mathbf{a}} \otimes \bar{1}_\chi \mid 0 \le a_i \le p-1\big\}$.*

Proof. The Kazhdan filtration of the $U(\mathfrak{g}_\Bbbk)$-module $Q_{\chi,\Bbbk}$ induces that on the factor-module $Q_\chi^\eta = Q_{\chi,\Bbbk}/I_\eta Q_{\chi,\Bbbk}$. For $k \ge 0$ denote by $(Q_\chi^\eta)_k$ the kth component of the Kazhdan filtration of Q_χ^η. Call a tuple $\mathbf{a} \in \mathbb{Z}_+^l$ *admissible* if $a_i \le p-1$ for all i. By Lemma 2.2(iv), the monomials $\theta^{\mathbf{a}} := \theta_1^{a_1} \cdots \theta_r^{a_r}$, where $\mathbf{a}$ runs through the admissible tuples in $\mathbb{Z}_+^r$, form a $\Bbbk$-basis of $U_\eta(\mathfrak{g}_\Bbbk, e)$. Using (1) and induction on the Kazhdan degree $k = \sum_{i=1}^r a_i(n_i+2)$ of $\Theta^{\mathbf{a}}$ it is easy to observe that

$$\theta^{\mathbf{a}}(\bar{1}_\chi) \equiv \bar{x}_1^{a_1} \cdots \bar{x}_r^{a_r} \otimes \bar{1}_\chi + \sum_{|(\mathbf{i},\mathbf{j})|_e = k,\ |\mathbf{i}|+|\mathbf{j}| > |\mathbf{a}|} \gamma_{\mathbf{i},\mathbf{j}}\, \bar{x}^{\mathbf{i}} \bar{z}^{\mathbf{j}} \otimes 1_\chi \pmod{(Q_\chi^\eta)_{k-1}}$$

for some $\gamma_{\mathbf{i},\mathbf{j}} \in \Bbbk$. This relation in conjunction with double induction on $|(\mathbf{i},\mathbf{j})|_e$ and $|\mathbf{i}| + |\mathbf{j}|$ (upward on $|(\mathbf{i},\mathbf{j})|_e$ and downward on $|\mathbf{i}| + |\mathbf{j}|$) yields that every $\bar{x}^{\mathbf{i}} \bar{z}^{\mathbf{j}} \otimes \bar{1}_\chi$ belongs to the $\Bbbk$-submodule of Q_χ^η spanned by the vectors $\bar{X}^{\mathbf{a}} \theta^{\mathbf{b}}(\bar{1}_\chi)$ with admissible $\mathbf{a} \in \mathbb{Z}_+^{d(e)}$ and $\mathbf{b} \in \mathbb{Z}_+^r$. Since $\dim_\Bbbk Q_\chi^\eta = p^{d(e)+r}$ by Lemma 2.2(i), these vectors are linearly independent. The result follows. □

Let $\mathfrak{a}_k$ be the $\Bbbk$-span of $\bar{X}_1, \dots, \bar{X}_{d(e)}$ in $\mathfrak{g}_\Bbbk$ and put $\widetilde{\mathfrak{a}}_\Bbbk := \mathfrak{a}_\Bbbk \oplus \mathfrak{z}_\chi$. By our assumptions on $x_{r+1}, \dots, x_m$ in (2.1) and the inclusion $\operatorname{Ker} \operatorname{ad} f \subset \bigoplus_{i \le 0} \mathfrak{g}_\Bbbk(i)$, we have that

$$\mathfrak{a}_\Bbbk = \{x \in \widetilde{\mathfrak{a}}_\Bbbk \mid (x, \operatorname{Ker} \operatorname{ad} f) = 0\}. \tag{5}$$

The bilinear form $(\,\cdot\,,\,\cdot\,)$ allows us to identify the symmetric algebra $S(\widetilde{\mathfrak{a}}_\Bbbk)$ with the coordinate ring $\Bbbk[\chi + \mathfrak{m}_\Bbbk^\perp]$. Given a subspace V in $\mathfrak{g}_\Bbbk$ we denote by $Z_p(V)$ the subalgebra of the p-centre $Z(\mathfrak{g}_k)$ generated by all $x^p - x^{[p]}$ with $x \in V$. Clearly, $Z_p(V)$ is isomorphic to a polynomial algebra in $\dim_\Bbbk V$ variables. Let $\rho_\Bbbk$ denote the representation of $U(\mathfrak{g}_\Bbbk)$ in $\operatorname{End}_\Bbbk Q_{\chi,\Bbbk}$.

Our next result is, in a sense, analogous to Velkamp's theorem [42] on the structure of the centre of $U(\mathfrak{g}_\Bbbk)$. Similarity becomes apparent when one takes for e a regular nilpotent element in $\mathfrak{g}_\Bbbk$ and observes that in this special case $U(\mathfrak{g}_\Bbbk, e)$ identifies with the invariant algebra $U(\mathfrak{g}_\Bbbk)^{G_\Bbbk}$.

Theorem 2.1. *The following hold for any nilpotent element $e \in \mathfrak{g}_\Bbbk$:*

(i) *the algebra $\widehat{U}(\mathfrak{g}_\Bbbk, e)$ is generated by its subalgebras $U(\mathfrak{g}_\Bbbk, e)$ and $\rho_\Bbbk(Z_p)$;*

(ii) *$\rho_\Bbbk(Z_p) \cong Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$ and $\widehat{U}(\mathfrak{g}_\Bbbk, e)$ is a free $\rho_\Bbbk(Z_p)$-module of rank p^r;*

(iii) *$\widehat{U}(\mathfrak{g}_\Bbbk, e) \cong U(\mathfrak{g}_\Bbbk, e) \otimes_\Bbbk Z_p(\mathfrak{a}_\Bbbk)$ as $\Bbbk$-algebras.*

Proof. (a) First note that $Z_p(\mathfrak{g}_\Bbbk) \cong Z_p(\mathfrak{m}_\Bbbk) \otimes_\Bbbk Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$ as algebras, and $Z(\mathfrak{m}_\Bbbk) \cap \mathrm{Ker}\ \rho_\Bbbk$ is an ideal of codimension 1 in $Z_p(\mathfrak{m}_\Bbbk)$. Hence $\rho_\Bbbk(Z_p) = \rho_\Bbbk(Z_p(\widetilde{\mathfrak{a}}_\Bbbk))$. As the monomials $\bar{x}^{\mathbf{i}}\bar{z}^{\mathbf{j}} \otimes 1_\chi$ with $(\mathbf{i},\mathbf{j}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s$ form a basis of $Q_{\chi,\Bbbk}$ and $Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$ is polynomial algebra in $\bar{z}_i^p - \bar{z}_i^{[p]}$ $(1 \le i \le s)$ and $\bar{x}_j^p - \bar{x}_j^{[p]}$ $(1 \le j \le m)$, we have that $Z_p(\widetilde{\mathfrak{a}}_\Bbbk) \cap \mathrm{Ker}\ \rho_\Bbbk = \{0\}$. It follows that $\rho_\Bbbk(Z_p) \cong Z_p(\widetilde{\mathfrak{a}}_\Bbbk) \cong \Bbbk[\chi + \mathfrak{m}_\Bbbk^\perp]$ as $\Bbbk$-algebras.

(b) Denote by $\mathbf{I}_l$ the set of all admissible tuples in $\mathbb{Z}_+^l$ and let $\mathbf{e}_i$ denote the tuple in $\mathbf{I}_l$ whose only nonzero component equals 1 and occupies the i-th position. As an immediate consequence of (1), for $1 \le k \le r$ we have that

$$\bar{\Theta}_k^p(1_\chi) - \Big(\bar{x}_k^p + \sum_{|(\mathbf{i},\mathbf{j})|_e = n_k+2} \mu_{\mathbf{i},\mathbf{j}}^k\, \bar{x}^{p\,\mathbf{i}}\bar{z}^{p\,\mathbf{j}}\Big) \otimes 1_\chi \in \big(Q_{\chi,\Bbbk}\big)_{p(n_k+2)-1} \tag{6}$$

for some $\mu_{\mathbf{i},\mathbf{j}}^k \in \mathbb{F}_p$. Also, $\mathrm{gr}(\bar{z}_i^p - \bar{z}_i^{[p]}) = \mathrm{gr}(\bar{z}_i)^p$ and $\mathrm{gr}(\bar{x}_i^p - \bar{x}_j^{[p]}) = \mathrm{gr}(\bar{x}_j)^p$ for all $1 \le i \le s$ and $1 \le j \le m$. On the other hand, [34, Lemma 4.2(i)] implies that the vectors $\bar{X}^{\mathbf{a}} \otimes 1_\chi$ with $\mathbf{a} \in \mathbb{Z}_+^{d(e)}$ form a free basis of the right $U(\mathfrak{g}_\Bbbk, e)$-module $Q_{\chi,\Bbbk}$. As $Q_{\chi,\Bbbk}$ is a Kazhdan filtered $U(\mathfrak{g}_\Bbbk)$-module, straightforward induction on filtration degree, based on (6), shows that $Q_{\chi,\Bbbk}$ is generated as a $Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$-module by the set $\{\bar{X}^{\mathbf{i}}\bar{\Theta}^{\mathbf{j}} \otimes 1_\chi \mid \mathbf{i} \in \mathbf{I}_{d(e)},\, \mathbf{j} \in \mathbf{I}_r\}$.

Let h be an arbitrary element of $\widehat{U}(\mathfrak{g}_\Bbbk, e)$. Then $h(1_\chi) = \sum_{(\mathbf{i},\mathbf{j}) \in \mathbf{I}_{d(e)} \times \mathbf{I}_r} f_{\mathbf{i},\mathbf{j}} \bar{X}^{\mathbf{i}}\bar{\Theta}^{\mathbf{j}}(1_\chi)$ for some $f_{\mathbf{i},\mathbf{j}} \in Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$. For every $\xi \in \chi + \mathfrak{m}_\Bbbk^\perp$ the image of $f_{\mathbf{i},\mathbf{j}}$ in $U_\xi(\mathfrak{g}_\Bbbk)$ is a scalar which shall be denoted by $\xi(\mathbf{i},\mathbf{j})$. Suppose $f_{\mathbf{a},\mathbf{b}} \ne 0$ for a nonzero $\mathbf{a} \in \mathbf{I}_{d(e)}$ and some $\mathbf{b} \in \mathbf{I}_r$. Then there exists $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$ such that $\eta(\mathbf{a},\mathbf{b}) \ne 0$. Let $h(\eta)$ be the image of h in $U_\eta(\mathfrak{g}_\Bbbk, e) = \big(\mathrm{End}_{\mathfrak{g}_\Bbbk} Q_\chi^\eta\big)^{\mathrm{op}}$. Lemma 2.2(iv) implies that $h(\eta)(\bar{1}_\chi)$ is a $\Bbbk$-linear combination of $\theta^{\mathbf{i}}(\bar{1}_\chi)$ with $\mathbf{i} \in \mathbf{I}_r$. By Lemma 2.3, the set $\{\bar{X}^{\mathbf{i}} \otimes \bar{1}_\chi \mid \mathbf{i} \in \mathbf{I}_{d(e)}\}$ is a free basis of the right $U_\eta(\mathfrak{g}_\Bbbk, e)$-module Q_χ^η. Since $\eta(\mathbf{a},\mathbf{b}) \ne 0$ and $\theta^{\mathbf{i}}$ is the image of $\bar{\Theta}^{\mathbf{i}}$ in $U_\eta(\mathfrak{g}_\Bbbk, e)$, it is now evident that $h(\eta)(\bar{1}_\chi)$ cannot be a $\Bbbk$-linear combination of $\theta^{\mathbf{i}}(\bar{1}_\chi)$ with $\mathbf{i} \in \mathbf{I}_r$. This contradiction shows that $f_{\mathbf{i},\mathbf{j}} = 0$ unless $\mathbf{i} = \mathbf{0}$. As a consequence, the set $\{\bar{\Theta}^{\mathbf{i}} \mid \mathbf{i} \in \mathbf{I}_r\}$ generates $\widehat{U}(\mathfrak{g}_\Bbbk, e)$ as a $Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$-module. Specialising at a suitable $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$ and applying Lemma 2.2(iv) one more time we deduce that the set $\{\bar{\Theta}^{\mathbf{i}} \mid \mathbf{i} \in \mathbf{I}_r\}$ is a free basis of the $Z_p(\widetilde{\mathfrak{a}}_\Bbbk)$-module $\widehat{U}(\mathfrak{g}_\Bbbk, e)$.

(c) Our next goal is to show that $\widehat{U}(\mathfrak{g}_\Bbbk, e) = U(\mathfrak{g}_\Bbbk, e) \cdot Z_p(\mathfrak{a}_\Bbbk)$. Every $\mathfrak{g}_\Bbbk$-endomorphism of $Q_{\chi,\Bbbk}$ is uniquely determined by its value at 1_χ. For a nonzero $u \in \widehat{U}(\mathfrak{g}_\Bbbk, e)$ write $u(1_\chi) = \sum_{|(\mathbf{i},\mathbf{j})|_e \le n} \lambda_{\mathbf{i},\mathbf{j}}\, \bar{x}^{\mathbf{i}}\bar{z}^{\mathbf{j}} \otimes 1_\chi$, where $n = n(u)$ and $\lambda_{\mathbf{i},\mathbf{j}} \ne 0$ for at least one $(\mathbf{i},\mathbf{j})$ with $|(\mathbf{i},\mathbf{j})|_e = n$. For $k \in \mathbb{Z}_+$ put $\Lambda^k(u) := \{(\mathbf{i},\mathbf{j}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s \mid \lambda_{\mathbf{i},\mathbf{j}} \ne 0 \ \&\ |(\mathbf{i},\mathbf{j})|_e = k\}$ and denote by $\Lambda^{\max}(u)$ the set of all $(\mathbf{a},\mathbf{b}) \in \Lambda^n(u)$ for which the quantity $n - |\mathbf{a}| - |\mathbf{b}|$ assumes its maximum value. This maximum value will be denoted by $n' = n'(u)$. For each $(\mathbf{a},\mathbf{b}) \in \Lambda^{\max}$ we have that

$$|(\mathbf{a},\mathbf{b})|_e - |\mathbf{a}| - |\mathbf{b}| \;=\; \textstyle\sum_{i=1}^m (n_i+2)a_i + \sum_{i=1}^s b_i - |\mathbf{a}| - |\mathbf{b}| \ge 0.$$

Consequently, $n(u)$, $n'(u) \in \mathbb{Z}_+$ and $n(u) \ge n'(u)$.

Put $\Omega := \{(a,b) \in \mathbb{Z}_+^2 \mid a \ge b\}$. By the preceding remark, $(n(u), n'(u)) \in \Omega$ for all nonzero $u \in \widehat{U}(\mathfrak{g}_\Bbbk, e)$. It is immediate from (1) and our discussion in part (b) that $\Lambda^{\max}(\bar{\Theta}_k) = \{(\mathbf{e}_k, \mathbf{0})\}$, $\Lambda^{\max}\big(\rho_\Bbbk(\bar{x}_i^p - \bar{x}_i^{[p]})\big) = \{(p\,\mathbf{e}_i, \mathbf{0})\}$ for $1 \le i \le m$, and $\Lambda^{\max}\big(\rho_\Bbbk(\bar{z}_j^p - \bar{z}_j^{[p]})\big) = \{(\mathbf{0}, p\,\mathbf{e}_j)\}$ for $1 \le j \le s$. Since $Q_{\chi,\Bbbk}$ is a Kazhdan filtered

$U(\mathfrak{g}_\Bbbk)$-module, this implies that

$$\Lambda^{\max}\Big(\prod_{i=1}^m \rho_\Bbbk(\bar{x}_i^p - \bar{x}_i^{[p]})^{a_i} \cdot \prod_{i=1}^s \rho_\Bbbk(\bar{z}_i^p - \bar{z}_i^{[p]})^{b_i} \cdot \bar{\Theta}^{\mathbf{c}}\Big) \;=\; \big\{\big(p\,\mathbf{a} + \sum_{i=1}^r c_i \mathbf{e}_i, p\,\mathbf{b}\big)\big\}$$

for all $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s$ and all $\mathbf{c} \in \mathbf{I}_r$. Since $\widehat{U}(\mathfrak{g}_\Bbbk, e)$ is generated as a $Z_p(\mathfrak{a}_\Bbbk)$-module by the set $\{\bar{\Theta}^{\mathbf{i}} \mid \mathbf{i} \in \mathbf{I}_r\}$, it follows that for every $u \in \widehat{U}(\mathfrak{g}_\Bbbk, e)$ with $(n(u), n'(u)) = (d, l)$ there exists a $\Bbbk$-linear combination u' of the endomorphisms

$$u(\mathbf{a},\mathbf{b},\mathbf{c}) := \prod_{i=1}^m \rho_\Bbbk(\bar{x}_i^p - \bar{x}_i^{[p]})^{a_i} \cdot \prod_{i=1}^s \rho_\Bbbk(\bar{z}_i^p - \bar{z}_i^{[p]})^{b_i} \cdot \bar{\Theta}^{\mathbf{c}}, \qquad (\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s,\ \mathbf{c} \in \mathbf{I}_r,$$

with $\Lambda^{\max}\big(u(\mathbf{a},\mathbf{b},\mathbf{c})\big) \subseteq \Lambda^{\max}(u)$ such that either $n(u-u') < d$ or $n(u-u') = d$ and $n'(u-u') < l$.

Order the tuples in Ω lexicographically and assume that $u \in U(\mathfrak{g}_\Bbbk, e) \cdot Z_p(\mathfrak{a}_\Bbbk)$ for all nonzero $u \in \widehat{U}(\mathfrak{g}_\Bbbk, e)$ with $(n(u), n'(u)) \prec (d, l)$ (when $(n(u), n'(u)) = (0,0)$ this is a valid assumption). Now let $u \in \widehat{U}(\mathfrak{g}_\Bbbk, e)$ be such that $(n(u), n'(u)) = (d, l)$. By the preceding remark, there exists $u' = \sum_{(\mathbf{a},\mathbf{b},\mathbf{c})} \lambda_{\mathbf{a},\mathbf{b},\mathbf{c}}\, u(\mathbf{a},\mathbf{b},\mathbf{c})$ with $\Lambda^{\max}\big(u(\mathbf{a},\mathbf{b},\mathbf{c})\big) \subseteq \Lambda^{\max}(u)$ for all $(\mathbf{a},\mathbf{b},\mathbf{c})$ with $\lambda_{\mathbf{a},\mathbf{b},\mathbf{c}} \neq 0$ such that $(n(u-u'), n'(u-u')) \prec (d, l)$. Set

$$v(\mathbf{a},\mathbf{b},\mathbf{c}) \;:=\; u((0,\dots,0,a_{r+1},\dots,a_m),\mathbf{b},\mathbf{0}) \cdot \textstyle\prod_{i=1}^r \bar{\Theta}^{pa_i} \cdot \bar{\Theta}^{\mathbf{c}}.$$

Using (6) it is easy to observe that $\Lambda^{\max}\big(u(\mathbf{a},\mathbf{b},\mathbf{c})\big) = \Lambda^{\max}\big(v(\mathbf{a},\mathbf{b},\mathbf{c})\big)$ and

$$\big(n(u(\mathbf{a},\mathbf{b},\mathbf{c}) - v(\mathbf{a},\mathbf{b},\mathbf{c})), n'(u(\mathbf{a},\mathbf{b},\mathbf{c}) - v(\mathbf{a},\mathbf{b},\mathbf{c}))\big) \prec \big(n(u(\mathbf{a},\mathbf{b},\mathbf{c})), n'(u(\mathbf{a},\mathbf{b},\mathbf{c}))\big).$$

We now put $u'' := \sum_{(\mathbf{a},\mathbf{b},\mathbf{c})} \lambda_{\mathbf{a},\mathbf{b},\mathbf{c}}\, v(\mathbf{a},\mathbf{b},\mathbf{c})$, an element of $U(\mathfrak{g}_\Bbbk, e) \cdot Z_p(\mathfrak{a}_\Bbbk)$. Because $(n(u-u''), n'(u-u'')) \prec (n(u), n'(u))$, the equality $\widehat{U}(\mathfrak{g}_\Bbbk, e) = U(\mathfrak{g}_\Bbbk, e) \cdot Z_p(\mathfrak{a}_\Bbbk)$ follows by induction on the length of (d, l) in the linearly ordered set $(\Omega, \prec)$.

