STABLE MODIFICATION OF RELATIVE CURVES

MICHAEL TEMKIN

Abstract. We generalize theorems of Deligne-Mumford and de Jong on semi-stable modifications of families of proper curves. The main result states that after a generically étale alteration of the base any (not necessarily proper) family of multipointed curves with semi-stable generic fiber admits a minimal semi-stable modification. The latter can also be characterized by the property that its geometric fibers have no certain exceptional components. The main step of our proof is uniformization of one-dimensional extensions of valued fields. Riemann-Zariski spaces are then used to obtain the result over any integral base.

1. Introduction

1.1. **The motivation.** The stable reduction theorem of Deligne-Mumford [DM, 2.7] states that for any smooth projective curve C over the fraction field K of a discrete valuation ring R there exists a finite separable extension L of K such that $C \otimes_K L$ can be extended to a stable curve over the integral closure of R in L. This theorem plays a key role in the proof of properness of the moduli space of stable n-pointed curves of genus g. In its turn, the latter implies the following generalization of the Deligne-Mumford theorem (stable extension theorem): for any proper stable curve C over an open dense subscheme of a quasi-compact quasi-separated integral scheme S there exists an alteration $S' \to S$ such that $C \times_S S'$ can be extended to a proper stable curve over S' (see [Del, 1.6]).

A stronger semi-stable modification theorem was proved by de Jong in [dJ]: for any proper curve C over an integral quasi-compact excellent scheme S there exist an alteration $S' \to S$ and a modification $C' \to C \times_S S'$ such that C' is a proper semi-stable curve over S'. De Jong's proof is also based on existence and properness of the moduli spaces. Naturally, de Jong's theorem leads to the following two questions. Is it true that the same result takes place for not necessarily proper curves C over S? (Of course, in that case C' is not required to be proper over S'.) And, is it true that there is a minimal semi-stable modification?

The main result of the current paper is stable modification theorem formulated in §1.2. This theorem strengthens de Jong's theorem in a few aspects; in particular, it answers affirmatively both above questions. In addition, our work is not based

Key words and phrases. Stable modification, stable reduction, relative curves.

This article is based on a part of my Ph.D. thesis, and I want to thank my advisor Prof. V. Berkovich. I am absolutely indebted to B. Conrad and I owe a lot to A. Ducros for pointing out various mistakes and inaccuracies, and for many suggestions that led to two revisions of the paper and improved the exposition drastically. I express my deep gratitude to Israel Clore Foundation for supporting my doctoral studies and to Max Planck Institute for Mathematics where a part of this paper was written. A final revision was made when the author was staying at IAS and supported by NFS grant DMS-0635607.

on [DM] and [dJ], and we thereby reprove their results including the stable reduction theorem. The main ingredient in proving the stable modification theorem are Theorems 2.1.8 and 6.3.1 on uniformization of valued fields. These Theorems are of their own interest; in particular, Theorem 6.3.1 is used in a subsequent work [Tem4] to establish inseparable local uniformization of varieties of positive characteristic. We refer to Remark 1.3.1 for more comments on the connections between stable reduction and uniformization of valued fields.

1.2. **The main results.** To formulate our main results we have to first introduce some terminology. Let S be a scheme. A *multipointed S-curve* (C,D) consists of flat finitely presented morphisms $C\to S$ and $D\to S$ of pure relative dimensions one and zero, respectively, and of a closed immersion $D\to C$ over S (the subscheme D may be empty). Note that C is not assumed to be S-proper or even S-separated. A *morphism* $f:(C',D')\to(C,D)$ is a compatible pair of S-morphisms $f_C:C'\to C$ and $f_D:D'\to D$. A *modification of* (C,D) is a morphism f in which both f_C and f_D are *modifications*, i.e. proper morphisms inducing isomorphisms between schematically dense open subschemes. Furthermore, a multipointed S-curve (C,D) is said to be *semi-stable* if $\phi:C\to S$ is *semi-stable* (i.e. ϕ is flat and its geometric fibers have at most ordinary double points as singularities), $D\to S$ is étale, and D is disjoint from the non-smoothness locus of $C\to S$. Note that (C,D) is semi-stable if and only if all its geometric fibers $(C_{\overline{s}},D_{\overline{s}})$ are semi-stable multipointed $\overline{s}$-curves. Indeed, obviously $C\to S$ is semi-stable if and only if its geometric fibers are so, and by the fiber criterion of étaleness the flat morphism $D\to S$ is étale if and only if its geometric fibers are étale.

A *semi-stable modification of* (C,D) is a modification $f:(C',D')\to(C,D)$ in which (C',D') is semi-stable. Finally, such a semi-stable modification is said to be *stable* if for any geometric point $\overline{s}\to S$ the fiber $C'_{\overline{s}}$ has no *exceptional components*, i.e. irreducible components Z which are isomorphic to the projective line, have at most two points of intersection with $D'_{\overline{s}}\cup(C'_{\overline{s}})_{sing}$, and are contracted to a point in $C_{\overline{s}}$.

Now we are going to introduce a sheaf $\omega_{C/S}$. Similarly to [DM] we will use Grothendieck's duality theory for the sake of speeding things up, though one can define $\omega_{C/S}$ in a lengthier but much more elementary way similarly to [Har, III.7.11] (after a Zariski localization, realize C as a complete intersection in $X=\mathbf{A}^n_S$ via $i:C\hookrightarrow X$, then set $\omega_{C/S}=\mathcal{E}xt^{n-1}_X(i_*(\mathcal{O}_C),\Omega^n_{X/S})$ and compute it explicitly by use of the Koszul complex via the ideal of C in $\mathcal{O}_X$). Any semi-stable curve $\phi:C\to S$ is a relative locally complete intersection, so if ϕ is separated then the complex $\phi^!(\mathcal{O}_S)$ has a unique non-zero cohomology sheaf which is invertible and is the dualizing sheaf. We denote the latter sheaf as $\omega_{C/S}$ and note that its definition is local on C, since the non-zero sheaf of $\phi^!(\mathcal{O}_S)$ is local on the source for any CM morphism (see [Con2, p.157]). Therefore, the definition of $\omega_{C/S}$ extends to non-separated semi-stable curves as well. The following well known properties of $\omega_{C/S}$ (mentioned, for example, in [DM, §1] in the case of proper ϕ) allow to compute the geometric fibers: (a) the sheaves $\omega_{C/S}$ are compatible with the base changes $S'\to S$, (b) if $S=\mathrm{Spec}(k)$ for an algebraically closed field k, $\pi:\widetilde{C}\to C$ is the normalization and $E=\pi^{-1}(C_{\mathrm{sing}})$ then $\omega_{C/S}$ is the subsheaf of $\pi_*(\omega_{\widetilde{C}/S}(E))=\pi_*(\Omega^1_{\widetilde{C}/S}(E))$ given by the conditions $\mathrm{Res}_{x_i}(\nu)=\mathrm{Res}_{y_i}(\nu)$ for all pairs of different points $x_i,y_i\in E$ with common image in C. If (C,D) is semi-stable then D is a Cartier divisor and we set $\omega_{(C,D)/S} := \omega_{C/S}(D)$. It is well known that for a proper semi-stable n-pointed S-curve (C,D) with geometrically connected fibers stability (over S) is equivalent to S-ampleness of the sheaf $\omega_{(C,D)/S}$. (This fact is used for constructing moduli spaces of n-pointed nodal curves.) We now prove an analog of this ampleness result for stable modifications by a slight adjustment of the classical proof.

Theorem 1.1 (Stable Modification Theorem: projectivity)**.** *For any multipointed S-curve (C,D) with a semi-stable modification (C',D'), the sheaf $\omega_{(C',D')/S}$ is C-ample if and only if the modification is stable. In particular, for any stable modification $(C_{\mathrm{st}},D_{\mathrm{st}}) \to (C,D)$ the modification $C_{\mathrm{st}} \to C$ is projective.*

Proof. Assume that the modification is stable. Since relative ampleness can be checked separately on each fiber by [EGA, IV$_3$, 9.6.5], it suffices to establish ampleness of the restriction of $\omega = \omega_{(C',D')/S}$ onto the fiber $Z := C'_x$ over a point $x \in C$. It is enough to consider the case when x is closed in its S-fiber because otherwise Z is zero-dimensional and so $\omega|_Z$ is ample. If $\overline{s}$ is a geometric point over $s = \phi(x)$ then $(C'_{\overline{s}}, D'_{\overline{s}}) \to (C_{\overline{s}}, D_{\overline{s}})$ is a proper morphism with semi-stable source and such that its fibers do not contain exceptional irreducible components. Also, $\overline{\omega} := \omega_{(C_{\overline{s}},D_{\overline{s}})/\overline{s}}$ is the pullback of ω by compatibility with base changes (property (a) above). Ampleness of the restriction of ω onto $Z \hookrightarrow C'_s$ is equivalent to ampleness of the restriction of $\overline{\omega}$ onto $\overline{Z} \hookrightarrow C'_{\overline{s}}$ where $\overline{Z}$ is the preimage of Z viewed as a reduced scheme.

Now, we have a reduced $\overline{s}$-proper subscheme $\overline{Z} \subset C'_{\overline{s}}$ not containing exceptional components of $(C'_{\overline{s}}, D'_{\overline{s}})$ and it suffices to show that $\overline{\omega}|_{\overline{Z}}$ is ample. Obviously, we can deal with connected components independently, so we can assume that $\overline{Z}$ is connected. Only the case when $\overline{Z}$ is a curve should be treated and then $\overline{Z}$ is a proper connected semi-stable curve. Let $\overline{D} = \overline{Z} \cap D'_{\overline{s}}$ and let $\overline{E}$ be the union of points that are smooth in $\overline{Z}$ but are singular in $C'_{\overline{s}}$. The assumption that an irreducible component $V \subset \overline{Z}$ is not an exceptional component of $(C'_{\overline{s}}, D'_{\overline{s}})$ is equivalent to the fact that V is not an exceptional component (in the classical sense) of the n-pointed $\overline{s}$-curve $(\overline{Z}, \overline{D} \sqcup \overline{E})$. Thus, $(\overline{Z}, \overline{D} \sqcup \overline{E})$ is stable and hence $\omega_{(\overline{Z},\overline{D}\sqcup\overline{E})/\overline{s}}$ is ample by the classical computation in the theory of stable n-pointed curves. It remains to note that by the property (b) above $\overline{\omega}|_{\overline{Z}}$ is isomorphic to $\omega_{(\overline{Z},\overline{D})/\overline{s}}(\overline{E}) = \omega_{(\overline{Z},\overline{D}\sqcup\overline{E})/\overline{s}}$.

If the modification is not stable then there exists a geometric fiber $(C'_{\overline{s}}, D'_{\overline{s}})$ with an exceptional irreducible component $\overline{Z}$ contracted to a point $\overline{x} \in C_{\overline{s}}$. Let $x \in C$ be the image of $\overline{x}$. Since $\overline{\omega}|_{\overline{Z}}$ is not ample in this case (by the same classical computation), one can use the above reasoning to show that already the restriction of ω onto C'_x is not ample. Thus, ω is not C-ample. □

Other parts of our main result are given below and they will be proved in §5. Starting with this point we assume that S is integral quasi-compact and quasi-separated with generic point η. By an *η-modification* $(C',D') \to (C,D)$ of multipointed S-curves we mean a modification whose η-fiber $(C'_\eta, D'_\eta) \widetilde{\to} (C_\eta, D_\eta)$ is an isomorphism.

Theorem 1.2 (Stable Modification Theorem: uniqueness)**.** *Assume that S is normal and a multipointed S-curve (C,D) admits stable η-modification $f : (C_{\mathrm{st}},D_{\mathrm{st}}) \to (C,D)$. Then this modification is minimal in the sense that any semi-stable modification $(C',D') \to (C,D)$ goes through a unique S-morphism $(C',D') \to (C_{\mathrm{st}},D_{\mathrm{st}})$.*

Corollary 1.3. *In the situation of Theorem 1.2, the stable η-modification f is*

(i) unique up to unique isomorphism;
(ii) an isomorphism over the semi-stable locus of $(C, D) \to S$.

Corollary 1.4. *Assume that* (C, D) *admits a stable* η*-modification* $(C_{\rm st}, D_{\rm st})$ *and a semi-stable modification* $(\overline{C}, \overline{D})$. *Then there exists a finite modification* $S' \to S$ *such that* $(\overline{C}, \overline{D}) \times_S S' \to (C, D) \times_S S'$ *goes through a unique* S'*-morphism* $(\overline{C}, \overline{D}) \times_S S' \to (C_{\rm st}, D_{\rm st}) \times_S S'$.

Theorem 1.5 (Stable Modification Theorem: existence)**.** *For any multipointed* S*-curve* (C, D) *with semi-stable generic fiber* (C_η, D_η) *there exist a generically étale alteration* $S' \to S$ *and a stable* η*-modification* $(C'_{\rm st}, D'_{\rm st}) \to (C, D) \times_S S'$.

Corollary 1.6. *Assume that* S' *is normal in Theorem 1.5. Then the natural actions of the groups* $\mathrm{Aut}(S)$ *and* $\mathrm{Aut}_S((C, D)) \times \mathrm{Aut}_S(S')$ *on* $(C, D) \times_S S'$ *lift equivariantly to* $(C'_{\rm st}, D'_{\rm st})$.

1.3. **Overview.** Our proof of the stable modification theorem originates in non-Archimedean analytic geometry. Namely, the stable reduction theorem of Deligne-Mumford (in the form of Bosch-Lütkebohmert [BL1]) was used in [Ber1, §4.3] and [Ber2, §3.6] to give a local description of smooth analytic curves. On the other hand, it was clear to experts that the stable reduction theorem is a consequence of such a description and that the latter is actually of a very elementary nature and would follow from a description of certain extensions of analytic fields (i.e. complete valued fields of height one).

In general outline, our proof goes as follows. The core is Theorem 6.3.1 which provides a description of extensions of analytic fields K/k with k algebraically closed (or, more generally, deeply ramified in the sense of Assumption 6.2.1) and K the completion of a finitely generated field of transcendence degree one over k. Since some readers may be interested in following the algebra-geometric arguments with taking the non-archimedean analytic results as a black box, we build our exposition accordingly. In §2 we formulate the only analytic black box we need. This is Theorem 2.1.10, which is a light version of Theorem 6.3.1. Then we postpone until §6 any work involving non-archimedean analytic geometry. As a next step in the paper, we deduce Theorem 2.1.8 from (temporarily black boxed) Theorem 2.1.10. Theorem 2.1.8 is the main result of §2 and it describes general extensions of valued fields of transcendence degree one.

Remark 1.3.1. (i) In this paper we only use the case when the ground valued field k is algebraically closed. It was known to experts that in this case the uniformization of valued fields is equivalent (up to some work) to the stable reduction theorem. We prove uniformization directly, thus obtaining a new proof of the stable reduction theorem. On the other side, one could use the stable reduction (over general valuation rings!) to get an indirect proof of Theorem 2.1.8. This would shorten our argument, nearly by eliminating long but elementary §6. Note for the sake of comparison that the only known alternative method for proving stable reduction theorem in such generality is by use of moduli spaces of curves (so this involves stable reduction over a DVR, Hilbert schemes, GIT and/or DM stacks, etc.).

(ii) The main advantage of uniformizing valued fields directly (in addition to making the proof much more elementary) is that we are able to treat some cases of deeply ramified analytic ground fields, including the perfect fields in the equicharacteristic case, in Theorem 6.3.1. It does not seem to be probable that the latter case could be deduced from the stable reduction theorem. In particular, we will prove local desingularization of varieties up to purely inseparable alteration in [Tem4], and the proof is ultimately based on Theorem 6.3.1 for perfect equicharacteristic ground fields.

(iii) In Appendix A.3, we briefly discuss other results on uniformization of one-dimensional valued fields due to Grauert-Remmert, Matignon and Kuhlmann. It seems that none of those approaches applies when the ground field is not algebraically closed.

In §4 we use Theorem 2.1.8 to prove the stable modification theorem in the case when the base scheme S is the spectrum of a valuation ring of finite height and with separably closed fraction field. This case of the stable modification theorem is rather similar to the theory of minimal desingularization of surfaces (see the Appendix). Curiously enough, we essentially use uniqueness of stable modification in order to prove its existence. The situation is similar to the desingularization theory, where local desingularizations are glued together using functoriality of the construction. Moreover, even when the initial C is S-proper we glue the stable modification from pieces that are not S-proper, and at this stage we exploit the fact that our theorem treats all S-curves, including the non-proper ones (see the proof of Proposition 4.5.1).

The general stable modification theorem (with an arbitrary base S) is deduced in §5 rather easily. The main idea is that if a multipointed S-curve (C, D) is chosen then for any valuation ring $\mathcal{O}$ of the separable closure of the field of rational functions on S, there exist a generically étale alteration $S' \to S$ and an open subscheme $U \subset S'$ such that $\mathcal{O}$ is centered on U and the main results for (C, D) hold already after base change to U. Then, using quasi-compactness of the Riemann-Zariski space of S introduced in §3 and the uniqueness of the constructed stable modifications over U's, we glue them together into a stable modification over sufficiently large generically étale alteration of S.

It seems that our method of proving the stable modification theorem can be applied to many other birational problems. So, we describe the method in an abstract general form in §3.4, and then use it to prove in Theorem 3.5.5 a particular case of the reduced fiber theorem [BLR, 2.1'] of Bosch-Lütkebohmert-Raynaud (in our case the base S is integral). Also, we show in §3.3 how to define relative Riemann-Zariski spaces generalizing the classical ones. In [Tem3], these spaces will play a critical role in re-proving the general reduced fiber theorem and generalizing the stable modification theorem to the case of not integral S. It should be noted also that our method of applying relative and absolute Riemann-Zariski spaces to birational geometry is very close to the method of K. Fujiwara and F. Kato, as explained in their survey [FK].

The paper contains two appendixes. In Appendix A we describe a similarity between our method and Zariski's desingularization of surfaces, and compare the new proof of the stable reduction theorem with other proofs (I know six published proofs). In Appendix B we collect some known results on curves over separably closed fields which are used in the paper.

We conclude the introduction with an interesting question which is not studied in the paper. Is stable modification of S-curves functorial with respect to all S-morphisms? For the sake of comparison, we remark that minimal desingularization of surfaces is functorial with respect to regular morphisms, but is not functorial with respect to all morphisms. Nevertheless, I expect that the answer to the above question is affirmative. Such result would indicate that despite large similarity between stable modification and minimal desingularization of surfaces (see Appendix A), the former is a "tighter" (and less subtle) construction.

Contents

2. Uniformization of one-dimensional valued fields

2.1. **Basic theory of valued fields.** Unfortunately, it is not so easy to give a good reference to the theory of valued fields, so we are going to recall in §2.1 some basic (and often well known) facts about valued fields. By a *valued field* we mean a field k provided with a non-Archimedean multiplicative valuation $|\ |: k^\times \to \Gamma_k$. Recall that this means that Γ_k is a totally ordered multiplicative group and $|\ |$ is a homomorphism satisfying the strong triangle inequality $|x+y| \le \max(|x|,|y|)$. We write also $|0| = 0$ and postulate that 0 is smaller than any element of Γ_k. The map $|\ | : k \to \Gamma_k \cup \{0\}$ is often called a non-Archimedean absolute value, but we will simply say *valuation* in the sequel. Two valuations $|\ |_i : k^\times \to \Gamma_i$, $i = 1, 2$ are called *equivalent* if there exists an ordered isomorphism between their images $|k^\times|_i$ compatible with the valuations.

The set $k^\circ := \{x \in k|\ |x| \le 1\}$ is a *valuation ring* (i.e. either $f \in k^\circ$ or $f^{-1} \in k^\circ$ for any $f \in k^\times$) called the ring of integers of k, its maximal ideal is $k^{\circ\circ} := \{x \in k|\ |x| < 1\}$ and its residue field is $\widetilde{k} := k^\circ/k^{\circ\circ}$. Note that $(k^\circ)^\times$ is the kernel of $|\ |$, $k^\times/(k^\circ)^\times \widetilde{\to} |k^\times|$ and the order on $|k^\times|$ is induced from the divisibility in $k^\times$ with respect to k°, i.e. $|a| \le |b|$ if and only if $a/b \in k^\circ$. In particular, k° defines the valuation up to an equivalence, and an alternative way to provide k with a structure of a valued field is to fix a valuation ring $k^\circ \subset k$ with $\mathrm{Frac}(k^\circ) = k$.

Any ring $A \subseteq k$ that contains k° is a valuation ring. Note that $A = k^\circ_m$ for the prime ideal $m = \{x \in k^\circ|\ x^{-1} \notin A\}$. This gives a one-to-one correspondence between the points of $\mathrm{Spec}(k^\circ)$ and the overrings $A \supseteq k^\circ$ in k. Also, there is a one-to-one correspondence $A \mapsto |A^\times|$ between the overrings and the convex subgroups $\Gamma \subseteq |k^\times|$. (Thus, A induces a valuation on k with the group of values $|k^\times|/\Gamma$.) Recall that the *height* of a valued field k is defined as the cardinality of the set $\mathrm{Spec}(k^\circ)$ decreased by 1 (following [Bou], we prefer the notion of height, though one usually says the rank or the convex rank of the valuation). We automatically provide any valued field of height one with the π-adic topology, where $\pi \in k^{\circ\circ} \setminus \{0\}$. (The topology is independent of the choice of π.) By an *analytic field* k we mean a complete valued field with a non-trivial valuation $|\ | : k \to \mathbf{R}_+$. Any analytic field is *henselian* in the sense that its ring of integers k° is henselian.

Note that the topological space of the scheme $S = \mathrm{Spec}(k^\circ)$ is totally ordered with respect to generalizations since the set of all prime ideals of k° (as well as the sets of all its ideals and fractional ideals) is totally ordered with respect to inclusion. So, k is of a finite height h if and only if $|S|$ is homeomorphic to the set $\{0, \dots, h\}$ with the topology given by open subsets $\{0, \dots, n\}$ for $-1 \le n \le h$.

Lemma 2.1.1. *Let R be a valuation ring of finite height and $S = \mathrm{Spec}(R)$. Then for any quasi-finite S-scheme X with a point $x \in X$, $X_x := \mathrm{Spec}(\mathcal{O}_{X,x})$ is an open subscheme of X. In particular, $\mathcal{O}_{X,x}$ is finitely generated over R. Likewise, if X is finitely presented over S then $\mathcal{O}_{X,x}$ is finitely presented over R.*

Proof. Since the set $|S|$ is finite, so is the set $|X|$. Replacing X with a neighborhood of x we can achieve that $X = \mathrm{Spec}(A)$ is affine. The ring $\mathcal{O}_{X,x}$ is a localization of A along an infinite set, but since X_x is obtained from X by removing finitely many points, $\mathcal{O}_{X,x}$ can be obtained by localizing A by a single element. The assertion of the Lemma follows. □

Given a valued field k, by a *valued k-field* we mean a field l containing k and provided with a valuation $|\ |_l : l^\times \to \Gamma_l$ such that $\Gamma_k \subset \Gamma_l$ and $|\ |_l$ extends $|\ |_k$. In the above situation we say that l/k is an *extension of valued fields*. We use the standard notation $e_{l/k} = \#(|l^\times|/|k^\times|)$ and $f_{l/k} = [\widetilde{l} : \widetilde{k}]$ (these cardinals can be infinite) and say that l/k is *immediate* if $e = f = 1$. If L/k is a finite extension of abstract fields, L° is the integral closure of k° in L and $m_1, \dots, m_n$ are its maximal ideals, then $L_i^\circ := L^\circ_{m_i}$ for $1 \le i \le n$ are the extensions of k° to a valuation ring of L. A classical argument shows that $\sum_{i=1}^n e_{L_i/k} f_{L_i/k} \le [L : k]$. It is obvious from the above that the valuation of k uniquely extends to all finite extensions L/k if and only if each L° is a local ring. It is easy to see that the latter happens if and only if any connected finite k°-scheme is local, that is, k° is henselian. In this case we will also say that the valued field k is *henselian*. For an extension l/k of valued fields we set also $F_{l/k} = \mathrm{tr.deg.}(\widetilde{l}/\widetilde{k})$ and $E_{l/k} = \dim_{\mathbf{Q}}((|l^\times|/|k^\times|)\otimes_{\mathbf{Z}}\mathbf{Q})$. The following Lemma is called Abhyankar's inequality.

Lemma 2.1.2. *Any extension l/k of valued fields satisfies* $\mathrm{tr.deg.}(l/k) \ge E_{l/k} + F_{l/k}$.

Proof. Choose a transcendence basis of $\widetilde{l}/\widetilde{k}$ and let $B_F \subset l^\circ$ be any lifting. Also, choose a subset of $B_E \subset |l^\times|$ which is mapped bijectively onto a $\mathbf{Q}$-basis of $|l^\times|/|k^\times| \otimes_{\mathbf{Z}} \mathbf{Q}$. Note that $B_E \cap B_F = \emptyset$ and set $B = B_E \sqcup B_F$. Then $|B_E| = E_{l/k}$ and $|B_F| = F_{l/k}$, and it is easy to see that the monomial basis of $k[B]$ is orthogonal with respect to the valuation on l. In particular, $k[B] \hookrightarrow l$ and hence $|B| \le \mathrm{tr.deg.}(l/k)$. □

Corollary 2.1.3. *(i) If a valued field k is of finite absolute transcendence degree d then the height of k is bounded by $d+1$.*

(ii) Any valuation ring is a filtered union of its valuation subrings of finite height.