(d) It remains to show that $\widehat{U}(\mathfrak{g}_\Bbbk, e) \cong U(\mathfrak{g}_\Bbbk, e) \otimes_\Bbbk Z_p(\mathfrak{a}_\Bbbk)$. We have already mentioned that the vectors $\bar{X}^{\mathbf{a}} \otimes 1_\chi$ with $\mathbf{a} \in \mathbb{Z}_+^{d(e)}$ form a free basis of the right $U(\mathfrak{g}_\Bbbk, e)$-module $Q_{\chi,\Bbbk}$. Since $\bar{X}_i^p$ and $\bar{X}_i^p - \bar{X}_i^{[p]}$ have the same Kazhdan degree in $U(\mathfrak{g}_\Bbbk)$ and $Q_{\chi,\Bbbk}$ is a Kazhdan filtered $U(\mathfrak{g}_\Bbbk)$-module, it follows that the vectors

$$\Big\{\textstyle\prod_{i=1}^{d(e)} \rho_\Bbbk(\bar{X}_i^p - \bar{X}_i^{[p]})^{a_i} \cdot \bar{\Theta}^{\mathbf{c}} \otimes 1_\chi \mid a_i \in \mathbb{Z}_+,\ \mathbf{c} \in \mathbb{Z}_+^r\Big\}$$

are linearly independent. This implies that $\widehat{U}(\mathfrak{g}_\Bbbk, e) \cong U(\mathfrak{g}_\Bbbk, e) \otimes_\Bbbk Z_p(\mathfrak{a}_\Bbbk)$ as algebras, completing the proof. □

2.8. As an immediate consequence of Theorem 2.1 we obtain:

Corollary 2.2. *$\widehat{U}(\mathfrak{g}_\Bbbk, e)^{\mathrm{ab}} \cong U(\mathfrak{g}_\Bbbk, e)^{\mathrm{ab}} \otimes Z_p(\mathfrak{a}_\Bbbk)$ as $\Bbbk$-algebras.*

Proof. If C is an associative commutative $\Bbbk$-algebra, then for any associative $\Bbbk$-algebra Λ we have that

$$[\Lambda \otimes_\Bbbk C, \Lambda \otimes_\Bbbk C] \cdot (\Lambda \otimes_\Bbbk C) \;=\; \big([\Lambda,\Lambda] \otimes_\Bbbk C\big) \cdot (\Lambda \otimes_\Bbbk C) \;=\; [\Lambda,\Lambda] \cdot \Lambda \otimes_\Bbbk C.$$

Hence $(\Lambda \otimes_\Bbbk C)^{\mathrm{ab}} \;\cong\; \Lambda^{\mathrm{ab}} \otimes_\Bbbk C$ as $\Bbbk$-algebras. In view of Theorem 2.1 the corollary obtains by setting $\Lambda := U(\mathfrak{g}_\Bbbk, e)$ and $C := Z_p(\mathfrak{a}_\Bbbk)$. □

We are now in a position to prove the main result of this section.

Theorem 2.2. *If the finite W-algebra $U(\mathfrak{g},e)$ affords a 1-dimensional representation, then for $p \gg 0$ the reduced enveloping algebra $U_\chi(\mathfrak{g}_\Bbbk)$ admits irreducible representations of dimension $p^{d(e)}$.*

Proof. (a) Suppose $U(\mathfrak{g},e)$ affords a 1-dimensional representation. Then $\mathcal{E}(\mathbb{C}) \neq \varnothing$. Since the affine variety $\mathcal{E}(\mathbb{C}) = \operatorname{Specm} U(\mathfrak{g},e)^{\mathrm{ab}}$ is defined over $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ is algebraically closed, it follows that $\mathcal{E}(\overline{\mathbb{Q}}) \neq \varnothing$. Hence $\mathcal{E}(K) \neq \varnothing$ for some finite Galois extension K of $\mathbb{Q}$. It follows that there exists $d \in \mathbb{N}$ such that $\mathcal{E}$ has a point with coordinates in $\mathcal{O}_K[d^{-1}]$, where $\mathcal{O}_K$ stands for the ring of algebraic integers of K. If $p \nmid d$, then there is $\mathfrak{P} \in \operatorname{Spec} \mathcal{O}_K[d^{-1}]$ such that $\mathcal{O}_K[d^{-1}]/\mathfrak{P} \cong \mathbb{F}_q$, where q is a power of p. Embedding $\mathbb{F}_q$ into $\Bbbk = \overline{\mathbb{F}}_p$ we see that $\mathcal{E}(\Bbbk) \neq \varnothing$ for all such p. In view of Lemma 2.1 this implies that $U(\mathfrak{g}_\Bbbk, e)$ affords 1-dimensional representations for all primes p satisfying $p \nmid d$.

(b) Now suppose that $p \gg 0$ and $U(\mathfrak{g}_\Bbbk, e)$ affords a 1-dimensional representation. Then Theorem 2.1(iv) yields that the $\Bbbk$-algebra $\widehat{U}(\mathfrak{g}_\Bbbk, e)$ affords a 1-dimensional representation too; we call it ν. By Theorem 2.1(ii), $\rho_\Bbbk(Z_p) \cap \operatorname{Ker} \nu$ is a maximal ideal of the algebra $\rho_\Bbbk(Z_p) \cong Z_p(\widetilde{\mathfrak{a}}_\Bbbk) \cong \Bbbk[\chi + \mathfrak{m}_\Bbbk^\perp]$. So there exists $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$ such that $\rho_\Bbbk(x^p - x^{[p]} - \eta(x)^p) \in \operatorname{Ker} \nu$ for all $x \in \mathfrak{g}_\Bbbk$. Our choice of η ensures that the $\Bbbk$-algebra $\widehat{U}_\eta(\mathfrak{g}_\Bbbk, e) := \widehat{U}(\mathfrak{g}_\Bbbk, e) \otimes_{Z_p(\widetilde{\mathfrak{a}}_\Bbbk)} \Bbbk_\eta$ affords a 1-dimensional representation. On the other hand, the canonical projection $Q_{\chi,\Bbbk} \twoheadrightarrow Q_{\chi,\Bbbk}/I_\eta Q_{\chi,\Bbbk} = Q_\chi^\eta$ gives rise to an algebra homomorphism $\rho_\eta\colon \widehat{U}_\eta(\mathfrak{g}_\Bbbk, e) \to \big(\operatorname{End}_{\mathfrak{g}_\Bbbk} Q_\chi^\eta\big)^{\mathrm{op}} = U_\eta(\mathfrak{g}_\Bbbk, e)$. As $\dim_\Bbbk \widehat{U}_\eta(\mathfrak{g}_\Bbbk, e) \leq p^r$ by Theorem 2.1(ii), applying Lemma 2.2(iv) yields that ρ_η is an algebra isomorphism. As $U_\eta(\mathfrak{g}_\Bbbk) \cong \operatorname{Mat}_{p^{d(e)}}\big(U_\eta(\mathfrak{g}_\Bbbk, e)\big)$ by Lemma 2.2(iii), it follows that the algebra $U_\eta(\mathfrak{g}_\Bbbk)$ has an irreducible representation of dimension $p^{d(e)}$.

(c) Let Ξ denote the set of all $\xi \in \mathfrak{g}_\Bbbk^*$ for which the algebra $U_\xi(\mathfrak{g}_\Bbbk)$ contains a two-sided ideal of codimension $p^{2d(e)}$. It is immediate from [35, Lemma 2.3] that the set Ξ is Zariski closed in $\mathfrak{g}_\Bbbk^*$. If $\xi' = (\operatorname{Ad}^* g)(\xi)$ for some $g \in G_\Bbbk$, then $U_\xi(\mathfrak{g}_\Bbbk) \cong U_{\xi'}(\mathfrak{g}_\Bbbk)$ as algebras. Hence Ξ is stable under the coadjoint action of $G_\Bbbk$.

We claim that $\Bbbk^\times \cdot \xi \subset \Xi$ for all $\xi \in \Xi$. To prove the claim we first recall that $\xi = (x, \cdot\,)$ for some $x \in \mathfrak{g}_\Bbbk$. Let $x = x_s + x_n$ be the Jordan–Chevalley decomposition of x in the restricted Lie algebra $\mathfrak{g}_\Bbbk$ and put $\xi_s := (x_s, \cdot\,)$, $\xi_n := (x_n, \ , \cdot\,)$, and $\mathfrak{l} := \mathfrak{z}(\chi_s)$. As $p \gg 0$ and x_s is semisimple, $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}_\Bbbk$. It $t \in \Bbbk^\times$, then $tx = tx_s + tx_n$ is the Jordan–Chevalley decomposition of tx. Obviously, $\mathfrak{z}(t\xi_s) = \mathfrak{l}$.

Put $d := \frac{1}{2}(\dim_\Bbbk \mathfrak{g}_\Bbbk - \dim_\Bbbk \mathfrak{l})$. It follows from the Kac–Weisfeiler theorem (or rather from its generalisation due to Friedlander–Parshall) that $U_\xi(\mathfrak{g}_\Bbbk) \cong \operatorname{Mat}_{p^d}\big(U_\xi(\mathfrak{l})\big)$ and $U_{t\xi}(\mathfrak{g}_\Bbbk) \cong \operatorname{Mat}_{p^d}\big(U_{t\xi}(\mathfrak{l})\big)$; see [31, 2.5], for example. Since $p \gg 0$, we have a direct sum decomposition $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{s} = [\mathfrak{l},\mathfrak{l}]$, and induced tensor product decompositions $U_\xi(\mathfrak{l}) \cong U_\xi(\mathfrak{s}) \otimes_\Bbbk U_\xi(\mathfrak{z}(\mathfrak{l}))$ and $U_{t\xi}(\mathfrak{l}) \cong U_{t\xi}(\mathfrak{s}) \otimes_\Bbbk U_{t\xi}(\mathfrak{z}(\mathfrak{l}))$. As $\mathfrak{z}(\mathfrak{l})$ is a toral subalgebra of $\mathfrak{g}_\Bbbk$, the reduced enveloping algebra $U_\psi(\mathfrak{z}(\mathfrak{l}))$ is commutative and semisimple for every $\psi \in \mathfrak{z}(\mathfrak{l})^*$. From this it is immediate that $U_\xi(\mathfrak{z}(\mathfrak{l})) \cong U_{t\xi}(\mathfrak{z}(\mathfrak{l}))$ as algebras.

Let L be the Levi subgroup of $G_\Bbbk$ with $\operatorname{Lie}(L) = \mathfrak{l}$. It acts on $\mathfrak{s}$ as restricted Lie algebra automorphisms. Note that $\xi_{|\mathfrak{s}} = \xi_n$. As x_n is nilpotent and L is reductive, all nonzero scalar multiples of x_n are conjugate under the adjoint action of L. This implies that the algebras $U_\xi(\mathfrak{s})$ and $U_{t\xi}(\mathfrak{s})$ are isomorphic. In view of our earlier

remarks this shows that $U_{t\xi}(\mathfrak{l}) \cong U_\xi(\mathfrak{l})$ and $U_{t\xi}(\mathfrak{g}_\Bbbk) \cong U_\xi(\mathfrak{g}_\Bbbk)$ for all $t \in \Bbbk^\times$. Our claim is an immediate consequence of the last isomorphism.

(d) Since Ξ is Zariski closed and $\Bbbk^\times \cdot \xi \subset \Xi$ for all $\xi \in \Xi$, the set Ξ is conical. As $U_\eta(\mathfrak{g}_\Bbbk)$ has a simple module of dimension $p^{d(e)}$, we have $\eta \in \Xi$. As $\eta \in \chi + \mathfrak{m}_\Bbbk^\perp$ we can write $\eta = (e + y, \cdot\,)$ for some $y = \sum_{i \le -1} y_i$ with $y_i \in \mathfrak{g}_\Bbbk(i)$. There is a cocharacter $\lambda\colon \Bbbk^\times \to G_\Bbbk$ such that $(\mathrm{Ad}\,\lambda(t))\, x = t^j x$ for all $x \in \mathfrak{g}_\Bbbk(j)$, $j \in \mathbb{Z}$ and $t \in \Bbbk^\times$. For $i \le -1$, set $\eta_i := (y_i, \cdot\,)$. Then $\eta = \chi + \sum_{i \le -1} \eta_i$ and $(\mathrm{Ad}^*\,\lambda(t))\,\eta = t^2\chi + \sum_{i \le 1} t^i \eta_i$. As Ξ is conical and $(\mathrm{Ad}^*\, G_\Bbbk)$-invariant, this implies that $\chi + \sum_{i \le 1} t^{2-i}\,\eta_i \in \Xi$ for all $t \in \Bbbk^\times$. Since Ξ is Zariski closed, this yields $\chi \in \Xi$.

Let I be a two-sided ideal of codimension $p^{2d(e)}$ in $U_\chi(\mathfrak{g}_\Bbbk)$. Then $U_\chi(\mathfrak{g}_\Bbbk)/I$ is a $U_\chi(\mathfrak{g}_\Bbbk)$-bimodule. Since $U_\chi(\mathfrak{g}_\Bbbk) \otimes_\Bbbk U_\chi(\mathfrak{g}_\Bbbk)^{\mathrm{op}} \cong U_{(\chi,-\chi)}(\mathfrak{g}_\Bbbk \oplus \mathfrak{g}_\Bbbk)$ as $\Bbbk$-algebras, it is immediate from [30, Thm. 3.10] that the bimodule $U_\chi(\mathfrak{g}_\Bbbk)/I$ is irreducible. But then $U_\chi(\mathfrak{g}_\Bbbk)/I \cong \mathrm{Mat}_{p^{d(e)}}(\Bbbk)$. This shows that $U_\chi(\mathfrak{g}_\Bbbk)$ has a simple module of dimension $p^{d(e)}$, completing the proof. □

2.9. We call a representation of $U_\xi(\mathfrak{g}_\Bbbk)$ *small* if it has dimension equal to $p^{(\dim G_\Bbbk \cdot \xi)/2}$. To prove that every reduced enveloping algebra $U_\xi(\mathfrak{g}_\Bbbk)$ has such a representation is a well-known open problem in the modular representation theory of Lie algebras; see [30, p. 114], [20], [17, p. 110], for example. This problem has a positive solution for Lie algebras type A due to the fact that all nilpotent elements in $\mathfrak{gl}_n$ are Richardson. This enables one to construct small representations by inducing up 1-dimensional representations of appropriate parabolic subalgebras. However, outside type A the problem of small representations is wide open, and in the most interesting cases it is impossible to obtain such representations by parabolic induction. Our next result solves the problem of small representations for Lie algebras of types B, C, D under the assumption that $p \gg 0$.

Corollary 2.3. *If $\mathfrak{g}_\Bbbk$ is of type* B, C *or* D*, then the problem of small representations for $\mathfrak{g}_\Bbbk$ has a positive solution for almost all primes. More precisely, if $\Bbbk = \overline{\mathbb{F}}_p$ and $p \gg 0$, then for every $\xi \in \mathfrak{g}_\Bbbk^*$ the reduced enveloping algebra $U_\xi(\mathfrak{g}_\Bbbk)$ has a simple module of dimension $p^{(\dim G_\Bbbk \cdot \xi)/2}$.*

Proof. If $\mathfrak{l}$ Levi subalgebra of $\mathfrak{g}_\Bbbk$, then $\mathfrak{l} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}(\mathfrak{l})$ and $[\mathfrak{l},\mathfrak{l}]$ decomposes as a direct sum of ideals each of which is a simple Lie algebra of type A, B, C, D (one should keep in mind here that $p \gg 0$). In view of the Kac–Weisfeiler theorem this reduces the problem of small representations to the case where $\xi = (\bar{n}, \cdot\,)$ for some nilpotent element $\bar{n} \in \mathfrak{g}_\Bbbk$; see [31, 2.5] or [17, p. 114]. Furthermore, it can be assumed that $\bar{n} = n \otimes 1$ for some nilpotent element $n \in \mathfrak{g}$. By [24, Thm. 1.2.3(1)], the finite W-algebra $U(\mathfrak{g}, n)$ admits a 1-dimensional representation (the argument in [24] relies on earlier results of McGovern on completely prime primitive ideals; see [27, Ch. 5]). Applying Theorem 2.2 we now see that the reduced enveloping algebra $U_\xi(\mathfrak{g}_\Bbbk)$ has a module of dimension $p^{(\dim G_\Bbbk \cdot \xi)/2}$. This module is irreducible thanks to [30, Thm. 3.10]. □

Remark 2.1. Applying successively [34, 4.3], Corollary 2.1, [33, Thm. 3.1(ii)], and [24, Prop. 3.4.6] one observes that if the problem of small representations for $\mathfrak{g}_\Bbbk$ has a positive solution for almost all primes, then for every nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ there

exists a completely prime primitive ideal I of $U(\mathfrak{g})$ such that $\mathcal{VA}(I) = \overline{\mathcal{O}}$ (here $\mathcal{VA}(I)$ stands for the associated variety of I).

Remark 2.2. It seems likely that Corollary 2.3 remains true for all $p > 2$. To relax the assumption on p in the statement of Corollary 2.3 by the methods of this paper one would need a more explicit presentation of $U(\mathfrak{g}, e)$ in the spirit of [7].

3. Sheets and commutative quotients of finite W-algebras

3.1. Our main goal in this section is to estimate the number of irreducible components of the affine variety $\operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$ and determine their dimensions. Since our arguments will rely on Corollary 2.3, we have to leave aside some nilpotent orbits in Lie algebras of type E_7 and E_8.

Because the field $\overline{\mathbb{Q}}$ is algebraically closed, all irreducible components of $\mathcal{E}(\mathbb{C}) = \operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$ are defined over an algebraic number field, K say. Let R denote the ring of algebraic integers of K. For any maximal ideal $\mathfrak{p}$ of R the residue field $R/\mathfrak{p}$ is finite. Denote by $\Bbbk(\mathfrak{p})$ the algebraic closure of $R/\mathfrak{p}$ and let $\varphi\colon R[X_1, \dots, X_r] \to (R/\mathfrak{p})[X_1, \dots, X_r]$ be the homomorphism of polynomial algebras induced by inclusion $R/\mathfrak{p} \hookrightarrow \Bbbk(\mathfrak{p})$. Given a Zariski closed set $V \subseteq \mathbb{A}^r_{\mathbb{C}}$ with defining ideal $J \subset K[X_1, \dots, X_r]$ we let $\mathfrak{p}(V)$ stand for the zero locus of $\varphi(J \cap R[X_1, \dots, X_r])$ in $\mathbb{A}^r_{\Bbbk(\mathfrak{p})}$.

Given an algebraic variety Y we let $\operatorname{Comp}(Y)$ denote the set of all irreducible components of Y. If is a regular function f on Y, we write $V(f)$ for the zero locus of f in Y.

Lemma 3.1. *For any $p \gg 0$ there exists a bijection $\sigma\colon \operatorname{Comp}(\mathcal{E}(\mathbb{C})) \xrightarrow{\sim} \operatorname{Comp}(\mathcal{E}(\Bbbk))$ such that $\dim_{\mathbb{C}} \mathcal{Y} = \dim_{\Bbbk} \sigma(\mathcal{Y})$ for all $\mathcal{Y} \in \operatorname{Comp}(\mathcal{E}(\mathbb{C}))$.*

Proof. Let $\mathcal{Y}_1, \dots, \mathcal{Y}_q$ be the irreducible components of $\mathcal{E}(\mathbb{C})$. Since the $\mathcal{Y}_i$'s are defined over K, it follows from [29, Satz XVII], [38, Ch. III, Prop. 17] and [37, Prop. 18 and Thm. 28] that for almost all $\mathfrak{p} \in \operatorname{Spec} R$ the affine varieties $\mathfrak{p}(\mathcal{Y}_1), \dots, \mathfrak{p}(\mathcal{Y}_q)$ are irreducible and nonempty, that $\dim_{\mathbb{C}} \mathcal{Y}_i = \dim_{\Bbbk} \mathfrak{p}(\mathcal{Y}_i)$ for all i, and that $\mathfrak{p}(\mathcal{E}(\mathbb{C})) = \mathfrak{p}(\mathcal{Y}_1) \cup \ldots \cup \mathfrak{p}(\mathcal{Y}_q)$.

Note that $A \subseteq S^{-1}R$ and $\mathcal{E}(\mathbb{C}) = \bigcap_{i,j} V(F_{ij})$. Passing to a finite extension of K if necessary, we may assume that all hypersurfaces $V(F_{ij})$ are defined over K and the sets $\mathcal{Y}_1(K), \dots, \mathcal{Y}_q(K)$ are pairwise distinct. By [38, Ch. III, Prop. 19], if Zariski closed sets V_1 and V_2 are defined over K, then $\mathfrak{p}(V_1 \cap V_2) = \mathfrak{p}(V_1) \cap \mathfrak{p}(V_2)$ for almost all $\mathfrak{p}$. This shows that $\mathfrak{p}(\mathcal{E}(\mathbb{C})) = \bigcap_{i,j} \mathfrak{p}(V(F_{ij}))$ for almost all $\mathfrak{p} \in \operatorname{Spec} R$. If $p = \operatorname{char} \Bbbk(\mathfrak{p})$, then $\Bbbk(\mathfrak{p}) = \Bbbk$ and $\mathfrak{p}(V(F_{ij})) = V({}^pF_{ij})$ for all i, j. As a consequence,

$$\mathfrak{p}(\mathcal{E}(\mathbb{C})) = \bigcap_{i,j} \mathfrak{p}(V(F_{ij})) = \bigcap_{i,j} V({}^pF_{ij}) = \mathcal{E}(\Bbbk)$$

for almost all $\mathfrak{p} \in \operatorname{Spec} R$ (see [14, pp. 28, 30] for a similar reasoning). Since the morphism $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ induced by inclusion $\mathbb{Z} \subset R$ is surjective, we obtain that $\operatorname{Comp}(\mathcal{E}(\Bbbk)) = \{\mathfrak{p}(\mathcal{Y}_1), \dots, \mathfrak{p}(\mathcal{Y}_q)\}$ for all but finitely many $p \in \pi(A)$. As $\mathfrak{p}(\mathcal{Y}_1), \dots, \mathfrak{p}(\mathcal{Y}_q)$ are pairwise distinct for almost all $\mathfrak{p}$ and $\dim_{\mathbb{C}} \mathcal{Y}_i = \dim_{\Bbbk} \mathfrak{p}(\mathcal{Y}_i)$ for all i, the statement follows. □

3.2. In what follows we are going to use the Lusztig–Spaltenstein theory of induced nilpotent orbits and the Borho–Kraft theory of sheets in $\mathfrak{g}_{\Bbbk}$; see [26] and [4]. Our main reference here is [1]. Although the base field in [1] is assumed to have characteristic

0, the results in *loc. cit.* that we actually need are valid over $\Bbbk$ under the assumption that $\operatorname{char}\Bbbk$ is a good prime for the root system of $G_\Bbbk$; see [26], [1, p. 289], [41, p. 33] and [28] for related discussions.

At some point, we are going to invoke Katsylo's results [21] on sections of sheets. The original argument in [21] involved Hausdorff neighbourhoods and holomorphic maps, but a purely algebraic proof was recently found by Im Hof; see [18, pp. 8–14]. Since all results of [1] used in [18, pp. 8–14] apply in good characteristic, one can see by inspection that Im Hof's arguments are valid in positive characteristic provided that $(\mathfrak{g}_\Bbbk)_f \cap [e, \mathfrak{g}_\Bbbk] = 0$. The latter holds for all $p \gg 0$.

From now on we assume that $p \gg 0$. Let F be either $\mathbb{C}$ or $\Bbbk$ and put $\mathfrak{g}_F := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} F$. Then $\mathfrak{g}_F = \operatorname{Lie}(G_F)$ and $(\mathfrak{g}_F, G_F)$ is either $(\mathfrak{g}, G)$ or $(\mathfrak{g}_\Bbbk, G_\Bbbk)$ (depending $\operatorname{char} F$). Let $\mathfrak{l}_F = \operatorname{Lie} L_F$ be a proper Levi subalgebra of G_F and let $\mathcal{O}_0$ be a nilpotent orbit in $\mathfrak{l}_F$. Let $\mathfrak{g}_F = \mathfrak{u}_{-,F} \oplus \mathfrak{l}_F \oplus \mathfrak{u}_{+,F}$ be a triangular decomposition of $\mathfrak{g}_F$ with $\mathfrak{l}_F \oplus \mathfrak{u}_{-,F}$ and $\mathfrak{l}_F \oplus \mathfrak{u}_{+,F}$ being conjugate parabolic subalgebras of $\mathfrak{g}_F$. Since the number of nilpotent orbits in $\mathfrak{g}_F$ is finite, there is a unique nilpotent orbit $\mathcal{O} \subset \mathfrak{g}_F$ which intersects densely with the irreducible Zariski closed set $\overline{\mathcal{O}_0} + \mathfrak{u}_{+,F}$. We say that the orbit $\mathcal{O}$ is *induced* from $\mathcal{O}_0$, written $\mathcal{O} = \operatorname{Ind}_{\mathfrak{l}_F}^{\mathfrak{g}_F} \mathcal{O}_0$. It is known that $\mathcal{O}$ is independent of the choice of a triangular decomposition of $\mathfrak{g}_F$ involving $\mathfrak{l}_F$, which justifies the notation; see [26, [1], [41]. If $e_0 \in \mathcal{O}_0$ and $e \in \operatorname{Ind}_{\mathfrak{l}_F}^{\mathfrak{g}_F} \mathcal{O}_0$, then e is said to be *induced* from e_0. If a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}_F$ is *not* induced from a nilpotent orbit in a proper Levi subalgebra of $\mathfrak{g}_F$, then $\mathcal{O}$ is said to be *rigid* and every $x \in \mathcal{O}$ is called a *rigid* nilpotent element of $\mathfrak{g}_F$.

Theorem 3.1. *Let $\mathcal{O}_0$ be a nilpotent orbit in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, and $\mathcal{O} = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_0$. Let $e_0 \in \mathcal{O}_0$ and $e \in \mathcal{O}$. If the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ affords a 1-dimensional representation, then so does the finite W-algebra $U(\mathfrak{g}, e)$.*

Proof. (a) By the Bala–Carter theory, we may assume that $\mathfrak{l} = \operatorname{Lie}(L)$ is a standard Levi subalgebra of $\mathfrak{g}$ and $e_0 \in \mathfrak{l}_\mathbb{Z}$, where $\mathfrak{l}_\mathbb{Z} = \mathfrak{l} \cap \mathfrak{g}_\mathbb{Z}$. Let $\mathfrak{p}_\mathbb{Z} = \mathfrak{l}_\mathbb{Z} \oplus \mathfrak{u}_\mathbb{Z}$ be a standard parabolic $\mathbb{Z}$-subalgebra of $\mathfrak{g}_\mathbb{Z}$ with nilradical $\mathfrak{u}_\mathbb{Z}$. By our earlier discussion, we may also assume that $\mathcal{O}$ intersects densely with $\overline{\mathcal{O}_0} + \mathfrak{u}$, where $\mathfrak{u} := \mathfrak{u}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$. Set $\bar{e}_0 := e_0 \otimes 1$, an element of $\mathfrak{l}_\Bbbk = \mathfrak{l}_\mathbb{Z} \otimes_\mathbb{Z} \Bbbk$. As explained in [34, 2.5], we may choose e_0 such that $\dim_\mathbb{C} \mathcal{O}_0 = \dim_\Bbbk \mathcal{O}_{\Bbbk,0}$, where $\mathcal{O}_{\Bbbk,0} := (\operatorname{Ad} L_\Bbbk) \cdot \bar{e}_0$.

Since $\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_0$ contains a nonempty Zariski open subset of $\overline{\mathcal{O}_0} + \mathfrak{u}$ and the set $\big((\operatorname{Ad} L(\mathbb{Q})\big) \cdot e_0 + \mathfrak{u}_\mathbb{Q}$ is dense in $\mathcal{O}_0 + \mathfrak{u}$, there is $e_1 \in \mathfrak{u}_\mathbb{Q}$ with $e := e_0 + e_1 \in \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_0$. Enlarging A if necessary, we may assume that $e_1 \in \mathfrak{u}_A$. For $p \in \pi(A)$ set $\bar{e} := \bar{e}_0 + \bar{e}_1$, an element of $\mathfrak{g}_\Bbbk = \mathfrak{g}_A \otimes_A \Bbbk$. It follows from [26, Thm. 1.3] that $\dim \mathfrak{g}_e = \dim \mathfrak{l}_{e_0}$ and $\mathfrak{g}_e \subset \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{p}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$. Therefore, $\dim [\mathfrak{p}, e] = \dim [\mathfrak{l}, e_0] + \dim \mathfrak{u}$, forcing $[\mathfrak{p}_\mathbb{Q}, e] = [\mathfrak{l}_\mathbb{Q}, e_0] + \mathfrak{u}_\mathbb{Q}$. Extending A further, we may assume that $[\mathfrak{l}_A, e_0]$ is a direct summand of $\mathfrak{l}_A$ and $[\mathfrak{p}_A, e] = [\mathfrak{l}_A, e_0] + \mathfrak{u}_A$. Then $[\mathfrak{p}_\Bbbk, \bar{e}] = [\mathfrak{l}_\Bbbk, \bar{e}_0] + \mathfrak{u}_\Bbbk$ for all $p \in \pi(A)$, implying that $(\operatorname{Ad} P_\Bbbk) \cdot \bar{e}$ is dense in $\overline{\mathcal{O}_{0\Bbbk}} + \mathfrak{u}_\Bbbk$ (here $P_\Bbbk$ is the parabolic subgroup of $G_\Bbbk$ with $\operatorname{Lie}(P_\Bbbk) = \mathfrak{p}_\Bbbk$). This shows that $\bar{e} \in \operatorname{Ind}_{\mathfrak{l}_\Bbbk}^{\mathfrak{g}_\Bbbk} \mathcal{O}_{0,\Bbbk}$ for all $p \gg 0$. Extending A even further we include e into an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}_A$ and then consider the finite W-algebra $U(\mathfrak{g}_A, e)$ as in (2.3).

(b) Put $\xi_0 := (\bar{e}_0, \cdot\,)$ and $\xi := (\bar{e}, \cdot\,)$, linear functions on $\mathfrak{l}_\Bbbk$ and $\mathfrak{g}_\Bbbk$, respectively. Note that ξ vanishes on $\mathfrak{u}_\Bbbk$ and the restriction of ξ to $\mathfrak{l}_\Bbbk$ equals ξ_0. As $[\mathfrak{l}_\Bbbk, \mathfrak{l}_\Bbbk]$ is a direct sum of simple ideals and $U_{\xi_0}(\mathfrak{l}_\Bbbk) \cong U_{\xi_0}([\mathfrak{l}_\Bbbk, \mathfrak{l}_\Bbbk]) \otimes_\Bbbk U_{\xi_0}(\mathfrak{z}(\mathfrak{l}_\Bbbk))$, it is immediate from Theorem 2.2 that for all $p \gg 0$ the reduced enveloping algebra $U_{\xi_0}(\mathfrak{l}_\Bbbk)$ has a simple

module of dimension $p^{d(e_0)}$, where $d(e_0) = (\dim \mathcal{O}_0)/2$. Given such a module V we regard it as a $U_\xi(\mathfrak{p}_\Bbbk)$-module with the trivial action of $\mathfrak{u}_\Bbbk$ and consider the induced $U_\xi(\mathfrak{g}_\Bbbk)$-module $\widetilde{V} := U_\xi(\mathfrak{g}_\Bbbk) \otimes_{U_\xi(\mathfrak{p}_\Bbbk)} V$. It follows from the PBW theorem that

$$\dim \widetilde{V} = p^{\dim \mathfrak{g}_\Bbbk - \dim \mathfrak{p}_\Bbbk} \cdot p^{d(e_0)} = p^{(\dim \mathfrak{g} - \dim \mathfrak{l} + \dim \mathcal{O}_0)/2} = p^{d(e)}.$$

Since $\dim_\Bbbk (\mathrm{Ad}^* G_\Bbbk) \cdot \xi = 2d(e)$ by our choice of e, Lemma 2.2(iii) entails that the algebra $U_\xi(\mathfrak{g}_\Bbbk, \bar{e})$ affords a 1-dimensional representation. Then so does the algebra $U(\mathfrak{g}_\Bbbk, e)$ thanks to Lemmas 2.2(iv) and 2.1. Since this holds for all $p \gg 0$, Corollary 2.1 yields that the finite W-algebra $U(\mathfrak{g}, e)$ affords a 1-dimensional representation too. This completes the proof. □

Corollary 3.1. *Let $\mathcal{O}_0$ and $\mathcal{O}$ be as in Theorem 3.1. If the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ affords a 1-dimensional representation, then the enveloping algebra $U(\mathfrak{g})$ has a completely prime primitive ideal I with $\mathcal{VA}(I) = \overline{\mathcal{O}}$.*

Proof. Let $\chi = (e, \cdot\,)$, a linear function on $\mathfrak{g}$. By Theorem 3.1, the finite W-algebra $U(\mathfrak{g}, e)$ has a 1-dimensional module, $\mathbb{C}_0$ say. By Skryabin's equivalence, the annihilator $I := \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g}, e)} \mathbb{C}_0\big)$ is a primitive ideal of $U(\mathfrak{g})$; see [39]. By [33, Thm. 3.1], the associated variety of I equals $\overline{\mathcal{O}}$. By [24, Prop. 3.4.6], the primitive quotient $U(\mathfrak{g})/I$ is a domain, that is I is completely prime. □

Remark 3.1. Corollary 3.1 reduces to rigid orbits the well-known open problem of assigning to any nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ a completely prime ideal primitive ideal I of $U(\mathfrak{g})$ with $\mathcal{VA}(I) = \overline{\mathcal{O}}$. Closely related results were recently obtained by Borho–Joseph through a careful study of the behaviour of Goldie rank under parabolic induction; see [2, 4.8 and 7.4]. Our arguments are completely different (and more elementary). We recall from the proof of Corollary 2.3 that if all components of the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$ are of type A, B, C, D, then $U([\mathfrak{l}, \mathfrak{l}], e_0)$ affords 1-dimensional representations (this follows from [27, Ch. 5] and [24, Thm. 1.2.3(1)]).

3.3. The group $G_\Bbbk$ contains a unique connected unipotent group $M_\Bbbk$ of dimension $d(e)$ with the property that $\exp \mathrm{ad}\, x \in \mathrm{Ad}\, M_\Bbbk$ for all $x \in \mathfrak{m}_\Bbbk$ (since $p \gg 0$ exponentiating nilpotent derivations of $\mathfrak{g}_\Bbbk$ does not cause us any problems). Note that $\mathrm{Lie}\, M_\Bbbk = \mathfrak{m}_\Bbbk$. The group $M_\Bbbk$ is a characteristic p analogue of the unipotent group M from [16] which, in turn, is a special instance of a group N_l for $l = \mathfrak{g}(-1)^0$ (the group N_l can be defined for any totally isotropic subspace $l \subset \mathfrak{g}(-1)$; see [13]).

In what follows we need a characteristic p version of [13, Lemma 2.1]. Let $\kappa\colon\ \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be the Killing isomorphism given by $x \mapsto (x, \cdot\,)$, so that $\chi = \kappa(e)$, and write $\mathbb{S}_\Bbbk$ for the Slodowy slice $\chi + \kappa(\mathrm{Ker}\, \mathrm{ad}\, f)$ to the coadjoint orbit $(\mathrm{Ad}^*\, G_\Bbbk) \cdot \chi$. Since χ vanishes on $[\mathfrak{m}_\Bbbk\, \mathfrak{m}_\Bbbk]$, the group $\mathrm{Ad}^*\, M_\Bbbk$ preserves the affine subspace $\chi + \mathfrak{m}_\Bbbk^\perp \subset \mathfrak{g}_\Bbbk^*$. Set $\mathfrak{g}_\Bbbk(1)^0 := \{x \in \mathfrak{g}_\Bbbk(1) \mid (x, \mathfrak{g}_\Bbbk(-1)^0) = 0\}$, an s-dimensional subspace of $\mathfrak{g}(1)$. Then

$$\kappa^{-1}(\mathfrak{m}_\Bbbk^\perp) \;=\; \mathfrak{g}_\Bbbk(1)^0 \oplus \textstyle\bigoplus_{i \le 0} \mathfrak{g}_\Bbbk(i).$$

Let $\lambda_e \in X_*(G_\Bbbk)$ be the cocharacter such that $(\mathrm{Ad}\, \lambda_e(x)) \cdot x = t^i x$ for all $x \in \mathfrak{g}_\Bbbk(i)$ and $i \in \mathbb{Z}$ and define a rational action $\rho_e\colon\ \Bbbk^\times \to \mathrm{GL}(\mathfrak{g}_\Bbbk)$ by setting $\rho_e(t)(x) := t^2(\mathrm{Ad}\, \lambda_e)(t^{-1})(x)$ for all $x \in \mathfrak{g}_\Bbbk$.

Lemma 3.2. (cf. [13, Lemma 2.1]) *The coadjoint action-map $\alpha\colon\ M_\Bbbk \times \mathbb{S}_\Bbbk \to \chi + \mathfrak{m}_\Bbbk^\perp$ is an isomorphism of affine varieties.*

Proof. As $M_{\Bbbk}$ is a connected unipotent group, we have that $M_{\Bbbk} \cong \mathbb{A}_{\Bbbk}^{d(e)}$ as affine varieties. Set $\widetilde{\mathfrak{m}}_{\Bbbk} := \kappa^{-1}(\mathfrak{m}_{\Bbbk}^{\perp})$. In order to prove the lemma we need to show that the adjoint action-map $\alpha\colon\ M_{\Bbbk} \times (e + \operatorname{Ker}\operatorname{ad} f) \longrightarrow e + \widetilde{\mathfrak{m}}_{\Bbbk}$ is an isomorphism. It is easy to see that both varieties have the same dimension.

The differential $\mathrm{d}_{(1,\,e)}\alpha\colon\ \mathfrak{m}_{\Bbbk} \oplus \operatorname{Ker}\operatorname{ad} f \longrightarrow \widetilde{\mathfrak{m}}_{\Bbbk}$ is given by $x + z \mapsto [x, e] + z$ for all $x \in \mathfrak{m}_{\Bbbk}$ and $z \in \operatorname{Ker}\operatorname{ad} f$. Since $\operatorname{ad} e$ is injective on $\mathfrak{m}_{\Bbbk}$ and $(\operatorname{Ker}\operatorname{ad} f) \cap (\operatorname{Im}\operatorname{ad} e) = 0$ under our assumptions on p, the map $\mathrm{d}_{(1,\,e)}\alpha$ is a linear isomorphism. As in [13], we define a $\Bbbk^{\times}$-action on the affine variety $M_{\Bbbk} \times (e + \widetilde{\mathfrak{m}}_{\Bbbk})$ by

$$t \cdot (g, x) := (\lambda_e(t)^{-1} g \lambda_e(t), \rho_e(t)(x)) \qquad (t \in \Bbbk^{\times},\ g \in M_{\Bbbk},\ x \in \widetilde{\mathfrak{m}}_{\Bbbk}).$$

As in [13, p. 246], we see that this $\Bbbk^{\times}$-action is contracting and the Zariski closure of the set $\{t \cdot (g, x) \mid t \in \Bbbk^{\times}\}$ contains $(1, e)$. Since the morphism α is $\Bbbk^{\times}$-equivariant, we can apply [40, Lemma 8.1.1] to complete the proof. □

Remark 3.2. Instead of applying [40, Lemma 8.1.1] we could finish the proof of Lemma 3.2 by a more geometric argument outlined in [15, p. 553]. This argument is purely algebraic and works in all characteristics.

3.4. Let $\widehat{\mathcal{E}}$ denote the maximal spectrum of $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}$. Composing the embedding $Z_{\mathfrak{p}}(\widetilde{\mathfrak{a}}_{\Bbbk}) \hookrightarrow \widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ with the canonical homomorphism $\widehat{U}(\mathfrak{g}_{\Bbbk}, e) \twoheadrightarrow \widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}$ we get a map $\Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}] \to \widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}$ which, in turn, gives rise to an algebra homomorphism

$$\beta^*\colon\ \Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}] \longrightarrow \widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}/\operatorname{nil}\widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}} = \Bbbk[\widehat{\mathcal{E}}]$$

(as in (2.7), we identify $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$ with the coordinate algebra $\Bbbk[\chi + \mathfrak{m}_{\chi}^{\perp}]$). Let $J_{\chi} = \operatorname{Ker}\beta^*$ and denote by Y_{χ} the zero locus of J_{χ} in $\chi + \mathfrak{m}_{\chi}^{\perp}$. As $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ is a finite $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$-module by Theorem 2.1(ii), $\Bbbk[\widehat{\mathcal{E}}] = \widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}/\operatorname{nil}\widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}$ is a finite module over $\Bbbk[Y_{\chi}]$. So β^* induces a finite (hence surjective) morphism of affine varieties

$$\beta\colon\ \widehat{\mathcal{E}} \longrightarrow Y_{\chi}.$$

The group $M_{\Bbbk}$ preserves the left ideal $U(\mathfrak{g}_{\Bbbk})N_{\chi,\,\Bbbk}$ and therefore acts on $\widehat{U}(\mathfrak{g}_{\Bbbk}, e) = \big(\operatorname{End}_{\mathfrak{g}_{\Bbbk}} U(\mathfrak{g}_{\Bbbk})/U(\mathfrak{g}_{\Bbbk})N_{\chi,\,\Bbbk}\big)^{\mathrm{op}}$ as algebra automorphisms. Hence $M_{\Bbbk}$ acts on $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}}$. As $M_{\Bbbk}$ preserves $\rho_{\Bbbk}(Z_p) \cong \Bbbk[\chi + \mathfrak{m}_{\chi}]$, the map β^* is a homomorphism of $M_{\Bbbk}$-modules. Thus, both $\widehat{\mathcal{E}}$ and Y_{χ} are $M_{\Bbbk}$-varieties and the morphism β is $M_{\Bbbk}$-equivariant. Thanks to Lemma 3.2, the action-map $M_{\Bbbk} \times \mathbb{S}_{\Bbbk} \to \chi + \mathfrak{m}_{\Bbbk}^{\perp}$ induces an isomorphism

$$Y_{\chi} \cong M_{\Bbbk} \times (\mathbb{S}_{\Bbbk} \cap Y_{\chi}). \tag{7}$$

Proposition 3.1. *The following statements hold:*

(1) $\widehat{\mathcal{E}} \cong M_{\Bbbk} \times \mathcal{E}(\Bbbk)$ *as affine varieties.*

(2) *The map* β *induces a finite morphism* $\bar{\beta}\colon\ \mathcal{E}(\Bbbk) \twoheadrightarrow \mathbb{S}_{\Bbbk} \cap Y_{\chi}$.