Proof. (i) holds for the prime fields $\mathbf{Q}$ and $\mathbf{F}_p$ since all their valuation rings are $\mathbf{Z}_{(p)}$, $\mathbf{Q}$ and $\mathbf{F}_p$. The general case of (i) follows by applying Abhyankar's inequality to the extension $k/\mathbf{F}$ where $\mathbf{F}$ is the valued prime subfield of k. Using (i) we see that any valuation ring k° is the union of its valuation subrings of finite height which are cut off from k° by finitely generated subfields of k. □

Many statements about schemes over arbitrary valuation rings can be reduced to the case of finite height using approximation from Corollary 2.1.3(ii). Next, we are going to introduce a technique of induction on height that often makes it possible to reduce a problem on valuations to the height one case. Assume that the field $\widetilde{k}$ is provided with a valuation $|\ |_{\widetilde{k}}$ and let $\widetilde{k}^\circ$ be its valuation ring. Then the preimage of $\widetilde{k}^\circ$ in k° is a valuation ring R, and we say that the corresponding valuation of k (unique up to equivalence) is *composed* from $|\ |_k$ and $|\ |_{\widetilde{k}}$. Topologically $\mathrm{Spec}(R)$ is obtained by gluing $\mathrm{Spec}(k^\circ)$ and $\mathrm{Spec}(\widetilde{k}^\circ)$ so that the closed point $\mathrm{Spec}(\widetilde{k})$ of the first space is pasted to the generic point of the second one. (We will not need it, but one can easily show that this is a scheme-theoretic gluing in the sense that the four morphisms $\mathrm{Spec}(\widetilde{k}) \to \mathrm{Spec}(k^\circ) \to \mathrm{Spec}(R)$, $\mathrm{Spec}(\widetilde{k}) \to \mathrm{Spec}(\widetilde{k}^\circ) \to \mathrm{Spec}(R)$ are monomorphisms that form a *universally bi-Cartesian square*, i.e. the square is both Cartesian and universally co-Cartesian; see [Tem3, §2.3] for more details.)

Lemma 2.1.4. *Let k be a valued field.*

(i) For any overring A with $k^\circ \subseteq A \subseteq k$ we have that A is a valuation ring, $m_A \subset k^\circ$, $A' := k^\circ/m_A$ is a valuation ring in A/m_A and k° is composed from A and A'.

(ii) If k is of height larger than one then there exists A as above so that both A and A' are of positive height.

(iii) If k is of finite height then it can be obtained by iterative composing valuations of height one.

Proof. (i) is obvious. In (ii) we choose a prime ideal p with $0 \subsetneq p \subsetneq k^\circ$ and set $A = k^\circ_p$. To prove (iii) we choose A as in (ii) and note that the heights of A and A' are smaller than that of k, hence we can apply induction on height. □

Next, we discuss basic ramification theory of algebraic extensions of valued fields. Such an extension l/k is called *unramified* if there exists a local injective k°-homomorphism $l^\circ \hookrightarrow (k^\circ)^{\mathrm{sh}}$ with target being the strict henselization of k°. In particular, $k^{\mathrm{nr}} := \mathrm{Frac}((k^\circ)^{\mathrm{sh}})$ is the maximal unramified extension of k and a finite extension l/k is unramified if and only if the embedding $k^\circ \to l^\circ$ is local-étale (i.e. l° is a localization of a k°-étale algebra). It follows that if k is henselian (e.g. analytic) then $\mathrm{Gal}(k^{\mathrm{nr}}/k)\widetilde{\to}\mathrm{Gal}(\widetilde{k}^s/\widetilde{k})$ and any subfield $\widetilde{k} \subseteq \widetilde{l} \subseteq \widetilde{k}^s$ is the residue field of a valued field l that is unramified over k and unique up to unique isomorphism . An extension l/k is called *totally ramified* if for any tower $k \subseteq k' \subsetneq l' \subseteq l$, the extension l'/k' is not unramified. An extension l/k is called *moderately ramified* (or *tamely ramified*) if it is a composition of an unramified extension l^n/k followed by an extension l/l^n such that any its finite subextension is of degree prime to p. Any other l/k is called *wildly ramified*, including the case when l/k is inseparable, as opposed to the usual convention.

Let k be henselian. By [GR, §6.2], k possesses unique maximal moderately ramified extension k^{mr}, $k^{\mathrm{mr}}/k^{\mathrm{nr}}$ is an abelian extension and $\mathrm{Gal}(k^s/k^{\mathrm{mr}})$ is a pro-p-group, where $p = \mathrm{exp.char}(\widetilde{k})$ is the *exponential characteristic* (i.e. $p = 1$ or p is the usual prime characteristic). In particular, any finite extension of k^{mr} is a p-extension, and $k^{\mathrm{mr}} = k^a$ when $p = 1$. We say that a finite extension l/k of henselian valued fields *defectless* if its *defect* $d_{l/k} := [l : k]/(e_{l/k}f_{l/k})$ equals to one (in other words, the defect is trivial). In this case one also says that l/k is *Cartesian* because this happens if and only if the k-vector space l has an *orthogonal* basis, i.e. a basis $v_1, \dots, v_n$ such that the non-archimedean inequality $|\sum_{i=1}^n a_i v_i| \le \max_{1\le i\le n} |a_i v_i|$ is an equality for any choice of $a_i \in k$.

Lemma 2.1.5. *Let l/k be a finite extension of henselian valued fields, $e = e_{l/k}$, $f = f_{l/k}$, $d = d_{l/k}$, $n = [l : k]$ and $p = \mathrm{exp.char}(\widetilde{k})$.*

(i) l/k splits uniquely into a tower $l/l^m/l^n/k$, where l^n/k is unramified, l^m/l^n is totally moderately ramified, and l/l^m is totally wildly ramified;

(ii) l/k is unramified if and only if $f = n$ and $\widetilde{l}/\widetilde{k}$ is separable;

(iii) l/k is moderately ramified if and only if $\widetilde{l}/\widetilde{k}$ is separable, $(e, p) = 1$ and $d = 1$ (i.e. l/k is Cartesian);

(iv) d is a power of p.

Proof. Find an unramified extension l^n/k with an isomorphism $\phi : \widetilde{l^n}\widetilde{\to}F$, where F is the separable closure of $\widetilde{k}$ in $\widetilde{l}$. Then ϕ lifts to a (necessarily local) homomorphism $(l^n)^\circ \hookrightarrow l^\circ$ by [EGA, IV$_4$, 18.8.4], so we identify $(l^n)^\circ$ and l^n with their images in l. The extension l/l^n is totally ramified because all intermediate fields are henselian and have purely inseparable residue field extension, so there is no non-trivial unramified subextension. By the maximality condition $(l^n)^{\mathrm{mr}}$ contains all its conjugates over l^n, hence $l^m = l \cap (l^n)^{\mathrm{mr}}$ is a well defined moderately ramified extension of k. Finally, l/l^m is a p-extension because $\mathrm{Gal}(k^s/(l^n)^{\mathrm{mr}})$ is a pro-p-group and so $k^a/(l^n)^{\mathrm{mr}}$ is a union of finite p-extensions. This proves existence in (i), and the method of proof gives uniqueness, so the remaining claims follow. □

Lemma 2.1.6. *A finite extension l/k of valued fields of finite height is unramified if and only if l°/k° is étale.*

Proof. By the very definition, l/k is unramified if and only if l°/k° is local-étale (or essentially étale), hence we have only to show that the local-étale extension l°/k° is étale. But this is a consequence of Lemma 2.1.1. □

Remark 2.1.7. By definition, l/k is unramified if and only if l° is local-étale over k°. However, if the heights are infinite the latter does not necessarily imply that l°/k° is étale.

An extension K/k of valued fields will be called *bounded* if any non-zero element of K° divides a non-zero element of k°. This condition is equivalent to requiring that kK°, which is the localization of K° at $k^\times \cap K^\circ$ and hence a valuation ring of K, coincides with K. The main result of §2 is the following statement, which will be called uniformization of one-dimensional valued fields.

Theorem 2.1.8. *Let K/k be a finitely generated extension of valued fields of transcendence degree one. Assume that k is separably closed and the valuation ring kK° is centered on a smooth point of the normal projective k-model of K. Then there exists a transcendence basis $\{x\}$ of K/k such that the finite extension $K/k(x)$ is unramified.*

Remark 2.1.9. I expect that the Theorem is true for any $n = \mathrm{tr.deg.}(K/k)$. We will see that already in our case the proof is difficult, and the case of $n > 1$ is absolutely open. Establishing the case of $n > 1$ would be a major breakthrough that (almost surely) would enable one to prove a local version of the higher dimensional semi-stable reduction theorem. Even when k is trivially valued, this is open for $n > 3$. This particular case follows from (conjectural) local desingularization of varieties along valuations (so called, Zariski local uniformization), and I expect it is not essentially easier than the Zariski local uniformization itself.

Theorem 2.1.8 will be proved in the next section. It admits the following analytic version, which we call uniformization of one-dimensional analytic fields.

Theorem 2.1.10. *Let k be an analytic algebraically closed field and K be an analytic k-field which is finite over a subfield $\overline{k(y)}$ topologically generated by an element. Then*

(i) K is finite and unramified over a subfield $\overline{k(x)}$ for some choice of $x \in K$,

(ii) moreover, there exists a positive ε such that for any $x' \in K$ with $|x - x'| < \varepsilon$ the extension $K/\overline{k(x')}$ is finite and unramified,

(iii) if $\widetilde{K} \neq \widetilde{k}$ and $x \in K^\circ$ is any element such that $\widetilde{x}$ is transcendental over $\widetilde{k}$ and $\widetilde{K}/\widetilde{k}(\widetilde{x})$ is separable then $K/\overline{k(x)}$ is finite and unramified.

Our proof of Theorem 2.1.10 involves methods from non-archimedean analytic geometry, and for expository reasons we postpone it until §6. For our current purposes the reader can view this Theorem as the only analytic black box we use.

2.2. **Reduction to uniformization of analytic fields.** The goal of §2.2 is to prove Theorem 2.1.8 modulo the black box given by Theorem 2.1.10. The proof contains two main steps: decompletion, i.e. proving the theorem for non-complete fields of height one, and composition of valuations which allows induction on height. We start with proving two criteria for an extension of valued fields to be unramified.

These are a decompletion criterion and a composition criterion. Note that in the decompletion criterion given below, the separability assumption is necessary only when $\widehat{K}$ is not separable over K.

Proposition 2.2.1. *A finite extension L/K of height one valued fields is unramified if and only if it is separable and the extension $\widehat{L}/\widehat{K}$ is unramified.*

Proof. Assume that the extension L/K is unramified. In particular, it is separable and the homomorphism $K^\circ \to L^\circ$ is étale by Lemma 2.1.6. It follows that $A := L^\circ \otimes_{K^\circ} \widehat{K}^\circ$ is étale over $\widehat{K}^\circ$ and hence A is normal. Note also that $L \otimes_K \widehat{K}$ is a separable $\widehat{K}$-algebra with a direct factor isomorphic to $\widehat{L}$. By normality of A it has a direct factor A' with fraction field isomorphic to $\widehat{L}$. Since A' is normal and $\widehat{L}^\circ$ is the integral closure of $\widehat{K}^\circ$ in $\widehat{L}$ because $\widehat{K}$ is henselian, we obtain that $A' \widetilde{\to} \widehat{L}^\circ$. So, $\widehat{L}^\circ$ is étale over $\widehat{K}^\circ$.

The converse implication is more involved. Let F denote the field L without the valued field structure (so, F/K is separable). Let F° be the integral closure of K° in F; it is a semi-local ring with maximal ideals $m_1, \dots, m_n$. The localizations $F_i^\circ = F_{m_i}$ are the valuation rings of F lying over K°, and without loss of generality $L^\circ = F_1^\circ$. Let F_i denote the valued field corresponding to F_i°.

Let y be a primitive element of the separable extension F/K and x_0 be an element in the ideal $m_2 \dots m_n \subset F^\circ$ such that its image in $F^\circ/m_1 \widetilde{\to} \widetilde{L}$ is non-zero and generates $\widetilde{L}$ over $\widetilde{K}$. Since all but finitely many linear combinations $x_0 + zy$ with $z \in K$ are primitive for F/K, we can find $z \in K^{\circ\circ}$ such that $x = x_0 + zy$ is primitive and $zy \in m_1 \dots m_n$, and so $x \in m_2 \dots m_n$ and the image of x is a non-zero primitive element of $\widetilde{L}/\widetilde{K}$. Let $f(T) \in K^\circ[T]$ be the minimal polynomial of x and $f = f_1 \dots f_n$ be its decomposition in $\widehat{K}[T]$. Since $\widehat{K} \otimes_K F \widetilde{\to} \prod_{i=1}^n \widehat{F}_i$, we can renumber f_i's so that $\widehat{F}_i \widetilde{\to} \widehat{K}[T]/(f_i(T))$. Now, we consider the situation in $\widehat{L}^\circ$. Let $\widehat{x}$ be the image of x in $\widehat{L}^\circ$. Since $f_1(\widehat{x}) = 0$, we have that $f'(\widehat{x}) = f_1'(\widehat{x}) f_2(\widehat{x}) \dots f_n(\widehat{x})$. The element $f_1'(\widehat{x})$ is invertible because $\widehat{L}^\circ$ is finite étale over $\widehat{K}^\circ$ with degree $[\widehat{L} : \widehat{K}]$ (so $\widetilde{f}_1 \in \widetilde{K}[T]$ is the minimal polynomial of $\widetilde{x}$). For any $2 \le i \le n$, $f_i(T)$ is an irreducible polynomial of degree $d_i = [\widehat{F}_i : \widehat{K}]$ whose roots are in $\widehat{F}_i^{\circ\circ}$ (since $\widehat{K}^\circ[T]/(f_i)$ in $\widehat{F}_i$ has T corresponding to x, so f_i has a root in $m\widehat{F}_i^\circ = \widehat{F}_i^{\circ\circ}$). It follows that $f_i(T) \in T^{d_i} + \widehat{K}^{\circ\circ}[T]$. The residue of $\widehat{x}$ in $\widetilde{L}$ is not zero, hence $\widehat{x}$ is invertible and therefore the elements $f_i(\widehat{x})$ are invertible in L° for $i > 1$.

We have proved that $f'(\widehat{x})$ is invertible in $\widehat{L}^\circ$, and therefore $f'(x)$ is invertible in L°. We conclude that the ring $A = K^\circ[x]_{f'(x)}$ is contained in L°. On the other hand, A is étale over K° and hence is integrally closed. Thus, A contains F°, so the localization of A at its (necessarily maximal) prime ideals over $K^{\circ\circ}$ are valuation rings, so one of these must be L°. It follows that L° is local-étale over K° (and it is even étale by Lemma 2.1.1). □

Next we prove a composition criterion. We formulate just the statement that will be used. Note, however, that much stronger results for "composed" rings can be found in [Tem3, §2.3] and further generalizations were obtained by D. Rydh.

Proposition 2.2.2. *Assume that l/k is a finite extension of valued fields of finite height. Suppose that $\widetilde{l}$ and $\widetilde{k}$ are provided with compatible structures of valued fields* *of finite height, and let L and K be the fields l and k provided with the composed valuations. If the extensions l/k and $\widetilde{l}/\widetilde{k}$ are unramified then L/K is unramified.*

Proof. To simplify the notation we rename the valuation rings and their maximal ideals as $A = k^\circ$, $B = \widetilde{k}^\circ$ and $C = K^\circ$ (so C is the composed valuation ring), $m = k^{\circ\circ}$ and $n = K^{\circ\circ}$. In the same manner, we set $A' = l^\circ$, $B' = \widetilde{l}^\circ$, $C' = L^\circ$, $m' = l^{\circ\circ}$, $n' = L^{\circ\circ}$. Finally, let m_B and m'_B be the maximal ideals of B and B'. By Lemma 2.1.6 the embeddings $f_A : A \to A'$ and $f_B : B \to B'$ are étale and our aim is to prove that $f_C : C \to C'$ is étale. For this it is enough to show that C' is C-flat, $n' = nC'$ and f_C is finitely presented. The first claim is obvious because any C-module without torsion is flat.

By Lemma 2.1.1 there exists an element $s \in C$ such that $A = C_s$. Clearly, we also have that $A' = C'_s$. By the obvious bijection between C-submodules $m \subseteq D \subseteq A$ and B-submodules $\widetilde{D} \subseteq K$ (given by $D \mapsto \widetilde{D} = D/m$) we see that $m = s^{-1}m \subset n$ and similarly $m' = s^{-1}m' \subset n'$. Using that $m' = mA'$ by étaleness of f_A we obtain that $m' = mC'_s = mC' \subset nC'$. Now, to prove that the inclusion $nC' \subseteq n'$ is an equality it is enough to show that it becomes an equality after quotient by m'. So, it remains to note that $n'/m' = m'_B$ and nC'/m' contains $(n/m)B' = m_B B'$, which is m'_B by étaleness of f_B.

The last (and the most subtle) check is that f_C is finitely presented. Let a be a finite subset of A' that generates it over A. Multiplying a by an appropriate s^n, we can achieve that $a \subset C'$. Note that $m' = mA' = mA[a] = m[a] \subset C[a]$, where $m[a]$ denotes the set of polynomials in a with coefficients in m. Pick up a finite subset $b \subset C'$ whose image generates B' over B, then one easily sees C' coincides with its subalgebra $C'' = C[a, b]$ because $C''_s = A' = C'_s$ and $C''/m' = B' = C'/m'$. This proves that f_C is of finite type. Since f_C is flat, [RG, 3.4.7] implies that f_C is also finitely presented. ☐

Proof of Theorem 2.1.8. The extension K/k is induced from an extension of absolutely finitely generated valued fields, that is there exist subfields $L \subset K$ and $l \subset k$ such that l and L are finitely generated over their prime subfields with L/l separable of transcendence degree 1 such that $L \otimes_l k$ is integral and $K = \mathrm{Frac}(L \otimes_l k)$. The valued field L and l are of finite height by Corollary 2.1.3(i). We claim that it suffices to prove the Theorem for the extension L/l. Indeed, assume that there exists $x \in L$ transcendental over k and such that $L/l(x)$ is unramified. Then L° is étale over $l(x)^\circ$ by Lemma 2.1.6, and since K° is a localization of its subring $L^\circ \otimes_{l(x)^\circ} k(x)^\circ$ (because both are localizations of the integral closure of $k(x)^\circ$ in K), we obtain that K is unramified over $k(x)$. Thus, we assume in the sequel that k is of finite height h.

Let C be the normal projective k-model of K, and consider the valuation ring $\mathcal{O} = kK^\circ$. If $\mathcal{O} \neq K$ (i.e. the extension K/k is not bounded) then $\mathcal{O}$ dominates the local ring $\mathcal{O}_z$ of a closed point z, which must be k-smooth by the assumption of the Theorem. In particular, $\mathcal{O}$ is the DVR $\mathcal{O}_z$, and hence for any uniformizer x at z, $\mathcal{O}$ is étale over $\mathcal{O} \cap k(x)$ (since x induces an étale morphism from a neighborhood of z to $\mathbf{A}^1_k$). By Lemma 2.1.4(i), the valuation of K is composed from the valuation induced by $\mathcal{O}$ and from the valuation on its residue field $k(z)$ which extends the valuation of k (the latter is uniquely defined because $k(z)/k$ is finite and purely inseparable). Notice that the residue field of the valuation ring $\mathcal{O} \cap k(x)$ is also $k(z)$. Hence applying Proposition 2.2.2 we obtain that K is unramified over $k(x)$.

In the sequel we assume that $\mathcal{O} = K$, and so it is centered on the generic point of C. Note that K/k is separable because the generic point of C is k-smooth by our assumptions. Our proof will run by induction on h. If $h = 0$, then K is necessarily of height 0 too, hence $K/k(x)$ is unramified if and only if x is a separable transcendence basis of K over k. It is well known that such a basis exists.

In the sequel we assume that $h > 0$ (i.e the valuations are non-trivial). We will need the following well known fact which hold for any extension K/k which is separable, finitely generated and of transcendence degree one: K/k possesses a separable transcendence basis, and $\{x\} \subset K$ is a such a basis if and only if $x \notin kK^p$, where $p = \mathrm{char}(k)$ and we agree on notation $K^p = 1$ when $p = 0$. Let k_1 and K_1 denote the fields k and K provided with the induced valuations of height one. We provide the residue fields $\widetilde{k}_1$ and $\widetilde{K}_1$ with the valuations induced from k and K. Since k_1 is separably closed, its completion $\widehat{k}_1$ is algebraically closed, and we obtain in particular that $\widetilde{k}_1$ is algebraically closed. We will also use the simple fact that $k_1K_1^p$ is a k_1-subspace of K_1 with empty interior. Indeed, it suffices to check that 0 is not in the interior, but for any $y \in K_1 \setminus k_1K_1^p$ the set $yk_1^\times$ is disjoint from $k_1K_1^p$ but contains 0 in its closure.

Assume first that $\widetilde{K}_1 = \widetilde{k}_1$. We claim that there exists $x \in K_1$ such that firstly $\widehat{K}_1$ is unramified over $\overline{k_1(x)}$, and secondly $x \notin k_1K_1^p$. If $\widehat{K}_1 = \widehat{k}_1$ (i.e. K_1 is a valued subfield in $\widehat{k}_1$) then this is obvious since the first condition is empty. Otherwise, we apply Theorem 2.1.10(i) to find $x \in \widehat{K}_1$ such that $\widehat{K}_1/\overline{k_1(X)}$ is finite and unramified. Moreover, part (ii) of the same Theorem implies that we can move x slightly, and since $k_1K_1^p$ is nowhere dense we can achieve that $x \in K_1$ and $x \notin k_1K_1^p$. This proves the above claim. Note that K_1 is a finite separable extension of $k_1(x)$, and hence K_1 is unramified over $k_1(x)$ by Proposition 2.2.1. Now, to prove that $K/k(x)$ is unramified it remains to use Proposition 2.2.2 and the assumption that $\widetilde{K}_1 = \widetilde{k}_1$.

Finally we consider the case when $\widetilde{K}_1 \neq \widetilde{k}_1$. Since $\widetilde{k}_1$ is algebraically closed, Abhyankar inequality (Lemma 2.1.2) implies that $\mathrm{tr.deg.}(\widetilde{K}_1/\widetilde{k}_1) = 1$. Thus, the extension $\widetilde{K}_1/\widetilde{k}_1$ satisfies the conditions of the Theorem, and we can use the induction assumption due to the fact that the height of $\widetilde{k}_1$ equals to the height of k decreased by one. Find $\widetilde{x} \in \widetilde{K}_1$ transcendental over $\widetilde{k}_1$ and such that the extension $\widetilde{K}_1/\widetilde{k}_1(\widetilde{x})$ is unramified (in particular, it is separable), and lift it to an element $x \in K_1^\circ$ not contained in $k_1K_1^p$. Then $\widehat{K}_1/\overline{k_1(x)}$ is finite and unramified by Theorem 2.1.10(iii). Now it remains to use Propositions 2.2.1 and 2.2.2, as earlier. □

3. Riemann-Zariski spaces

We describe the classical Riemann-Zariski spaces in §3.2 and indicate in §3.3 how they can be generalized to a relative case. In §3.4 we describe a method of proving modification theorems, which we illustrate in §3.5 by proving a particular case of the reduced fiber theorem that will be used later in the paper.

If X is a scheme then by X^0 we denote the set of generic points of X. We refer the reader to [EGA I], 6.10.1 and 6.10.2, for the definitions of schematical image and density. Let us fix the notions of modification and alteration (which can differ from paper to paper). By a *modification* (resp. *quasi-modification*) we mean a proper (resp. separated finite type) morphism which induces an isomorphism of open schematically dense subschemes. By an *alteration* (resp. *quasi-alteration*) of an integral scheme X we mean a proper (resp. separated finite type) morphism

$f : Y \to X$ with integral Y and finite field extension $k(Y)/k(X)$. If the latter extension is separable then we say that the (quasi-) alteration is *generically étale*. To justify this terminology, we note such morphism $f : Y \to X$ is of finite presentation over an open subscheme of X, and hence an easy approximation argument (see §3.1) implies that f is étale over a dense open in X when f is generically étale in the preceding sense.

The combination of words "quasi-compact quasi-separated" will appear very often in the paper, so we will abbreviate it with a single "word" *qcqs*. Note that Grothendieck called qcqs schemes *coherent*, but this might be too confusing. Until the end of this paper S is a scheme. We assume that S is integral, qcqs and with the generic point $\eta = \mathrm{Spec}(K)$ if not said explicitly. For an S-scheme X, by X_η and X_s we denote its fibers over η and s, respectively, where $s \in S$ is any point. A modification $f : X' \to X$ is called an *η-modification* if its η-fiber $f_\eta : X'_\eta \to X_\eta$ is an isomorphism. A reduced S-scheme X is called *η-normal* if it does not admit non-trivial finite η-modifications. Note that η-normality is a sort of partial normality condition "semi-orthogonal" to normality of X_η because X is normal if and only if X_η is normal and X is η-normal.