Proof. (a) Let $\widehat{\mathcal{E}}_0 := \beta^{-1}(\mathbb{S}_{\Bbbk} \cap Y_{\chi})$, a Zariski closed subset of $\widehat{\mathcal{E}}$. Since β is $M_{\Bbbk}$-equivariant, we have a natural morphism $\gamma\colon\ M_{\Bbbk} \times \widehat{\mathcal{E}}_0 \to \widehat{\mathcal{E}}$. As β is surjective, (7) entails that so is γ. If p_1 is the first projection $Y_{\chi} \xrightarrow{\sim} M_{\Bbbk} \times (\mathbb{S}_{\Bbbk} \cap Y_{\chi}) \twoheadrightarrow M_{\Bbbk}$, then $p_1(x)^{-1}(x) \in \widehat{\mathcal{E}}_0$ for every $x \in \widehat{\mathcal{E}}$, and the the morphism

$$\widehat{\mathcal{E}} \longrightarrow M_{\Bbbk} \times \widehat{\mathcal{E}}_0, \qquad x \mapsto \big(p_1(x), p_1(x)^{-1}(x)\big),$$

is the inverse of γ. Hence $\widehat{\mathcal{E}} \cong M_{\Bbbk} \times \widehat{\mathcal{E}}_0$ as affine varieties.

(b) By Corollary 2.2, $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}} \cong U(\mathfrak{g}_{\Bbbk}, e)^{\mathrm{ab}} \otimes Z_p(\mathfrak{a}_{\Bbbk})$. Since $Z_p(\mathfrak{a}_{\Bbbk})$ is a domain, it follows that $\Bbbk[\widehat{\mathcal{E}}] \cong \Bbbk[\mathcal{E}(\Bbbk)] \otimes Z_p(\mathfrak{a}_{\Bbbk})$ as algebras. Therefore, $Z_p(\mathfrak{a}_{\Bbbk})$ embeds into $\Bbbk[\widehat{\mathcal{E}}]$. It also follows that the ideal $\Bbbk[\widehat{\mathcal{E}}]\mathfrak{a}_{\Bbbk}$ of $\Bbbk[\widehat{\mathcal{E}}]$ is radical and its zero locus, $\mathcal{V}$ say, is isomorphic to $\mathcal{E}(\Bbbk)$. On the other hand, it is evident from (5) that the ideal of $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk}) = \Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}]$ generated by $\mathfrak{a}_{\Bbbk}$ is nothing but the defining ideal of $\mathbb{S}_{\Bbbk}$ in $\Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}]$. As a consequence, $\beta(\mathcal{V}) \subseteq \mathbb{S}_{\Bbbk} \cap Y_{\chi}$, implying $\mathcal{V} \subseteq \widehat{\mathcal{E}}_0$.

Now $\widehat{\mathcal{E}} \cong \widehat{\mathcal{E}}_0 \times M_{\Bbbk}$ by part (a) and $\widehat{\mathcal{E}} \cong \mathcal{E}(\Bbbk) \times \mathbb{A}_{\Bbbk}^{d(e)}$ by our earlier remarks in this part. As $M_{\Bbbk} \cong \mathbb{A}_{\Bbbk}^{d(e)}$ and $\mathcal{E}(\Bbbk) \cong \mathcal{V}$, we deduce that there exists a bijection τ between $\mathrm{Comp}(\mathcal{V})$ and $\mathrm{Comp}(\widehat{\mathcal{E}}_0)$ such that $\dim X = \dim \tau(X)$ for all $X \in \mathrm{Comp}(\mathcal{V})$. As $\mathcal{V} \subseteq \widehat{\mathcal{E}}_0$, this yields $\mathcal{V} = \widehat{\mathcal{E}}_0$ and statement (1) follows.

(c) Let I_1 be the augmentation ideal of the Hopf algebra $\Bbbk[M_{\Bbbk}]$. By part (b), we can identify $\Bbbk[M_{\Bbbk}] \otimes \Bbbk[\mathcal{E}(\Bbbk)]$ and $\Bbbk[M_{\Bbbk}] \otimes \Bbbk[\mathbb{S}_{\Bbbk} \cap Y_{\chi}]$ with $\Bbbk[\widehat{\mathcal{E}}]$ and $\Bbbk[Y_{\chi}]$, respectively, in such a way that $\widetilde{I}_1 := I_1 \otimes \Bbbk[\mathcal{E}(\Bbbk)]$ identifies with the defining ideal of the closed subset $\widehat{\mathcal{E}}_0 \cong \mathcal{E}(\Bbbk)$ of $\widehat{\mathcal{E}}$. Since β is $M_{\Bbbk}$-equivariant, composing β^* with the canonical homomorphism $\Bbbk[\widehat{\mathcal{E}}] \twoheadrightarrow \Bbbk[\widehat{\mathcal{E}}]/\widetilde{I}_1$ induces an algebra map $\bar{\beta}^*\colon \Bbbk[Y_{\chi}] \longrightarrow \Bbbk[\mathcal{E}(\Bbbk)]$ whose kernel equals $I_1 \otimes \Bbbk[\mathbb{S}_{\Bbbk} \cap Y_{\chi}]$. Since β is a finite morphism and $\mathrm{Ker}\,\bar{\beta}^*$ identifies with the defining ideal of $\{1\} \times (\mathbb{S}_{\Bbbk} \cap Y_{\chi}) \cong \mathbb{S}_{\Bbbk} \cap Y_{\chi}$, we thus obtain a finite morphism $\bar{\beta}\colon \mathcal{E}(\Bbbk) \twoheadrightarrow \mathbb{S}_{\Bbbk} \cap Y_{\chi}$. This completes the proof. □

3.5. In order to obtain a good lower bound on the number of irreducible components of $\mathcal{E}(\mathbb{C})$ we now need more information the affine variety $\mathbb{S}_{\Bbbk} \cap Y_{\chi}$.

For $d \in \mathbb{N}$, define $\mathfrak{g}_{\Bbbk}^{(d)} := \{x \in \mathfrak{g}_{\Bbbk} \mid \dim(\mathfrak{g}_{\Bbbk})_x = d\}$. When $p \gg 0$, the centraliser $(\mathfrak{g}_{\Bbbk})_x$ coincides with the Lie algebra of $(G_{\Bbbk})_x = Z_{G_{\Bbbk}}(x)$ and $\dim(\mathfrak{g}_{\Bbbk})_x = \dim(G_{\Bbbk})_x$ for all $x \in \mathfrak{g}_{\Bbbk}$; see [19], for instance. Since the set $\mathfrak{g}_{\Bbbk}^{(d)}$ is quasi-affine, it decomposes as a union of finitely many irreducible components $\mathfrak{g}_{\Bbbk}$. The irreducible components of the $\mathfrak{g}_{\Bbbk}^{(d)}$'s are called *sheets* of $\mathfrak{g}_{\Bbbk}$. The sheets are $(\mathrm{Ad}\, G_{\Bbbk})$-stable, locally closed subsets of $\mathfrak{g}_{\Bbbk}$. By one of the main result of [1], there is a bijection between the sheets of $\mathfrak{g}_{\Bbbk}$ and the $G_{\Bbbk}$-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O}_0)$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}_{\Bbbk}$ and $\mathcal{O}_0$ is a rigid nilpotent orbit in $[\mathfrak{l}, \mathfrak{l}]$. Borho's classification of sheets remains valid over $\Bbbk$ under the assumption that $\mathrm{char}\,\Bbbk$ is a good prime for the root system of G; see (3.2) for related references. By [4, 5.8], every sheet of $\mathfrak{g}_{\Bbbk}$ contains a unique nilpotent orbit. However, outside type A sheets are *not* disjoint, and when two sheets overlap, they always contain the same nilpotent orbit.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}_{\Bbbk}$. The centre $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is a toral subalgebra of $\mathfrak{g}_{\Bbbk}$, and $(\mathfrak{g}_{\Bbbk})_z \supseteq \mathfrak{l}$ for all $z \in \mathfrak{z}(\mathfrak{l})$. We denote by $\mathfrak{z}(\mathfrak{l})_{\mathrm{reg}}$ the set of all $z \in \mathfrak{z}(\mathfrak{l})$ for which the equality $(\mathfrak{g}_{\Bbbk})_z = \mathfrak{l}$ holds; this is a nonempty Zariski open subset of $\mathfrak{z}(\mathfrak{l})$. For a nilpotent element $e_0 \in [\mathfrak{l}, \mathfrak{l}]$ define $\mathcal{D}(\mathfrak{l}, e_0) := (\mathrm{Ad}\, G_{\Bbbk}) \cdot (e_0 + \mathfrak{z}(\mathfrak{l})_{\mathrm{reg}})$, a locally closed subset of $\mathfrak{g}_{\Bbbk}$. We call $\mathcal{D}(\mathfrak{l}, e_0)$ a *decomposition class* of $\mathfrak{g}_{\Bbbk}$ (this term has to do with the Jordan–Chevalley decomposition in $\mathfrak{g}_{\Bbbk}$). Each sheet $\mathcal{S} \subset \mathfrak{g}_{\Bbbk}$ is a finite union of decomposition classes and contains a unique open such class; see [1, 3.7]. Moreover, if $\mathcal{D}(\mathfrak{l}, e_0)$ is open in $\mathcal{S}$, then $\mathcal{O}_0 := (\mathrm{Ad}\, L) \cdot e_0$ is rigid in $[\mathfrak{l}, \mathfrak{l}]$, the orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}_{\Bbbk}}(\mathcal{O}_0)$ is contained in $\mathcal{S}$, and $\dim(\mathcal{S}/G_{\Bbbk}) = \dim \mathfrak{z}(\mathfrak{l})$. These results, established in [1, 3.2, 4.3 and 5.6], are valid under our assumption on p.

Let $C(e) := (G_{\Bbbk})_e \cap (G_{\Bbbk})_f$. This is a reductive group and its finite quotient $\Gamma(e) := C(e)/C(e)^\circ$ identifies naturally with the component group $\Gamma(e) := (G_{\Bbbk})_e/(G_{\Bbbk})_e^\circ$; see [32], for instance. If $\mathcal{S}(e)$ is a sheet containing e, then the set $X := \mathcal{S}(e) \cap (e + \operatorname{Ker} \operatorname{ad} f)$ is Zariski closed and connected. Indeed, since $e \in X$, this follows from the fact that X is preserved by the contracting action of the 1-dimensional torus $\rho_e(\Bbbk^\times)$ introduced in (3.3). Clearly, X is stable under the adjoint action of $C(e)$.

Assume for a moment that $\Bbbk = \mathbb{C}$. In [21], Katsylo proved that the connected group $C(e)^\circ$ acts trivially on X and the irreducible components of X are permuted transitively by the component group $\Gamma(e)$. The action-morphism $\varphi\colon G_{\Bbbk} \times X \longrightarrow \mathcal{S}(e)$ is smooth, surjective of relative dimension $\dim (\mathfrak{g}_{\Bbbk})_e$. By [21], it gives rise to an open morphism $\psi\colon \mathcal{S}(e) \longrightarrow X/\Gamma(e)$, whose fibres are $(\operatorname{Ad} G_{\Bbbk})$-orbits, such that for any open set $U \subseteq X/\Gamma(e)$ the induced map $\Bbbk[U] \longrightarrow \Bbbk[\psi^{-1}(U)]^{G_{\Bbbk}}$ is an isomorphism. In brief, ψ is a geometric quotient. Since $\Gamma(e)$ acts transitively on $\operatorname{Comp}(X)$, it is straightforward to see that $X/\Gamma(e) = \operatorname{Specm} \Bbbk[X]^{\Gamma(e)}$ is an irreducible affine variety.

A purely algebraic (and rather short) proof of Katsylo's results was given in [18]. It is a matter of routine to check that this proof works under our assumption on p.

Summarising, if $\mathcal{D}(\mathfrak{l}, e_0)$ is the open decomposition class in $\mathcal{S}(e)$, then $e \in \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}_{\Bbbk}} \mathcal{O}_0$, the orbit $\mathcal{O}_0 = (\operatorname{Ad} L) \cdot e_0$ is rigid in $[\mathfrak{l}, \mathfrak{l}]$, and

$$\dim \mathfrak{z}(\mathfrak{l}) = \dim \mathcal{S}(e)/G_{\Bbbk} = \dim X_i \qquad \forall\, X_i \in \operatorname{Comp}(X). \tag{8}$$

3.6. Let $\mathcal{S}_1, \dots, \mathcal{S}_t$ be the pairwise distinct sheets of $\mathfrak{g}_{\Bbbk}$ containing our nilpotent element e. For $1 \leq i \leq t$ set $X_i := \mathcal{S}_i \cap (e + \operatorname{Ker} \operatorname{ad} f)$ and denote by $\mathcal{D}(\mathfrak{l}_i, e_i)$ the open decomposition classes of $\mathcal{S}_i$. Recall from (3.3) the Killing isomorphism $\kappa\colon \mathfrak{g}_{\Bbbk} \xrightarrow{\sim} \mathfrak{g}_{\Bbbk}^*$ and put $Y_i := \kappa(X_i) = \kappa(\mathcal{S}_i) \cap \mathbb{S}_{\Bbbk}$, where $1 \leq i \leq t$.

Proposition 3.2. *The following are true for all $p \gg 0$:*

(i) $Y_\chi \cap \mathbb{S}_{\Bbbk} \subseteq Y_1 \cup \ldots \cup Y_t$.

(ii) $\dim \mathcal{E}(\mathbb{C}) = \dim \mathcal{E}(\Bbbk) \leq \max_{1 \leq i \leq t} \dim \mathfrak{z}(\mathfrak{l}_i)$.

(iii) *If e is rigid, then $\mathcal{E}(\Bbbk)$ and $\mathcal{E}(\mathbb{C})$ are finite sets of the same cardinality.*

Proof. If $\eta \in Y_\chi$, then the definition of β^* in (3.4) shows that the algebra $\widehat{U}_\eta(\mathfrak{g}_{\Bbbk}, e) = \widehat{U}(\mathfrak{g}_{\Bbbk}, e) \otimes_{Z_p(\widehat{\mathfrak{a}}_{\Bbbk})} \Bbbk_\eta$ affords a 1-dimensional representation. In part (b) of the proof of Theorem 2.2 we have shown that this algebra is isomorphic to $U_\eta(\mathfrak{g}_{\Bbbk}, e)$. By Lemma 2.2(iii), the reduced enveloping algebra $U_\eta(\mathfrak{g}_{\Bbbk})$ affords a representation of dimension $p^{d(e)}$. Then [30, Thm. 3.10] yields $\dim \mathfrak{z}(\eta) \leq d(e)$.

On the other hand, our discussion in (3.3) shows that $\eta = \kappa(e + x)$ for some $x \in \bigoplus_{i \leq 1} \mathfrak{g}_{\Bbbk}(i)$. Since e lies in the Zariski closure of $\rho_e(\Bbbk^\times)(e + x)$ and the centralisers of $\rho_e(t)(e + x)$ and $e + x$ in $\mathfrak{g}_{\Bbbk}$ have the same dimension for all $t \in \Bbbk^\times$, it must be that $\dim \mathfrak{z}(\eta) \geq r$. As a result, $e + x \in \mathfrak{g}_{\Bbbk}^{(r)}$. Every irreducible component of $\mathfrak{g}_{\Bbbk}^{r)}$ containing $e + x$ must contain $\rho_e(\Bbbk^\times)(e + x)$ and hence e. This yields

$$Y_\chi \subseteq \bigcup\nolimits_{1 \leq i \leq t} \big(\kappa(\mathcal{S}_i) \cap (\chi + \mathfrak{m}_{\Bbbk}^{\perp})\big),$$

from which statement (i) is immediate. Since $\dim(Y_\chi \cap \mathbb{S}_{\Bbbk}) = \dim \mathcal{E}(\Bbbk)$ by Proposition 3.1(2) and $\dim \mathcal{E}(\Bbbk) = \dim \mathcal{E}(\mathbb{C})$ by Lemma 3.1, statement (ii) now follows from (8). When e is rigid, there is only one sheet containing e, namely, the orbit $\mathcal{O} = (\operatorname{Ad} G_{\Bbbk}) \cdot e$. So (8) implies that $X = \mathcal{O} \cap (e + \operatorname{Ker} \operatorname{ad} f) = \{e\}$ (for X is

connected). Then (i) shows that either $Y_\chi \cap \mathbb{S}_\Bbbk = \{\chi\}$ or $Y_\chi \cap \mathbb{S}_\Bbbk = \varnothing$. By Proposition 3.1(2) and Lemma 3.1, the sets $\mathcal{E}(\mathbb{C})$ and $\mathcal{E}(\Bbbk)$ are finite and have the same cardinality. □

We say that $\mathfrak{g}$ is *well-behaved* if for any proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and any nilpotent element $e_0 \in \mathfrak{l}$ the finite W-algebra $U([\mathfrak{l},\mathfrak{l}], e_0)$ admits a 1-dimensional representation. Thanks to [27, Ch. 5] and [24, Thm. 1.2.3(1)] the Lie algebras of types A_ℓ, B_ℓ, C_ℓ, D_ℓ, G_2, F_4, E_6 are well-behaved (in these cases all irreducible components of the proper subsets of Π have type A, B, C, D).

Proposition 3.3. *If $\mathfrak{g}$ is well-behaved and e is not rigid, then $Y_\chi \cap \mathbb{S}_\Bbbk = Y_1 \cup \ldots \cup Y_t$ for all $p \gg 0$.*

Proof. Since β is a closed morphism, we just need to show that $\beta(\widehat{\mathcal{E}})$ contains an open dense subset of each Y_i. By (3.5), the adjoint action-map $\varphi\colon G_\Bbbk \times X_i \longrightarrow \mathcal{S}_i$ is surjective. As $\mathcal{D}(\mathfrak{l}_i, e_i)$ is open in $\mathcal{S}_i$ and $C(e)$ permutes the components of X_i transitively, the set $\varphi^{-1}(\mathcal{D}(\mathfrak{l}_i, e_i))$ is open dense in $G_\Bbbk \times X_i$. Looking at the image of $\varphi^{-1}(\mathcal{D}(\mathfrak{l}_i, e_i))$ under the second projection $G_\Bbbk \times X_i \twoheadrightarrow X_i$ we observe that the set

$$X_i^{\mathrm{reg}} := \mathcal{D}(\mathfrak{l}_i, e_i) \cap (e + \operatorname{Ker} \operatorname{ad} f)$$

contains an open dense subset of X_i. We are thus reduced to show that for every $\eta \in \kappa(X_i^{\mathrm{reg}})$ the algebra $\widehat{U}_\eta(\mathfrak{g}_\Bbbk, e)$ has a 1-dimensional representation. As explained in part (b) of the proof of Theorem 2.2 this is equivalent to showing that the reduced enveloping algebra $U_\eta(\mathfrak{g}_\Bbbk)$ has a module of dimension $p^{d(e)}$. Note that $\mathfrak{l}_i$ is a proper Levi subalgebra of $\mathfrak{g}_\Bbbk$ (otherwise e would be rigid in $\mathfrak{g}_\Bbbk$).

As every element of $\mathcal{D}(\mathfrak{l}_i, e_i)$ is $(\operatorname{Ad} G_\Bbbk)$-conjugate to an element in $e_i + \mathfrak{z}(\mathfrak{l}_i)_{\mathrm{reg}}$, no generality will be lost by assuming that $\eta = \eta_s + \eta_n$, where $\eta_n = (e_i, \cdot\,)$ and $\eta_s = (z, \cdot\,)$ for some $z \in \mathfrak{z}(\mathfrak{l}_i)_{\mathrm{reg}}$. Since $\eta = \eta_s + \eta_n$ is the Jordan decomposition of η and $\mathfrak{z}(\eta_s) = (\mathfrak{g}_\Bbbk)_z = \mathfrak{l}_i$, applying the Kac–Weisfeiler theorem (as generalised by Friedlander–Parshall) we derive that $U_\eta(\mathfrak{g}_\Bbbk) \cong \operatorname{Mat}_{p^{m_i}}\big(U_\eta(\mathfrak{l}_i)\big)$, where $m_i = (\dim \mathfrak{g}_\Bbbk - \dim \mathfrak{l}_i)/2$; see [31, 2.5], for instance.

As $U_\eta(\mathfrak{l}_i) \cong U_{\eta_n}([\mathfrak{l}_i, \mathfrak{l}_i]) \otimes U_\eta(\mathfrak{z}(\mathfrak{l}_i))$ and $\dim (\operatorname{Ad} L_i) \cdot e_i = d(e) - m_i$, it remains to show that the reduced enveloping algebra $U_{\eta_n}([\mathfrak{l}_i, \mathfrak{l}_i])$ has a module of dimension $p^{(d(e)-m_i)/2}$. But this follows from Theorem 2.2 by our assumption on $\mathfrak{g}$. □

3.7. We are now in a position to state and prove the main result of this section:

Theorem 3.2. *Suppose $\mathfrak{g}$ is well-behaved and let e be any nonrigid nilpotent element of $\mathfrak{g}$. Let $\mathcal{S}_1, \ldots, \mathcal{S}_t$ be the pairwise distinct sheets of $\mathfrak{g}$ containing e. Let $\mathcal{D}(\mathfrak{l}_i, e_i)$ be the open decomposition class of $\mathcal{S}_i$ and $X_i = \mathcal{S}_i \cap (e + \operatorname{Ker} \operatorname{ad} f)$, where $1 \le i \le t$. Then there exists a surjection*

$$\operatorname{Comp}(\mathcal{E}(\mathbb{C})) \twoheadrightarrow \operatorname{Comp}(X_1) \sqcup \ldots \sqcup \operatorname{Comp}(X_t)$$

such that for every component $\mathcal{Y}$ of $\mathcal{E}(\mathbb{C})$ lying over $\operatorname{Comp}(X_i)$ the equality $\dim \mathcal{Y} = \dim \mathfrak{z}(\mathfrak{l}_i)$ holds.