3.1. **Noetherian approximation.** In this section we recall some results from [EGA, IV$_3$, §8] on projective limits of schemes. These results will be used very often in the sequel. Let $\{S_i\}$ be a filtered projective family of schemes with affine transition morphisms and initial object S_0 (the latter assumption does not really restrict the generality), then $S = \mathrm{proj}\lim S_i$ exists by [EGA, IV$_3$, 8.2.3]. We assume that S_0 is qcqs, then by [EGA, IV$_3$, 8.8.2 and 8.5.2], any finitely presented morphism $f : X \to S$ (resp. finitely presented quasi-coherent $\mathcal{O}_S$-module) is induced from a finitely presented morphism $f : X_i \to S_i$ (resp. finitely presented quasi-coherent $\mathcal{O}_{S_i}$-module), and for any pair of finitely presented morphisms $X_i \to S_i$, $Y_i \to S_i$ we have that

$$\mathrm{inj}\lim_{j \geq i} \mathrm{Hom}_{S_j}(Y_i \times_{S_i} S_j, X_i \times_{S_i} S_j) \widetilde{\to} \mathrm{Hom}_S(Y_i \times_{S_i} S, X_i \times_{S_i} S).$$

Moreover, by [EGA, IV$_3$, 11.2.6] f is flat if and only if there exists i_0 such that each f_i with $i \geq i_0$ is flat. We will refer to all these statements by the word "approximation". By "noetherian approximation" we mean these statements combined with [TT, C.9], which asserts that any qcqs scheme S can be represented as the projective limit of a filtered family of schemes of finite type over $\mathbf{Z}$ such that the transition morphisms are affine. A typical example of an application of noetherian approximation is that any relative S-curve C is induced from a relative curve C_0 over a scheme S_0 of finite type over $\mathbf{Z}$ in the sense that $C \widetilde{\to} C_0 \times_{S_0} S$.

Remark 3.1.1. (i) Many properties of $f : X \to S$ descend to f_i's with $i \geq i_0$. Let $\mathbf{P}$ be a property of schemes of finite type over a field such that (a) for any field extension l/k, a finite type morphism $f : X \to \mathrm{Spec}(k)$ satisfies $\mathbf{P}$ if and only if $f \otimes_k l$ satisfies $\mathbf{P}$, (b) for any finitely presented morphism $X \to S$ with qcqs S the set of points $s \in S$ with the fiber X_s satisfying $\mathbf{P}$ is constructible. For example, $\mathbf{P}$ can be geometric reducedness since (a) is obvious and (b) is proved in [EGA, IV$_3$, 9.7.7(iii)]. Then f is a $\mathbf{P}$-morphism (i.e. it is flat and its fibers satisfy $\mathbf{P}$) if and only if each f_i is a $\mathbf{P}$-morphism for $i \geq i_0$. Indeed, this is true for flatness by [EGA, IV$_3$, 11.2.6], hence without restriction of generality we can assume that f_0 is flat. Now, if $E_i \subset S_i$ is the constructible set of points s with non-$\mathbf{P}$ fiber $(X_i)_s$ then each E_i is the preimage of E_0. Since the preimage of E_0 in S is empty, already some E_i is empty by [EGA, IV$_3$, 8.3.3].

(ii) A similar claim holds for suitable properties of diagrams of S-flat schemes, e.g. for modifications $(C', D') \to (C, D)$ of multipointed S-curves.

We will not care about it, but in all cases when noetherian approximation is used in this paper, it can be easily seen that the case of an affine S suffices. In this particular case noetherian approximation is easier and appeared already in [EGA, IV$_3$, 8.9.1].

3.2. **Absolute Riemann-Zariski spaces.** We adopt the exposition of [Tem1, §1] to the case of general schemes, but we will use different notation. Fix an integral scheme S (temporarily not necessarily qcqs) and a dominant morphism $\overline{\eta} : \mathrm{Spec}(\overline{K}) \to S$, where $\overline{K}$ is a field. Let η be the generic point of S and $K = k(\eta)$. By a *$\overline{K}$-modification* (resp. *$\overline{K}$-quasi-modification*) we mean a splitting of $\overline{\eta}$ into a composition of a schematically dominant morphism $\mathrm{Spec}(\overline{K}) \to S_i$ (so S_i is integral) and a proper (resp. separated finite type) morphism $S_i \to S$. By use of schematical images of $\mathrm{Spec}(\overline{K})$ in fiber products over S one shows that the projective family of all $\overline{K}$-modifications (resp. $\overline{K}$-quasi-modifications) is filtered. If $K \widetilde{\to} \overline{K}$, $K^a \widetilde{\to} \overline{K}$ or $K^s \widetilde{\to} \overline{K}$ then S_i's are modifications, alterations or generically étale alterations of S, respectively (resp. quasi-versions of these notions). The topological space $\mathfrak{S} = \mathrm{RZ}_{\overline{K}}(S) = \mathrm{proj\,lim}\, S_i$, where S_i's are the $\overline{K}$-modifications, is called the *Riemann-Zariski space* of S with respect to $\overline{K}$.

Proposition 3.2.1. *The space $\mathfrak{S}$ is qcqs if and only if the scheme S is qcqs.*

Proof. Assume that S is qcqs. Let $f_i : \mathfrak{S} \to S_i$ be the projections and $f_{ji} : S_j \to S_i$ with $j \geq i$ be the transition maps. An open subscheme $U \hookrightarrow S_i$ is qcqs if and only if it is compact in the constructible topology of U. If this is the case then each $f_{ji}^{-1}(U)$ is compact in the constructible topology and hence $\mathfrak{U} = f_i^{-1}(U)$ is compact in the constructible projective limit topology. Since the usual (Zariski) topology of $\mathfrak{S}$ is weaker, $\mathfrak{U}$ is quasi-compact. In particular, $\mathfrak{S}$ is quasi-compact. Preimages of the sets U as above form a basis of the topology of $\mathfrak{S}$, hence for quasi-separatedness of $\mathfrak{S}$ it is enough to show that the intersection of two such preimages $\mathfrak{U}$ and $\mathfrak{U}'$ is quasi-compact. Enlarging i enough we can assume that $\mathfrak{U} = f_i^{-1}(U)$ and $\mathfrak{U}' = f_i^{-1}(U')$, but then $\mathfrak{U} \cap \mathfrak{U}' = f_i^{-1}(U \cap U')$ is quasi-compact by what we proved above.

Now, let us assume that $\mathfrak{S}$ is qcqs. Let us assume for a moment that the projection $f : \mathfrak{S} \to S$ is surjective (that is not automatic even though f_{ji}'s are surjective). Then S is obviously quasi-compact, and quasi-separatedness of S follows from the fact that the preimage in $\mathfrak{S}$ of an open quasi-compact set $U \subset S$ is open and quasi-compact (by the same argument as we used in the direct implication). It remains for a point $x \in S$ to show that $f^{-1}(x)$ is non-empty. Find a valuation ring $\mathcal{O}$ of $\overline{K}$ which dominates $\mathcal{O}_{S,x} \subset \overline{K}$ (it exists by Zorn's lemma). Then the morphism $\mathrm{Spec}(\mathcal{O}) \to S$ factors through each S_i by the valuative criterion of properness, hence the images of the closed point of $\mathrm{Spec}(\mathcal{O})$ in each S_i give rise to a compatible family of points $x_i \in S_i$. The point $\mathfrak{x} \in \mathfrak{S}$ corresponding to the family $\{x_i\}$ sits over $x \in S$, and we are done. $\square$

In the sequel, we assume again that S is qcqs. We provide $\mathfrak{S}$ with a sheaf $\mathcal{O}_{\mathfrak{S}} = \operatorname{inj\,lim} \pi_i^{-1}(\mathcal{O}_{S_i})$, where $\pi_i : \mathfrak{S} \to S_i$ are the projections. The following easy approximation lemma will be very useful in the sequel.

Lemma 3.2.2. *For any point $\mathfrak{x} \in \mathfrak{S}$, the scheme* $\operatorname{Spec}(\mathcal{O}_{\mathfrak{S},\mathfrak{x}})$ *is isomorphic to a projective limit of $\overline{K}$-quasi-modifications of S.*

Proof. Since $\mathcal{O}_{\mathfrak{S},\mathfrak{x}} = \operatorname{inj\,lim} \mathcal{O}_{S_i,x_i}$ where $x_i = \pi_i(\mathfrak{x})$, $\operatorname{Spec}(\mathcal{O}_{\mathfrak{S},\mathfrak{x}})$ is isomorphic to the projective limit of the schemes $\operatorname{Spec}(\mathcal{O}_{S_i,x_i})$. It remains to notice that each $\operatorname{Spec}(\mathcal{O}_{S_i,x_i})$ is isomorphic to the projective limit of its open neighborhoods in S_i, and the latter are obviously $\overline{K}$-quasi-modifications of S. □

This Lemma combined with an approximation argument often allows to reduce certain birational problems on S to problems over the local rings $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$. These rings are valuation rings, as we are going to prove.

Lemma 3.2.3. *For any element $f \in \overline{K}$, there exists a $\overline{K}$-modification $S' \to S$ such that f is a rational function on S' giving rise to a morphism $S' \to \mathbf{P}^1_{\mathbf{Z}}$.*

Proof. Consider the morphism $\operatorname{Spec}(\overline{K}) \stackrel{(\overline{\eta},f)}{\to} S \times_{\operatorname{Spec}(\mathbf{Z})} \mathbf{P}^1_{\mathbf{Z}}$, and take S' to be its schematical image. □

Corollary 3.2.4. *For any point $\mathfrak{x} \in \mathfrak{S}$, the ring $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$ is a valuation ring with the fraction field $\overline{K}$.*

Proof. Since $\mathcal{O}_{\mathfrak{S},\mathfrak{x}} = \operatorname{inj\,lim} \mathcal{O}_{S_i,x_i}$, where $\{S_i\}$ is the set of all $\overline{K}$-modifications, including the one in Lemma 3.2.3, it follows that for any element $f \in \overline{K}^{\times}$, either f or f^{-1} is contained in $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$. Hence $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$ is a valuation ring and $\operatorname{Frac}(\mathcal{O}_{\mathfrak{S},\mathfrak{x}}) = \overline{K}$. □

Let $Val_{\overline{K}}(S)$ be the set of morphisms $\phi_{\mathfrak{x}} : \operatorname{Spec}(\mathcal{O}_{\mathfrak{x}}) \to S$ such that $\mathcal{O}_{\mathfrak{x}}$ is a valuation ring of $\overline{K}$ and $\phi_{\mathfrak{x}}$ has $\overline{\eta}$ as the generic fiber. By the above Corollary, we have a natural map $\mathfrak{S} \to Val_{\overline{K}}(S)$. Conversely, any morphism $\phi_{\mathfrak{x}}$ as above factors uniquely through any $\overline{K}$-modification of S by the valuative criterion of properness. The images in all S_i's of the closed point of $\operatorname{Spec}(\mathcal{O}_{\mathfrak{x}})$ give rise to a point $\mathfrak{x}$ of the projective limit $\mathfrak{S}$. We have constructed an opposite map $Val_{\overline{K}}(S) \to \mathfrak{S}$ which is inverse because $\mathcal{O}_{\mathfrak{S},\mathfrak{x}} \widetilde{\to} \mathcal{O}_{\mathfrak{x}}$. Indeed, $\mathcal{O}_{\mathfrak{x}}$ dominates the local rings $\mathcal{O}_{S_i,x_i}$, hence it dominates their union $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$, but both are valuation rings with common fraction field, so they must coincide. We have thereby proved the following statement.

Corollary 3.2.5. *The sets $\mathfrak{S}$ and $Val_{\overline{K}}(S)$ are naturally bijective.*

Any quasi-compact open subset $\mathfrak{S}' \subset \mathfrak{S}$ is induced from a quasi-compact open subscheme S' of some modification of S, and by Corollary 3.2.5 the natural map $\mathfrak{S}' \to \mathrm{RZ}_{\overline{K}}(S')$ is bijective. More generally, by the valuative criterion of separatedness we obtain a natural injective map $\mathrm{RZ}_{\overline{K}}(S') \hookrightarrow \mathfrak{S}$ for any $\overline{K}$-quasi-modification $S' \to S$.

Proposition 3.2.6. *Assume that $S_1, \dots, S_n$ are $\overline{K}$-quasi-modifications of S. Then there exists a $\overline{K}$-modification $S' \to S$ such that S' contains open subschemes S'_i which are S-isomorphic to $\overline{K}$-modifications of S_i's.*

Proof. Since S is quasi-compact, S and each S_i possess a finite affine covering. Note that we can replace each S_i with its open subschemes $U_1, \dots, U_m$ which cover S_i. For this reason it suffices to treat the case when each S_i is affine and its image is contained in an affine open subscheme of S. Next, we claim that it suffices to find a $\overline{K}$-modification $X_i \to S$ which satisfies the assertion of the Proposition for a single S_i, because then we can take S' to be any $\overline{K}$-modification of S which dominates all X_i's. So, we can assume that $n = 1$, $S_1 = \mathrm{Spec}(B)$ and the image of S_1 is contained in $\mathrm{Spec}(A) = V \hookrightarrow S$. Let $f_1, \dots, f_l \in \overline{K}$ be generators of B over A. By Lemma 3.2.3 we can find a $\overline{K}$-modification $\phi : S' \to S$ such that each f_j induces a morphism $F_j : S' \to \mathbf{P}^1_{\mathbf{Z}}$. Then S' is as required because it is easy to check that $(\cap_{j=1}^l F_j^{-1}(\mathbf{A}^1_{\mathbf{Z}})) \cap \phi^{-1}(V) \hookrightarrow S'$ is a $\overline{K}$-modification of S_1. □

Corollary 3.2.7. *For any $\overline{K}$-quasi-modification $S' \to S$, the injection $\mathrm{RZ}_{\overline{K}}(S') \to \mathrm{RZ}_{\overline{K}}(S)$ is a homeomorphism onto an open subspace.*

Proof. It would suffice to know that there exists a cofinal family of $\overline{K}$-modifications $S'_i \to S'$ such that each S'_i admits an open immersion $S'_i \hookrightarrow S_i$ compatible with $\overline{K}$ into a $\overline{K}$-modification of S. But the latter is an obvious consequence of Proposition 3.2.6. □

Remark 3.2.8. (i) Corollaries 3.2.5 and 3.2.7 imply that Riemann-Zariski spaces of affine schemes (we call them *affine*) admit a usual valuation-theoretic description. Namely, if $X = \mathrm{Spec}(A)$ then $\mathfrak{S}$ is the set of all valuation rings of $\overline{K}$ that contain A. The sets $\mathrm{RZ}(A[f_1, \dots, f_n])$ with $f_i \in \overline{K}$ form a basis of open subsets of the topology of $\mathfrak{S}$.

(ii) In particular, $\mathrm{RZ}_{\overline{K}} = \mathrm{RZ}_{\overline{K}}(\mathbf{Z})$ (resp. $\mathrm{RZ}_{\overline{K}}(k)$ for a subfield $k \subset \overline{K}$) is the classical Riemann-Zariski space of $\overline{K}$ whose points are valuations on $\overline{K}$ (resp. trivial on k).

(iii) By Proposition 3.2.1, a general Riemann-Zariski space is pasted from finitely many affine ones via a finite gluing data.

We will not need the following result, but analogously to [Tem1, 1.3] one can strengthen our Corollary 3.2.5 as follows.

Lemma 3.2.9. *The topology on $\mathfrak{S}$ is the weakest topology for which the natural maps $\phi : \mathfrak{S} \to \mathrm{RZ}_{\overline{K}}$ and $\mathfrak{S} \to S$ are continuous. If S is separated then ϕ is a homeomorphism onto its image (so, the topology is generated only by ϕ).*

3.3. **Relative Riemann-Zariski spaces.** This section will not be used in this paper, but the relative spaces will play important role in [Tem3]. Throughout §3.3 we fix a qcqs morphism of schemes $f : Y \to X$. In particular, the sheaf $f_*(\mathcal{O}_Y)$ is quasi-coherent by [EGA I, 6.7.1]. Consider the family of all factorizations of f into a composition of a schematically dominant qcqs morphism $f_i : Y \to X_i$ followed by a proper morphism $\pi_i : X_i \to X$. We call the pair (f_i, π_i) a *Y-modification* of X; usually it will be denoted simply X_i. Given two Y-modifications of X, we say that X_j dominates X_i if there exists an X-morphism $\pi_{ji} : X_j \to X_i$ compatible with f_i, f_j, π_i and π_j. Notice that if π_{ji} exists then it is unique. The family of Y-modifications of X is filtered because two Y-modifications X_i, X_j are dominated by the schematical image of Y in $X_i \times_X X_j$. Also, this family has an initial object corresponding to the schematical image of Y in X. The same facts are valid for the more restrictive class of finite Y-modifications. The projective limit of finite Y-modifications of X exists in the category of schemes. We will denote it $\mathcal{N}r_Y(X)$ and call the *Y-normalization of X*. We define the *Riemann-Zariski space of X with respect to Y* to be the projective limit of the underlying topological spaces of all

Y-modifications of X; this space will be denoted $\mathrm{RZ}_Y(X)$. The proof of Proposition 3.2.1 carries over verbatim to prove the following Proposition.

Proposition 3.3.1. *The space* $\mathfrak{X} = \mathrm{RZ}_Y(X)$ *is qcqs if and only if the scheme* X *is qcqs.*

Let $\pi : \mathfrak{X} \to X$ and $i : Y \to \mathfrak{X}$ be the natural maps. We provide $\mathfrak{X}$ with the sheaf $\mathcal{M}_{\mathfrak{X}} = i_*(\mathcal{O}_Y)$ of "meromorphic functions" and the sheaf $\mathcal{O}_{\mathfrak{X}} = \mathrm{inj}\lim \pi_i^{-1}(\mathcal{O}_{X_i})$ of "regular functions".

Example 3.3.2. (i) The classical (absolute) version of the above notions is obtained when X is integral and Y is the generic point of X.

(ii) Another example is obtained when X is of finite presentation over a valuation ring R and $Y = X_\eta$ is the generic fiber of X. Then $\mathfrak{X} := \mathrm{RZ}_Y(X)$ is the projective limit of all η-modifications of X, so this space arise naturally when one studies some problems involving η-modifications.

(iii) Assume that R is of height one in (ii). One can show that if $\widehat{X}$ is the formal completion of X along the special fiber and $\widehat{X}_\eta$ is its "generic" fiber in the category of adic spaces, then the special R-fiber of $\mathrm{RZ}_Y(X)$ is homeomorphic to $\widehat{X}_\eta$ (the generic R-fiber of $\mathrm{RZ}_Y(X)$ is, obviously, Y).

(iv) Although f is a monomorphism (probably not of finite type) in (i)–(ii), there exist other interesting examples. In [Tem1, §1] and in the previous section, we considered the case when Y is a point and f is dominant but not necessarily a monomorphism.

3.4. **A method of proving P-modification theorems.** This is the only section in the paper where we weaken our assumptions on η. We only assume that S is a qcqs scheme and $|\eta| \subset |S|$ is a quasi-compact subset which is closed under generalizations and such that $\eta = (|\eta|, \mathcal{O}_S|_{|\eta|})$ is a scheme. Then η is isomorphic to the scheme-theoretical projective limit of its open neighborhoods. In particular, the natural embedding morphism $i_\eta : \eta \to S$ is a quasi-compact monomorphism and any morphism $X \to S$ with image in $|\eta|$ factors through η uniquely. (So, η is in a sense a "pro-open subscheme" of S.) We assume in addition that i_η is schematically dominant, i.e. η is not contained in a proper closed subscheme of S.

Let P be a property of morphisms of schemes. By a *P-modification statement* over S we mean a statement that if the "generic fiber" $\phi_\eta : X_\eta = X \times_S \eta \to \eta$ of a flat finitely presented morphism $\phi : X \to S$ satisfies P then there exist a base change morphism $S' \to S$ of a certain class Q and a modification $\psi : X' \to X \times_S S'$ such that $X' \to S'$ is flat and satisfies P. (Such kind of statements is called a permanence principle in the introduction to [BLR].) The class Q can be, for example, the class of all η-modifications (i.e. proper morphisms $S' \to S$ such that $\eta' = \eta \times_S S'$ is schematically dense in S' and is mapped isomorphically onto η), the class of all generically étale alterations, of all finite flat morphisms, etc.

Example 3.4.1. In the following two examples Q is the class of morphisms of the form $S' \to S'' \to S$, where $S'' \to S$ is an η-modification and $S' \to S''$ is a finitely presented flat surjective morphism which is étale over η.

(i) The *semi-stable modification theorem* is obtained when P is being a semi-stable curve.

(ii) The *reduced fiber theorem* is obtained when P is having geometrically reduced fibers.

Note that any morphism $f : S' \to S$ from Q is étale over η. This is rather restrictive and as a drawback, one has to allow reducible S''s even when S is irreducible. In the particular case when η is a point, one can impose an extra-condition that $f^{-1}(\eta)$ is a point. Then Q reduces to the class of all generically étale alterations.

Remark 3.4.2. The reduced fiber theorem was proved by Bosch, Lütkebohmert and Raynaud in [BLR, 2.1'] (the main theorem 2.1 of loc.cit. deals with its formal version). Also, it was conjectured (or hoped) in loc.cit. in the end of the introduction that a semi-stable modification exists in all relative dimensions. It follows from simple examples with two-dimensional bases, see [AK], conjecture 0.2, that semi-stable modification does not exist in general. One can also construct analogous examples over the "two-dimensional" base $S = \mathrm{Spec}(K^\circ)$, where K is a valued field with $|K^\times|\xrightarrow{\sim}\mathbf{Z}^2$. A possible salvage of the situation is to extend the class of semi-stable morphism. For example, one can consider a wider class of poly-stable morphisms from [Ber3, 1.2]. The author expects (or hopes) that poly-stable modification is possible over any qcqs base scheme.

We will prove the two above modification theorems only when η is the spectrum of a field (so, S is integral) and Q is the class of generically étale alterations. The case of an arbitrary qcqs scheme S will be deduced in a subsequent work [Tem3] by use of relative Riemann-Zariski spaces $\mathrm{RZ}_\eta(S)$. Both theorems are proved via a similar method which, as the author hopes, can be useful when studying other P-modification problems. So, it seems plausible to describe this method briefly. Note also that our method seems to be very close to the approach of K. Fujiwara and F. Kato, as outlined in [FK].

(i) Uniqueness: add extra conditions to your problem, so that the required modification ψ becomes uniquely defined or functorial in S'.

(ii) Analytic input: prove the theorem over the valuation ring of a compete algebraically closed field K of height one.

(iii) Decompletion: use approximation to deduce the theorem over valuation rings of height one.

(iv) Induction on height: deduce the theorem over valuation rings of finite height.

(v) Limit: deduce the theorem over valuation rings.

(vi) The general case: use the Riemann-Zariski space $\mathrm{RZ}_\eta(S)$ and the uniqueness property from (i) to deduce the general case.

The first two steps are critical. Naturally, one can hope to incorporate some non-Archimedean analytic geometry over K into the second step (it will be so in our two cases). We stress that it is necessary to consider the case of an arbitrary analytic K, including the cases when K is not isomorphic to the completed algebraic closure of a discretely valued field, e.g. the case when $\mathrm{rk}_{\mathbf{Q}}(|K^\times|) > 1$.

In the case of the reduced fiber theorem, we will take the Grauert-Remmert finiteness theorem (see Step 2 in the proof of Theorem 3.5.5) as an analytic input. To achieve functoriality we will consider η-normalizations instead of arbitrary modifications ψ. For the semi-stable modification theorem, we achieve uniqueness by considering stable modifications rather then semi-stable ones. Our main analytic input here is the uniformization of analytic fields established by Theorem 6.3.1 from which we deduced in §2.2 uniformization of valued fields via steps (iii)-(iv). In §4 we will deduce stable modification over valuation rings, and in §5 we will work out step (vi) for stable modification (with (v) obtained as a by-product). For the sake of completeness, we note that alternatively one could deduce from Theorem 6.3.1 analytic stable modification theorem of Bosch-Lütkebohmert (it is easy, and was done in unpublished master thesis of the author), and then steps (iii)-(vi) of the method could be worked out for the assertion of the stable modification theorem itself.

3.5. **Reduced fiber theorem.** In this section we apply the method from §3.4 to the reduced fiber theorem [BLR, 2.1'], see Theorem 3.5.5. Recall that in this paper we treat only the particular case when $\eta = \mathrm{Spec}(K)$. So, we will show that up to a generically étale alteration of the base, any finitely presented morphism $X \to S$ with geometrically reduced η-fiber can be *η-modified* to a flat morphism $X' \to S$ with geometrically reduced fibers (i.e. $X' \to X$ is proper and induces an isomorphism $X'_\eta \widetilde{\to} X_\eta$ on η-fibers).

Lemma 3.5.1. *Let R be a valuation ring and A be an R-algebra. Then A is R-flat if and only if A has no π-torsion for any non-zero element $\pi \in R$. If A is R-flat then it is finitely presented over R if and only if it is finitely generated over R.*

Proof. The first part is easy, so we omit the proof. The second statement is much deeper. It holds more generally over any integral ring R, as proved in [RG, 3.4.7]. □

The following result is critical for the proof of the reduced fiber theorem; it ensures uniqueness of the η-modification in the theorem. The author is indebted to [BLR, 2.3(v)] and [BL2, 2.5(c)] for an elegant idea of a proof based on the theory of depth and Z-closures as developed in [EGA, IV_2,§§5.9-5.10].

Proposition 3.5.2. *Assume that S is normal. Let $\phi : X \to S$ be a flat finitely presented morphism with reduced geometric fibers. Then X is η-normal in the sense that any finite η-modification of X is an isomorphism.*

Proof. Clearly, X is reduced. Assume, on the contrary, that X is not η-normal and pick up a non-trivial finite η-modification $f : X' \to X$. Let us check that it is harmless to assume that S, X and X' are noetherian. By noetherian approximation, see §3.1, there exists a normal noetherian scheme S_0 with a morphism $S \to S_0$ such that f is induced from a modification $f_0 : X'_0 \to X_0$ of finitely presented S_0-schemes. Moreover, by Remark 3.1.1(i) we can achieve that X_0 has geometrically reduced S_0-fibers. It now suffices to contradict that f_0 is not an isomorphism, so replacing S and f with S_0 and f_0, respectively, we can assume that S, X and X' are noetherian.

Lemma 3.5.3. *Let $\phi : X \to S$ be a flat finitely presented morphism of schemes. Then the set $(X/S)^0$ of the generic points of the S-fibers of X is closed under generalizations.*

Proof. The claim is local on S, so we can assume that it is qcqs. Then by noetherian approximation we reduce the question to the case of a universally catenary noetherian S (e.g. of finite type over $\mathbf{Z}$). For any point $x \in X$ with $s = \phi(x)$ the inequality $\dim(\mathcal{O}_{X,x}) \geq \dim(\mathcal{O}_{S,s})$ holds, and x is in $(X/S)^0$ if and only if this is an exact equality. For any generalization $x' \succ x$ with $s' = \phi(x')$ the dimension drop $\dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{X,x'})$ cannot be smaller than $\dim(\mathcal{O}_{S,s}) - \dim(\mathcal{O}_{S,s'})$ because any chain $s' \succ s_1 \succ \cdots \succ s$ can be lifted to a chain $x' \succ x_1 \succ \cdots \succ x$ by the going down theorem [Mat, 9.5] applied to the homomorphism $\mathcal{O}_{S,s}/m_{s'} \to \mathcal{O}_{X,x}/m_{s'}\mathcal{O}_{X,x}$.

Therefore, if $x \in (X/S)^0$ and x' is any its generalization then the two dimension drops are equal and $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{S,s'})$. Thus, $x' \in (X/S)^0$, as claimed. $\square$

The Lemma implies that the set $U = X_\eta \cup (X/S)^0$ is closed under generalizations. By [EGA, IV$_4$, 17.5.1] ϕ is smooth at $(X/S)^0$, hence X is normal at the points of $(X/S)^0$ by [EGA, IV$_2$, 6.8.3(i)], and therefore f is an isomorphism over $(X/S)^0$. Since f is an η-modification, it is an isomorphism over the whole U. Let z be a point of $Z = X \setminus U$, $s = \phi(z)$ and $Y = X_s$ the s-fiber of X. Then, $\mathrm{prof}(\mathcal{O}_{X,z}) = \mathrm{prof}(\mathcal{O}_{Y,z}) + \mathrm{prof}(\mathcal{O}_{S,s})$ by [EGA, IV$_2$, 6.3.1] (set $M = A = \mathcal{O}_{S,s}$ and $N = B = \mathcal{O}_{X,z}$ in loc.cit.). Since $z \notin Y^0$ and Y has no embedded components, $\mathrm{prof}(\mathcal{O}_{Y,z}) \geq 1$. Also, $\mathrm{prof}(\mathcal{O}_{S,s}) \geq 1$ because $s \neq \eta$ and S is integral. Hence $\mathrm{prof}(\mathcal{O}_{X,z}) \geq 2$, and [EGA, IV$_2$, 5.10.4] implies that $\mathcal{O}_X$ is Z-closed. Recall that the latter means that $\mathcal{O}_X \widetilde{\to} \mathcal{H}^0_{X/Z} = \mathrm{inj}\lim(\pi_i)_*(\mathcal{O}_{U_i})$, where $\{U_i\}$ is the family of open neighborhoods of U and $\pi_i : U_i \to X$ are the open immersions. It remains to notice that $f_*(\mathcal{O}_{X'}) \subset \mathcal{H}^0_{X/Z}$ because f is an isomorphism near each $u \in U$. So, $f_*(\mathcal{O}_{X'}) = \mathcal{O}_X$, and therefore f is an isomorphism. $\square$

Proposition 3.5.2 gives a new insight on the reduced fiber theorem. Note that η-normality does not have to be preserved by base changes $f : S' \to S$ with normal S', but the Proposition implies that if an η-normal S-scheme X has reduced geometric fibers then its base changes with normal S''s are η-normal. Thus, an η-normal X with reduced geometric S-fibers can be viewed as *stably η-normal* with respect to S. Then the reduced fiber theorem can be interpreted as a stabilization theorem which states that if X is finitely presented over S and has geometrically reduced generic fiber, then it can be made η-normal by a generically étale alteration of the base S and subsequent η-normalization of the base change of X. This also explains why the heart of the proof is the finite presentation result of Grauert-Remmert (see Step 2 in the proof of Theorem 3.5.5) – we have to assure that stabilization can be achieved after a reasonably small base change (e.g. after an alteration of the base).

Corollary 3.5.4. *Let S be an integral qcqs scheme, $\phi : X \to S$ be a flat finitely presented morphism and $f_i : X_i \to X$, $i = 1, 2$, be two finite η-modifications. Assume that X_1 and X_2 are S-flat with geometrically reduced fibers. Then there exists a finite modification $S' \to S$ such that $X_1 \times_S S'$ and $X_2 \times_S S'$ are $X \times_S S'$-isomorphic and such an isomorphism is unique.*

Proof. Let $\widetilde{S}$ be the normalization of S. Then each $\widetilde{X}_i = X_i \times_S \widetilde{S}$ is a finite η-modification of $\widetilde{X} = X \times_S \widetilde{S}$, which is η-normal by Proposition 3.5.2. So, each $\widetilde{X}_i$ is the η-normalization of $\widetilde{X}$, i.e. $\widetilde{X}_1 \widetilde{\to} \widetilde{X}_2$. By approximation, this isomorphism descends to an isomorphism $X_1 \times_S S' \widetilde{\to} X_2 \times_S S'$ for a finite modification $S' \to S$. $\square$

Probably, X_i are already X-isomorphic, but proving this will require a new argument (similarly to the situation with stable modifications, see Remark 5.3).

Theorem 3.5.5. *Let $X \to S$ be a dominant finitely presented morphism with a geometrically reduced generic fiber. Then there exists a generically étale alteration $S' \to S$, and a finite η-modification $X' \to X \times_S S'$ such that X' is flat, finitely presented and has reduced geometric fibers over S'.*

Proof. Step 0. *Flattening. There exists a modification $S' \to S$ and a finite η-modification $X' \to X \times_S S'$ such that X' is S'-flat.* It is a particular case of the flattening by blow ups theorem of Raynaud-Gruson, see [RG, 5.2.2]. Thus, we can assume that X is S-flat.

Step 1. *Localization. We can assume that* $S = \mathrm{Spec}(R)$ *and* $X = \mathrm{Spec}(A)$ *are affine.* Find a finite affine covering $\{S_i\}$ of S and finite affine coverings $\{X_{ij}\}$ of $\phi^{-1}(S_i)$. If the affine case is established then we can find generically étale alterations $S'_{ij} \to S_i$ such that $X_{ij} \times_{S_i} S'_{ij}$ admit finitely presented η-modifications X'_{ij} with geometrically reduced fibers. Notice that the same properties hold for any further alteration $S''_{ij} \to S'_{ij}$, hence by Proposition 3.2.6 we can alter S'_{ij} so that they are open subschemes of a generically étale alteration S' of S. Note that $\mathrm{RZ}_{K^s}(S'_{ij}) = \mathrm{RZ}_{K^s}(S_i)$ form an open covering of $\mathrm{RZ}_{K^s}(S) = \mathrm{RZ}_{K^s}(S')$, hence the schemes S'_{ij} form a covering of S'. We did not rule out the possibility that X'_{ij} do not agree on intersections (i.e. that the preimages of $X_{ij} \cap X_{kl}$ in X'_{ij} and X'_{kl} are not isomorphic), but we know from Corollary 3.5.4 that X'_{ij} do agree after an additional finite modification $S'' \to S'$ of the base. Then $X''_{ij} = X'_{ij} \times_{S'} S''$ glue to a required η-modification $X'' \to X \times_S S''$. This completes the step, and in the sequel we assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$.

Step 2. *Analytic input. Grauert-Remmert finiteness theorem.* We make ultimate use of the following fact, which is a consequence of Grauert-Remmert theorem, see [BGR, §6.4 and 6.4.1/4]: assume that K is an algebraically closed complete field of height one, $\mathcal{A}$ is a reduced affinoid K-algebra and $A \subset \mathcal{A}^\circ$ is a topologically finitely generated K°-subalgebra with $A \otimes_{K^\circ} K \widetilde{\to} \mathcal{A}$, then $\mathcal{A}^\circ$ is finite over A (where $\mathcal{A}^\circ \subset \mathcal{A}$ denotes the subalgebra of power-bounded elements).

Step 3. *Decompletion. The Theorem holds when* K *is a separably closed valued field of height one and* $R = K^\circ$. Set $A_K = A \otimes_R K$, and let $A^\circ = \mathcal{N}r_{A_K}(A)$ denote the integral closure of A in A_K. It suffices to show that A° is finitely presented over R and has geometrically reduced fibers (actually we know from Proposition 3.5.2 that this is the only way the Theorem can hold for $S = \mathrm{Spec}(R)$ since such S has no non-trivial separable alterations). Choose a non-zero $\pi \in m_R$ and provide A and R with the π-adic topology. We can assume that $\pi^{-1} \notin A$ because otherwise $A_K = A$ and there is nothing to prove. Let $\widehat{A}$ and $\widehat{R}$ be the completions, then $\widehat{A}$ is topologically finitely presented and flat over $\widehat{R}$, $\widehat{K} = \widehat{R}[\pi^{-1}]$ is the completion of K, and the $\widehat{K}$-affinoid algebra $\mathcal{A} = \widehat{A}[\pi^{-1}]$ is reduced by the following argument. It suffices to prove reducedness of the formal completion of $\mathcal{A}$ along a maximal ideal m. Set $A_{\widehat{K}} = A \otimes_R \widehat{K}$, then $\mathcal{M}(\mathcal{A})$ is an affinoid domain in the analytification of the $\widehat{K}$-variety $\mathrm{Spec}(A_{\widehat{K}})$ given by $|f_i(x)| \le 1$ where $f_1, \dots, f_n$ is a set of generators of A over R. Since m corresponds to a Zariski closed (or rigid) point of $\mathcal{M}(\mathcal{A})$, the ideal $p = m \cap A_{\widehat{K}}$ is a maximal ideal and $\widehat{(A_{\widehat{K}})}_p \widetilde{\to} \widehat{\mathcal{A}}_m$. But the excellent ring $A_{\widehat{K}}$ is reduced by our assumptions, hence so is any its completion along a maximal ideal.

By Step 2, $\mathcal{A}^\circ$ is finite over $\widehat{A}$, and since $\mathcal{A}^\circ \subseteq \mathcal{A} = (\widehat{A})_\pi$ we obtain that $\mathcal{A}^\circ = \widehat{A}[a_1/\pi_1 \dots a_n/\pi_n]$ for some choice of $a_i \in \widehat{A}$ and $\pi_i \in R$. We can replace a_i with any element a'_i with $a_i - a'_i \in \pi_i \widehat{A}$. Since A is dense in $\widehat{A}$, we can achieve that $a_i \in A$. Now, we will prove that A° is finitely presented over R by showing that it coincides with $B = A[a_1/\pi_1, \dots, a_n/\pi_n]$, which is finitely presented by Lemma 3.5.1. Note that for any finitely presented and flat R-algebra C and an element $\omega \in R$ we have that $\omega\widehat{C} \cap C = \omega C$. Indeed, if a sequence ωx_j of elements of C converges to $x \in C$ then $x - \omega x_j$ is divisible by ω for sufficiently large j, hence x is divisible by ω as well. Now fix $1 \le i \le n$. Since a_i/π_i is integral over $\widehat{A}$, there exist

$m \in \mathbf{N}$ and $b_j \in \widehat{A}$ such that $x := a_i^m + b_1 a_i^{m-1}\pi_i + \cdots + b_{m-1}a_i\pi_i^{m-1} \in \pi_i^m\widehat{A}$. The inclusion survives when we move b_j's slightly, hence we can achieve that $b_j \in A$. It then follows that $x \in \pi_i^m\widehat{A} \cap A = \pi_i^m A$, and therefore a_i/π_i is integral over A. We obtain that $B \subseteq A^\circ$, so B is integral over A. Since B is finitely generated over A it is finite. Moreover, since B is finitely presented over R, it is finitely presented over A as an algebra, and then it is finitely presented over A as a module. The latter implies that $\widehat{B} = \widehat{A}[a_1/\pi_1 \dots a_n/\pi_n] = \mathcal{A}^\circ$. Let $b \in B$ and $\omega \in R$ be such that $b/\omega \in A^\circ$, then $b/\omega \in \widehat{B}$ and, as we saw earlier (with $B = C$), this implies that $b \in \omega B$. Thus, $b/\omega \in B$, and we have proved that $B = A^\circ$, as required.

It remains to show that $A^\circ \otimes_R \widetilde{K}$ is geometrically reduced. Since $\widetilde{K}$ is algebraically closed we have to prove that any non-zero element $\widetilde{a} \in A^\circ \otimes_R \widetilde{K}$ is not nilpotent. If it is not so then there exists an element $a \in A^\circ \setminus m_R A^\circ$ such that a^n is divisible by an element $x \in m_R$. Since $|K^\times|$ is divisible, we can replace x with y^n such that $|x| = |y|^n$. Then a/y is in A° because A° is integrally closed in $A^\circ[\pi^{-1}]$, and we obtain a contradiction to the assumption that $a \notin m_R A^\circ$.

Step 4. *Composition and induction on height. The Theorem holds when K is a separably closed valued field of finite height and $R = K^\circ$.* We will use Lemma 2.1.4 to carry out induction on the height of K. (Note that this process respects the separably closed hypothesis on fraction fields of valuation rings of finite height under consideration.) An easy inductive argument on height proves that the group $|K^\times|$ is divisible, and then the same proof as in the end of the previous step shows that $A^\circ \otimes_R \widetilde{K}$ is reduced, where $A^\circ = \mathcal{N}r_{A_K}(A)$, as earlier. Thus, we have only to prove that A° is finitely presented over R, and by Lemma 3.5.1 it is enough to check that A° is finitely generated. While proving the latter we can obviously replace A with any finitely generated ring B with $A \subseteq B \subseteq A^\circ$. If m is the minimal non-zero prime ideal of R then $R_1 := R_m$ is the localization of R of height one and its maximal ideal mR_m coincides with m. Furthermore, $\widetilde{R} = R/m$ is a valuation ring with the separably closed fraction field $L := R_1/m$, and the valuation induced by R is composed from those induced by R_1 and $\widetilde{R}$.

Since the heights of $\widetilde{R}$ and R_1 are smaller than the height of R, we can assume that the Theorem holds for $\widetilde{R}$ and R_1. So, for $A_1 := A \otimes_R R_1$ the algebra $A_1^\circ := \mathcal{N}r_{A_K}(A_1)$ is R_1-finitely presented with a geometrically reduced special fiber $\widetilde{A}_1 = A_1^\circ/mA_1^\circ$. Set $T = R \setminus m$, then $R_1 = R_T$ and $A_1 = A_T$. Note also that $A_1^\circ = (A^\circ)_T$ because normalization is compatible with localization. Let $a_1, \dots, a_k$ be R_1-generators of A_1°. Multiplying a_i's by elements from T we can achieve that $a_i \in A^\circ$. Then we replace A with $A[a_1, \dots, a_k]$ achieving that $A_1 = A_1^\circ$. Now, A and A° coincide after inverting T and next we will study the situation modulo m. Since m is T-divisible due to the structure of $\mathrm{Spec}(R)$, mA_1° equals to both mA and mA°, and hence $\widetilde{A} := A/mA$ is embedded into $\widetilde{A}^\circ := A^\circ/mA^\circ$. Note that it is enough to prove that $\widetilde{A}^\circ = \widetilde{A}[\widetilde{b}_1, \dots, \widetilde{b}_l]$ because it would follow immediately that $A^\circ = A[b_1, \dots, b_l]$ for any choice of liftings $b_i \in A^\circ$ of $\widetilde{b}_i$.

Note that $\widetilde{A}^\circ$ is integral over $\widetilde{A}$ because A° is integral over A. Note also that applying $\otimes_{\widetilde{R}} L$ to both rings we obtain the geometrically reduced L-algebra $\widetilde{A}_1$. It follows, that $\widetilde{A}^\circ$ is contained in $\mathcal{N}r_{\widetilde{A}_1}(\widetilde{A})$, which is finite over $\widetilde{A}$ by the induction assumption (applied to $\widetilde{R}$). We claim that actually, $\widetilde{A}^\circ = \mathcal{N}r_{\widetilde{A}_1}(\widetilde{A})$ and proving this will finish Step 4. Any element of $\mathcal{N}r_{\widetilde{A}_1}(\widetilde{A})$ is of the form $\widetilde{a}/\widetilde{\pi}$ where $a \in A$,

$\pi \in T$ and $\widetilde{a} \in \widetilde{A}$, $\widetilde{\pi} \in \widetilde{R} \setminus \{0\}$ are their reductions modulo m. Furthermore, there exist $\widetilde{b}_1, \dots, \widetilde{b}_n \in \widetilde{A}$ such that $\widetilde{b}_n = 1$ and $\sum_{i=1}^n \widetilde{b}_i \widetilde{a}^i \widetilde{\pi}^{n-i} \in \widetilde{\pi}^n \widetilde{A}$. Lifting $\widetilde{b}_i$'s to some elements $b_i \in A$ with $b_n = 1$ and using that $m \subset \pi^n A$ we obtain that $\sum_{i=1}^n b_i a^i \pi^{n-i} \in \pi^n A$ and hence $a/\pi \in A^\circ$. Thus, the reduction $\widetilde{a}/\widetilde{\pi}$ lies already in $\widetilde{A}^\circ$ and we are done.

Step 5. *A limit argument. The Theorem holds in general.* Since any valuation ring coincides with the union of all its valuation subrings of finite height by Lemma 2.1.1(ii), the Theorem holds when K is an arbitrary valued field and $R = K^\circ$. Let us pass to the general case. The Riemann-Zariski space $\mathfrak{S} = \mathrm{RZ}_{K^s}(S)$ introduced in §3.2 is homeomorphic to the projective limit of all generically étale alterations of $S = \mathrm{Spec}(R)$. To give a point $\mathfrak{x} \in \mathfrak{S}$ is equivalent to give a valuation ring $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$ of K^s which contains R. For any point $\mathfrak{x} \in \mathfrak{S}$ set $A_\mathfrak{x} = A \otimes_R \mathcal{O}_{\mathfrak{S},\mathfrak{x}}$. We know from the previous step that the $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$-algebra $A'_\mathfrak{x} = \mathcal{N}r_{A_K}(A_\mathfrak{x})$ is finitely presented and has geometrically reduced fibers. It follows from Lemma 3.2.2 by approximation that the morphism $\mathrm{Spec}(\mathcal{O}_{\mathfrak{S},\mathfrak{x}}) \to S$ factors through a generically étale quasi-alteration $S_\mathfrak{x} = \mathrm{Spec}(R_\mathfrak{x}) \to S$ satisfying the following condition: there exists a finite η-modification $X'_\mathfrak{x} \to X \times_S S_\mathfrak{x}$ such that the geometric fibers of the morphism $X'_\mathfrak{x} \to S_\mathfrak{x}$ are reduced.

By Corollary 3.2.7, the Riemann-Zariski space $\mathfrak{S}_\mathfrak{x} = \mathrm{RZ}_{K^s}(S_\mathfrak{x})$ can be naturally identified with an open subspace of $\mathfrak{S}$ containing $\mathfrak{x}$. Since $\mathfrak{S}$ is quasi-compact by Proposition 3.2.1, we can find finitely many points $\mathfrak{x}_i \in \mathfrak{S}$ such that the corresponding quasi-alterations $S_{\mathfrak{x}_i}$ are such that their Riemann-Zariski spaces cover $\mathfrak{S}$. Now we act exactly as in Step 1. By Proposition 3.2.6, replacing $S_{\mathfrak{x}_i}$ with generically étale alterations, we can achieve that they glue to a generically étale alteration $S' \to S$. Then after an additional finite modification of the base, the schemes $X'_{\mathfrak{x}_i}$ glue to an η-modification $X' \to X \times_S S'$ which is as required. □

4. Desingularization of curves over valuation rings

Throughout §4 we assume that $S = \mathrm{Spec}(R)$ for a valuation ring R of finite height and with separably closed fraction field K. In particular, $|K^\times|$ is divisible and each point $s \in S$ has separably closed residue field $k(s)$, which is even algebraically closed if $s \neq \eta$ (this is easily seen for analytic fields, and the general case follows by decompletion and induction on height). For such S we will prove Theorems 1.2 and 1.5. The first proof is easy, and the second one runs in two main stages. The first stage is standard, but slightly technical: we prove that any semi-stable modification of (C, D) can be blown down successively until a stable modification is obtained. The heart of the second stage is Proposition 4.3.3 which is an analog of local uniformization. It asserts that locally along a valuation (which is interpreted as a point in a Riemann-Zariski space) (C, D) admits a semi-stable quasi-modification. We deduce this Proposition from uniformization of valued fields established in Theorem 2.1.8. In the sequel, (C, D) is a multipointed S-curve with reduced C and D and structure morphism $(\phi : C \to S, \phi_D : D \to S)$. Other multipointed S-curves will be denoted as (C', D'), $(\overline{C}, \overline{D})$, etc.

Remark 4.1. Note that any η-modification $f : C' \to C$ extends uniquely to an η-modification $(C', D') \to (C, D)$ by taking D' to be the schematical closure of D_η in $C'_\eta \widetilde{\to} C_\eta$ (we use that D' is finitely presented by Lemma 3.5.1). For this reason, we will often denote an η-modification $(f_C, f_D) : (C', D') \to (C, D)$ only by use of the η-modification $f = f_C : C' \to C$.

4.1. **Reduction to the case of a smooth generic fiber.** Since K is separably closed, it follows that any étale morphism $S' \to S$ is Zariski locally an isomorphism. Also, normalization $\overline{C}$ of C is finitely presented over S by Theorem 3.5.5 (and Proposition 3.5.2), and so it is an S-curve. The semi-stable generic fiber C_η can be obtained from its normalization $\overline{C}_\eta$ by gluing together pairs of points $(x_1, y_1), \dots, (x_n, y_n)$ where $z = \{z_1, \dots, z_n\} = (C_\eta)_{\mathrm{sing}}$ and $\overline{z} = x \cup y$ is the preimage of z. In other words, C_η is the pushout of the diagram $\overline{C}_\eta \leftarrow \overline{z} \to z$ with the second map taking x_i and y_i to z_i. Let X, X_i, Y, Y_i denote the closures of x, x_i, y, y_i in $\overline{C}$, and define $Z \hookrightarrow C$ similarly (everything is reduced, so we can use the usual Zariski closure instead of the schematical one). Also, let $\pi : \overline{C} \to C$ be the projection and $\overline{D} = \pi^{-1}(D) \cup X \cup Y$. Note that if $(C', D') \to (C, D)$ is a semi-stable η-modification then D' is disjoint from Z' (which is the closure of z in C') and hence the normalization $\overline{C}'$ of C' underlies a semi-stable modification $\overline{f} : (\overline{C}', \overline{D}') \to (\overline{C}, \overline{D})$. We will show that the converse is true when $\overline{f}$ is projective, that is for any projective semi-stable modification $(\overline{C}', \overline{D}') \to (\overline{C}, \overline{D})$ there exists a unique semi-stable η-modification $f : (C', D') \to (C, D)$ such that $\overline{C}'$ is the normalization of C'.

Let $\overline{X}_i$ and $\overline{Y}_i$ be the closures of x_i and y_i in $\overline{C}'$. Obviously, the only possible way for defining C' is to glue the closed subschemes $\overline{X} = \sqcup \overline{X}_i$ and $\overline{Y} = \sqcup \overline{Y}_i$ in $\overline{C}'$ (disjointness follows from semi-stability of $(\overline{C}', \overline{D}')$). First, we note that indeed each $\overline{X}_i$ is isomorphic to $\overline{Y}_i$ because they are S-étale and proper over Z_i. So, set $\overline{Z} = \overline{X} \sqcup \overline{Y}$ and let $\overline{Z} \to Z$ be the morphism identifying $\overline{X}$ and $\overline{Y}$. Our aim now is to paste $\overline{X}$ and $\overline{Y}$, that is to define a scheme C' as the pushout of $\overline{C}' \leftarrow \overline{Z} \to Z$. (We will not need this, but it is well known that such pushout C' always exists in the category of algebraic spaces. In our case, C' will be a scheme due to projectivity of $\overline{C}' \to \overline{C}$.) We can work locally over C, so assume that C, and hence, $\overline{C}$ are affine. Furthermore, it suffices to construct the pushout $\pi' : \overline{C}' \to C'$ on a neighborhood of $\overline{Z}$ in $\overline{C}'$. By our assumption, $\overline{C}'$ is projective over the affine scheme $\overline{C}$, hence the finite set $\overline{Z}_0$ of closed points of $\overline{Z}$ possesses an affine neighborhood. Since $\overline{Z}$ is a semi-local scheme, we can replace $\overline{C}'$ with this affine neighborhood. We will no longer need the projectivity, so our problem reduces to the case when $\overline{C}' = \mathrm{Spec}(A)$, and then the affine pushout is defined as $C' = \mathrm{Spec}(B)$, where B consists of all elements $h \in A$ such that the restrictions of h on the closed subschemes $\overline{X}$ and $\overline{Y}$ are compatible with the isomorphism $\overline{X} \widetilde{\to} \overline{Y}$. It is well known that C' is the pushout in the category of schemes, so we omit this check. We claim that (C', D') is a semi-stable η-modification of (C, D), where $D' = \pi'(\overline{D}' \setminus \overline{Z})$. Indeed, gluing along S-flat closed subschemes commutes with base changes, so we can check semi-stability on the S-fibers, and then the claim is trivial.

Since any stable modification morphism is projective by Theorem 1.1, it now suffices to prove Theorems 1.2 and 1.5 for the multipointed curve $(\overline{C}, \overline{D})$ and the case of (C, D) will follow by the above pushout construction (we leave it to interested reader to check on S-fibers that the pushout of $\overline{C}_{\mathrm{st}}$ produces C_{st} and the normalization of C_{st} is $\overline{C}_{\mathrm{st}}$). Thus, we reduced the problem to the case of curves with smooth generic fiber.