Proof. We may assume that $e \in \mathfrak{g}_\mathbb{Z}$, that $\mathfrak{l}_1, \ldots, \mathfrak{l}_t$ are standard parabolic subalgebras of $\mathfrak{g}$, and that $e_i \in \mathfrak{l}_\mathbb{Z}$ for all i. We then may regard e and e_i as nilpotent elements of $\mathfrak{g}_\Bbbk$ and $\mathfrak{l}_{i,\Bbbk}$, respectively. Arguing as in part (a) of the proof of Theorem 3.1, one

observes that for $p \gg 0$ each e_i is rigid in $\mathfrak{l}_{i,\Bbbk}$ and e is not rigid in $\mathfrak{g}_\Bbbk$. By Lemma 3.1, there is a dimension preserving bijection between $\mathrm{Comp}(\mathcal{E}(\mathbb{C}))$ and $\mathrm{Comp}(\mathcal{E}(\Bbbk))$.

Let $\mathcal{S}_\Bbbk$ be a sheet of $\mathfrak{g}_\Bbbk$ containing e and let $\mathcal{D}(\mathfrak{l}, e_0)$ be the open decomposition class of $\mathcal{S}_k$. Since $\mathfrak{l}$ is $(\mathrm{Ad}\, G_\Bbbk)$-conjugate to a standard Levi subalgebra and $e \in \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}_k} \mathcal{O}_0$ for some rigid nilpotent orbit $\mathcal{O}_0 \subset [\mathfrak{l}, \mathfrak{l}]$, our discussion in (3.5) shows that there is a dimension preserving bijection between the sheets of $\mathfrak{g}$ containing e and those of $\mathfrak{g}_\Bbbk$ containing its image in $\mathfrak{g}_\Bbbk$. Moreover, every sheet of $\mathfrak{g}_\Bbbk$ containing $e \in \mathfrak{g}_\Bbbk$ has the form

$$\mathcal{S}_{i,\Bbbk} := \overline{\mathcal{D}(\mathfrak{l}_{i,\Bbbk}, e_i)} \cap \mathfrak{g}_\Bbbk^{(r)}, \qquad 1 \le i \le t.$$

By our discussion in (3.5), each variety $X_{i,\Bbbk} := \mathcal{S}_{i,\Bbbk} \cap (e + \mathrm{Ker}\, \mathrm{ad}\, f)$ is equidimensional of dimension $\dim \mathfrak{z}(\mathfrak{l}_i)$. To finish the proof it suffices now to apply Theorem 3.3 and Proposition 3.1(ii). □

Remark 3.3. In [33, 3.4] the author made the following conjecture:

1. Every finite W-algebra $U(\mathfrak{g}, e)$ has an ideal of codimension 1.
2. The ideals of codimension 1 in $U(\mathfrak{g}, e)$ are finite in number if and only if the orbit $(\mathrm{Ad}\, G) \cdot e$ is rigid.
3. For any ideal I of codimension 1 in $U(\mathfrak{g}, e)$ the annihilator of the $U(\mathfrak{g})$-module $Q_\chi \otimes_{U(\mathfrak{g}, e)} (U(\mathfrak{g}, e)/I)$ is a completely prime ideal of $U(\mathfrak{g})$.

Theorem 3.1 reduces part 1 of this conjecture to the case where e is rigid in $\mathfrak{g}$, whereas Theorem 3.2 and Proposition 3.2(iii) show that part 1 implies part 2. Part 3 was recently proved by Losev, who also confirmed part 1 for the Lie algebras of classical types; see [24]. As far as I am aware, part 1 remains open for some rigid nilpotent orbits in Lie algebras of types F_4, E_6, E_7, E_8. There are indications that these open cases will soon be tackled by computational methods.

3.8. As an application of Theorem 3.2 we now wish to describe the commutative quotient $U(\mathfrak{g}, e)^{\mathrm{ab}}$ for $\mathfrak{g} = \mathfrak{gl}(N)$. We are going to use the explicit presentation of $U(\mathfrak{g}, e)$ obtained by Brundan–Kleshchev in [7]. Given a partition $\mu = (q_1 \ge \cdots \ge q_m)$ of N with m parts we denote by $\mathfrak{g}(\mu)$ the standard Levi subalgebra $\mathfrak{gl}(q_1) \oplus \cdots \oplus \mathfrak{gl}(q_m)$ of $\mathfrak{gl}(N)$. Note that the centre of $\mathfrak{gl}(\mu)$ has dimension m.

Let $\lambda = (p_n \ge p_{n-1} \ge \cdots \ge p_1)$ be a partition of N with n parts. As in [7], we associate with λ a nilpotent element $e = e_\lambda \in \mathfrak{gl}(N)$ of Jordan type $(p_1, p_2, \ldots, p_n)$. By [7, Thm. 10.1], the finite W-algebra $U(\mathfrak{g}, e)$ is isomorphic to the shifted truncated Yangian $Y_{n,l}(\sigma)$ of level $l := p_n$. Here σ is an upper triangular matrix of order n with nonnegative integral entries; see [7, § 7] for more detail. It follows from the main results of [7] that $U(\mathfrak{g}, e)$ is generated by elements

$$\begin{aligned} &\{D_i^{(r)} \in U(\mathfrak{g}, e) \mid \ 1 \le i \le n;\ r \ge 1\}, \\ &\{E_i^{(r)} \in U(\mathfrak{g}, e) \mid \ 1 \le i \le n-1;\ r > p_{i+1} - p_i\}, \\ &\{F_i^{(r)} \in U(\mathfrak{g}, e) \mid \ 1 \le i \le n-1;\ r \ge 1\}, \end{aligned} \tag{9}$$

with $D_1^{(r)} = 0$ for $r > p_1$, subject to certain relations; see [7, (2.4) – (2.15)].

Recall from [33, p. 524] that the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ identifies canonically with the centre of $U(\mathfrak{g}, e)$ (this holds for for any simple Lie algebra $\mathfrak{g}$ and any nilpotent element $e \in \mathfrak{g}$).

Theorem 3.3. *If $\mathfrak{g} = \mathfrak{gl}(N)$ and $e = e_\lambda$, then $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $l = p_n$ variables.*

Proof. If $n = 1$, then e is regular and $l = N$. Hence $U(\mathfrak{g}, e) \cong Z(\mathfrak{g}) \cong \mathbb{C}[X_1, \dots, X_l]$. So assume from now that $n \geq 2$ and denote by $d_i^{(r)}$, $e_i^{(r)}$, $f_i^{(r)}$ the images of $D_i^{(r)}$, $E_i^{(r)}$, $F_i^{(r)}$ in $U(\mathfrak{g}, e)^{\mathrm{ab}}$. Applying [7, (2.6) and (2.7)] with $r = 1$ we see that $e_i^{(s)} = f_i^{(s)} = 0$ for all $1 \leq i \leq n-1$ and $s \geq 1$. By [7, (2.4)], the elements $D_i^{(r)}$ and $D_j^{(s)}$ commute for all $i, j \leq n$ and all r, s.

As in [7], we set $D_i^{(0)} := 1$ and $D_i(u) := \sum_{r \geq 0} D_i^{(r)} u^{-r}$, an element of $Y_{n,l}(\sigma)[u^{-1}]$, and define $\widetilde{D}_i^{(r)}$ from the equation $\widetilde{D}_i(u) = \sum_{r \geq 0} \widetilde{D}_i^{(r)} u^{-r} := -D_i(u)^{-1}$. Since

$$D_i(u)^{-1} = \Big(1 + \sum_{r \geq 1} D_i^{(r)} u^{-r}\Big)^{-1} = 1 + \sum_{k \geq 1} (-1)^k \Big(\sum_{r \geq 1} D_i^{(r)} u^{-r}\Big)^k,$$

it is easy to see that $\widetilde{D}_i^{(r)} - D_i^{(r)}$ is a polynomial in $D_i^{(1)}, \dots, D_i^{(r-1)}$ with initial form of degree ≥ 2. In particular, $\widetilde{D}_i^{(0)} = -1$, $\widetilde{D}_i^{(1)} = D_i^{(1)}$ and $\widetilde{D}_i^{(2)} = D_i^{(2)} - D_i^{(1)} D_i^{(1)}$. Let $\tilde{d}_i^{(r)}$ denote the image of $\widetilde{D}_i^{(r)}$ in $U(\mathfrak{g}, e)$. Since $[e_j^{(r)}, f_j^{(r)}] = 0$, applying [7, (2.5)] yields

$$\sum_{t=0}^{r} \tilde{d}_j^{(t)} d_{j+1}^{(r-t)} = 0 \qquad (1 \leq j \leq n-1,\ r > p_{i+1} - p_i). \tag{10}$$

Set $p_0 := 0$ and denote by $\mathcal{A}'$ the subalgebra of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ generated by all $d_j^{(k)}$ with $1 \leq k \leq p_j - p_{j-1}$. We claim that $d_j^{(k)} \in \mathcal{A}'$ for all (j, k) with $1 \leq j \leq n$ and $k \geq 0$. The claim is certainly true when $j + k = 2$. Suppose $d_j^{(k)} \in \mathcal{A}'$ for all (j, k) with $j + k \leq d$ and let (i, r) be such that $D_i^{(r)} \neq 0$ and $i + r = d + 1$. If $r \leq p_i - p_{i-1}$, then $d_i^{(r)} \in \mathcal{A}'$ by the definition of $\mathcal{A}'$. If $r > p_i - p_{i-1}$, then $i \geq 2$, for otherwise $D_i^{(r)} = 0$. Applying (10) with $j = i - 1$ we obtain

$$d_i^{(r)} \in \mathbb{C}\big[\tilde{d}_{i-1}^{(1)}, \dots, \tilde{d}_{i-1}^{(r)}, d_i^{(1)}, \dots, d_i^{(r-1)}\big].$$

Since $d_i^{(1)}, \dots, d_i^{(r-1)} \in \mathcal{A}'$ by our induction assumption and $\tilde{d}_{i-1}^{(m)} - d_{i-1}^{(m)}$ is a polynomial in $d_{i-1}^{(1)} \dots, d_{i-1}^{(m-1)}$, the claim follows by induction on d. Since $d_i^{(0)} = 1$, we thus deduce that the algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is generated by $p_1 + (p_2 - p_1) + \dots + (p_n - p_{n-1}) = p_n = l$ elements.

As a result, there is a surjective algebra map $\gamma \colon \mathbb{C}[X_1, \dots, X_l] \to U(\mathfrak{g}, e)^{\mathrm{ab}}$. If γ is not injective, then the morphism induced by γ identifies $\mathcal{E}(\mathbb{C}) = \operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$ with a *proper* Zariski closed subset of $\mathbb{A}^l_{\mathbb{C}}$. Then $\dim \mathcal{E}(\mathbb{C}) < l$. On the other hand, [22, Satz 2.2] says that e is Richardson in a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g} = \mathfrak{gl}(N)$ with $\mathfrak{l} \cong \mathfrak{g}(\lambda')$, where λ' is the partition of N conjugate to λ. In other words, $(\mathrm{Ad}\ \mathrm{GL}(n)) \cdot e = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}(\lambda')} \{0\}$. As λ' has l parts, Theorem 3.2 then yields $\dim \mathcal{E}(\mathbb{C}) \geq \dim \mathfrak{z}(\mathfrak{g}(\lambda')) = l$. This contradiction shows that $U(\mathfrak{g}, e)^{\mathrm{ab}} \cong \mathbb{C}[X_1, \dots, X_l]$. □

Question 3.1. *Is it true that for any simple Lie algebra $\mathfrak{g}$ and any nilpotent element $e \in \mathfrak{g}$ the algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ has no nonzero nilpotent elements?*

3.9. Recall from [33, p. 524] that the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ can be identified with the centre of $U(\mathfrak{g}, e)$ (this holds for any simple Lie algebra $\mathfrak{g}$ and any nilpotent element $e \in \mathfrak{g}$). In [31, Rem. 2], the author asked whether it is true that the centre of any factor-algebra $\mathcal{A}$ of $U(\mathfrak{g}, e)$ coincides with the image of

$Z(\mathfrak{g})$ in $\mathcal{A}$. The aim of this subsection is to show that the answer to this question is negative already for $\mathcal{A} = U(\mathfrak{g},e)^{\mathrm{ab}}$ and $\mathfrak{g} = \mathfrak{gl}(4)$. We keep the notation introduced in (3.8).

The centre of $U(\mathfrak{g},e)$ was determined in [8] and [6]. Let $Z_1,\dots,Z_N$ be the central elements of $U(\mathfrak{g},e)$ introduced in [6, Sect. 3] and denote by $z_1,\dots,z_N$ their images in $U(\mathfrak{g},e)^{\mathrm{ab}}$. Set $Z_0 = z_0 = 1$ and define $Z(u) := \sum_{i=0}^{N} Z_i u^{N-i}$ and $z(u) := \sum_{i=0}^{N} z_i u^{N-i}$, elements of $U(\mathfrak{g},e)[u]$ and $U(\mathfrak{g},e)^{\mathrm{ab}}[u]$, respectively. From the explicit presentation of $Z(u)$ given in [6, Sect. 3] it follows that $z(u)$ equals the determinant of the diagonal matrix

$$\mathrm{diag}\Big(u^{p_1}d_1(u), (u-1)^{p_2}d_2(u-1), \cdots, (u-n+1)^{p_n}d_n(u-n+1)\Big).$$

Now suppose $N=4$ and $\lambda=(2,2)$. Then $n=2$ and $p_1=p_2=2$. Combining [8, Thm. 3.5] with the equalities $f_1^{(r)} = e_1^{(r)} = 0,\ r\geq 1$ and $d_1^{(r)}=0$, $r>2$, it is not hard to observe that $d_2^{(r)}=0$ for all $r>2$. This implies that

$$z(u) \;=\; \big(u^2 + d_1^{(1)}u + d_1^{(2)}\big)\big((u-1)^2 + d_2^{(1)}(u-1) + d_2^{(2)}\big).$$

It was mentioned in (3.8) that $\tilde{d}_i^{(1)} = d_i^{(1)}$ and $\tilde{d}_i^{(2)} = d_i^{(2)} - d_i^{(1)}d_i^{(1)}$ for $i=1,2$. The proof of Theorem 3.3 shows that $U(\mathfrak{g},e)^{\mathrm{ab}} = \mathbb{C}[d_1^{(1)}, d_1^{(2)}]$, whilst from (10) we get $d_1^{(1)} + \tilde{d}_2^{(1)} = 0$ and $\tilde{d}_1^{(2)} + \tilde{d}_1^{(1)}d_2^{(1)} + d_2^{(2)} = 0$. This yields $d_2^{(1)} = \tilde{d}_2^{(1)} = -d_1^{(1)}$ and $d_2^{(2)} = -d_1^{(2)}$. Setting $X := d_1^{(1)}$ and $Y := d_1^{(2)}$ we obtain

$$\begin{aligned} z(u) &= \big(u^2 + Xu + Y\big)\big((u-1)^2 - X(u-1) - Y\big) \\ &= \big(u^2 + Xu + Y\big)\big((u^2 - (X+2)u + (X - Y + 1)\big) \\ &= u^4 - 2u^3 - \big(X^2 + X - 1\big)u^2 \\ &+ \big(X^2 - 2XY - 2Y + X\big)u + \big(XY - Y^2 + Y\big). \end{aligned}$$

According to [6], the image of $Z(\mathfrak{g})$ in $U(\mathfrak{g},e)$ is generated by Z_1, Z_2, Z_3, Z_4. Suppose for a contradiction that $\mathcal{A} = U(\mathfrak{g},e)^{\mathrm{ab}}$ coincides with the image of $Z(\mathfrak{g})$ in $\mathcal{A}$. As $X^2 - 2XY - 2Y + X = (X^2 + X) - 2Y(X+1)$, we then have the equality

$$\mathbb{C}[X,Y] \;=\; \mathcal{A} \;=\; \mathbb{C}[z_1, z_2, z_3, z_4] \;=\; \mathbb{C}[X^2 + X, Y(X+1), Y(X-Y+1)].$$

It follows that $\mathbb{C}[X,Y]/(Y)$ is generated by the image of $X(X+1)$ in $\mathbb{C}[X,Y]/(Y)$. Since $\mathbb{C}[X,Y]/(Y) \cong \mathbb{C}[X]$, this is impossible, however. This shows that the image of $Z(\mathfrak{g})$ in $U(\mathfrak{g},e)^{\mathrm{ab}}$ is a proper subalgebra of $U(\mathfrak{g},e)^{\mathrm{ab}}$.

4. Generalised Whittaker models for primitive ideals

4.1. We denote by $L(\lambda)$ the irreducible $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}^*$. Recall that $L(\lambda)$ is the simple quotient of the Verma module $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}\tilde{v}_\lambda$, where $\mathbb{C}\tilde{v}_\lambda$ is a 1-dimensional $(\mathfrak{h}\oplus\mathfrak{n}_+)$-module with $h\cdot\tilde{v}_\lambda = \lambda(h)\tilde{v}_\mu$ for all $h\in\mathfrak{h}$. Given a primitive ideal P of $U(\mathfrak{g})$ we write $\mathcal{VA}(P)$ for the associated variety of P. The affine variety $\mathcal{VA}(P) \subset \mathfrak{g}^*$ is the zero locus of the $(\mathrm{Ad}\,G)$-invariant ideal $\mathrm{gr}\,P$ of $S(\mathfrak{g}) = \mathrm{gr}\,U(\mathfrak{g})$. By the Irreducibility Theorem, $\mathcal{VA}(P)$ coincides with the Zariski closure of a coadjoint nilpotent orbit in $\mathfrak{g}^*$. By Duflo's Theorem, $P = \mathrm{Ann}_{U(\mathfrak{g})}\,L(\lambda)$ for some $\lambda\in\mathfrak{h}^*$. In general, such a λ is not unique, but if $\mathrm{Ann}_{U(\mathfrak{g})}\,L(\lambda) = \mathrm{Ann}_{U(\mathfrak{g})}\,L(\lambda')$ then $\lambda' + \rho = w(\lambda+\rho)$ for some $w\in W$ (here $W = \langle s_\alpha \,|\, \alpha\in\Phi\rangle$ is the Weyl group of $\mathfrak{g}$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is the half-sum of positive roots).

By [33, Thm. 3.1(ii)], if $P = \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},\,e)} V\big)$ for some finite dimensional irreducible $U(\mathfrak{g},e)$-module V, then $\mathcal{VA}(\mathcal{J}) = \overline{\mathcal{O}}_\chi$, where $\chi = (e,\,\cdot\,)$. A few years ago the author conjectured that the converse should also be true; that is, for every primitive ideal P of $U(\mathfrak{g})$ with $\mathcal{VA}(P) = \overline{\mathcal{O}}_\chi$ there should exist a finite dimensional irreducible $U(\mathfrak{g},e)$-module V such that $P = \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},\,e)} V\big)$; see [33, Conjecture 3.2]. In [34], this conjecture was proved under the assumption that the infinitesimal character of $P = \mathrm{Ann}_{U(\mathfrak{g})}\, L(\lambda)$ is rational, i.e. $\langle \lambda, \alpha^\vee\rangle \in \mathbb{Q}$ for all $\alpha \in \Pi$. In proving [33, Conjecture 3.2] under this assumption the author relied almost entirely on characteristic p methods.

In the meantime, two different proofs of the author's conjecture have appeared in the literature. The first proof, based on equivariant Fedosov quantisation, was obtained by Losev; see [24, Thm. 1.1.2(viii)]. The second proof, relying on Harish-Chandra bimodules for quantised Slodowy slices, was recently found by Ginzburg; see [16, Thm. 4.5.1].

The main goal of this section is to revisit the proof of [34, Thm. 1.1] and make a few amendments sufficient for confirming [33, Conjecture 3.2] in full generality (this was announced [34, p. 745]).

4.2. Given a Lie algebra $\mathfrak{L}$ over a commutative ring A, which is free as an A-module, we denote by $U_n(\mathfrak{L})$ the nth component of the canonical filtration of the universal enveloping algebra $U(\mathfrak{L})$. By the PBW theorem, the corresponding graded algebra $\mathrm{gr}\, U(\mathfrak{L})$ is isomorphic to the symmetric algebra $S(\mathfrak{L})$ of the free A-module $\mathfrak{L}$. Given a commutative Noetherian ring R we write $\dim R$ for the Krull dimension of R.

Let $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}\, L(\mu)$ be a primitive ideal of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_\chi$. From now on we shall always assume that our admissible ring A contains all elements $\langle \mu, \alpha^\vee\rangle$ with $\alpha \in \Pi$. In this case, $M_A(\mu) := U(\mathfrak{n}^-_A)\tilde{v}_\mu$ is a $\mathfrak{g}_A$-stable A-lattice in the Verma module $M(\mu)$ (here $\mathfrak{n}^-_A$ stands for the A-span of the e_γ with $\gamma \in \mathfrak{n}_-$).

Denote by $M^{\max}(\mu)$ the unique maximal submodule of $M(\mu)$, so that $L(\mu) = M(\mu)/M^{\max}(\mu)$, and let v_μ be the image of $\tilde{v}_\mu$ under the canonical homomorphism $M(\mu) \twoheadrightarrow L(\mu)$. Put $M^{\max}_A(\mu) := M^{\max}(\mu) \cap M_A(\mu)$ and define

$$L_A(\mu) \;:=\; M_A(\mu)/M^{\max}_A(\mu).$$

Since $M_A(\mu)$ is a Noetherian $U(\mathfrak{g}_A)$-module, so are $M^{\max}(\mu)$ and $L_A(\mu)$. For $n \in \mathbb{Z}_+$, put $L_n(\mu) := U_n(\mathfrak{g})v_\mu = U_n(\mathfrak{n}^-)v_\mu$ and $L_{A,n}(\mu) := U_n(\mathfrak{g}_A)v_\mu = U_n(\mathfrak{n}^-_A)v_\mu$, and let

$$\mathrm{gr}\, L(\mu) = \textstyle\bigoplus_{n\geq 0} L_n(\mu)/L_{n-1}(\mu) \quad \text{and} \quad \mathrm{gr}\, L_A(\mu) = \bigoplus_{n\geq 0} L_{A,n}(\mu)/L_{A,n-1}(\mu)$$

(here $L_{-1}(\mu) = L_{A,-1}(\mu) = 0$). Note that $\mathrm{gr}\, L(\mu)$ and $\mathrm{gr}\, L_A(\mu)$ are generated by $v_\mu = \mathrm{gr}_0\, v_\mu$ as modules over $S(\mathfrak{g}) = \mathrm{gr}\, U(\mathfrak{g})$ and $S(\mathfrak{g}_A) = \mathrm{gr}\, U(\mathfrak{g}_A)$, respectively.

We now define

$$\mathcal{J} := \mathrm{Ann}_{S(\mathfrak{g})}\, \mathrm{gr}\, L(\mu) \;=\; \mathrm{Ann}_{S(\mathfrak{g})}\, v_\mu \quad \text{and} \quad \mathcal{J}_A := \mathrm{Ann}_{S(\mathfrak{g}_A)}\, \mathrm{gr}\, L(\mu) \;=\; \mathrm{Ann}_{S(\mathfrak{g}_A)}\, v_\mu.$$

These are graded ideals of $S(\mathfrak{g})$ and $S(\mathfrak{g}_A)$, respectively. Put

$$\mathcal{R} := S(\mathfrak{g})/\mathcal{J} \quad \text{and} \quad \mathcal{R}_A := S(\mathfrak{g}_A)/\mathcal{J}_A.$$

The zero locus of the ideal $\mathcal{J} \subset S(\mathfrak{g})$ in $\mathfrak{g}^*$ is called the *associated variety* of $L(\mu)$ and denoted by $\mathcal{V}_{\mathfrak{g}}L(\mu)$. By a result of Gabber, all irreducible components of the variety $\mathcal{V}_{\mathfrak{g}}L(\mu)$ have dimension $d(e)$; see [34, 2.2] for more detail. In particular, $\dim \mathcal{R} = d(e)$. Since $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}(n)$, where $\mathcal{R}(n) \cong L_n(\mu)/L_{n-1}(\mu)$, is a graded Noetherian algebra

with $\mathcal{R}(0) = \mathbb{C}$, we have that $d(e) = \dim \mathcal{R} = 1 + \deg P_{\mathcal{R}}(t)$, where $P_{\mathcal{R}}(t)$ is the Hilbert polynomial of $\mathcal{R}$; see [12, Corollary 13.7].

First we note that $\mathcal{R}_A = \bigoplus_{n\geq 0} \mathcal{R}_A(n)$ is a finitely generated graded A-algebra and all $\mathcal{R}_A(n) \cong L_{A,n}(\mu)/L_{A,n-1}(\mu)$ are finitely generated A-modules. Also, $A \subset \mathbb{C}$ is a Noetherian domain. If $0 \neq b \in A$, then standard properties of localisation [5, Ch. II, 2.4] yield that $\mathcal{J}_{A[b^{-1}]} = \mathcal{J} \otimes_A A[b^{-1}]$ and

$$\mathcal{R}_{A[b^{-1}]} = S(\mathfrak{g}_{A[b^{-1}]})/\mathcal{J}_{A[b^{-1}]} \cong \big(S(\mathfrak{g}_A) \otimes_A A[b^{-1}]\big)/\big(\mathcal{J}_A \otimes_A A[b^{-1}]\big) \cong \mathcal{R}_A \otimes_A A[b^{-1}].$$

Denote by F the quotient field of A. Since $\mathcal{R}_F := \mathcal{R}_A \otimes_A F$ is a finitely generated algebra over a field, the Noether Normalisation Theorem says that there exist homogeneous, algebraically independent $y_1, \dots, y_d \in \mathcal{R}_F$, such that $\mathcal{R}_F$ is a finitely generated module over its graded polynomial subalgebra $F[y_1, \dots, y_d]$; see [12, Thm. 13.3]. Let $v_1, \dots, v_D$ be a generating set of the $F[y_1, \dots, y_d]$-module $\mathcal{R}_F$ and let $x_1, \dots, x_{m'}$ be a generating set of the A-algebra $\mathcal{R}$. Then

$$\begin{aligned} v_i \cdot v_j &= \textstyle\sum_{k=1}^{D} p_{i,j}^k(y_1, \dots, y_d) v_k \qquad (1 \leq i, j \leq D) \\ x_i &= \textstyle\sum_{j=1}^{D} q_{i,j}(y_1, \dots, y_d) v_j \qquad (1 \leq i \leq m') \end{aligned}$$

for some polynomials $p_{i,j}^k$, $q_{i,j} \in F[X_1, \dots, X_d]$. The algebra $\mathcal{R}_A$ contains an F-basis of $\mathcal{R}_F$. The coordinate vectors of the x_i's, y_i's and v_i's relative to this basis and the coefficients of the polynomials $q_{i,j}$ and $p_{i,j}^k$ involve only finitely many scalars in $\mathbb{Q}$. Replacing A by $A[b^{-1}]$ for a suitable $0 \neq b \in A$ if necessary, we may assume that all y_i and v_i are in $\mathcal{R}_A$ and all $p_{i,j}^k$ and $q_{i,j}$ are in $A[X_1, \dots, X_d]$. In conjunction with our earlier remarks this shows that no generality will be lost by assuming that

$$\mathcal{R}_A = A[y_1, \dots, y_d]v_1 + \cdots + A[y_1, \dots, y_d]v_D \tag{11}$$

is a finitely generated module over the polynomial algebra $A[y_1, \dots, y_d]$. We may assume without loss that $D!$ is invertible in A.

Lemma 4.1. *There exists an admissible ring $A \subset \mathbb{C}$ such that each graded component $\mathcal{R}_A(n)$ of $\mathcal{R}_A$ is a free A-module of finite rank.*

Proof. Since $\mathcal{R}_A$ is a finitely generated $A[y_1, \dots, y_d]$-module and A is a Noetherian domain, a graded version of the Generic Freeness Lemma shows that there exists a nonzero $a \in A$ such that each $\big(\mathcal{R}_A(n)\big)[a^{-1}]$ is a free $A[a^{-1}]$-module of finite rank; see (the proof of) Theorem 14.4 in [12]. Since it follows from [5, Ch. II, 2.4] that $\big(\mathcal{R}_A(n)\big)[a^{-1}] \cong \mathcal{R}_{A[a^{-1}]}(n)$ for all $n \in \mathbb{Z}_+$, the result follows. □

4.3. Denote by $L_F(\mu)$ the highest weight module $L_A(\mu) \otimes_A F$ over the split Lie algebra $\mathfrak{g}_F$, where $F = \mathrm{Quot}(A)$. Since $L(\mu) \cong L_F(\mu) \otimes_F \mathbb{C}$, each subspace $\mathfrak{I} \cap U_n(\mathfrak{g})$ is defined over F. It follows that the graded ideal

$$\mathrm{gr}\,\mathfrak{I} = \textstyle\bigoplus_{n\geq 0} \big(\mathfrak{I} \cap U_n(\mathfrak{g})\big)/\big(\mathfrak{I} \cap U_{n-1}(\mathfrak{g})\big) \subset S(\mathfrak{g})$$

is defined over F as well. Hence, for every $n \in \mathbb{Z}_+$ the F-subspace $S^n(\mathfrak{g}_F) \cap \mathrm{gr}\,\mathfrak{I}$ is an F-form of the graded component $\mathrm{gr}_n \mathfrak{I} \subset S^n(\mathfrak{g})$. Since $S(\mathfrak{g})$ is Noetherian, the ideal $\mathrm{gr}\,\mathfrak{I}$ is generated by its F-subspace $\mathrm{gr}\,\mathfrak{I}_{F,n'} := \mathrm{gr}\,\mathfrak{I} \cap \bigoplus_{k\leq n'} S^k(\mathfrak{g}_F)$ for some $n' = n'(\mu) \in \mathbb{Z}_+$. From this it follows that $\mathfrak{I}$ is generated over $U(\mathfrak{g})$ by its F-subspace $\mathfrak{I}_{F,n'} := U_{n'}(\mathfrak{g}_F) \cap \mathfrak{I}$. Since $\mathfrak{I}$ is a two-sided ideal of $U(\mathfrak{g})$, all subspaces $\mathfrak{I} \cap U_n(\mathfrak{g})$

and $\mathrm{gr}_n \mathfrak{I}$ are invariant under the adjoint action of G on $U(\mathfrak{g})$. It follows that the F-subspaces $\mathrm{gr}\,\mathfrak{I}_{F,n'}$ and $\mathfrak{I}_{F,n'}$ are invariant under the adjoint action of the distribution algebra $U_F := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} F$. Since $\mathfrak{h}_K := \mathfrak{h} \cap \mathfrak{g}_F$ is a split Cartan subalgebra of $\mathfrak{g}_F$, the adjoint $\mathfrak{g}_F$-modules $\mathrm{gr}\,\mathfrak{I}_{F,n'}$ and $\mathfrak{I}_{F,n'}$ decompose into a direct sum of absolutely irreducible $\mathfrak{g}_F$-modules with integral dominant highest weights. Consequently, these $\mathfrak{g}_F$-modules possess $\mathbb{Z}$-forms invariant under the adjoint action of the Kostant $\mathbb{Z}$-form $U_{\mathbb{Z}}$; we call them $\mathrm{gr}\,\mathfrak{I}_{\mathbb{Z},n'}$ and $\mathfrak{I}_{\mathbb{Z},n'}$.

Let $\{\psi_i \mid i \in I\}$ be a homogeneous basis of the free $\mathbb{Z}$-module $\mathrm{gr}\,\mathfrak{I}_{\mathbb{Z},n'}$ and let $\{u_i \mid i \in I\}$ be any basis of the free $\mathbb{Z}$-module $\mathfrak{I}_{\mathbb{Z},n'}$. Expressing the u_i and ψ_i via the PBW bases of $U(\mathfrak{g}_F)$ and $S(\mathfrak{g}_F)$ associated with our Chevalley basis $\mathcal{B}$ involves only finitely many scalars in F. Thus, no generality will be lost by assuming that all ψ_i are in $S(\mathfrak{g}_A)$ and all u_i are in $U(\mathfrak{g}_A)$.

Let K be an algebraically closed field whose characteristic is a good prime for the root system Φ. Let $\mathfrak{g}_K = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ and let G_K be the simple, simply connected algebraic K-group with hyperalgebra $U_K := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. Let $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{g}_K)$ denote the nilpotent cones of $\mathfrak{g}$ and $\mathfrak{g}_K$, respectively. As explained in [32] and [34, 2.5], there are nilpotent elements $e_1, \dots, e_t \in \mathfrak{g}_{\mathbb{Z}}$ such that

(i) $\{e_1, \dots, e_t\}$ is a set of representatives for $\mathcal{N}(\mathfrak{g})/G$;

(ii) $\{e_1 \otimes 1, \dots, e_t \otimes 1\}$ is a set of representatives for $\mathcal{N}(\mathfrak{g}_K)/G_K$;

(iii) $\dim_{\mathbb{C}} (\mathrm{Ad}\, G)e_i = \dim_K (\mathrm{Ad}\, G_K)(e_i \otimes 1)$ for all $i \le t$.

For $1 \le i \le t$ set $\chi_i := (e_i, \cdot\,)$. As in [34], we assume that $e = e_k$ for some $k \le t$ and $\mathcal{O}(e_i) \subset \overline{\mathcal{O}(e)}$ for $i \le k$. Since $\mathcal{VA}(\mathfrak{I})$ is the zero locus of $\mathrm{gr}\,\mathfrak{I}$ and $\mathrm{gr}\,\mathfrak{I}$ is generated by the set $\{\psi_i \mid i \in I\}$, we have that $\overline{\mathcal{O}(\chi)} = \bigcap_{i \in I} V(\psi_i)$. It follows that the ψ_i vanish on all χ_j with $j \le k$. Since all ψ_i are in $S(\mathfrak{g}_A)$, all e_j are in $\mathfrak{g}_{\mathbb{Z}}$, and the form $(\,\cdot\,,\,\cdot\,)$ is A-valued, we also have that $\psi_i(\chi_j) \in A$. Localising further if necessary we may assume that all nonzero $\psi_i(\chi_j)$ are invertible in A.

4.4. Now suppose that A satisfies all the conditions mentioned above. Take $p \in \pi(A)$ and let $\nu\colon A \to \mathbb{F}_p$ be the algebra homomorphism with kernel $\mathfrak{P} \in \mathrm{Specm}\, A$. Write $\Bbbk$ for the algebraic closure of $\mathbb{F}_p$ and set $L_{\mathfrak{P}}(\mu) := L_A(\mu) \otimes_A \Bbbk$, where it is assumed that A acts on $\Bbbk$ via ν. Clearly, $L_{\mathfrak{P}}(\mu)$ is a module over the Lie algebra $\mathfrak{g}_\Bbbk = \mathfrak{n}_\Bbbk^- \oplus \mathfrak{h}_\Bbbk \oplus \mathfrak{n}_\Bbbk^+$, where $\mathfrak{n}_\Bbbk^\pm := \mathfrak{n}^\pm \otimes_A \Bbbk$ and $\mathfrak{h}_\Bbbk := \mathfrak{h}_A \otimes_A \Bbbk$. Furthermore, $\bar{v}_\mu := v_\mu \otimes 1 \in L_{\mathfrak{P}}(\mu)$ is a highest weight vector for the Borel subalgebra $\mathfrak{h}_\Bbbk \oplus \mathfrak{n}_\Bbbk^+$ of $\mathfrak{g}_\Bbbk$, and $L_{\mathfrak{P}}(\mu) = U(\mathfrak{n}_\Bbbk^-) \cdot \bar{v}_\mu$. Denote by $\bar{\mu}$ the $\mathfrak{h}_\Bbbk$-weight of $\bar{v}_\mu$. Since $\mu(h_\alpha) \in A$ for all $\alpha \in \Pi$ and $\nu(a) \in \mathbb{F}_p$ for all $a \in A$, we have that $\bar{\mu}(\bar{h}_\alpha) \in \mathbb{F}_p$ for all $\alpha \in \Pi$.

Recall the notation and conventions of Section 2. Similar to [34, 3.1], we now set $I_{\mathfrak{P}}(\mu) := \{z \in Z_p \mid z \cdot \bar{v}_\mu = 0\}$, an ideal of the p-centre Z_p of $U(\mathfrak{g}_\Bbbk)$, and denote by $V_{\mathfrak{P}}(\mu)$ the zero locus of $I_{\mathfrak{P}}(\mu)$ in $\mathfrak{g}_\Bbbk^*$. It is immediate from the preceding remark that $\bar{e}_\gamma^{\,p} \in I_{\mathfrak{P}}(\mu)$ for all $\gamma \in \Phi^+$ and $\bar{h}_\alpha^{\,p} - \bar{h}_\alpha \in I_{\mathfrak{P}}(\mu)$ for all $\alpha \in \Pi$. Consequently,

$$V_{\mathfrak{P}}(\mu) \subseteq \{\eta \in \mathfrak{g}_\Bbbk^* \mid \eta(\mathfrak{h}_\Bbbk) = \eta(\mathfrak{n}_\Bbbk^+) = 0\}. \tag{12}$$

As the $U(\mathfrak{g}_\Bbbk)$-module $L_{\mathfrak{P}}(\mu)$ is generated by $\bar{v}_\mu$, we have that $I_{\mathfrak{P}}(\mu) = \mathrm{Ann}_{Z_p} L_{\mathfrak{P}}(\mu)$. Given $\eta \in \mathfrak{g}_\Bbbk^*$ we set $L_{\mathfrak{P}}^\eta(\mu) := L_{\mathfrak{P}}(\mu)/I_\eta \cdot L_{\mathfrak{P}}(\mu)$. By construction, $L_{\mathfrak{P}}^\eta(\mu)$ is a $\mathfrak{g}_\Bbbk$-module with p-character η. It follows from (12) that every $\xi \in V_{\mathfrak{P}}(\mu)$ has the form $\xi = (x, \cdot\,)$ for some $x \in \mathfrak{n}_\Bbbk^+$.

Lemma 4.2. *If $\eta \in V_{\mathfrak{P}}(\mu)$, then $L^{\eta}_{\mathfrak{P}}(\mu)$ is a nonzero $U_{\eta}(\mathfrak{g}_{\Bbbk})$-module.*

Proof. Replace $L_p(\mu)$ by $L_{\mathfrak{P}}(\mu)$ and $I_p(\mu)$ by $I_{\mathfrak{P}}(\mu)$, and argue as in the proof of [34, Lemma 3.1]. □

Set $L_{\mathfrak{P},n}(\mu) := U_n(\mathfrak{g}_{\Bbbk})\bar{v}_{\mu}$ and $\operatorname{gr} L_{\mathfrak{P}}(\mu) := \bigoplus_{n\geq 0} L_{\mathfrak{P},n}(\mu)/L_{\mathfrak{P},n-1}(\mu)$, where $n \in \mathbb{Z}_+$. Note that $\operatorname{gr} L_{\mathfrak{P}}(\mu)$ is a cyclic $S(\mathfrak{g}_{\Bbbk})$-module generated by $\bar{v}_{\mu} = \operatorname{gr}_0 \bar{v}_{\mu}$. Also,

$$L_{\mathfrak{P},n}(\mu) = U_n(\mathfrak{g}_k)\bar{v}_{\mu} = \big(U_n(\mathfrak{g}_A)v_{\mu}\big) \otimes_A \Bbbk = L_{A,n}(\mu) \otimes_A \Bbbk.$$

We put $\mathcal{J}_{\mathfrak{P}} := \operatorname{Ann}_{S(\mathfrak{g}_{\Bbbk})} \operatorname{gr} L_{\mathfrak{P}}(\mu) = \operatorname{Ann}_{S(\mathfrak{g}_{\Bbbk})} \bar{v}_{\mu}$ and $\mathcal{R}_{\mathfrak{P}} := S(\mathfrak{g}_{\Bbbk})/\mathcal{J}_{\mathfrak{P}}$, and denote by $\mathcal{V}_{\mathfrak{g}}L_{\mathfrak{P}}(\mu)$ the zero locus of $\mathcal{J}_{\mathfrak{P}}$ in $\operatorname{Specm} S(\mathfrak{g}_{\Bbbk}) = \mathfrak{g}^*_{\Bbbk}$. Since $\bar{v}_{\mu}$ is a highest weight vector for $\mathfrak{h}_{\Bbbk} \oplus \mathfrak{n}^+_{\Bbbk}$, all linear functions from $\mathcal{V}_{\mathfrak{g}}L_{\mathfrak{P}}(\mu)$ vanish on $\mathfrak{h}_{\Bbbk} \oplus \mathfrak{n}^+_{\Bbbk}$.

By Lemma 4.1, all graded components $\mathcal{R}_{A,n} \cong L_{A,n}(\mu)/L_{A,n-1}(\mu)$ of $\mathcal{R}_A$ are free A-modules of finite rank. From this it is immediate that so are the A-modules $L_{A,n}$, and $\mathcal{R}_{\mathfrak{P}} \cong \mathcal{R}_A \otimes_A \Bbbk$ as graded $\Bbbk$-algebras. Comparing the Hilbert polynomials of $\mathcal{R} = \mathcal{R}_A \otimes_A \mathbb{C}$ and $\mathcal{R}_{\mathfrak{P}} \cong \mathcal{R}_A \otimes_A \Bbbk$ we see that $\dim \mathcal{R}_{\mathfrak{P}} = \dim \mathcal{R} = d(e)$; see [12, Corollary 13.7]. As a consequence,

$$\dim_{\Bbbk} \mathcal{V}_{\mathfrak{g}}L_{\mathfrak{P}}(\mu) = \dim \mathcal{R}_{\mathfrak{P}} = d(e). \tag{13}$$

Recall from (11) the generators $v_1, \dots, v_D$ of the $A[y_1, \dots, y_d]$-module $\mathcal{R}_A$. Since $\mathcal{R} = \mathcal{R}_A \otimes_A \mathbb{C}$, the above discussion also shows that $d = \dim \mathcal{R} = d(e)$. We stress that $D = D(\mu)$ depends on μ, but not on $\mathfrak{P}$.