4.2. **Modifications of curves over valuation rings.** Until the end of §4 we assume that C_η is smooth. This assumption will slightly simplify the terminology because it implies that any modification of C is an η-modification. Nevertheless, all intermediate results of §4 can be easily generalized at cost of replacing $\mathrm{RZ}_{k(C)}(C)$, normality, modifications, etc., with $\mathrm{RZ}_{C_\eta}(C)$, η-normality, η-modifications, etc. In this section we also assume that D is empty. We can restrict ourselves to the case of a connected C, and then C_η is also connected by S-flatness of C. Thus, C is irreducible, and we denote by L the field of rational functions of C. Note that any modification C' of C is an S-curve because $\mathcal{O}_{C'}$ has no R-torsion (and so C' is S-flat and hence finitely presented).

Let $\mathfrak{C} = \mathrm{RZ}_L(C)$ be the Riemann-Zariski space of C as defined in §3.2. By Proposition 3.2.1 $\mathfrak{C}$ is qcqs. The space $\mathfrak{C}$ is provided with a sheaf of rings $\mathcal{O}_\mathfrak{C}$ whose stalks are valuation rings of L. To give a point $\mathfrak{x} \in \mathfrak{C}$ is equivalent to give a valuation ring $\mathcal{O}_{\mathfrak{C},\mathfrak{x}}$ containing R and a morphism $\psi_\mathfrak{x} : \mathrm{Spec}(\mathcal{O}_{\mathfrak{C},\mathfrak{x}}) \to C$ respecting L.

Next, we attach to C the set $(C/S)^0 = \cup_{s\in S} C^0_s$, which is closed with respect to generalization by Lemma 3.5.3. For any modification $C' \to C$, by $\Gamma(C')$ we denote the preimage of $(C'/S)^0$ in $\mathfrak{C}$. Note that $\Gamma(C')$ is contained in the subset $\mathfrak{C}^0 \subset \mathfrak{C}$ which is defined as follows: a point $\mathfrak{x} \in \mathfrak{C}$ is in $\mathfrak{C}^0$ if the valuation ring $\mathcal{O} = \mathcal{O}_{\mathfrak{C},\mathfrak{x}}$ satisfies $\mathrm{tr.deg.}((\mathcal{O}/m_\mathcal{O})/(R'/m_{R'})) = 1$, where $R' = \mathcal{O} \cap K$. In this case the valuation ring $\mathcal{O}$ is bounded over R' in the sense of the definition above Theorem 2.1.8. We remark that analogs of $\mathfrak{C}^0$ are the set of type 2 points of a non-Archimedean analytic curve, or the set of divisorial valuations in the Riemann-Zariski space of an algebraic surface. Let C_0 denote the set of closed points of C, and define $\mathfrak{C}^0_0$ and $\Gamma_0(C')$ to be the preimages of C_0 in $\mathfrak{C}^0$ and $\Gamma(C')$, respectively. Although we will not use that, we note that it is not difficult to show that $\mathfrak{C}^0$ and $\mathfrak{C}^0_0$ are the unions of the sets $\Gamma(C')$ and $\Gamma_0(C')$, respectively, where C' runs over all modifications of C.

Given a modification $f : Y \to X$, by the *modification locus* of f we mean the minimal closed set $Z \subset X$ such that f is an isomorphism over $X \setminus Z$. Sometimes, we will treat Z as a reduced closed subscheme.

Lemma 4.2.1. *Assume that C is normal, $x \in (C/S)^0$ is a point and $f : C' \to C$ is a modification with the modification locus Z. Then*

(i) the set $f^{-1}(x)$ consists of a single point x' and $\mathcal{O}_{C,x} \widetilde{\to} \mathcal{O}_{C',x'}$;

(ii) Z is quasi-finite over S, and so $|Z|$ is a finite set.

Proof. Note that for any point $y \in (C/S)^0$, the fiber $f^{-1}(y)$ is a finite set of generic points of $C'_{\phi(y)}$. Since $Y := \mathrm{Spec}(\mathcal{O}_{C,x})$ is contained in $(C/S)^0$ by Lemma 3.5.3, we obtain that $Y' = Y \times_C C'$ has finite fibers over Y, i.e. the morphism $f' : Y' \to Y$ is quasi-finite. By [EGA, IV_3, 8.11.1], the modification $f' : Y' \to Y$ is finite. Since Y is normal, f' is an isomorphism. This proves (i), and (ii) follows. □

The Lemma implies that any point $x \in (C/S)^0$ possesses a unique preimage $\mathfrak{x}$ in $\mathfrak{C}$ and $\mathcal{O}_{C,x} \widetilde{\to} \mathcal{O}_{\mathfrak{C},\mathfrak{x}}$. In particular, $\mathcal{O}_{C,x}$ is a valuation ring. The following obvious observation will often be used in the sequel: for any neighborhood U of Z the modification f is defined by its restriction $f_U : f^{-1}(U) \to U$.

Lemma 4.2.2. *Assume that C' is normal. Then $\Gamma(C') \widetilde{\to} (C'/S)^0$, the set $\Gamma_0(C')$ is finite, and $\Gamma(C') = gen(\Gamma_0(C')) \cup \Gamma(C)$, where $gen(\Gamma_0(C')) \subset \mathfrak{C}^0$ is the set of all generalizations of the points of $\Gamma_0(C')$.*

Proof. The bijection is explained above and obviously the sets $\Gamma(C')$ and $\Gamma_0(C')$ are finite. Furthermore, the set $(C'/S)^0$ is closed under generalizations by Lemma 3.5.3, hence its preimage $\Gamma(C') \subset \mathfrak{C}$ is also closed under generalizations. In particular, $gen(\Gamma_0(C')) \cup \Gamma(C) \subseteq \Gamma(C')$. To establish an equality it suffices to prove that if $x \in (C'/S)^0$ is *contracted* in C (i.e. is mapped to a closed point in some C_s) then x possesses a specialization $y \in (C'/S)^0$ which is mapped to a closed point of C. The closure X of x has one-dimensional S-fibers (i.e. any non-empty S-fiber is one-dimensional), but the image of X in C has zero-dimensional S-fibers by Lemma 3.5.3. Let $s \in S$ be the closed point of the image of X, and find an irreducible curve $Y \subset X_s$. Then Y is closed in X, and it is contracted to a point in C which is closed by properness of the morphism $C' \to C$. Thus, the generic point y of Y is as required. □

Suppose that $f_i : C_i \to C$, $i = 1, 2$, are two modifications. If f_1 factors through C_2 then we say that C_1 *dominates* C_2. Since f_i are modifications, the domination morphism $C_1 \to C_2$ is unique.

Proposition 4.2.3. *Let C' and C'' be two modifications of C with $\Gamma_0(C') \subseteq \Gamma_0(C'')$, and assume that C'' is normal. Then C'' dominates C'.*

Proof. Lemma 4.2.2 implies that $\Gamma(C') \subseteq \Gamma(C'')$. Let $\overline{C}$ be the schematical closure of $\mathrm{Spec}(L)$ in $C' \times_C C''$; it is a modification of C which dominates both C' and C''. If $\Gamma(\overline{C}) = \Gamma(C'')$ then the modification $f' : \overline{C} \to C''$ is finite because its S-fibers $f'_s : \overline{C}_s \to C''_s$ do not contract components. By normality of C'', f' is an isomorphism, hence C'' dominates C', as required.

It remains to prove that indeed $\Gamma(\overline{C}) = \Gamma(C'')$. Suppose on the contrary that $\mathfrak{x} \in \Gamma(\overline{C}) \setminus \Gamma(C'')$. Let s and x be the images of $\mathfrak{x}$ in S and $\overline{C}$ then obviously $x \in (\overline{C}/S)^0$. From other side the images of x in C'_s and C''_s are closed points because $\mathfrak{x}$ is not in $\Gamma(C'')$ and $\Gamma(C')$. Therefore x has to be a closed point of $\overline{C}_s$, and the contradiction finishes the proof. □

4.3. **Local uniformization.** For any affine scheme $\overline{S} = \mathrm{Spec}(A)$ consider two examples of multipointed semi-stable $\overline{S}$-curves: $C' = \mathrm{Spec}(A[T])$ and $D' = \{T = 0\}$, or $C' = \mathrm{Spec}(A[U,V]/(UV - a))$, $a \in A$ and $D' = \emptyset$. A multipointed S-curve C is called *strictly semi-stable*, if locally on C and $\overline{S}$, C admits an étale morphism $f : (C, D) \to (C', D')$ (i.e. both f_C and f_D are étale) to one of the above curves. Obviously, strict semi-stability implies semi-stability. If $\overline{S} = \mathrm{Spec}(k)$ for a separably closed field then the strictness condition means that the irreducible components of C are smooth.

Proposition 4.3.1. *Assume that C is a strictly semi-stable S-curve with smooth generic fiber C_η. Then strictly semi-stable blow ups of C (i.e. blow ups $C' \to C$ with C' strictly semi-stable over S) are cofinal in the family of all its modifications.*

Proof. We claim that it is enough to prove the following claim: for any element $\mathfrak{x} \in \mathfrak{C}_0^0$ there exists a strictly semi-stable blow up $h : C' \to C$ such that $\mathfrak{x} \in \Gamma_0(C')$. Indeed, it is well known that blow ups are preserved under compositions (see, for example, [Con1, 1.2]), hence given any finite subset $F \subset \mathfrak{C}_0^0$ we can apply the claim iteratively to construct a strictly semi-stable blow up $C' \to C$ with $F \subset \Gamma_0(C')$. Then Proposition 4.3.1 would follow from Proposition 4.2.3 and normality of semi-stable S-curves with smooth generic fiber.

Now, let $x \in C$ be the center of $\mathfrak{x} \in \mathfrak{C}_0^0$. We will act in two stages. At first stage we will blow C up so that it becomes S-smooth at the center of $\mathfrak{x}$ and at the second stage we will make the center a generic point of its S-fiber. Note that if at some stage x is not a closed point of its fiber then it is already a generic point of its S-fiber and we are done. We will build these blow ups so that x is the only closed point of the modification locus; this allows to work locally at x since each such blow up extends trivially from a neighborhood of x to all of C. In particular, we can replace R with its localization beneath x, so now x is in the special fiber.

At first stage we assume that C is not S-smooth at x. Localizing we can assume that there exists an étale morphism $C \to \overline{C}$ with $\overline{C} = \operatorname{Spec}(A)$, $A = R[u,v]/(uv-a)$, $a \in R$, $a \neq 0$ and such that the image of x is the point $\overline{x}$ with $u(\overline{x}) = v(\overline{x}) = 0$. Let $\overline{\mathfrak{x}}$ be the image of $\mathfrak{x}$ in $\mathrm{RZ}_{k(\overline{C})}(\overline{C})$, that is $\mathcal{O}_{\overline{\mathfrak{x}}} = \mathcal{O}_{\mathfrak{x}} \cap k(\overline{C})$. It is enough to find a blow up $\overline{f} : \overline{C}' \to \overline{C}$ such that $\overline{\mathfrak{x}}$ is centered on a smooth point $\overline{x}' \in \overline{C}'$ and $\overline{x}$ is the only closed point of the modification of $\overline{f}$. Indeed, if such $\overline{f}$ exists then $C' := \overline{C}' \times_{\overline{C}} C$ is a strictly semi-stable blow up of C and the center $x' \in C'$ of $\mathfrak{x}$ is smooth since it is the preimage of $\overline{x}'$ under the étale morphism $C' \to \overline{C}'$. Thus, replacing C and $\mathfrak{x}$ with $\overline{C}$ and $\overline{\mathfrak{x}}$, we can assume that $C = \operatorname{Spec}(A)$. Consider the valued fields L and K with the valuations induced by $\mathcal{O} = \mathcal{O}_{\mathfrak{C},\mathfrak{x}}$ and R. Since $\mathrm{tr.deg.}(L/K) = 1 = \mathrm{tr.deg.}(\widetilde{L}/\widetilde{K})$, $E_{L/K} = 0$ by the Abhyankar inequality. In particular, the group $G = |L^\times|/|K^\times|$ is a torsion group. By our assumption K is separably closed, hence $|K^\times|$ is divisible, $G = 1$ and $|R| = |\mathcal{O}|$. In particular, we can find an element $\pi \in R$ such that $|u| = |\pi|$ in L. Note that $|uv| = |a|$ and $|v| \le 1$, hence $|\pi| \ge |a|$ and so $\pi | a$ in R. Let $C' = \mathrm{Bl}_{(u,\pi)}(C)$ be the blow up of C along the ideal (u,π). Since C' is covered by the charts $\operatorname{Spec}(A[v, \frac{u}{\pi}]/(v\frac{u}{\pi} - \pi^{-1}a))$ and $\operatorname{Spec}(A[u, \frac{\pi}{u}]/(u\frac{\pi}{u} - \pi))$ (where the fractions serve as indeterminants, as is customary for blow up formulas), we obtain that C' is semi-stable. Finally, x is the only closed point of the modification locus of $C' \to C$, and the center $x' \in C'$ of $\mathfrak{x}$ is contained in the S-smooth open subscheme $\operatorname{Spec}(A[\frac{u}{\pi}, \frac{\pi}{u}]/(\frac{u}{\pi}\frac{\pi}{u} - 1))$ of C' because $\frac{\pi}{u} \in \mathcal{O}^\times$.

Now we assume that the center x is smooth and we want to make it the generic point of an irreducible component in the fiber. Let $s \in S$ be the image of x. Locally at x there exists an étale morphism $C \to \operatorname{Spec}(A)$ with $A = R[T]$, and the same argument as in the first stage shows that we can actually assume that $C = \operatorname{Spec}(A)$. Consider the valued fields L and K as earlier. Note that $\widetilde{L}$ is generated over $\widetilde{K}$ by the residues of the elements $f(T)/\pi$, where f is an irreducible monic polynomial and $\pi \in R$. Indeed, any element $f(T) \in L \setminus R$ with $|f(T)| = 1$ can be represented as $\prod (f_i(T_i)/\pi_i)^{\varepsilon_1}$, where f_i are irreducible over K, $\pi_i \in R$, $\varepsilon_i \in \{\pm 1\}$ and $|f_i/\pi_i| = 1$. Moreover, we can take only separable irreducible polynomials because the residues of $f(T)/\pi$ and $(f(T) + \omega T)/\pi$ coincide for any $\omega \in \pi m_R$. Since any separable irreducible polynomial of $K[T]$ is linear, we can find $a \in K$ and $\pi_a \in R$ such that the residue of $(T-a)/\pi_a$ is transcendental over $\widetilde{K}$. Note that $|\pi_a| < 1$ as otherwise $\pi_a \in R^\times$ and $\mathfrak{x}$ would be centered on the generic point of $C_s = \operatorname{Spec}(k(s)[T])$, contradicting the assumption that x is closed. Also, if $\widetilde{a} \in k(s)$ is such that x is the point of C_s given by $T = \widetilde{a}$ and $a' \in R$ is a lifting of $\widetilde{a}$, then $|T - a'| < 1$ and hence $|a - a'| < 1$. So, $a \in R$ and we can consider the blow up $C' = \mathrm{Bl}_{(\pi_a, T-a)}(C)$ covered by the charts $\operatorname{Spec}(A[\frac{T-a}{\pi_a}])$ and $\operatorname{Spec}(A[T, \frac{\pi_a}{T-a}]/((T-a)\frac{\pi_a}{T-a} - \pi_a))$. Now one checks straightforwardly that C' is a strictly semi-stable modification of C,

x is the closed point given by $T - a = 0$ and it is the only closed point of the modification locus of $h : C' \to C$, $Z = h^{-1}(x)$ is a $\mathbf{P}^1_{k(s)}$-component of C' and $\mathfrak{x}$ is centered at the generic point of Z, as required. □

We will also need the following Lemma, where a nonempty D is allowed.

Lemma 4.3.2. *Assume that for a multipointed S-curve (C, D) with a semi-stable generic fiber (C_η, D_η) the S-curve C is strictly semi-stable with smooth generic fiber C_η. Then there exists a blow up $C' \to C$ underlying a strictly semi-stable modification $(C', D') \to (C, D)$.*

Proof. We will construct the required blow up by composing few intermediate blow ups. The assumptions that K is separably closed and D_η is K-smooth imply that D_η is a union of few smooth K-points. It is well known that there exists a modification $f : C' \to C$ which separates the irreducible components D_i of D. For example, since D' is the strict transform of D it follows from [Con1, 1.4] that one can separate each pair D_{i_1} and D_{i_2} by blowing up C along $D_{i_1} \times_C D_{i_2}$. Applying this procedure few times we can separate all components. By Proposition 4.3.1 C' admits a strictly semi-stable modification C'' which is a blow up of C, hence we can replace (C, D) with a modification (C'', D'') achieving that C is strictly semi-stable over S and the irreducible components of D are disjoint. Using Lemma 3.5.1 to show that $D \to S$ is finitely presented, we obtain that D is Zariski locally isomorphic to S. Thus, the only problem can arise if D intersects the singular locus of $\phi : C \to S$.

We will show that there exists a semi-stable blow up $(C', D') \to (C, D)$ whose center is contained in D. Note that it suffices to prove this stronger claim locally at a closed point $d \in D \cap C_{\mathrm{sing}}$. In particular, similarly to the argument in Proposition 4.3.1 it suffices to study the following model situation: $C = \mathrm{Spec}(R[u, v]/(uv - a))$ and $D \widetilde{\to} S$ is given by $u = \pi$ for an element $\pi \in R$ with $|a| < |\pi| < 1$. Then (again, similarly to the proof of Proposition 4.3.1) the blow up $C' = \mathrm{Bl}_{(\pi,u)}(C)$ is a strictly semi-stable modification of C such that the pair (C', D') is as required because D' is contained in $\mathrm{Spec}(R[\frac{u}{\pi}, \frac{\pi}{u}]/(\frac{u}{\pi}\frac{\pi}{u} - 1))$ and hence is disjoint form the singular locus of C'. □

Finally, we establish a local uniformization of an S-curve along a valuation.

Proposition 4.3.3. *Let $C \to S$ be a curve with smooth C_η. Then there exist quasi-modifications $C_1 \to C, \dots, C_m \to C$ such that C_i are strictly semi-stable S-curves and any point of $\mathfrak{C}$ is centered on some C_i.*

Proof. We can assume that $C = \mathrm{Spec}(A)$ is affine. Since $\mathfrak{C}$ is quasi-compact and $\mathrm{RZ}_L(C_i) \hookrightarrow \mathfrak{C}$ is an open space embedding by Corollary 3.2.7, it suffices to show that any point $\mathfrak{x} \in \mathfrak{C}$ can be centered on a strictly semi-stable quasi-modification of C. To simplify notation we set $\mathcal{O} = \mathcal{O}_{\mathfrak{C},\mathfrak{x}}$. Also, we provide $L = k(C)$ with a valuation induced by $\mathcal{O}$.

By Theorem 2.1.8 the one-dimensional valued K-field L is uniformizable because $K\mathcal{O}$ is centered on the smooth K-curve C_η. Therefore there exists an element $T \in L$ such that L is unramified over $k(T)$, and so $\mathcal{O}$ is étale over $\mathcal{O} \cap k(T)$ by Lemma 2.1.1. Replacing T with T^{-1}, if necessary, we can assume that $T \in \mathcal{O}$. Furthermore, replacing A with $A[T] \subset L$ and C with its quasi-modification $\mathrm{Spec}(A[T])$ we achieve that $T \in A$ while $\mathcal{O}$ is still centered on C. In particular, we obtain a morphism

$T : C \to \overline{C} = \mathrm{Spec}(\overline{A})$, where $\overline{A} = k[T]$. Let $\overline{\mathfrak{C}} = \mathrm{RZ}_{k(T)}(\overline{C})$ be the Riemann-Zariski space of $\overline{C}$, then a natural map $\mathfrak{C} \to \overline{\mathfrak{C}}$ arises, and we denote the image of $\mathfrak{x}$ by $\overline{\mathfrak{x}}$. Notice that $\overline{\mathcal{O}} = \mathcal{O}_{\overline{\mathfrak{C}},\overline{\mathfrak{x}}}$ coincides with $\mathcal{O} \cap k(T)$, so $\mathcal{O}$ is étale over $\overline{\mathcal{O}}$. In the sequel, we will few times replace C and $\overline{C}$ with their affine quasi-modifications whose preimages in $\mathfrak{C}$ and $\overline{\mathfrak{C}}$ contain $\mathfrak{x}$ and $\overline{\mathfrak{x}}$, respectively. For simplicity, the new curves will be also denoted $C = \mathrm{Spec}(A)$ and $\overline{C} = \mathrm{Spec}(\overline{A})$.

Let $T_1, \dots, T_l$ be $\overline{\mathcal{O}}$-generators of $\mathcal{O}$. Replacing C with a quasi-modification if necessary, we can assume that $T_i \in A$. By flattening theorem, there exists a modification $\overline{C}' \to \overline{C}$ such that the schematical closure of $\mathrm{Spec}(L)$ in $C \times_{\overline{C}} \overline{C}'$ is flat over $\overline{C}'$. Thus, replacing C and $\overline{C}$ with their quasi-modifications, we can achieve that A is $\overline{A}$-flat. It follows that $A \otimes_{\overline{A}} \overline{\mathcal{O}}$ is a subalgebra of L containing T_i's, hence $A \otimes_{\overline{A}} \overline{\mathcal{O}} \widetilde{\to} \mathcal{O}$. Let $\{\overline{C}_i\}_{i \in I}$ denote the family of modifications of $\overline{C}$, and let $\overline{x}_i$ denote the centers of $\overline{\mathfrak{x}}$ on $\overline{C}_i$, then $\cup_{i \in I} \mathcal{O}_{\overline{C}_i,\overline{x}_i} \widetilde{\to} \overline{\mathcal{O}}$. By [EGA, IV$_4$, 17.7.8], $A \otimes_{\overline{A}} \mathcal{O}_{\overline{C}_i,\overline{x}_i}$ is étale over $\mathcal{O}_{\overline{C}_i,\overline{x}_i}$ for some $i \in I$. Moreover, by Proposition 4.3.1 enlarging i we can achieve that $\overline{C}_i$ is strictly semi-stable. Then the center of $\mathfrak{x}$ on $C_i = C \times_{\overline{C}} \overline{C}_i$ is contained in the strictly semi-stable locus of C_i. So, this locus is a required quasi-modification of C. □

4.4. **Blowing down to a stable modification.** In this section we will prove that any semi-stable modification can be blown down to a stable modification. In addition, we will prove Theorem 1.2 (in the case of $S = \mathrm{Spec}(K^\circ)$). Similarly to Castelnuovo's contraction of exceptional curves on surfaces, this involves some cohomological technique. We will need the notion of $\mathbf{P}^1_k$-trees introduced in Appendix B.

Lemma 4.4.1. *Let C be an S-curve with geometrically reduced C_η.*

(i) $\mathcal{N}r_{C_\eta}(C)$ is an S-curve (i.e. it is finitely presented over S) with geometrically reduced fibers, and $\mathcal{N}r_{C_\eta}(C) = \mathcal{N}r(C)$ if C_η is smooth.

(ii) If C is normal and affine then it can be embedded into a projective S-curve $\overline{C}$ with geometrically reduced S-fibers.

Proof. We will prove (ii) and the proof of (i) is similar (using Proposition 3.5.2). Since a reduced K-scheme is geometrically reduced if and only if it is generically so, we can easily find an S-projective compactification $C \hookrightarrow C'$ with geometrically reduced C'_η. Recall that any separable alteration of S is an isomorphism, hence by Theorem 3.5.5 there exists a finite η-modification $\overline{C} \to C'$ that has geometrically reduced S-fibers. This map is an isomorphism over C, so $\overline{C}$ is as required. □

Lemma 4.4.2. *Let $f : (C', D') \to (C, D)$ be a modification of a multipointed S-curve (C, D), $x \in C$ be a point, $Z = f^{-1}(x)$ and $s = \phi(x)$. Assume that C is normal and Z is a curve contained in the semi-stable locus of C'. Then (C, D) is semi-stable at x if and only if Z is an exceptional $\mathbf{P}^1_{k(s)}$-tree in the sense of Appendix B (i.e. $|\partial_{C'_s}(Z)| + |D'_s \cap Z| \le 2$).*

Proof. The question is local in x. Obviously, x is closed in its fiber C_s. Localizing R we can assume that s is the closed point of S, and then x is closed in C. Shrinking C we can assume that C is connected, normal and affine. The modification locus V of f is quasi-finite over S, hence by Lemma 2.1.1 $V_x := \mathrm{Spec}(\mathcal{O}_{V,x})$ is an open neighborhood of x in V. Thus, shrinking C again, we can achieve that x is the only closed point of V. By Lemma 4.4.1, we can embed C into a connected normal

S-projective curve $\overline{C}$. Define $\overline{D}$ as the closure of D in $\overline{C}$. Since V is closed in $\overline{C}$, we can extend f trivially outside of C obtaining a modification $f' : \overline{C}' \to \overline{C}$. So, it suffices to solve our problem for the projective multipointed S-curves $\overline{C}$ and $\overline{C}'$.

Now, we can assume that C is S-projective. By Lemma 4.4.1(i), we can replace C' with $\mathcal{N}r_{C'_\eta}(C')$ (no change near Z by Proposition 3.5.2) achieving that C' has geometrically reduced S-fibers. Note that any finite, connected and dominant S-scheme maps bijectively onto S. Applying the Zariski connectedness theorem, we obtain that the S-fibers of C' are connected (usually, Zariski connectedness theorem is formulated for a noetherian base scheme S, e.g. [EGA, III_1, 4.3.2], but C' comes from a curve defined over a noetherian base). It follows that $h^0(C'_t) = 1$ for any $t \in S$. Since the Euler-Poincare characteristic of the fibers is constant on S (again, [EGA, III_2, 7.9.4] is formulated for noetherian schemes, but the general case follows by noetherian approximation), we see that the arithmetic genus $h^1(C'_t)$ of the fibers is constant on S. Hence $h^1(C'_s) = h^1(C_\eta) = h^1(C_s)$, and the case of connected Z follows from Corollary B.2. Finally, Z is connected because otherwise C is not normal at x by Zariski connectedness theorem. $\square$

Corollary 4.4.3. *Assume that C_η is smooth and (C, D) has a stable modification $(C_{\mathrm{st}}, D_{\mathrm{st}})$ and a semi-stable modification (C', D').*

(i) If both modifications are dominated by a semi-stable modification $(\overline{C}, \overline{D})$ then C' dominates C_{st}.

(ii) The assumption of (i) is satisfied when C is a modification of a strictly semi-stable S-curve C_1.

Proof. For (i), we note that Lemmas 4.4.2 and B.3 imply for any $s \in S$ the following claim: if an irreducible component $Z \subset \overline{C}_s$ is contracted in C'_s then it is contracted in $(C_{\mathrm{st}})_s$ too. Therefore, C' dominates C_{st} by Proposition 4.2.3. To prove (ii) we note that both C' and C_{st} are modifications of C_1 and so they are dominated by a strictly semi-stable modification $\widetilde{C}$ by Proposition 4.3.1. Let $\widetilde{D}$ be the Zariski closure of D_η in $\widetilde{C}$. Then the multipointed curve $(\widetilde{C}, \widetilde{D})$ possesses a semi-stable modification $(\overline{C}, \overline{D})$ by Lemma 4.3.2. Obviously, the latter dominates both $(C_{\mathrm{st}}, D_{\mathrm{st}})$ and (C', D'). $\square$

Let us assume that (C', D') is a semi-stable modification of (C, D). We will show that by successive blowing down exceptional components of S-fibers of C' one can construct a stable modification $(C_{\mathrm{st}}, D_{\mathrm{st}})$ of (C, D). Let $E = E(C', D')$ be the set of exceptional components of the fibers C'_s. We identify E with a subset of $\Gamma(C')$ and set $E_0 = E_0(C', D') = E \cap \Gamma_0(C')$.

Lemma 4.4.4. *Assume that C_η is smooth. A semi-stable modification $f : (C', D') \to (C, D)$ is stable if and only if E_0 is empty.*

Proof. Only the converse implication needs a proof. Any semi-stable modification factors through the normalization of C, hence we can replace C with its normalization (and update D accordingly). Assume that $Z \subset C'_s$ is an exceptional component. Acting as in the proof of Lemma 4.4.2 one establishes the following claim: if $t \in S$ is a specialization of s then any irreducible component $Z' \subset C'_t$ lying in the Zariski closure of Z is exceptional. Indeed, Z' lies in the modification locus of f, hence it is contracted to a point $x \in C$ by normality of C. Now similarly to the mentioned proof we shrink C about x, compactify it and compute genera. It remains to notice that the generic point of Z has a specialization $z \in \Gamma_0(C')$, and therefore z is the generic point of an exceptional component belonging to E_0. The contradiction shows that E is empty, i.e. the modification f is stable. $\square$

In the sequel, by *exceptional blow down* of (C', D') we mean a normal modification $(C'', D'') \to (C, D)$ such that C' dominates C'', the morphism $C' \to C''$ contracts exactly one point of $\Gamma_0(C')$ and that point is in $E = E(C', D')$ (so it is even in E_0).

Lemma 4.4.5. *If (C'', D'') is an exceptional blow down of (C', D') then (C'', D'') is semi-stable.*

Proof. Semi-stability is an open condition by Lemma 5.1(i) below (no circular reasoning occurs here), hence it suffices to prove that (C'', D'') is semi-stable at closed points. Let $Z \in E_0$ be the component contracted in C'' and $x \in C''$ be its image, then x is the only closed point of the modification locus of $C' \to C''$. It remains to note that (C'', D'') is semi-stable at x by Lemma 4.4.2. $\square$

Proposition 4.4.6. *Assume that C_η is smooth. If (C, D) admits a semi-stable modification (C', D') then it admits a stable modification $(C_{\mathrm{st}}, D_{\mathrm{st}})$.*

Proof. Assume that (C', D') is not stable. Let $Z \in E_0$ be an exceptional component of C'_s; it exists by Lemma 4.4.4. It suffices to find an exceptional blow down $C' \to C''$ which contracts Z. Indeed: (C'', D'') is semi-stable by the above Lemma, and if $E_0(C'', D'')$ is not empty then we can blow down C'' further, etc. This process must stop because we can perform at most $|\Gamma_0(C')|$ contractions.

Now we will reduce to the case of S-projective C as in the proof of Lemma 4.4.2. Since $f : C' \to C$ factors through $\mathcal{N}r(C)$, we can assume that C is normal. Let x be the image of Z in C and $s = \phi(x)$. The problem is local in x, hence we can localize S and shrink C, so that s is closed, C is connected and affine and x is the only closed point of the modification locus V of f (V is quasi-finite over S and we use Lemma 2.1.1). Embed C into a connected normal S-projective curve $\overline{C}$ (and take $\overline{D}$ to be the closure of D) and define $(\overline{C}', \overline{D}') \to (\overline{C}, \overline{D})$ as the trivial extension of $(C', D') \to (C, D)$. It now suffices to find an exceptional blow down $\overline{C}' \to \overline{C}''$ which contracts Z. To simplify the notation we replace $\overline{C}' \to \overline{C}$ with $C' \to C$ achieving that C is S-projective.

Let $Z_1, \dots, Z_n$ be other irreducible components of C'_s. For any $1 \le i \le n$, find a point $P_i \in C'_\eta$ such that the s-fiber of its closure $\overline{P}_i$ is a smooth point $p_i \in Z_i$, and so $\overline{P}_i$ lies in the smooth locus of C'. In particular, $\cup \overline{P}_i$ is a Cartier divisor. Note that $Y := C'_s = (\cup Z_i) \cup Z$, $Z \widetilde{\to} \mathbf{P}^1_{k(s)}$ intersects $\cup Z_i$ transversally, and the intersection is exactly $\partial_{C'_s}(Z)$, so it contains at most two points. Using that $h^1(Z, \mathcal{O}_Z) = 0$ it follows easily that $H^1(\mathcal{O}_Y(m \sum p_i)) = 0$ for sufficiently large m.

Consider the R-flat sheaf $\mathcal{L} = \mathcal{O}_{C'}(m \sum \overline{P}_i)$. Then $h^1(\mathcal{L}_s) = 0$ and therefore $h^1(\mathcal{L}_\eta) = 0$ by semi-continuity. Since the Euler-Poincare characteristic of $\mathcal{L}$ is constant, $h^0(\mathcal{L}_\eta) = h^0(\mathcal{L}_s)$. Applying the theorem of Grauert and Grothendieck on base changes and direct images, see [Har, III.12.9], or [EGA, III_2, 7.6.9 and 7.7.5], we obtain that the homomorphism $H^0(\mathcal{L}) \to H^0(\mathcal{L}_s)$ is onto (the cited results are formulated in noetherian setting, so we use noetherian approximation).

For sufficiently large m, there exists a section $h_s \in H^0(\mathcal{L}_s)$ which does not vanish at p_i's. Find a lifting $h \in H^0(\mathcal{L})$ and consider it as a meromorphic function on C'. The pole divisor of h is at most $\overline{P} = m \sum \overline{P}_i$, and the zero divisor V does not pass through the points p_i because h_s has poles of order exactly m at p_i. Thus, $V\cap\overline{P}\cap C'_s=\emptyset$, and therefore the intersection of V and $\overline{P}$ is empty. It follows that h defines a morphism $C'\to\mathbf{P}^1_S$, the induced morphism $\overline{h}:C'\to\mathbf{P}^1_C$ contracts Z to a point and Z is the only component of C'_s contracted by $\overline{h}$. Hence, one can take C'' to be the normalization of $\overline{h}(C')$ and define D'' accordingly (with semi-stability of (C',D') following from Lemma 4.4.2). □

4.5. **Gluing local models.**

Proposition 4.5.1. *Theorems 1.2 and 1.5 hold for $S=\mathrm{Spec}(K^\circ)$ where K is a separably closed valued field of a finite height.*

Proof. We showed in §4.1 that the case of smooth C_η implies the general case by a pushout procedure. So, we assume in the sequel that C_η is smooth. Note that if C is a modification of a strictly semi-stable S-curve then Theorem 1.2 holds true for (C,D) by Corollary 4.4.3. Next, let us prove Theorem 1.5. It suffices to prove that (C,D) possesses a semi-stable modification because then we can construct the stable modification by Proposition 4.4.6. By Proposition 4.3.3 there exist strictly semi-stable quasi-modifications $C_1,\dots,C_m$ of C such that any element of $\mathfrak{C}$ is centered on some C_i. Let D_i be the closure of $D_\eta\cap(C_i)_\eta$ in C_i.

By Proposition 3.2.6 there exists a modification $C'\to C$ and open subschemes $C'_i\subset C'$ such that C'_i are C-isomorphic to modifications of C_i. Let $D'_i\hookrightarrow C'_i$ be the closure of $(D_i)_\eta$ in C'_i. By our assumptions, $\mathfrak{C}$ is covered by the subspaces $\mathrm{RZ}_L(C_i)$, where $L=k(C)$. Hence C'_i's cover C' by the valuative criterion of properness and surjectivity of the projection $\mathfrak{C}\to C'$. Consider the curve (C_i,D_i) with the modification (C'_i,D'_i). Applying to them first Proposition 4.3.1 and then Lemma 4.3.2 we can construct a semi-stable modification of (C'_i,D'_i). Then Proposition 4.4.6 implies that each (C'_i,D'_i) admits a stable modification $f_i:(C''_i,D''_i)\to(C'_i,D'_i)$. In this case, stable modification is unique by Corollary 4.4.3 (with C_i being a ground strictly semi-stable curve), hence f_i's agree over the intersections $C'_i\cap C'_j$, and so they glue to a stable modification $(C'',D'')\to(C',D')$. Thus, (C'',D'') is a semi-stable modification of (C,D), and it remains to use Proposition 4.4.6 once again to construct a stable blow down of $(C'',D'')\to(C,D)$.

Finally, let us prove Theorem 1.2 in general. Given a stable modification $(C_{\mathrm{st}},D_{\mathrm{st}})$ and a semi-stable modification (C',D') of (C,D), find a modification (C'',D'') which dominates them. By Theorem 1.5 there exists a stable modification $(\overline{C},\overline{D})$ of (C'',D'') which obviously dominates both (C',D') and $(C_{\mathrm{st}},D_{\mathrm{st}})$. Hence (C',D') dominates $(C_{\mathrm{st}},D_{\mathrm{st}})$ by Corollary 4.4.3, and we are done. □

We will not need the following aside remark, so we omit a detailed proof of its assertion.

Remark 4.5.2. It follows from Proposition 4.5.1 that if $\mathbf{P}$ denotes semi-stability then for a multipointed relative S-curve (C,D) with a $\mathbf{P}$ η-fiber the family of its $\mathbf{P}$ modifications is cofinal in the family of all its modifications. (Actually, we established particular cases of this result while proving the Proposition.) Using pushout technique of §4.1 one can show that a similar cofinality result holds also for strict semi-stability and for ordinary relative curves, where we say that (C,D) is *ordinary* over S if the geometric fibers $(C_{\overline{s}},D_{\overline{s}})$ are ordinary in the sense of Appendix B. We will not make any use of ordinary curves, but it is described in Appendix A.3 how they appear in some proofs of the stable reduction theorem.

5. Proof of the main results

In this section we only assume that S is integral qcqs and with generic point η. Also, $K = k(\eta)$.

Lemma 5.1. *Let $(C', D') \to (C, D)$ be a modification of multipointed S-curves then:*

(i) the set of points $x \in C'$ at which (C', D') is semi-stable is open;

(ii) the set of points $s \in S$ for which (C'_s, D'_s) is semi-stable and $(C'_{\overline{s}}, D'_{\overline{s}})$ has no exceptional components for a geometric point $\overline{s}$ lying over s is constructible.

Proof. By approximation we can assume that S is of finite type over $\mathbf{Z}$. It is well known that the semi-stable locus of $C' \to S$ is open (e.g. use the local description from [dJ, 2.23]). It follows that the semi-stable locus of (C', D') is also open because étaleness of $D' \to S$ and disjointness of D' from $(C'/S)_{\text{sing}}$ are open conditions. This proves (i), and we also obtain that (C', D') is semi-stable over a constructible set whose complement is the image of the not semi-stable locus of (C', D').

It suffices now to prove that exceptional components show up in precisely the geometric fibers over a constructible subset $T \subset S$ when (C', D') is semi-stable. Let Z be the union of all irreducible components in the fibers C'_s that are contracted in C to a point. Thus, Z has proper S-fibers and is closed in C' by [EGA, IV_3, 13.1.3]. To each geometric fiber $Z_{\overline{s}}$ we associate the combinatorial data consisting of its incidence graph – vertex per generic point and edge per self-intersection, and also we provide each vertex with the weight equal to the arithmetic genus of its irreducible component. We claim that the combinatorial data of the geometric fiber over a point $s \in S$ is a locally constructible function of S. For the topological data without weights this follows from the results of [EGA, IV_3, §9.7], see loc.cit. 9.7.9 and 9.7.12; and for the genera this follows from existence of a stratification of S (by reduced locally closed subschemes) which flattens the morphism $Z \to S$ and from the semi-continuity theorem [EGA, III_2, 7.6.9] (to prove that flattening exists use that the flat locus is open by [EGA, IV_3, 11.1.1]). Since each exceptional component is an irreducible component in some $Z_{\overline{s}}$ which satisfies obvious combinatorial properties, it follows that T is constructible. $\square$

Recall that the Riemann-Zariski space $\mathfrak{S} = \text{RZ}_{K^s}(S)$ is homeomorphic to the projective limit of all generically étale alterations of S, and a point $\mathfrak{x} \in \mathfrak{S}$ is defined by a valuation ring $\mathcal{O}_{\mathfrak{S},\mathfrak{x}}$ of K^s and a morphism $\phi_{\mathfrak{x}} : \mathfrak{S}_{\mathfrak{x}} = \text{Spec}(\mathcal{O}_{\mathfrak{S},\mathfrak{x}}) \to S$ which agrees with $\text{Spec}(K^s) \to S$. By $C_{\mathfrak{x}} = (C_{\mathfrak{x}}, D_{\mathfrak{x}})$ we will denote the multipointed $\mathfrak{S}_{\mathfrak{x}}$-curve $(C, D) \times_S \mathfrak{S}_{\mathfrak{x}}$. We start with an analog of Proposition 4.2.3.

Proposition 5.2. *Let S be an integral noetherian scheme with generic point $\eta = \text{Spec}(K)$, and let $X \to C$ and $Y \to C$ be η-modifications, where C, X, Y are reduced flat S-schemes of finite type. Assume that Y is η-normal, and for any $\mathfrak{x} \in \mathfrak{S} = \text{RZ}_K(S)$ the modification $Y_{\mathfrak{x}} \to C_{\mathfrak{x}}$ factors through $X_{\mathfrak{x}}$. Then Y dominates X.*

Proof. By Lemma 3.2.2 and approximation, for any point $\mathfrak{x} \in \mathfrak{S}$ there exists a quasi-modification $S_{\mathfrak{x}} \to S$ such that $Y_{\mathfrak{x}} = Y \times_S S_{\mathfrak{x}}$ dominates $X_{\mathfrak{x}} = X \times_S S_{\mathfrak{x}}$, i.e. the modification $Y_{\mathfrak{x}} \to C_{\mathfrak{x}}$ factors through $X_{\mathfrak{x}}$. Since $\text{RZ}_K(S)$ is quasi-compact and $\text{RZ}_K(S_{\mathfrak{x}}) \hookrightarrow \text{RZ}_K(S)$ is a neighborhood of $\mathfrak{x}$ by Corollary 3.2.7, we can find a finite set of quasi-modifications $S_i \to S$ such that $\text{RZ}_K(S) = \cup_i \text{RZ}_K(S_i)$ and $Y \times_S S_i \to C \times_S S_i$ factors through $X \times_S S_i$. By Proposition 3.2.6, replacing S_i with their modifications we can achieve that S_i are open subschemes of a modification

S' of S. Set $C' = C \times_S S'$, $X' = X \times_S S'$ and $Y' = Y \times_S S'$, then $Y' \to C'$ factors through X'.

We want now to pull down the η-modification $f' : Y' \to X'$ to an η-modification $Y \to X$. For an S-scheme T let $(T/S)_0 \subset T$ denote the set of points closed in the S-fibers of T. Choose a closed point $y \in Y$ and let s be its image in S, so y is closed in the fiber Y_s. Since Y is η-normal, $Y' \to Y$ is its own Stein factorization and so the preimage Y'_y of y in Y' is connected. Clearly $(Y'/S')_0$ contains the preimage of $(Y/S)_0$ under the projection $Y' \to Y$, and similarly for X' and X. So $Y'_y \subset (Y'/S')_0$, hence $f'(Y'_y) \subset (X'/S')_0$, yet it is a closed set in $X_s \times S'_s$, so it must be quasi-finite over S'_s. Thus its image in X_s must be s-quasi-finite. By connectedness of Y'_y this image must be a single point $x \in X_s$. Obviously, $\mathcal{O}_{X,x} \subset \mathcal{O}_{Y'}(Y'_y)$, where $\mathcal{O}_{Y'}(Y'_y) = \operatorname{inj\,lim}_U \mathcal{O}_{Y'}(U)$ for U running through all open neighborhoods of Y'_y, and also $\mathcal{O}_{Y,y} = \mathcal{O}_{Y'}(Y'_y)$ by Stein factorization. Thus, we have proved that for any closed point $y \in Y$ there exists a point $x \in X$ whose local ring is contained in the local ring of y. It follows easily that the modification $Y \to C$ factors through X (and y is taken to x). □

Proof of Theorem 1.2. First we reduce to the case of S of finite type over $\mathbf{Z}$. Indeed, (C, D), $(C_{\mathrm{st}}, D_{\mathrm{st}})$ and (C', D') are induced from multipointed curves $(\overline{C}, \overline{D})$, $(\overline{C}_{\mathrm{st}}, \overline{D}_{\mathrm{st}})$ and $(\overline{C}', \overline{D}')$ over a scheme $\overline{S}$ of finite type over $\mathbf{Z}$, and by Lemma 5.1(ii) and Remark 3.1.1(ii) one can also achieve that $(\overline{C}_{\mathrm{st}}, \overline{D}_{\mathrm{st}})$ is stable and $(\overline{C}', \overline{D}')$ is semi-stable. Now in order to show that $C' \to C$ factors through $C_{\mathrm{st}} \to C$ it is enough to show that $\overline{C}' \to \overline{C}$ factors through $\overline{C}_{\mathrm{st}} \to \overline{C}$. So, we can replace S, C, D, etc., with $\overline{S}, \overline{C}, \overline{D}$, etc., achieving that S is of finite type over $\mathbf{Z}$. In particular, any valuation in $\mathfrak{S}$ is of finite height by Abhyankar inequality 2.1.2. Since C' is semi-stable over S, it is η-normal by Proposition 3.5.2. For any $\mathfrak{x} \in \mathfrak{S}$, the modification $((C_{\mathrm{st}})_{\mathfrak{x}}, (D_{\mathrm{st}})_{\mathfrak{x}}) \to (C_{\mathfrak{x}}, D_{\mathfrak{x}})$ is stable and the $\mathfrak{S}_{\mathfrak{x}}$-curve $(C'_{\mathfrak{x}}, D'_{\mathfrak{x}})$ is semi-stable. By Corollary 4.4.3, $C'_{\mathfrak{x}}$ dominates $(C_{\mathrm{st}})_{\mathfrak{x}}$, hence C' dominates C_{st} by Proposition 5.2. □

Next we deduce the Corollaries from the Introduction. To prove Corollary 1.3 we note that if $(C'_{\mathrm{st}}, D'_{\mathrm{st}})$ is another stable modification then C_{st} and C'_{st} dominate each other. Hence $C_{\mathrm{st}} \widetilde{\to} C'_{\mathrm{st}}$, and we get (i). The semi-stable locus U of $(C, D) \to S$ is an open subscheme of C by Lemma 5.1(i), so $f^{-1}(U)$ is a stable modification of U and $U_{\mathrm{st}} \widetilde{\to} U$ by minimality of the stable modification.

Corollary 1.4 follows from Theorem 1.2 as follows. The normalization S'' of S is the projective limit of finite modifications. Hence by approximation, the domination morphism $\overline{C} \times_S S'' \to C_{\mathrm{st}} \times_S S''$ can be defined already over a finite modification S' of S.

Remark 5.3. It is not clear if the above modification $S' \to S$ is necessary at all. Simple examples, see [AO, 3.17], show that extension of a stable curve C_η to a stable S-curve $C \to S$ can be not unique when S is not normal. However, stable modification is a more subtle creature (for example, it exists when $C \to S$ is not proper). The author does not know examples where stable modification is not unique.

Proof of Theorem 1.5. Since (C, D) is induced from a multipointed curve over some S_0 of finite type over $\mathbf{Z}$, it suffices to build a stable modification over S_0. So, it suffices to deal with S of finite type over $\mathbf{Z}$, and we will assume that this is the case. In particular, each valuation from $\mathfrak{S} = \mathrm{RZ}_{K^s}(S_0)$ is now of finite height. For any point $\mathfrak{x} \in \mathfrak{S}$, the relative multipointed curve $(C_\mathfrak{x}, D_\mathfrak{x}) \to \mathfrak{S}_\mathfrak{x}$ possesses a stable modification $(C'_\mathfrak{x}, D'_\mathfrak{x})$ by Proposition 4.5.1. By Lemma 3.2.2 and approximation, there exist a generically étale quasi-alteration $S' \to S$ and an η-modification $(C', D') \to (C, D) \times_S S'$ such that $\mathfrak{x}$ is centered on a point $x \in S'$ and $(C'_\mathfrak{x}, D'_\mathfrak{x}) \to \mathfrak{S}_\mathfrak{x}$ is the base change of $(C', D') \to S'$. Moreover, by Lemma 5.1(ii) and Remark 3.1.1(ii) we can achieve that (C', D') is a stable modification of $(C, D) \times_S S'$.

Now we can act similarly to the proof of Proposition 5.2. Since $\mathrm{RZ}_{K^s}(S') \hookrightarrow \mathfrak{S}$ is a neighborhood of $\mathfrak{x}$ and $\mathfrak{S}$ is quasi-compact, we can find a finite set of generically étale quasi-alterations $S_i \to S$ such that $\mathfrak{S} = \cup_i \mathrm{RZ}_{K^s}(S_i)$ and the multipointed curves $(C, D) \times_S S_i$ admit stable modifications (C_i, D_i). Replacing S_i with their generically étale alterations, we can achieve that S_i are open subschemes of a generically étale alteration S' of S. Let $\overline{S}$ be the normalization of S' and $\overline{S}_i$ be the preimages of S_i in $\overline{S}$ then the curves $(C, D) \times_S \overline{S}_i$ admit stable modifications $(\overline{C}_i, \overline{D}_i)$ which agree over the intersections $\overline{S}_i \cap \overline{S}_j$ by Corollary 1.3 (i). So, they glue to a stable modification of the $\overline{S}$-curve $(C, D) \times_S \overline{S}$. □

Finally, if S' is a normal alteration of S and a stable modification $(C'_{\mathrm{st}}, D'_{\mathrm{st}}) \to (C, D) \times_S S'$ exists then it is unique, whence Corollary 1.6.

6. Uniformization of one-dimensional analytic fields

Throughout §6 we fix an analytic ground field k and set $p = \mathrm{char}(\widetilde{k})$. All analytic fields are understood to be k-fields. Our general goal in §6 is to establish analytic one-dimensional local uniformization. In particular, we will fulfil our earlier promise to prove Theorem 2.1.10 (in the very end of §6). We will see that the main difficulty is in treating immediate extensions, and we propose a way to control them in §6.1.

6.1. Immediate extensions of degree p.

Our aim is to gain some control on immediate algebraic extensions of an analytic field K. For example, we would like to obtain a criterion when K is *stable*. Recall that stability means that any finite extension L/K is Cartesian, i.e. $e_{L/K} f_{L/K} = [L : K]$ or $d_{L/K} = 1$ (see also [BGR, §3.6]). In particular, if K is a DVR or $p = \mathrm{char}(\widetilde{K}) = 0$ then K is stable. It easily follows from simple ramification theory that K is not stable if and only if there exists a finite extension L/K such that L admits an immediate extension of degree p (it may happen, though, that K itself does not admit finite immediate extensions but K is not stable). We will assume that $p > 0$ until the end of §6.1 (note, however, that many statements become trivial but make sense if one takes p to be the exponential characteristic).

If L/K is Cartesian wildly ramified of degree p then there exists $b \in K$ such that $\min |b + K^p| = |b|$ and $\inf |b + L^p| < |b|$. Indeed, either $e_{L/K} = p$ and then we take b such that $|b|^{1/p} \in |L^\times| \setminus |K^\times|$, or $\widetilde{L}/\widetilde{K}$ is inseparable and then we take $b \in K$ such that $|b| = 1$ and $(-\widetilde{b})^{1/p} \in \widetilde{L} \setminus \widetilde{K}$. This fact has the following analog.