Lemma 4.3. *For every $\eta \in \mathfrak{g}^*_{\Bbbk}$ we have that $\dim_{\Bbbk} L^{\eta}_{\mathfrak{P}}(\mu) \leq Dp^{d(e)}$.*

Proof. Repeat verbatim the proof of Lemma 3.2 in [34]. □

4.5. Since $D!$ is invertible in A, we have that $p > D$ for all $p \in \pi(A)$. As before, we identify $\mathfrak{g}_{\Bbbk}$ with $\mathfrak{g}^*_{\Bbbk}$ by using the $G_{\Bbbk}$-equivariant map $x \mapsto (x, \cdot\,)$. Then $V_{\mathfrak{P}}(\mu) \subseteq \mathfrak{n}^+_{\Bbbk}$; see (12). The p-centre $Z_p(\mathfrak{n}^-_{\Bbbk}) = Z_p \cap U(\mathfrak{n}^-_{\Bbbk})$ of $U(\mathfrak{n}^-_{\Bbbk})$ is isomorphic to the polynomial algebra in $\bar{e}^p_{\gamma}$, where $\gamma \in \Phi^-$, hence can be identified with the subalgebra $S(\mathfrak{n}^-_{\Bbbk})^p$ of all p-th powers in $S(\mathfrak{n}^-_{\Bbbk})$. Therefore, we may regard $I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})$ as an ideal of the graded polynomial algebra $S(\mathfrak{n}^-_{\Bbbk})^p = \Bbbk[\bar{e}^p_{\gamma} \mid \gamma \in \Phi^-]$. It follows from our discussion in (4.4) and the above identifications that

$$V_{\mathfrak{P}}(\mu) = V(I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})) \cap \mathfrak{n}^+_{\Bbbk}. \tag{14}$$

Let $\operatorname{gr}\big(I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})\big)$ be the homogeneous ideal of $S(\mathfrak{n}^-_{\Bbbk})^p$ spanned by the highest components of all elements in $I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})$. From (14) it follows that the zero locus of $\operatorname{gr}\big(I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})\big)$ in $\mathfrak{n}^+_{\Bbbk}$ coincides with $\mathbb{K}(V_{\mathfrak{P}}(\mu))$, the associated cone to $V_{\mathfrak{P}}(\mu)$ (associated cones are defined in [4, §3], for instance). Since $I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})$ is contained in $\operatorname{Ann}_{Z_p} \bar{v}_{\mu}$, all elements of $\operatorname{gr}\big(I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})\big)$ annihilate $\operatorname{gr}_0 \bar{v}_{\mu} \in \operatorname{gr} L_{\mathfrak{P}}(\mu)$. Then $\operatorname{gr}\big(I_{\mathfrak{P}}(\mu) \cap Z_p(\mathfrak{n}^-_{\Bbbk})\big) \subset \mathcal{J}_{\mathfrak{P}} \cap S(\mathfrak{n}^-_{\Bbbk})$, which yields

$$\mathcal{V}_{\mathfrak{g}}L_{\mathfrak{P}}(\mu) = V(\mathcal{J}_{\mathfrak{P}} \cap S(\mathfrak{n}^-_{\Bbbk})) \cap \mathfrak{n}^+_{\Bbbk} \subseteq \mathbb{K}(V_{\mathfrak{P}}(\mu)). \tag{15}$$

Theorem 4.1. *Under the above assumptions on A, the variety $V_{\mathfrak{P}}(\mu)$ contains an irreducible component of maximal dimension which coincides with the Zariski closure of an irreducible component of $\mathfrak{n}^+_{\Bbbk} \cap (\operatorname{Ad} G_{\Bbbk})e$.*

Proof. This is a slight generalisation of [34, Thm. 3.1]. In view of (15) and (13) one just needs to replace $V_p(\mu)$ by $V_{\mathfrak{P}}(\mu)$, $\mathcal{V}_{\mathfrak{g}}L_p(\mu)$ by $\mathcal{V}_{\mathfrak{g}}L_{\mathfrak{P}}(\mu)$ and $\mathcal{J}_p$ by $\mathcal{J}_{\mathfrak{P}}$, and repeat the argument used in [34]. □

Recall from (4.3) the generating set $\{u_i \mid i \in I\}$ of the primitive ideal $\mathfrak{I}$. By construction, $u_i \in U(\mathfrak{g}_A)$ for all i and the A-span of the u_i's is invariant under the adjoint action of $\mathfrak{g}_A$. Let $\bar{u}_i$ be the image of u_i in $U(\mathfrak{g}_{\Bbbk}) = U(\mathfrak{g}_A) \otimes_A \Bbbk$. Clearly, the $\Bbbk$-span of the $\bar{u}_i$'s is invariant under the adjoint action of $\mathfrak{g}_{\Bbbk}$. Let $\varphi_\chi \colon U(\mathfrak{g}_A) \twoheadrightarrow Q_{\chi,A} = U(\mathfrak{g}_A)/N_{\chi,A}$ be the canonical homomorphism, and denote by $\bar{\varphi}_\chi$ the induced epimorphism from $U_\chi(\mathfrak{g}_{\Bbbk})$ onto Q_χ^χ; see (2.2) and Lemma 2.2(i). By Lemmas 2.2 and 2.3, there exists a finite subset C of $\mathbb{Z}_+^{d(e)}$ such that

$$\varphi_\chi(u_i) \;=\; \sum_{\mathbf{c}\in C} X^{\mathbf{c}} h_{i,\mathbf{c}}(1_\chi) \qquad\qquad \big(h_{i,\mathbf{c}} \in U(\mathfrak{g}_A, e),\;\; i \in I\big). \tag{16}$$

On the other hand, it follows from Lemma 2.1 that the $\Bbbk$-algebra $U_\chi(\mathfrak{g}_{\Bbbk}, e)$ is a homomorphic image of the $\Bbbk$-algebra $U(\mathfrak{g}_{\Bbbk}, e)$. Let $\bar{h}_{i,\mathbf{c}}$ denote the image of $h_{i,\mathbf{c}} \otimes 1$ in $U_\chi(\mathfrak{g}_{\Bbbk}, e)$. From(16) we get

$$\bar{\varphi}_\chi(\bar{u}_i) \;=\; \sum_{\mathbf{c}\in C} \bar{X}^{\mathbf{c}} \bar{h}_{i,\mathbf{c}}(\bar{1}_{\bar{\chi}}) \qquad\qquad (\forall\, i \in I). \tag{17}$$

Put $c := \max_{\mathbf{c}\in C} |\mathbf{c}|$. From now on we shall assume that $c!$ is invertible in A. This will ensure that the components of all tuples in C are smaller that any prime in $\pi(A)$.

Proposition 4.1. *Under the above assumptions on A, for every $\mathfrak{P} \in \operatorname{Specm} A$ with $A/\mathfrak{P} \cong \mathbb{F}_p$ there is a positive integer $k = k(\mathfrak{P}) \le D = D(\mu)$ such that the algebra $U_\chi(\mathfrak{g}_{\Bbbk}, e)$ has an irreducible k-dimensional representation ρ with the property that $\rho(\bar{h}_{i,\mathbf{c}}) = 0$ for all $\mathbf{c} \in C$ and all $i \in I$.*

Proof. Let $\mathfrak{P} \in \operatorname{Specm} A$ be such that $A/\mathfrak{P} \cong \mathbb{F}_p$. By Lemma 4.2 and Theorem 4.1, there exists $g \in G_{\Bbbk}$ such that $L_{\mathfrak{P}}^{g\cdot\chi}(\mu) \neq 0$, where $g\cdot\chi = (\operatorname{Ad}^* g)\chi$. By [30, Thm. 3.10] and Lemma 4.3, every composition factor V of the $\mathfrak{g}_{\Bbbk}$-module $L_p^{g\cdot\chi}(\mu)$ has dimension $kp^{d(e)}$ for some $k = k(V) \le D$. Since $u_i \in \operatorname{Ann}_{U(\mathfrak{g}_A)} L_A(\mu)$ for all $i \in I$, the elements $\bar{u}_i \in U(\mathfrak{g}_{\Bbbk})$ annihilate $L_p(\mu) = L_A(\mu) \otimes_A \Bbbk$. Consequently, all $\bar{u}_i$ annihilate $L_{\mathfrak{P}}^{g\cdot\chi}(\mu) = L_{\mathfrak{P}}(\mu)/I_{g\cdot\chi}L_{\mathfrak{P}}(\mu)$, and hence V.

Since $(\operatorname{Ad} g)(I_\chi) = I_{g\cdot\chi}$, the map $\operatorname{Ad} g \colon U(\mathfrak{g}_{\Bbbk}) \to U(\mathfrak{g}_{\Bbbk})$ gives rise to an algebra isomorphism $U_\chi(\mathfrak{g}_{\Bbbk}) \xrightarrow{\sim} U_{g\cdot\chi}(\mathfrak{g}_{\Bbbk})$. Let $V' = \{v' \mid v \in V\}$, a vector space copy of V. Give V' a $\mathfrak{g}_{\Bbbk}$-module structure by setting $x \cdot v' := ((\operatorname{Ad} g)^{-1}x \cdot v)'$ for all $x \in \mathfrak{g}_{\Bbbk}$ and $v' \in V'$. Since all elements $((\operatorname{Ad} g)x)^p - ((\operatorname{Ad} g)x)^{[p]} - \chi(x)^p 1$ annihilate V, the $\mathfrak{g}_{\Bbbk}$-module V' has p-character χ. Furthermore, all elements $(\operatorname{Ad} g)\bar{u}_i$ annihilate V'. The $\mathbb{Z}$-span of $\{u_i \mid i \in I\}$ is invariant under the adjoint action of $U_{\mathbb{Z}}$ on $U(\mathfrak{g}_{\mathbb{Z}})$; see (4.3). Since $U_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ is the hyperalgebra of $G_{\Bbbk}$, the $\Bbbk$-span of the $\bar{u}_i$'s is invariant under the adjoint action of $G_{\Bbbk}$ on $U(\mathfrak{g}_{\Bbbk})$. In conjunction with our preceding remark this implies that $\bar{u}_i \in \operatorname{Ann}_{U(\mathfrak{g}_{\Bbbk})} V'$ for all $i \in I$. Let

$$V_0' \;=\; \{v' \in V' \mid\; x \cdot v' = \chi(x)v' \;\text{ for all }\; x \in \mathfrak{m}_{\Bbbk}\}.$$

Since $U_\chi(\mathfrak{g}_{\Bbbk}, e) \cong \big(U_\chi(\mathfrak{g}_{\Bbbk})/U_\chi N_{\chi,\Bbbk}\big)^{\operatorname{ad}\mathfrak{m}_{\Bbbk}}$ by Lemma 2.2(ii), the algebra $U_\chi(\mathfrak{g}_{\Bbbk}, e)$ acts on V_0'. Since $\mathfrak{m}_{\Bbbk}$ is a χ-admissible subalgebra of dimension $d(e)$ in $\mathfrak{g}_{\Bbbk}$, it follows from

[31, Thm. 2.4] that V_0' is an irreducible k-dimensional $U_\chi(\mathfrak{g}_\Bbbk, e)$-module. We let ρ stand for the corresponding representation of $U_\chi(\mathfrak{g}_\Bbbk, e)$.

Denote by V'' the $U_\chi(\mathfrak{g}_\Bbbk)$-module $Q_\chi^\chi \otimes_{U_\chi(\mathfrak{g}_\Bbbk, e)} V_0'$ and let $v_1', \dots, v_k'$ be a basis of V_0'. It follows from Lemma 2.3 that the vectors $\bar{X}^{\mathbf{a}} \otimes v_j'$ with $0 \leq a_i \leq p-1$ and $1 \leq j \leq k$ form a basis of V'' over $\Bbbk$. Since V' is an irreducible $\mathfrak{g}_\Bbbk$-module, there is a $\mathfrak{g}_\Bbbk$-module epimorphism $\tau\colon V'' \twoheadrightarrow V'$ sending $v' \otimes 1$ to v' for all $v' \in V_0'$. Since $\dim_\Bbbk V' = kp^{d(e)}$, the map τ is an isomorphism. Let $\tilde{\rho}$ stand for the representation of $U_\chi(\mathfrak{g}_\Bbbk)$ in $\operatorname{End}_\Bbbk V''$. As $N_{\chi,\Bbbk}$ annihilates $V_0' \otimes 1 \subseteq V''$, it follows from (17) that

$$0 \,=\, \tilde{\rho}(\bar{u}_i)(v' \otimes 1) \,=\, \tilde{\rho}(\bar{\varphi}_\chi(\bar{u}_i))(v' \otimes 1) \,=\, \textstyle\sum_{\mathbf{c} \in C} \bar{X}^{\mathbf{c}} \otimes \rho(\bar{h}_{i,\mathbf{c}})(v')$$

for all $v' \in V_0'$. As the nonzero vectors of the form $\bar{X}^{\mathbf{c}} \otimes \rho(\bar{h}_{i,\mathbf{c}})(v')$ with v' fixed are linearly independent by our assumption on A, we see that $\rho(\bar{h}_{i,\mathbf{c}}) = 0$ for all $\mathbf{c} \in C$ and all $i \in I$. This completes the proof. $\square$

4.6. By our discussion in (2.3), there are polynomials $H_{i,\mathbf{c}} \in A[X_1, \dots, X_r]$ such that $h_{i,\mathbf{c}} = H_{i,\mathbf{c}}(\Theta_1, \dots, \Theta_r)$ for all $\mathbf{c} \in C$ and $i \in I$. Let $\mathcal{I}_\mathcal{W}$ be the two-sided ideal of $U(\mathfrak{g}, e)$ generated by the $h_{i,\mathbf{c}}$'s. In view of (2) and [34, Lemma 4.1], the algebra $U(\mathfrak{g}, e)/\mathcal{I}_\mathcal{W}$ is isomorphic to the quotient of the free associative algebra $\mathbb{C}\langle X_1, \dots, X_r\rangle$ by its two-sided ideal generated by all elements $[X_i, X_j] - F_{ij}(X_1, \dots, X_r)$ with $1 \leq i < j \leq r$ and all elements $H_{\mathbf{c},l}(X_1, \dots, X_r)$ with $\mathbf{c} \in C$ and $l \in I$. Given a natural number d we denote by $\mathcal{M}_d$ the set of all r-tuples $(M_1, \dots, M_r) \in \operatorname{Mat}_d(\mathbb{C})^r$ satisfying the relations

$$\begin{aligned} [M_i, M_j] - F_{ij}(M_1, \dots, M_r) &= 0 \qquad (1 \leq i < j \leq r) \\ H_{\mathbf{c},l}(M_1, \dots, M_r) &= 0 \qquad (\mathbf{c} \in C,\ l \in I) \end{aligned}$$

(the monomials in $M_1, \dots, M_r$ involved in $F_{ij}(M_1, \dots, M_r)$ and $H_{\mathbf{c},l}(M_1, \dots, M_r)$ are evaluated by using the matrix product in $\operatorname{Mat}_d(\mathbb{C})$). The preceding remark shows that $\mathcal{M}_d$ is nothing but the variety of all matrix representations of degree d of the algebra $U(\mathfrak{g}, e)/\mathcal{I}_\mathcal{W}$.

Lemma 4.4. *The set $\pi(A)$ of all primes p such that $A/\mathfrak{P} \cong \mathbb{F}_p$ for some $\mathfrak{P} \in \operatorname{Specm} A$ is infinite for any finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}$.*

Proof. By Hilbert's Nullstellensatz, there is an algebra homomorphism $A \to \overline{\mathbb{Q}}$. Thus, in proving the lemma we may assume that $A \subset \overline{\mathbb{Q}}$. Then A is a finitely generated $\mathbb{Z}$-subalgebra of an algebraic number field $K = \mathbb{Q}[X]/(f)$, where $f \in \mathbb{Z}[X]$ is a polynomial of positive degree irreducible over $\mathbb{Q}$. Then $A \subseteq \mathbb{Z}[b^{-1}][X]/(f)$ for some $b \in \mathbb{Z}^\times$. Since b has only finitely many prime divisors, we may assume without loss of generality that $A = \mathbb{Z}[X]/(f)$ and $\deg f > 1$.

Given $x \in \mathbb{R}$ denote by $\pi(x)$ the number of primes $\leq x$. If p is a prime, let $N_p(f)$ be the number of zeros of f in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. As explained in [36], for instance, it follows from Burnside's Lemma and Chebotarev's Density Theorem that

$$\lim_{x \to \infty} \frac{\sum_{p \leq x} N_p(f)}{\pi(x)} \,=\, 1. \tag{18}$$

Because $A = \mathbb{Z}[X]/(f)$, the set $\pi(A)$ consists of all primes p with $N_p(f) \neq 0$. In view of (18) this implies that $|\,\pi(A)| = \infty$. $\square$

4.7. Let $\mathbf{J}_d$ be the ideal of $\mathcal{P} := \mathbb{C}\,[x_{ab}^{(k)} \mid 1 \le a,b \le d,\ 1 \le k \le r]$ generated by the matrix coefficients of all $[M_i, M_j] - F_{ij}(M_1, \dots, M_r)$ and $H_{\mathbf{c},l}(M_1, \dots, M_r)$, where M_k is the generic matrix $\big(x_{ab}^{(k)}\big)_{1\le a,b\le d}$. Note that $\mathcal{M}_d$ is nothing but the zero locus of $\mathbf{J}_d$ in $\operatorname{Specm}\mathcal{P} = \mathbb{A}^{rd^2}(\mathbb{C})$. In particular, $\mathcal{M}_d$ as a Zariski closed subset of $\mathbb{A}^{rd^2}(\mathbb{C})$. As all F_{ij} and $H_{\mathbf{c},l}$ are in $A[X_1, \dots, X_r]$, the ideal $\mathbf{J}_d$ is generated by a finite set of polynomials in $\mathcal{P}_A = A[x_{ab}^{(k)} \mid 1 \le a,b \le d,\ 1 \le k \le r]$, say $\{f_1, \dots, f_N\}$. Given $g \in \mathcal{P}_A$ and an algebra homomorphism $\nu\colon A \to \mathbb{F}_p$, we write ${}^{\nu}g$ for the image of g in $\mathcal{P}_A \otimes_A (A/\operatorname{Ker}\nu) \subset \mathcal{P}_A \otimes_A \overline{\mathbb{F}}_p$ and denote by $\mathcal{M}_d(\overline{\mathbb{F}}_p)$ the zero locus of ${}^{\nu}f_1, \dots, {}^{\nu}f_N$ in $\mathbb{A}^{rd^2}(\overline{\mathbb{F}}_p)$.

Proposition 4.2. *The algebra $U(\mathfrak{g},e)/\mathfrak{I}_{\mathcal{W}}$ has an irreducible representation of dimension at most $D = D(\mu)$.*

Proof. We need to show that $\mathcal{M}_d(\mathbb{C}) \neq \varnothing$ for some $d \le D$. Suppose this is not the case. Then $g_1 f_1 + \dots + g_N f_N = 1$ for some $g_1, \dots, g_N \in \mathcal{P}$. Let B be the A-subalgebra of $\mathbb{C}$ generated by the coefficients of $g_1', \dots, g_N'$. By Lemma 4.4, the set $\pi(B)$ is infinite. Take $p \in \pi(B)$ and let $\nu\colon B \twoheadrightarrow \mathbb{F}_p$ be an algebra map such that $B/\operatorname{Ker}\nu \cong \mathbb{F}_p$. Denote by $\tilde{\nu}$ the composite

$$\mathcal{P}_B \twoheadrightarrow \mathcal{P}_B/(\operatorname{Ker}\nu)\mathcal{P}_B \hookrightarrow \overline{\mathbb{F}}_p[x_{ab}^{(k)} \mid 1 \le a,b \le d,\ 1 \le k \le r] \cong \mathcal{P}_A \otimes_A \overline{\mathbb{F}}_p.$$

Since $\tilde{\nu}(F) = {}^{\nu}F$ for all $F \in \mathcal{P}_B$, we have that ${}^{\nu}g_1\,{}^{\nu}f_1 + \dots + {}^{\nu}g_N\,{}^{\nu}f_N = 1$. But then $\mathcal{M}_d(\overline{\mathbb{F}}_p) = \varnothing$ for all $d \le D$. Since this contradicts Proposition 4.1, we conclude that $\mathcal{M}_d(\mathbb{C}) \neq \varnothing$ for some $d \le D$. □

We are ready to prove the main results of this section.

Theorem 4.2. *For any primitive ideal $\mathfrak{I}$ of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathfrak{I}) = \overline{\mathcal{O}}_\chi$ there is a finite dimensional irreducible $U(\mathfrak{g},e)$-module V such that $\mathfrak{I} = \operatorname{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{U(\mathfrak{g},e)} V)$.*

Proof. By Proposition 4.2, there is an irreducible finite dimensional representation $\rho\colon U(\mathfrak{g},e) \to \operatorname{End} V$ such that $\mathfrak{I}_{\mathcal{W}} \subseteq \operatorname{Ker}\rho$. Associated with ρ is a representation of $U(\mathfrak{g})$ in $\operatorname{End}\big(Q_\chi \otimes_{U\mathfrak{g},e)} V\big)$; call it $\tilde{\rho}$. It follows from Skryabin's theorem [39] and [33, Thm. 3.1(ii)] that $\operatorname{Ker}\tilde{\rho}$ is a primitive ideal of $U(\mathfrak{g})$ with $\mathcal{VA}(\operatorname{Ker}\tilde{\rho}) = \overline{\mathcal{O}}_\chi$. From (16) it follows that

$$\tilde{\rho}(u_i)(1_\chi \otimes v) \;=\; \tilde{\rho}(\varphi_\chi(u_i))(1_\chi \otimes v) \;=\; \textstyle\sum_{\mathbf{c}\in C} X^{\mathbf{c}} \otimes \rho(h_{i,\mathbf{c}})(v)$$

for all $v \in V$ and $i \in I$. Since $\mathfrak{I}_{\mathcal{W}} \subseteq \operatorname{Ker}\rho$, all $\tilde{\rho}(u_i)$ annihilate $1_\chi \otimes V \subset \widetilde{V}$. Since $1_\chi \otimes V$ generates the $\mathfrak{g}$-module $Q_\chi \otimes_{U(\mathfrak{g},e)} V$ and the span of the u_i's is stable under the adjoint action of $\mathfrak{g}$, we have that $u_i \in \operatorname{Ker}\tilde{\rho}$ for all $i \in I$. Since the u_i's generate the ideal $\mathfrak{I}$, it must be that $\mathfrak{I} \subseteq \operatorname{Ker}\tilde{\rho}$. Since the primitive ideals $\mathfrak{I}$ and $\operatorname{Ker}\tilde{\rho}$ have the same associated variety, applying [3, Korollar 3.6] gives $\mathfrak{I} = \operatorname{Ker}\tilde{\rho}$. □

4.8. A more invariant definition of the algebra $U(\mathfrak{g},e)$ was given by Gan–Ginzburg in [13]. Let $\mathfrak{n}_\chi = \bigoplus_{i\le -1} \mathfrak{g}(i)$ and $\mathfrak{n}_\chi' := \bigoplus_{i\le -2} \mathfrak{g}(i)$, and denote by $\widehat{Q}_\chi$ the Kazhdan-filtered $\mathfrak{g}$-module $U(\mathfrak{g})/U(\mathfrak{g})N_\chi'$, where N_χ' is the left ideal of $U(\mathfrak{g})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{n}_\chi'$. Note that $\widehat{Q}_\chi$ is a $U(\mathfrak{n}_\chi)$-bimodule and $\widehat{Q}_\chi^{\operatorname{ad}\mathfrak{n}_\chi}$ carries a natural algebra structure. By [13], the algebra $\widehat{Q}_\chi^{\operatorname{ad}\mathfrak{n}_\chi}$ is canonically isomorphic to $U(\mathfrak{g},e)$. Denote by $\widehat{\varphi}_\chi$ and $\varphi_{\mathfrak{m}}$ the canonical projections $U(\mathfrak{g}) \twoheadrightarrow \widehat{Q}_\chi$ and $\widehat{Q}_\chi \twoheadrightarrow Q_\chi$,

respectively. The adjoint action of G on $U(\mathfrak{g})$ gives rise to a rational action of the reductive part $C(e) = G_e \cap G_f$ of the centraliser G_e on $\widehat{Q}_\chi$. Clearly, the $\mathfrak{g}$-module map $\widehat{\varphi}_\chi$ is $C(e)$-equivariant and $\varphi_\mathfrak{m} \circ \widehat{\varphi}_\chi = \varphi_\chi$.

Recall from (2.1) the Witt basis $\{z'_1, \dots, z'_s, z_1, \dots, z_s\}$ of $\mathfrak{g}(-1)$ and write $Z'^{\mathbf{b}}$ for the monomial $z_1'^{b_1} \cdots z_s'^{b_s} \in U(\mathfrak{g})$, where $\mathbf{b} = (b_1, \dots, b_s) \in \mathbb{Z}_+^s$. Let $\hat{1}_\chi$ be the image of 1 in $\widehat{Q}_\chi$. Arguing as in [39] it is easy to observe that the monomials $X^{\mathbf{a}} Z'^{\mathbf{b}}(\hat{1}_\chi)$ with $\mathbf{a} \in \mathbb{Z}_+^{d(e)}$ and $\mathbf{b} \in \mathbb{Z}_+^s$ form a free basis of the right $U(\mathfrak{g}, e)$-module $\widehat{Q}_\chi$. Note that for any $h_{\mathbf{a},\mathbf{b}} \in U(\mathfrak{g}, e)$ we have that $\widehat{\varphi}_\chi\big(X^{\mathbf{a}} Z'^{\mathbf{b}} h_{\mathbf{a},\mathbf{b}}(\hat{1}_\chi)\big) = X^{\mathbf{a}} h_{\mathbf{a},\mathbf{b}}(1_\chi)$ if $\mathbf{b} = \mathbf{0}$ and 0 otherwise.

Lemma 4.5. *Let M be any $U(\mathfrak{g}, e)$-module and $u \in \operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g}, e)} M\big)$. Then $\widehat{\varphi}_\chi(u) = \sum_{\mathbf{a},\mathbf{b}} X^{\mathbf{a}} Z'^{\mathbf{b}} h_{\mathbf{a},\mathbf{b}}(\hat{1}_\chi)$ for some $h_{\mathbf{a},\mathbf{b}} \in \operatorname{Ann}_{U(\mathfrak{g}, e)} M$.*

Proof. Set $\Omega(u) = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^{d(e)} \times \mathbb{Z}_+^s \mid h_{\mathbf{a},\mathbf{b}} \notin \operatorname{Ann}_{U(\mathfrak{g}, e)} M\}$ and denote by $\Omega_{\max}(u)$ the set of all $(\mathbf{a}, \mathbf{b}) \in \Omega(u)$ for which the Kazhdan degree of $X^{\mathbf{a}} \in U(\mathfrak{g})$ is maximal possible. Suppose $\Omega_{\max} \neq \varnothing$ and denote bt $\Delta(u)$ the set of all $\mathbf{b} \in \operatorname{pr}_2(\Omega_{\max}(u))$ for which $h_{\mathbf{a},\mathbf{b}} \notin \operatorname{Ann}_{U(\mathfrak{g}, e)} M$ (here pr_2 is the second projection $\mathbb{Z}_+^{d(e)} \times \mathbb{Z}_+^s \twoheadrightarrow \mathbb{Z}_+^s$). Order the elements in $\mathbb{Z}_+^s$ lexicographically and denote by $\mathbf{m}$ the largest element in $\Delta(u)$. Let $\mathbf{a}_1, \dots, \mathbf{a}_l$ be all elements in $\mathbb{Z}_+^{d(e)}$ for which $(\mathbf{a}_i, \mathbf{m}) \in \Omega_{\max}(u)$.

Set $u' := \prod_{i=1}^s (\operatorname{ad} z_i)^{m_i}(u)$, an element of $\operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g}, e)} M\big)$. Since $\widehat{Q}_\chi$ is a Kazhdan-filtered $\mathfrak{g}$-module, we have that $\Omega_{\max}(u') \subseteq \Omega_{\max}(u)$, while it is immediate from the definition of $\{\mathbf{a}_1, \dots, \mathbf{a}_l\}$ that $(\mathbf{a}_i, \mathbf{0}) \in \Omega_{\max}(u')$ for all $i \leq l$. Furthermore,

$$\widehat{\varphi}_\chi(u') = \sum_{i=1}^l \lambda_i X^{\mathbf{a}_i} h_{\mathbf{a}_i,\mathbf{m}}(\hat{1}_\chi) + \sum_{(\mathbf{a},\mathbf{b}) \notin \Omega_{\max}(u')} X^{\mathbf{a}} Z'^{\mathbf{b}} h'_{\mathbf{a},\mathbf{b}}(\hat{1}_\chi), \qquad \lambda_i \neq 0,$$

by our choice of $\mathbf{m}$. Hence $(\varphi_\mathfrak{m} \circ \hat{\varphi}_\chi)(u') = \sum_{i=1}^l \lambda_i X^{\mathbf{a}_i} h_{\mathbf{a}_i,\mathbf{m}}(1_\chi) + \sum_{\mathbf{a} \neq \mathbf{a}_0} X^{\mathbf{a}} h'_{\mathbf{a}}(1_\chi)$ for some $h'_{\mathbf{a}} \in U(\mathfrak{g}, e)$. As $h_{\mathbf{a}_i,\mathbf{m}} \notin \operatorname{Ann}_{U(\mathfrak{g}, e)} M$ and $\lambda_i \neq 0$ for all $i \leq l$, we obtain $u' \notin \operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g}, e)} M\big)$, a contradiction. This completes the proof. □

Corollary 4.1. *Let M be as in Lemma 4.5 and denote by $\mathfrak{I}_M$ the $U(\mathfrak{g})$-submodule of $\widehat{Q}_\chi$ generated by $\operatorname{Ann}_{U(\mathfrak{g}, e)} M \subseteq \widehat{Q}_\chi^{\operatorname{ad} \mathfrak{n}_\chi}$. Then*

$$\operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g}, e)} M\big) = \bigcap_{g \in G} (\operatorname{Ad} g)\big(\widehat{\varphi}_\chi^{-1}(\mathfrak{I}_M)\big).$$

Proof. Let $I = \operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g}, e)} M\big)$ and $I' = \bigcap_{g \in G} (\operatorname{Ad} g)\big(\widehat{\varphi}_\chi^{-1}(\mathfrak{I}_M)\big)$. It follows from Lemma 4.5 that $I \subseteq \widehat{\varphi}_\chi^{-1}(\mathfrak{I}_M)$. Since I is a two-sided ideal of $U(\mathfrak{g})$, it is invariant under the adjoint action of G. Hence $I \subseteq I'$. On the other hand, I' is a left ideal of $U(\mathfrak{g})$ contained in $\widehat{\varphi}_\chi^{-1}(\mathfrak{I}_M)$ and invariant under the adjoint action of G. Therefore, I' is $(\operatorname{ad} \mathfrak{g})$-stable and annihilates the subspace $1_\chi \otimes M$ of $Q_\chi \otimes_{U(\mathfrak{g}, e)} M$ (one should keep in mind that $\mathfrak{n}'_\chi \subseteq \mathfrak{m}$). Since the latter generates the $\mathfrak{g}$-module $Q_\chi \otimes_{U(\mathfrak{g}, e)} M$, we deduce that $I = I'$. □

Since $C(e)$ stabilises both $\mathfrak{n}_\chi$ and $\mathfrak{n}'_\chi$, it acts on $U(\mathfrak{g}, e) = \widehat{Q}_\chi^{\operatorname{ad} \mathfrak{n}_\chi}$ as algebra automorphisms; see [33, 2.1] for more detail. Thus, we can twist the module structure $U(\mathfrak{g}, e) \times M \to M$ of any $U(\mathfrak{g}, e)$-module M by an element $g \in C(e)$ to obtain a new $U(\mathfrak{g}, e)$-module, M^g, with underlying vector space M and the $U(\mathfrak{g}, e)$-action given by

$u \cdot m = g(u) \cdot m$ for all $u \in U(\mathfrak{g}, e)$ and $m \in M$. Since the map $\widehat{\varphi}_\chi$ is $C(e)$-equivariant and $g(\mathfrak{I}_M) = \mathfrak{I}_{M^g}$, it follows from Lemma 4.1 that

$$\operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},\, e)} M\big) \;=\; \operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},\, e)} M^g\big) \qquad \big(\forall\, g \in C(e)\big). \tag{19}$$

4.9. As explained in [31, 6.2] and [33, p. 524], the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ maps isomorphically onto the centre of $U(\mathfrak{g}, e)$. Thus, we may identify $Z(\mathfrak{g})$ with the centre of $U(\mathfrak{g}, e)$. Given an algebra map $\lambda\colon\, Z(\mathfrak{g}) \to \mathbb{C}$ denote by $\operatorname{Irr}_\lambda U(\mathfrak{g}, e)$ the set of all isoclasses of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules with central character λ. As we recalled in (4.8), the reductive part $C(e) = G_e \cap G_f$ of the centraliser G_e acts on $U(\mathfrak{g}, e)$ as algebra automorphisms. Since $\operatorname{Ad} G$ acts trivially on $Z(\mathfrak{g})$, the group $C(e)$ acts on each set $\operatorname{Irr}_\lambda U(\mathfrak{g}, e)$.

By [33, Sect. 2], the Lie algebra $\mathfrak{g}_e(0)$ of $C(e)$ embeds into $U(\mathfrak{g}, e)$ in such a way that the adjoint action of $\mathfrak{g}_e(0) \subset U(\mathfrak{g}, e)$ on $U(\mathfrak{g}, e)$ coincides with the differential of the above-mentioned action of $C(e)$ on $U(\mathfrak{g}, e)$. This implies that twisting the module structure $U(\mathfrak{g}, e) \times M \to M$ of a finite dimensional $U(\mathfrak{g}, e)$-module M by an element of the connected component of $C(e)$ does not affect the isomorphism type of M. We thus obtain, for any $d \in \mathbb{N}$, a natural action of the component group $\Gamma(e) = G_e/G_e^\circ \;\cong\; C(e)/C(e)^\circ$ on the set of all isoclasses of d-dimensional $U(\mathfrak{g}, e)$-modules. By the same token, $\Gamma(e)$ acts on each set $\operatorname{Irr}_\lambda U(\mathfrak{g}, e)$.

Let $\mathfrak{X}$ be the primitive spectrum of $U(\mathfrak{g})$ and denote by $\mathfrak{X}^\lambda$ the set of all $I \in \mathfrak{X}$ with $I \cap Z(\mathfrak{g}) = \operatorname{Ker} \lambda$. Given a coadjoint nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ we write $\mathfrak{X}_\mathcal{O}$ for the set of all $I \in \mathfrak{X}$ with $\mathcal{VA}(I) = \overline{\mathcal{O}}$, and set $\mathfrak{X}^\lambda_\mathcal{O} := \mathfrak{X}^\lambda \cap \mathfrak{X}_\mathcal{O}$. It follows from Theorem 4.2 that for any algebra homomorphism $\lambda\colon\, Z(\mathfrak{g}) \to \mathbb{C}$ the map

$$\psi_\lambda\colon\, \operatorname{Irr}_\lambda U(\mathfrak{g}, e) \longrightarrow\; \mathfrak{X}^\lambda_{\mathcal{O}(\chi)}, \qquad [V] \longmapsto\; \operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},\, e)} V\big)$$

is surjective (here $[V]$ stands for the isomorphism class of a $U(\mathfrak{g}, e)$-module V and $\mathcal{O}(\chi) = (\operatorname{Ad}^* G)\chi$). For any finite dimensional irreducible $U(\mathfrak{g}, e)$-module M, the $\mathfrak{g}$-modules $Q_\chi \otimes_{U(\mathfrak{g},\, e)} M^g$, where $g \in C(e)$, have the same annihilator in $U(\mathfrak{g})$; see (19). From this it is immediate that that all fibres of ψ_λ are $\Gamma(e)$-stable.

In his talk at the MSRI workshop on Lie Theory in March 2008, the author conjectured that $\Gamma(e)$ acts transitively on the fibres of ψ_λ; that is, the fibres of ψ_λ are precisely the $\Gamma(e)$-orbits in $\operatorname{Irr}_\lambda U(\mathfrak{g}, e)$. This conjecture was known to hold in some special cases; see [34] and [8]. Very recently, the author's conjecture was proved in full generality by Losev; see [25, Thm. 1.2.2]. We would like to finish this paper by putting on record the following interesting consequence of Losev's result. For $\mathfrak{g}$ semisimple, it solves an old problem posed by Borho and Dixmier in the early 70s; see [11, Problem 2].

Theorem 4.3. *For any complex semisimple Lie algebra $\mathfrak{g}$ the primitive spectrum of $U(\mathfrak{g})$ is a countable disjoint union of quasi-affine algebraic varieties.*

Proof. Let $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ be the simple ideals of the Lie algebra $\mathfrak{g}$. Let I be a primitive ideal of $\mathfrak{g}$ and set $I_j := I \cap U(\mathfrak{g}_j)$, where $1 \le j \le k$. Since I is the annihilator in $U(\mathfrak{g})$ of a simple highest weight module, by Duflo's Theorem, it is straightforward to see that each I_j is a primitive ideal of $U(\mathfrak{g}_j)$ and $I = \sum_{j=1}^k U(\mathfrak{g}) I_j$. From this it is immediate that the primitive spectrum of $U(\mathfrak{g})$ is the direct product of the primitive spectra of the $U(\mathfrak{g}_j)$'s. Thus, in proving the theorem we may assume that $\mathfrak{g}$ is simple.

Since there are finitely many coadjoint nilpotent orbits in $\mathfrak{g}^*$, it suffices to show that $\mathfrak{X}_{\mathcal{O}(\chi)}$ is a countable union of quasi-affine algebraic varieties.

The group $\mathrm{GL}(d)$ acts on $\mathrm{Mat}_d(\mathbb{C})^r$ by simultaneous conjugations and preserves its Zariski closed subset $\mathcal{M}_d$ defined in (4.6). Since $\mathrm{GL}(d)$ is a reductive group, the invariant algebra $\mathbb{C}[\mathcal{M}_d]^{\mathrm{GL}(d)}$ is finitely generated and the $\mathbb{C}$-points of the affine variety $R_d := \mathrm{Specm}\big(\mathbb{C}[\mathcal{M}_d]^{\mathrm{GL}(d)}\big)$ parametrise the closed $\mathrm{GL}(d)$-orbits in $\mathcal{M}_d$. Moreover, the morphism $\pi_d\colon\ \mathcal{M}_d \to R_d$ induced by inclusion $\mathbb{C}[\mathcal{M}_d]^{\mathrm{GL}(d)} \hookrightarrow \mathbb{C}[\mathcal{M}_d]$ is surjective and takes the $\mathrm{GL}(d)$-stable closed subsets of $\mathcal{M}_d$ to closed subsets of R_d; see [23, Ch. II, 3.2], for example. It follows from Procesi's results on invariants of r-tuples of $d \times d$ matrices that the closed $\mathrm{GL}(d)$-orbits in $\mathcal{M}_d$ are in 1-1 correspondence with the equivalence classes of *semisimple* d-dimensional matrix representations of $U(\mathfrak{g}, e)$. Thus, the $\mathbb{C}$-points of R_d can be identified with the isoclasses of semisimple d-dimensional $U(\mathfrak{g}, e)$-modules.

It is well-known (and easily seen) that the set of all reducible d-dimensional matrix representations of $U(\mathfrak{g}, e)$ is Zariski closed in $\mathcal{M}_d$. Since this set is also $\mathrm{GL}(d)$-stable, the above-mentioned properties of π_d show that the subset $\mathrm{Irr}_d \subseteq R_d$ consisting of the isoclasses of *irreducible* d-dimensional $U(\mathfrak{g}, e)$-modules is Zariski open in R_d. As we mentioned earlier, the component group $\Gamma(e)$ acts on R_d and preserves its open subset Irr_d. As $\Gamma(e)$ is a finite group, the quotient space $R_d/\Gamma(e)$ is an affine variety (the coordinate algebra of $R_d/\Gamma(e)$ is nothing but the invariant algebra $\mathbb{C}[R_d]^{\Gamma(e)}$). Furthermore, the quotient morphism $\pi_{\Gamma(e)}\colon\ R_d \twoheadrightarrow R_d/\Gamma(e)$ is open in the Zariski topology. Since Irr_d is open in R_d, the set $\pi_{\Gamma(e)}(\mathrm{Irr}_d) = \mathrm{Irr}_d/\Gamma(e)$ is open in $R_d/\Gamma(e)$.

Thus, each orbit space $\mathrm{Irr}_d/\Gamma(e)$ is a quasi-affine variety. On the other hand, it follows from Theorem 4.2 and [25, Thm. 1.2.2] that there is a bijection

$$\bigsqcup_{d\geq 0} \big(\mathrm{Irr}_d/\Gamma(e)\big) \xrightarrow{\sim} \mathfrak{X}_{\mathcal{O}(\chi)}.$$

Since this holds for any coadjoint nilpotent orbit in $\mathfrak{g}^*$, the primitive spectrum of $U(\mathfrak{g})$ is a countable union of quasi-affine algebraic varieties. □

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School of Mathematics, University of Manchester, Oxford Road, M13 9PL, UK
E-mail address: sashap@maths.man.ac.uk