Proposition 6.1.1. *Let L/K be an immediate extension of degree p. Then there exist elements $a, b \in K$ and $\alpha \in L$ such that the infimum $s = \inf_{x \in K} |x^p - ax + b|$ is not achieved on K, $|pb| < s$ and $|\alpha^p - a\alpha + b| < s$. In addition, one can achieve that either $|a| = s^{\frac{p-1}{p}}$, or $a = 0$.*

The role of the oddly looking condition $|pb| < s$ will be seen later; it allows to replace b with $c^p - ac + b$ for any $c \in K$ such that $|pc^p| < s$. Before proving the Proposition we prefer to reformulate it in a less intuitive form that is more convenient for applications.

Let K be an analytic field of positive characteristic p. Given an element $a \in K$, the set $S_a = S_a(K) = \{c^p - ac | c \in K\}$ is an additive group whose cosets $b + S_a$ contain some information about K. If K is of mixed characteristic then we have a group structure only approximately. One can remedy the problem by switching to characteristic p and studying the ring K°/pK° and its modules xK°/pxK°. We prefer another approach which is nearly equivalent. Given an element $a \in K$ and a positive number s, the set $S_{a,s}(K) = \{c^p - ac + d| \ c, d \in K, |pc^p| < s, |d| < s\}$ is an additive subgroup of K because $|(c_1 + c_2)^p - c_1^p - c_2^p| < s$ as soon as $|pc_i|^p < s$. We will study its cosets $b + S_{a,s}(K)$, and we say that a coset is *trivial* or *split* if it contains zero. Note that because of the d-term a coset is split if and only if it contains an element x with $|x| < s$. We say that a non-split coset $b + S_{a,s}(K)$ is *critical* if $\inf |b + S_{a,s}(K)| = s > 0$, that is the coset contains elements of absolute value arbitrary close to the lowest possible bound. If in addition either $a = 0$, or $a = 1$ and $s = 1$ then the coset is called *special*. Note that for any non-split coset we can ignore the d-term when finding the infimum and so a coset is critical if and only if $s = \inf_{c\in K, |pc^p|<s} |c^p - ac + b|$ (with $s > 0$).

Lemma 6.1.2. *Assume that the valuation on K is not discrete.*

(i) A coset $b + S_{a,s}(K)$ is critical if and only if $|pb| < s = \inf_{c\in K} |c^p - ac + b|$.

(ii) Any critical coset satisfies $|a| \le s^{\frac{p-1}{p}}$.

Proof. First, we claim that given an element b' of a critical coset $b+S_{a,s}(K)$ and an element $c \in K$ such that $|pc^p| < s$ and $|c^p - ac + b'| < |b'|$ one necessarily has that $|b'| > |ac|$ and, in particular, $|b'| = |c^p|$. Without loss of generality $b = b'$. Then to justify our claim it is enough to show that if, to the contrary, $a, b, c \in K$ satisfy $|c^p - ac + b| < |b|$ and $|b| \le |ac|$ then there exists a root c_0 of $f(T) = T^p - aT + b$ with $|c - c_0| < |c|$. Indeed, $|pc_0^p| = |pc^p| \le s$ and so $b+S_{a,s}(K)$ is split, contradicting our assumption. Now let us prove that c_0 exists. Note that we can re-scale a, b, c by replacing them with $c' = uc$, $a' = u^{p-1}a$ and $b' = u^p b$ for any $u \in K^\times$ because $|c'^p - a'c' + b'| < |b'|$, $|b'| \le |a'c'|$ and any root c_0' of $T^p - a'T + b'$ corresponds to a root $c_0 = c_0'/u$ of $f(T)$. In particular, taking $u = c^{-1}$ we can make $c' = 1$. To simplify notation we will denote the new triple as a, b, c. Since $|1 - a + b| < |b|$ and $|b| \le |a|$ one of the following is true: (a) $|a| = 1 \ge |b|$, (b) $|a| = |b| > 1$. In case (a) the residue polynomial $\widetilde{f}(T) = T^p + \widetilde{a}T + \widetilde{b}$ is separable with root 1. In particular, it has a root c_0 by Hensel's lemma. In case (b) we consider the non-monic polynomial $g(T) = -a^{-1}f(T)$. The reduction $\widetilde{g}(T) = T - \widetilde{(b/a)}$ has a simple root, hence $g(T)$ has a root by the non-monic version of Hensel's lemma (the usual proof with lifting a root works fine).

Remark 6.1.3. A more elegant proof avoiding re-scalings can be given by use of graded Hensel's lemma. In such case one considers the graded reduction $\widetilde{k} \to \widetilde{k}_{\rm gr} = \oplus_{r\in \mathbf{R}_+^\times} \{x \in k| \ |x| \le r\}/\{x \in k| \ |x| < r\}$ as defined in [Tem2, §1] and lifts homogeneous roots of the graded reductions of $f(T)$ (one reduction per each value of $|T|$). The graded version of Hensel's lemma is proved very similarly to its classical version, and we refer to [Duc, 1.9] for details

Now let us prove the Lemma. We start with (ii). Suppose to the contrary that $|a| > s^{\frac{p-1}{p}}$. Replacing b with another element of the coset we can assume that

$s < |b| < |a|^{\frac{p}{p-1}}$. Then there exists c with $|c^p - ac + b| < |b|$ and we proved above that $|ac| < |b| = |c^p|$. In particular, $|a^p| < |b|^{p-1}$ and we obtain a contradiction.

Next we prove the direct implication in (i). Assume that a coset $b + S_{a,s}(K)$ is critical. If $|pb| \geq s$ then $p \neq 0$ and $|b| > s$. So, the norm of b can be decreased by adding an element of the form $c^p - ac$ with $|pc^p| < s$, and then $|c^p| < |p|^{-1}s \leq |b|$. In particular, we must have $|ac| = |b|$ contradicting the proved above claim that $|b| > |ac|$. This proves that $|pb| < s$. If $\inf_{c\in K} |c^p - ac + b| < s$ then there exists $c \in K$ with $|pc^p| \geq s$ and $|c^p - ac + b| < s$. We proved in (ii) that $|a| \leq s^{\frac{p-1}{p}}$ and it follows that for any c with $|c^p| > s$ we have that $|c^p| > |ac|$. In particular, $|c^p| = |b|$ and therefore $|pc^p| = |pb| < s$. This contradiction establishes the second claim of the direct implication.

It remains to prove the converse implication in (i), so assume that $|pb| < s = \inf_{c\in K} |c^p - ac + b|$. We should only prove that $s' = \inf_{c\in K, |pc^p|<s} |c^p - ac + b|$ equals to s. Assume to the contrary that $s' > s$, and let $c \in K$ be such that $|pc^p| \geq s$ and $|c^p - ac + b| < s' \leq |b|$. Certainly, $|c^p - ac| = |b|$, hence either $|c^p| = |ac| > |b|$ or $|c^p|, |ac| \leq |b|$ with at least one an equality. In the first case,

$$|a| = |c|^{p-1} > |b|^{(p-1)/p} \geq s'^{(p-1)/p} > s^{(p-1)/p}$$

which gives a contradiction with (ii). In the second case, we cannot have that $|c^p| = |b|$ because $|pc^p| \geq s > |pb|$. Thus, the only remaining possibility is that $|c^p| < |b| = |ac|$. But then $|a| > |c|^{p-1}$, and hence $s = |a|^{p/(p-1)} > |c|^p > |pc^p|$, contrary to our assumption. □

Now we can give a more precise version of Proposition 6.1.1. It is easily checked that the old statement follows from the new one.

Proposition 6.1.4. *Let L/K be a wildly ramified extension of degree p.*

(i) There exists a critical coset $b + S_{a,s}(K)$ which splits over L (i.e. $b + S_{a,s}(L)$ is split) and such that one of the following possibilities holds:

(a) $a = 0$;

(b) $|a| = s^{\frac{p-1}{p}}$ and $s \notin |b + S_{a,s}(K)|$.

(ii) In case (ia) the coset is special by the definition and in case (ib) the wildly ramified extension $L(a^{\frac{1}{p-1}})/K(a^{\frac{1}{p-1}})$ (obtained from L/K by the moderately ramified base field extension $K(a^{\frac{1}{p-1}})/K$) admits a special coset satisfying the conditions of (ib).

(iii) In case (ib) $L \widetilde{\to} K[T]/(T^p - aT + b)$.

(iv) The invariants $d = d_{L/k}$, $e = e_{L/K}$ and $f = f_{L/K}$ satisfy $def = p$ and in the situation of (i) they can be found as follows:

(a) The extension L/K is immediate (i.e. $d = p$) if and only if $s \notin |b + S_{a,s}(K)|$. In particular, it is the case in (ib).

(b) $e = p$ if and only if $s \in |b + S_{a,s}(K)|$ and $s \notin |K^\times|^p$. In this case $s^{1/p}$ generates $|L^\times|$ over $|K^\times|$.

(c) $f = p$ if and only if $s \in |b + S_{a,s}(K)|$ and $s \in |K^\times|^p$. In this case for any $y \in b + S_{a,s}(K)$ and $c \in K$ such that $|y| = s$ and $|c| = s^{1/p}$ the element $(\widetilde{y/c^p})^{1/p}$ generates the purely inseparable extension $\widetilde{L}/\widetilde{K}$.

Before proving the Proposition we will make few aside remarks that will not be used later.

Remark 6.1.5. (i) The Cartesian cases of the Proposition are classical and very easy. Although we will not need them in the sequel, they are included for the sake of comparison and completeness.

(ii) In the immediate case our method is to start with any $\alpha \in L \setminus K$ and to partially orthogonalize it with respect to K by subtracting elements $c \in K$. When $|\alpha - c|$ is close enough to its infimum, the minimal polynomial of α is essentially affected only by the terms T^p, aT and b, so it can be used to construct a critical coset over K that is split over L. As was explained to me by F.-V. Kuhlmann, this method was used already by Kaplansky, though the formulation of the Proposition seems to be new.

(iii) The Proposition admits various generalizations and complements which we plan to discuss elsewhere. Here we only remark that using the method of our proof one can easily show that the case (ib) (which has no Cartesian analog) happens exactly when the L/K is almost unramified in the sense of Faltings (or L°/K° is almost étale in the sense of [GR, Ch. 6]). Other equivalent conditions for case (ib) are that L/K has zero different, or $\Omega^1_{L^\circ/K^\circ} = 0$ (the latter a priori could be stronger than vanishing of the different).

Proof. If L/K is Cartesian then there exists non-zero $\alpha \in L$ orthogonal to K, i.e. $|\alpha| \le |\alpha - c|$ for any $c \in K$. Obviously, α^p is "orthogonal" to the set K^p in the sense that $|\alpha^p| \le |\alpha^p - c^p|$ for $c \in K$. On the other hand, α^p is not orthogonal to K because either $e_{L/K} = p$ and then $|\alpha^p| \in |K^\times|$ and $\widetilde{L} = \widetilde{K}$, or $f_{L/K} = p$ and then $|\alpha^p| = |c^p|$ for $c \in K$, and $\widetilde{\alpha^p/c^p} \in \widetilde{K}$. All in all, $|\alpha^p - b| < |\alpha^p| = s$ for some $b \in K$ and then b is orthogonal to K^p. In particular, $b + S_{0,s}(K)$ is a special coset split by L. This proves (i) for Cartesian extensions. Also, we note that $def = [L : K] = p$. Hence there are three cases which exclude each other: $d = p$, $e = p$ and $f = p$, and therefore it is enough to prove only converse implications in (iva), (ivb) and (ivc). The Cartesian cases (b) and (c) are done similarly to the above argument, so we skip the details.

Next and main, we are going to establish the remaining case of (i), so assume that L/K is immediate. If L/K is inseparable then set $a = 0$ and choose $b \in K$ such that $L = K(b^{1/p})$. Since the extension is immediate the infimum $r = \inf_{c \in K} |c + b^{1/p}|$ is not achieved. Also, $r > 0$ by completeness of K. Since $\mathrm{char}(K) = p$ we have that $s := \inf_{c \in K} |c^p + b| = r^p > 0$, hence $b + S_{0,s}(K)$ is a special coset. Finally, this coset is split by L because $-b = (-b^{1/p})^p \in S_{0,s}(L)$.

In the sequel, we assume that L/K is separable. Choose an element $\alpha \in L \setminus K$ and set $r = \inf_{x \in K} |\alpha - x|$ and $r_0 = |\alpha|$. As earlier, the infimum r is not attained on K and $r > 0$. Let $f(T) = T^p + \sum_{i=1}^{p-1} a_i T^i + b$ be the minimal monic polynomial of α. Recall that the *closed disc* $E_{\widehat{K}^a}(\alpha, r)$ of radius r and with center at α in the Berkovich affine line $\mathbf{A}^1_{\widehat{K}^a}$ with a fixed coordinate T is the affinoid domain given by the condition $|T - \alpha| \le r$ (see [Ber1, 1.4.4]). By the K-*disc* E of radius r and with center at α we mean the image of $E_{\widehat{K}^a}(\alpha, r)$ under the morphism $\mathbf{A}^1_{\widehat{K}^a} \to \mathbf{A}^1_K$, so the preimage E' of E in $\mathbf{A}^1_{\widehat{K}^a}$ is the union of discs of radii r with centers at the conjugates of α (i.e. the roots of $f(T)$). It is well known that for any polynomial $f(T)$ the Weierstrass domain $W_s = \mathbf{A}^1_{\widehat{K}^a}\{s^{-1} f(T)\}$ is the union of closed discs with centers at the roots of f, and by the symmetry in our case all these discs are of equal radius which monotonically depends on s. Thus, $E' = W_s$ for a certain s, and hence $E = \mathbf{A}^1_k\{s^{-1} f(T)\}$. (Note that K-discs are also introduced in [Ber2, 3.6], but the radius there is defined to be $s^{1/\deg(f(T)}$, and in general it does not equal to the radius in our sense.) We say that a K-disc is *split* (or K-split) if it has a K-point. Note that by our assumption on K and α the disc $E = E_K(\alpha, r)$ is not split, but any larger disc $E_K(\alpha, r+\varepsilon)$ is split.

For any point $x \in E$, the infimum $\inf_{z\in K} |(T-z)(x)|$ equals to r and is not achieved. If x is Zariski closed then we obtain that the finite extension $\mathcal{H}(x)/K$ is not Cartesian, in particular, $[\mathcal{H}(x) : K] \geq p$. On the other side, the derivative $f'(T)$ does not vanish identically and is of degree smaller than p, hence it is invertible on E. It follows that f' is invertible on a disc $E_0 = E(c, r_1) \supset E$ for $c \in K$ and $r_1 > r$. Replacing T with $T-c$, α with $\alpha - c$ and r_0 with $|\alpha - c|$, we can assume that $E_0 = E(0, r_0)$ with $r_0 > r$. By the same reasoning we can assume that if $1 \leq i < p$ and the higher derivative $f^{(i)}$ is not identically zero then $f^{(i)}$ is invertible on E_0.

The condition on the derivatives of f implies the following important property: if $c \in K$ satisfies $|c| \leq r_0$ and $f(T) = \sum_{i=0}^{p} a'_i(T-c)^i$ then $|a_i - a'_i| < |a_i|$ for $i < p$. Indeed, for a K-splits disc $\mathcal{M}(k\{s^{-1}T\})$ a function $f(T) = \sum_{i=0}^{\infty} b_i T^i$ is invertible if and only if $|b_0| > |b_i|s^i$ for $i > 0$, and hence $|b_0 - f(x)| < |b_0|$ for any point x in that disc. In our situation, $a'_i = f^{(i)}(c)/i!$ and a_i is the constant coefficient of the polynomial $f^{(i)}(T)/i!$ which is invertible on the K-split disc E_0.

Thus, the values of $|a_i|$ are fixed when we shrink E_0 and change the coordinate accordingly. The condition on f' implies that $|ia_i|r_0^i = |a_i|r_0^i < |a_1|r_0$ for $1 < i < p$. Since $f(T)$ is irreducible, we have that $|a_1\alpha| \leq |\alpha^p|$ by [BGR, 3.2.4/3], hence $|a_1| \leq r_0^{p-1}$. Shrinking E_0 we can make r_0 arbitrary close to r while $|a_i|$'s remain fixed. Therefore, we can assume that $|a_1| \leq r^{p-1}$ and $|a_i| < \frac{|a_1|}{r_0^{i-1}} < r^{p-i}$ for $i > 1$. By additional shrinking of E_0 we can, furthermore, achieve that $|a_i| < \frac{r^p}{r_0^i}$ for $i > 1$ and $|p|r_0^p < r^p$.

Set $a = -a_1$, then $g(T) = T^p - aT + b$ is obtained from $f(T)$ by removing the a_iT^i terms for $1 < i < p$. We will show that a, b and $s := r^p$ are as required. Notice that α is a root of f and $|a_i\alpha^i| < \frac{r^p}{r_0^i} r_0^i = s$ for any $1 < i < p$, hence $|g(\alpha)| < s$. It follows that $-b \in S_{a,s}(L)$. Moreover, if $|a_1| < r^{p-1}$ then we remove the $-aT$ term using the same argument, achieving that either (a) or (b) in (i) is satisfied. Notice that $|f(T) - g(T)| < s$ on E, $\max_{x\in E} |f(x)| \geq |b| = r_0^p > s$ and f has a root in E. It follows that $|g(T)|$ is not a constant on E and, therefore, it is not constant on the disc $E' := E_{\widehat{K}^a}(\alpha, r)$ (which is a connected component of the preimage of E in $\mathbf{A}^1_{\widehat{K}^a}$). It follows that $g(T)$ has a zero on E', i.e. it has a root β with $|\alpha - \beta| \leq r$. The latter inequality and the definition of r imply that $\inf |\beta - K| = r$ and the infimum is not attained. In particular, $K(\beta)/K$ is not Cartesian, hence its degree is divided by p. Since β is annihilated by the polynomial $g(T)$ of degree p, we obtain that $g(T)$ is the minimal polynomial of β and, in particular, $g(T)$ is irreducible. The distance between $\beta_1 = \beta$ and other roots $\beta_2, \ldots, \beta_p$ of $g(T)$ does not exceed $\inf |\beta - K| = r$ by Krasner's lemma, hence for any $c \in K$ the numbers $|c - \beta_i|$ are equal for $1 \leq i \leq p$. In particular, $|g(c)| = |c - \beta|^p$, and we obtain that $\inf_{c\in K} |g(c)| = (\inf_{c\in K} |c - \beta|)^p = r^p = s$, and the infimum is not achieved. Finally, the inequality $|pb| = |p|r_0^p < s$ implies that $b + S_{a,s}(K)$ is critical by Lemma 6.1.2(i). This finishes the proof of (i).

The claim of (ii) for $a = 0$ follows from the definition, and for $a \neq 0$ set $b' = b/a^{\frac{p}{p-1}}$ and $K' = K(a^{\frac{1}{p-1}})$ and observe that the coset $b' + S_{1,1}(K')$ is a required special coset. In (iii) the inequality $|g(\alpha)| < s$ implies that $\inf |b + S_{a,s}(L)| < s =$

$|a|^{\frac{p}{p-1}}$. As we saw in the proof of Lemma 6.1.2, it then follows from Hensel's lemma that $0 \in b + S_{a,s}(L)$, i.e. $g(T)$ has a root in L. Therefore $K(\beta)\widetilde{\to}L$ as claimed. It remains to prove (iv) and we earlier reduced this to proving that if the infimum s is not attained on the coset $b + S_{a,s}(K)$ then the extension is immediate. Set $f(T) = T^p - aT + b$, then the disc $E := \mathbf{A}^1_K\{s^{-1}f(T)\}$ is a non-split K-disc such that any larger disc $\mathbf{A}^1_K\{(s+\varepsilon)^{-1}f(T)\}$ is split. Therefore, for any element $\beta \in K^a$ which belongs to E (i.e. satisfies $|f(\beta)| \le s$) the infimum $\inf_{c\in K}|c-\beta|$ equals to the radius of E and is not achieved. Since the coset is split over L, the disc E contains an L-point β. Since $\inf|\beta - K|$ is not achieved the extension L/K is not Cartesian, and so $d_{L/K} = p$ as claimed. □

Corollary 6.1.6. *An analytic field K is stable if and only if either $p = \mathrm{char}(\widetilde{K})$ is zero or for any finite Cartesian extension K'/K any special coset $b + S_{a,s}(K')$ contains an element of minimal absolute value.*

Proof. Since any finite extension is Cartesian when $p = 0$, we have only to deal with the case of non-zero p. Set $f(T) = T^p - aT + b \in K'[T]$ and assume that $s = \inf_{z\in K'}|f(z)|$ is not achieved. Then, as we observed in the end of proof of Proposition 6.1.4, $E := \mathbf{A}^1_{K'}\{s^{-1}f(T)\}$ is a non-split K'-disc such that any larger disc is split, and for any $\beta \in K^a$ contained in E the extension $K'(\beta)/K'$ is not Cartesian. In particular, K is not stable then.

Conversely, assume that K is not stable. We claim that some its Cartesian extension F admits an immediate extension of degree p. Indeed, if L/K is not Cartesian then for sufficiently large moderately ramified extension K_1/K the extension LK_1/K_1 splits into a tower of p-extensions $K_1 \subset K_2 \subset \cdots \subset LK_1$ (we use that $\mathrm{Gal}(K^s/K^{\mathrm{mr}})$ is a pro-p-group and hence any (maybe inseparable) extension of K^{mr} splits into a tower of p-extensions). Then there exists i such that K_i/K is Cartesian and K_{i+1}/K_i is not Cartesian. Since the latter is of degree p, it is immediate and we can take $F = K_i$. By Proposition 6.1.4(ii), a field K' of the form $F(a^{\frac{1}{p-1}})$ possesses a special coset without minimal element. It remains to note that K'/F is Cartesian because $[K' : F] < p$, and hence K'/K is Cartesian. □

6.2. **Analytic fields topologically generated by an element.** An analytic k-field K is *topologically generated* by an element T if K coincides with the topological closure $\overline{k(T)}$ of the subfield $k(T)$ in K. If K is finite over a subfield of the form $\overline{k(T)}$ and $T \notin \widehat{k^a}$ then we say that K is *one-dimensional*. For example, if x is a point on a k-analytic curve C and the preimages of x in $C\widehat{\otimes}_k\widehat{k^a}$ are not Zariski closed then the analytic k-field $\mathcal{H}(x)$ is one-dimensional. We claim that the sum of $F = F_{K/k} = \mathrm{tr.deg.}(\widetilde{K}/\widetilde{k})$ and $E = E_{K/k} = \dim_{\mathbf{Q}}((|K^\times|/|k^\times|)\otimes_{\mathbf{Z}}\mathbf{Q})$ does not exceed one. Indeed, this follows from Abhyankar's inequality (see Lemma 2.1.2) applied to $k(T)$ and the fact that the numbers $E_{K/k}$ and $F_{K/k}$ are preserved by replacing K with the completion or a finite extension. Thus, similarly to [Ber1, 1.4.4] we divide one-dimensional fields to three types as follows: K is of type 2 (resp. 3, resp. 4) if $E = 0, F = 1$ (resp. $E = 1, F = 0$, resp. $E = F = 0$). (Type 1 points from [Ber1] correspond to subfields of $\widehat{k^a}$.) Note that the type of a one-dimensional field is preserved by passing to a finite extension. We say that K is *k-split* if for any $T \in K$ we have that $\inf|T-k| = \inf|T-k^a|$, where the right hand side makes sense due to uniqueness (up to an automorphism) of the isometric embedding $\widehat{k^a} \hookrightarrow \widehat{K}^a$. The next two sections are devoted to uniformization of one-dimensional analytic fields of the following form.

Assumption 6.2.1. Assume that K is a one-dimensional analytic k-field and one of the following conditions is satisfied:

(i) $k = k^a$;

(ii) K is of type 4 and k-split, $k = k^{\mathrm{mr}}$, and either $p = 0$, or $p > 0$ and $k^\circ = pk^\circ + (k^\circ)^p$. If the last condition is satisfied then we say that k is *deeply ramified*.

Remark 6.2.2. Note that the assumption implies that $\widetilde{k}$ is algebraically closed and $|k^\times|$ is divisible. Indeed, $\widetilde{k}$ is separably closed and $|k^\times|$ is divisible by any prime l with $(l,p) = 1$ because $k = k^{\mathrm{mr}}$. On the other side, the deep ramification condition implies that $\widetilde{k}$ is perfect and $|k^\times|$ is p-divisible.

Remark 6.2.3. We will need only case (i) in this paper, but case (ii) does not require any extra-work and it will play a central role in a subsequent work [Tem4] on inseparable local uniformization. Our definition of deeply ramified fields agrees with its analog in [GR, 6.6.1] by [GR, 6.6.6]. It is proved there that deeply ramified valued fields can be characterized by many other equivalent properties. For example, if k is not discrete valued then two other equivalent properties are that $\Omega^1_{(k^s)^\circ/k^\circ} = 0$, or any separable algebraic extension of k is almost unramified (i.e. has trivial different).

In this section we always assume in addition to 6.2.1 that $p > 0$ and K is topologically generated by an element, say $K = \overline{k(z)}$. Equivalently, $K \widetilde{\to} \mathcal{H}(x)$ for a not Zariski closed point $x \in \mathbf{A}^1_k = \mathcal{M}(k[z])$. It follows from the classification of points, see [Ber1, 1.4.4] and [Ber2, 3.6] for details, that the type of x is the type of K as defined above. So, if x is of type 2 or 3 then $k = k^a$ by the assumption, and x is the generic point of a disc of rational or irrational radius, respectively (i.e. the radius is or is not contained in $|k^\times| = \sqrt{|k^\times|}$). Then $K \widetilde{\to} \widehat{\mathrm{Frac}(k\{T\})}$ or $K \widetilde{\to} k\{r^{-1}T, rT^{-1}\}$ for some $r \notin \sqrt{|k^\times|}$, respectively, where the valuation used to complete $\mathrm{Frac}(k\{T\})$ is the extension of the Gauss (or spectral) norm on $k\{T\}$. In the situation of 6.2.1(ii), x is the intersection of a decreasing sequence $E_0 \supsetneq E_1 \supsetneq E_2 \supsetneq \dots$ of closed discs in $\mathbf{A}^1_k$. If $E_i = E(\alpha_i, r_i)$ then $r := \lim_{i\to\infty} r_i$ equals to $\inf|z - k^a|$ (where the absolute value is computed in $\widehat{K^a}$), and the infimum is not attained on k^a. If some E_i is not split then $r_i \le \inf|\alpha_i - k|$ and for each $j > i$ we already have that $r_j < \inf|\alpha_j - k|$. In particular, $|(z - \alpha_j)(x)| \le r_j < \inf|\alpha_j - k|$, and we obtain that $K = \mathcal{H}(x)$ is not k-split. The contradiction proves that all discs E_i are k-split, hence we can re-choose the centers so that $\alpha_i \in k$. Now it is also clear that $r = \lim_{i\to\infty}|z - \alpha_i| = \inf|z - k|$ and the infimum is not achieved since it is not attained even on k^a.

Proposition 6.2.4. *Assume that $p > 0$ and $K = \overline{k(z)}$ is as in 6.2.1, and let L be a finite moderately ramified extension of K. If K is of type 2 or 3 then any critical coset in L contains an element of minimal absolute value. If K is of type 4 then any special coset in L contains either an element of k or a topological generator of L over k.*

It is for the sake of simplicity that in the case of type 4 fields we consider only special cosets. To prove the Proposition (in the end of §6.2) will need a good explicit description of analytic k-fields topologically generated by an element. Recall that a subset B of a normed k-vector space V is called orthogonal (resp. orthonormal) Schauder basis if any element $v \in V$ admits a unique representation of the form $v = \sum_{b\in B} a_b b$ and $\|v\| = \max_{b\in B} |a_b|\|b\|$ (resp. $\|v\| = \max_{b\in B} |a_b|$). It is easy to see that any analytic field $\widehat{k(z)}$ of type 2 or 3 admits a Schauder basis over k (e.g. for $K = \widehat{\mathrm{Frac}(k\{T\})}$ one can take the union of the sets $B_\infty = \{T^i\}_{i\ge 0}$ and $B_{\widetilde{a}} = \{(T-a)^{-i}\}_{i\ge 1}$ where $\widetilde{a}$ runs over $\widetilde{k}$ and a is a fixed lifting of $\widetilde{a}$ to k), but we will need a Schauder basis of a special form.

Proposition 6.2.5. *Let k, K and L be as in Proposition 6.2.4 and assume that K is of type 2 or 3. Then there exists a set $U \subset L$ such that the set $B = \{1\} \sqcup U \sqcup U^p \sqcup U^{p^2} \dots$ is an orthogonal Schauder basis of L over k and any element $u \in \mathrm{Span}_k(U)$ is orthogonal to L^p, i.e. $|u + c^p| \ge |u|$ for any $c \in L$.*

Proof. If K is of type 3 then $K \widetilde{\to} k\{r^{-1}S, rS^{-1}\}$ for some $r \notin |k^\times|$. In particular, $\widetilde{K} = \widetilde{k}$ is algebraically closed and hence L is totally ramified. Note also that $\sqrt{|K^\times|}/|K^\times| \widetilde{\to} r^{\mathbf{Q}}/r^{\mathbf{Z}}$. A well known description of moderately ramified extensions (see for example [Ber2, 3.4.4(iii)]) implies that K possesses a unique totally ramified extension of degree $n := [L : K]$. Clearly L and $K(S^{1/n})$ are two such extensions, hence $L \widetilde{\to} K(S^{1/n}) \widetilde{\to} k\{s^{-1}T, sT^{-1}\}$ for $s = r^{1/n}$ and $T = S^{1/n}$. Therefore, we can take $U = T^{\mathbf{Z}\setminus p\mathbf{Z}}$ (i.e. all integral powers of T with exponent prime to p) and then $B = T^{\mathbf{Z}}$. Assume now that $K = \widehat{\mathrm{Frac}(k\{T\})}$ (i.e. T topologically generates K over k and the induced norm on $k[T]$ is the Gauss norm); then $|K^\times| = |k^\times|$ is divisible, hence L is unramified over K. We will need the following Lemma.

Lemma 6.2.6. *Let E be a perfect field of characteristic $p > 0$ and F be a finitely generated extension of E such that E is algebraically closed in F. Then there exists a set $U \subset F \setminus F^p$ such that $\{1\} \sqcup U \sqcup U^p \sqcup \dots$ is a basis of F over E and $\mathrm{Span}_E(U) \cap F^p = 0$.*

Proof. A naive attempt would be to take any basis $\widetilde{U}$ of F/F^p and to lift it to a subset $U \subset F$ arbitrarily. The set $B = \{1\} \sqcup U \sqcup U^p \dots$ is indeed linearly independent, and $F = F^{p^n} + \mathrm{Span}_E(B)$ for any n. Nevertheless, this is not enough for B being a basis. In particular, we must take finite generatedness of F into account to exclude the possibility that $\cap_{n=1}^\infty F^{p^n}$ is strictly larger than E. For this reason, we refine the above strategy by introducing a norm on F related to finite generatedness of F/E and lifting $\widetilde{U}$ by use of an orthogonal complement procedure.

Choose a proper normal model X of F over E. For any $f \in F$, let $\|f\|$ be the maximal order of poles of f on the points of X of codimension one. Then $\|\ \|$ induces on the E-vector space F a non-Archimedean semi-norm whose kernel coincides with E. The residue semi-norm on $V = F/F^p$ is actually a norm. Indeed, the kernel consists of the images of elements $f \in F$ such that $\|f - g^p\| = 0$, but then $c = f - g^p$ is a constant and by perfectness of E we obtain that $c = b^p$, whence $f \in F^p$ and its image in V is zero. The residue norm induces an increasing exhausting filtration $V_1 \subset V_2 \subset \dots$ on V by balls of radii less than n for $n = 1, 2, \dots$. Find a subset $U = U_1 \sqcup U_2 \sqcup \dots$ of $F \setminus F^p$ such that each U_n consists of elements of norm n and the image of $\sqcup_{i=1}^n U_i$ in V is a basis of V_n. We will see that U is as required. For any $f \in F$ there exists a unique element $f_0 \in \mathrm{Span}_E(U)$ such that $f - f_0 \in F^p$. In particular, $\mathrm{Span}_E(U) \cap F^p = 0$. Moreover, it follows from the construction that $\|f_0\| \le \|f\|$. Next, there exists a unique $f_1 \in \mathrm{Span}_E(U^p)$ such that $f - f_0 - f_1 \in F^{p^2}$, and then $\|f_1\| \leq \|f\|$, etc. It remains to notice that at some stage we obtain $f_i \in F^{p^i}$ which satisfies $\|f_i\| \leq \|f\| < p^i$, and then f_i is necessarily a constant. So, $\{1\} \sqcup U \sqcup U^p \sqcup \dots$ is indeed a basis of F. $\square$

We use Lemma 6.2.6 to find $\widetilde{U} \in \widetilde{L} \setminus \widetilde{L}^p$ such that 1 and p^n-th powers of $\widetilde{U}$ for $n \geq 1$ form a basis $\widetilde{B}$ of $\widetilde{L}$ over $\widetilde{k}$. Though one could expect that it suffices to lift $\widetilde{U}$ to L in an arbitrary way, some care should be exercised at this point. We will use in the sequel that any analytic field F is henselian, and so, as we noted in §2.1, for any finite separable extension $E^{\sim}/\widetilde{F}$ there exists a unique unramified extension E/F with $\widetilde{E} = E^{\sim}$. Furthermore, by [EGA, IV$_4$, 18.8.4] for any analytic F-field E' with an $\widetilde{F}$-embedding $\widetilde{i} : \widetilde{E} \hookrightarrow \widetilde{E}'$ there exists a lifting $i : E \hookrightarrow E'$ to an embedding of analytic F-fields.

Recall that $k = k^a$ by 6.2.1(i). Set $\pi = p$ in the mixed characteristic case, and choose any non-zero $\pi \in k^{\circ\circ}$ in the equal characteristic case. It is easy to find a discrete valued field $k_0 \subset k$ such that $\widetilde{k}_0 = \widetilde{k}$ and π is a uniformizer. For example, let $\widetilde{S} = \{\widetilde{S}_i\}_{i \in I}$ be a transcendence basis of $\widetilde{k}/\mathbf{F}_p$ with a lifting $S \subset k$. Then k contains a discrete valued field k_1 topologically generated over the prime field by π and the elements S_i^{1/p^n} for $n \in \mathbf{N}$ and $i \in I$ and we have that $\widetilde{k}_1 = \mathbf{F}_p(S^{1/p^\infty})$. Since $\widetilde{k} = \widetilde{k}_1^s$, the extension $\widetilde{k}/\widetilde{k}_1$ lifts to an unramified extension k_0/k_1 with $k_0 \subset k$, and the field k_0 is as required.

Let K_0 be the closure of $k_0(T)$ in K, so $\widetilde{K}_0 = \widetilde{k}(\widetilde{T}) = \widetilde{K}$, and let L_0 be the subfield of L such that L_0/K_0 is the unramified extension corresponding to $\widetilde{L}/\widetilde{K}$. By [BGR], 2.7.3/2 and 2.7.5/2, any lifting of a $\widetilde{k}$-basis of $\widetilde{L} = \widetilde{L}_0$ gives an orthonormal Schauder basis of L_0 over k_0. So, we take $U \subset L_0$ to be any lifting of $\widetilde{U}$, obtaining an orthonormal Schauder basis $B = \{1\} \sqcup U \sqcup U^p \sqcup U^{p^2} \sqcup \dots$ of L_0 lifting the basis $\widetilde{B}$. Note that $K_0 \widetilde{\to} \widehat{\mathrm{Frac}(k_0\{T\})}$ because K induces the Gauss norm on $k_0[T]$, hence $K_0 \widehat{\otimes}_{k_0} k \widetilde{\to} K$ and so $L_0 \widehat{\otimes}_{k_0} k \widetilde{\to} L$. The latter isomorphism implies that B is also an orthonormal Schauder basis of L over k. It remains to check that $\mathrm{Span}_k(U)$ is orthogonal to L^p. Assume to the contrary that there exists $u = \sum a_i u_i$ with $a_i \in k^\times$, $u_i \in U$ and $v \in L$ such that $|u - v^p| < |u|$. Since $|L^\times| = |k^\times|$ is divisible, we can re-scale this using an element from k^p so that $|u| = |v| = 1$, and then by the orthonormality $|a_i| \leq 1$. It follows that $\sum \widetilde{a}_i \widetilde{u}_i = \widetilde{v}^p$, which is an absurd since $\mathrm{Span}_{\widetilde{k}}(\widetilde{U}) \cap \widetilde{L}^p = 0$. $\square$

Next, we assume that $K = \overline{k(z)}$ satisfies 6.2.1(ii). We have observed earlier that $r = \inf_{\alpha \in k} |z - \alpha| = \inf_{\alpha \in k^a} |z - \alpha|$ and neither infimum is achieved. Consider a sequence $0 = \alpha_0, \alpha_1, \dots$ of elements of k such that the sequence $r_i = |z_i|$, where $z_i = z - \alpha_i$, monotonically decreases and tends to r. Clearly, the discs $E_i = E_k(\alpha_i, r_i) \subset \mathbf{A}_k^1$ have a unique common point x and $\mathcal{H}(x) \widetilde{\to} K$. The field K contains a dense subfield $\kappa(x) = \cup k\{r_i^{-1} z_i\}$, in particular, $K = \overline{k[z]}$. Caution: the elements $1, z_i, z_i^2 \dots$ do not form a Schauder basis, even worse, they do not form a topological generating system in the sense of [BGR, 2.7.2]. Any non-zero element $b \in k[z]$ is invertible in a neighborhood of x, hence replacing z with some z_j we can achieve that $b = \sum_{i=0}^m b_i z^i$ and the element $b(T) = \sum_{i=0}^m b_i T^i \in k\{|z|^{-1} T\}$ is invertible. We will need the following Lemma, which implies in particular that $|b| \geq |b_m| r^m$.

Lemma 6.2.7. *Let L be an analytic k-field, $b = \sum_{i=0}^m b_i z^i$ be an element of L such that $b_i \in k$ and $z \in L$ satisfies* $\inf |z - k^a| = r > 0$, *where the distance is measured* *in $\widehat{L^a}$. Then $|b| \geq |b_m| r^m$ and the inequality is strict if $m > 0$ and $|z-a| > r$ for any $a \in k^a$.*

Proof. Since $r > 0$, we have that $\inf |z - k^a|$ is achieved if and only if $\inf |z - \widehat{k^a}|$ is achieved. So, we can replace k and L with $\widehat{k^a}$ and $\widehat{k^a L}$, achieving that k is algebraically closed. Note that when replacing z with $z-a$ for $a \in k$ we do not change b_m. If there exists $a \in k$ with $|z-a| = r$ then the induced norm on $k[z-a]$ is the Gauss norm of radius r, hence $|b| \geq |b_m| r^m$. If the infimum $\inf_{a \in k} |z-a|$ is not achieved then $\overline{k(z)}$ is of type 4 over k, and we saw before the Lemma that replacing z with some $z-a$ we can achieve that $\sum b_i T^i$ is invertible in $k\{|z|^{-1}T\}$. Then $|b| = |b_0| > |b_i z^i|$ for any $i > 0$. In particular, $|b| > |b_m z^m| > |b_m| r^m$ for $m > 0$. □

A type 4 field K is not Cartesian over k (and if it is split then it contains finite dimensional subspaces which have no orthogonal bases). Sometimes it is convenient to enlarge the ground field so that orthogonalization becomes possible. Let l be an analytic k-field with an isometric embedding $\phi : K \to l$, and let $\overline{z} = \phi(z)$. Set $L = K \widehat{\otimes}_k l$ and $w = z - \overline{z}$ (i.e. $w = z \otimes 1 - 1 \otimes \overline{z}$), then we claim that $L \widetilde{\to} l\{r^{-1}w\}$. Recall that $K = \overline{k[z]}$ and the norm on $k[z]$ is the infimum of the Gauss norms on $E_k(\alpha_i, r_i)$, hence $L = \overline{l[z]}$ and the norm on $l[z]$ is the infimum of the Gauss norms on $E_l(\alpha_i, r_i)$. But the latter discs have common Zariski closed point corresponding to $\overline{z}$, hence $E_l(\alpha_i, r_i) = E_l(\overline{z}, r_i)$ and the infimum norm is the Gauss norm of the closed disc $E_l(\overline{z}, r)$. (On the geometric side, we showed that the preimage of the point $x = \cap_{i=1}^{\infty} E_k(\alpha_i, r_i)$ under the projection $\mathbf{A}^1_l \to \mathbf{A}^1_k$ is the closed disc $E(\overline{z}, r)$.) In particular, we see that the set $w^{\mathbf{N}}$ is an orthogonal Schauder basis of L over l. We remark that L is an l-affinoid algebra with a multiplicative norm, and it admits an isometric embedding into the l-field $\widehat{\mathrm{Frac}(L)}$ of type 2 or 3, depending on whether $r \in \sqrt{|l^\times|}$ or not.

Lemma 6.2.8. *Keep the above notation and assume that $|pz| < \inf |z - k|$. Then $z^{1/p} \notin K$.*

Proof. Assume, on the contrary, that $z^{1/p} \in K$. Choose an embedding $\phi : K \to l$ with an algebraically closed l and set $\overline{z} = \phi(z)$. As earlier, let $w = z - \overline{z} \in L = K \widehat{\otimes}_k l$, and let also $u = z^{1/p} + (-\overline{z})^{1/p}$. Using that $\binom{p}{i}$ is divided by p for $0 < i < p$ one easily obtains the inequality $|u^p - w| \leq |pz| < r = |w|$ in $L \widetilde{\to} l\{r^{-1}w\}$. The latter is impossible. Indeed, if $r \notin |l^\times|$ then $r = |w|$ is not a p-th power in $|L^\times| = r^{\mathbf{N}} |l^\times|$, and if $r \in |l^\times|$ then after re-scaling by an element of $l^p = l$ we can achieve that $r = 1$ and then $\widetilde{w}$ is not a p-th power in $\widetilde{L} = \widetilde{l}[\widetilde{w}]$. □

The following unpleasant Lemma plays a key role in our treating of type 4 fields.

Lemma 6.2.9. *Keep the above notation, and let $b + S_{a,s}(K)$ be a special coset. Assume that $b = \sum_{i=0}^m b_i z^i \in k[z]$ and $(p, m) = 1$. If $a = 1$ then we assume in addition that $|pb_i z^i| < 1$ for $i \geq 1$. Then $|b_m| r^m \leq s$, and the inequality is strict when $m > 1$.*

Proof. Recall that $r = \inf |z - k^a| = \inf |z - k|$. Shifting z by an element of k we can achieve that $|pz| < r$. We will later need the following two claims, which follow easily from Lemma 6.2.10 below: (*) for any n with $p^n \leq m$ one has that $\inf |z^{1/p^n} - k^a| = r^{1/p^n}$, (**) if k is as in 6.2.1(ii) then $\inf |k - c^{1/p^n}| \leq |pc|^{1/p^n}$ for any $c \in k$.

Lemma 6.2.10. *Given elements $x_1, \dots, x_m \in \widehat{k^a}$ one has that*
(i) $|(\sum_{i=1}^m x_i)^{p^n} - \sum_{i=1}^m x_i^{p^n}| \le |p| \max_{1\le i\le n} |x_i|^{p^n}$*;*
(ii) $|(\sum_{i=1}^m x_i)^{1/p^n} - \sum_{i=1}^m x_i^{1/p^n}| \le |p|^{1/p^n} \max_{1\le i\le n} |x_i|^{1/p^n}$*, where the choice of the root is not important since* $\xi_1^{p^n} = \xi_2^{p^n} = 1$ *implies that* $|\xi_1 - \xi_2| < |p|^{1/p^n}$.

Proof. (i) is clear and to prove (ii) we estimate the p^n-th power of the difference $(\sum_{i=1}^n x_i)^{1/p^n} - \sum_{i=1}^n x_i^{1/p^n}$ by use of (i). □

Choose an embedding $\phi : K \to l$, where l is an algebraically closed analytic k-field, and set $\overline{z} = \phi(z)$, $L = K\widehat{\otimes}_k l$ and $w = z - \overline{z}$. Recall that $|w| = r$ and $L\widetilde{\to}l\{r^{-1}w\}$, and so $w^{\mathbf{N}}$ is an orthogonal Schauder basis. Note that $b = \sum_{i=0}^m \overline{b}_i w^i$, where

$$\overline{b}_i = \overline{b}_i(\overline{z}) = \sum_{j=i}^m \binom{j}{i} b_j \overline{z}^{j-i} \in \phi(K) \subset l \tag{1}$$

Recall that $|pb| < s$ by Lemma 6.1.2(i), so each term of b satisfies $|p\overline{b}_i w^i| < s$. Similarly to the case when L is a field, the set $S_{a,s}(L) := \{c^p - ac + d|\ c, d \in L, |pc^p| < s, |d| < s\}$ is an additive group, and obviously $s' := \inf |b + S_{a,s}(L)| \le \inf |b + S_{a,s}(K)| = s$. To ease the exposition we separate the cases (i) $a = 0$, and (ii) $a = 1$ and $s = 1$.

We start with (i) since it is easier. We claim that in this case $|\overline{b}_1 w| \le s$. Indeed, for any $c = \sum c_i w^i \in L$ with $|pc^p| < s$ the first term of $b + c^p = \sum c'_i w^i$ is $c'_1 w = (\overline{b}_1 + pc_1 c_0^{p-1})w$. So, if $|\overline{b}_1 w| > s$ then there exists c as above with $|\overline{b}_1 w + pc_1 c_0^{p-1} w| < |\overline{b}_1 w|$ and the latter would imply that $|pc^p| \ge |pc_1 c_0^{p-1} w| = |\overline{b}_1 w| > s$. The contradiction proves the claim, and hence $|\overline{b}_1| \le s/|w| = s/r$. On the other hand we can estimate $|\overline{b}_1|$ by applying Lemma 6.2.7 to $\phi(K)$. Since $|m| = 1$ in L and $\overline{b}_1 = mb_m \overline{z}^{m-1} + \dots$ by (1), the Lemma implies that $|b_m| r^{m-1} \le |\overline{b}_1|$ and the inequality is strict when $m > 1$. Thus, $|b_m| r^{m-1} \le s/r$ and the first case is established.

Now let us assume that $a = 1$ and $s = 1$. Let $\overline{b}_{pi} w^{pi}$ be the non-zero term of b with largest i. Since $|p\overline{b}_{pi} w^{pi}| < 1$ and l is algebraically closed, $c := \overline{b}_{pi}^{1/p} w^i$ is an element of L satisfying $|pc^p| < 1$, and replacing b with $b - c^p + c$ we find another element of $b + S_{1,1}(L)$ with smaller value of i. Iterating this procedure we obtain an element $B = \overline{b}w + \sum_{i>1,(i,p)=1} x_i w^i \in b + S_{1,1}(L)$ where $\overline{b} = \overline{b}_1 + \overline{b}_p^{1/p} + \dots + \overline{b}_{p^N}^{1/p^N}$ and $N = [\log_p(m)]$. Similarly to the case (i), we will now prove that $|\overline{b}w| \le 1$. Assume that the inequality fails. Then there exists $c = \sum c_i w^i$ with $|pc^p| < 1$ and $|B + c^p - c| < |\overline{b}w|$. Note that c^p coincides with $C := \sum c_i^p w^{pi}$ up to terms of absolute value smaller than 1, hence the inequality $|B + c^p - c| < |\overline{b}w|$ can hold only when $|B + C - c| < |\overline{b}w| = |c_1 w|$. Choose the maximal i with $|c_i w^i| \ge |\overline{b}w|$, then the pi-th term of $B + C - c$ is $(c_i^p - c_{pi})w^{pi}$. Since $|c_i^p w^{pi}| \ge |\overline{b}w|^p > |\overline{b}w| > |c_{pi} w^{pi}|$, we obtain that $|B + C - c| \ge |\overline{b}w|^p > |\overline{b}w|$. The contradiction proves that $|\overline{b}w| \le 1$. Now, it suffices to show that $|b_m| r^{m-1} \le |\overline{b}|$ and the inequality is strict when $m > 1$. This will be done below to accomplish the proof.

As in (i), we would like to apply Lemma 6.2.7 to $\overline{b}$. Since $\overline{b}$ does not have to be in $\phi(K)$, we will also have to approximate it with an element $\overline{b}' \in \phi(K)(\overline{z}^{1/p^N})$. It will be convenient to assume that $|z|^{-1} < |b_m| r^{m-1}$, and this is harmless because otherwise $1 \ge |b_m z| r^{m-1} > |b_m| r^m$, which is the assertion of the Lemma. Recall that in the mixed characteristic case $|b_j z^j| < |p^{-1}|$ for $j \geq 1$ (by assumption of the Lemma). Therefore, in the formula (1) for $\overline{b}_{p^n}$ the absolute value of the summands is strictly smaller than $R := |p|^{-1}|z|^{-p^n}$. Using Lemma 6.2.10(i) to approximate the p^n-th root we obtain that

$$(\overline{b}_{p^n})^{1/p^n} = W + \sum_{j=p^n}^{m} \binom{j}{p^n}^{1/p^n} b_j^{1/p^n} \overline{z}^{(j-p^n)/p^n} \tag{2}$$

where the error term W satisfies $|W| < |p|^{1/p^n} R^{1/p^n} = |z|^{-1} < |b_m| r^{m-1}$. Recall that $\inf |k - b_j^{1/p^n}| \leq |pb_j|^{1/p^n}$ by claim (**). In particular, each b_j^{1/p^n} in (2) can be replaced by an element of k without increase in the error term (use that $|b_j| < |p|^{-1}|z|^{-j} \leq R$ for $j \geq p^n$). Summing up such approximations of the elements $(\overline{b}_{p^n})^{1/p^n}$ for $1 \leq n \leq N$ (without the error terms) we obtain an approximation $\overline{b}'$ of $\overline{b}$. More specifically, we obtain an element $\overline{b}' \in k[\overline{z}^{1/p^N}]$ such that $|\overline{b}' - \overline{b}| < |b_m| r^{m-1}$ and $\overline{b}'$ is a polynomial of degree $(m-1)p^N$ in $\overline{z}$ and with the highest degree term $mb_m\overline{z}^{m-1}$, which is the contribution of $\overline{b}_1$. Recall that $\inf |z^{1/p^N} - k^a| = r^{1/p^N}$ by claim (*), and hence $\inf |\overline{z}^{1/p^N} - k^a| = r^{1/p^N}$. Applying Lemma 6.2.7 to the field $\overline{k(\overline{z}^{1/p^N})}$, we obtain that $|\overline{b}'| \geq |b_m|(r^{1/p^N})^{(m-1)p^N} = |b_m| r^{m-1}$ and the inequality is strict when $m > 1$. It follows that $\overline{b}$ satisfies the same inequalities and we are done. $\square$

Proof of Proposition 6.2.4. Let $b + S_{a,s}(L)$ be a critical coset. The strategy of the proof is very simple: we gradually improve b moving it inside $b + S_{a,s}(L)$ until either $|b| = s$ or b is a topological generator. If K is of type 2 or 3 (so $k = k^a$) then L possesses a Schauder basis $B = 1 \sqcup U \sqcup U^p \sqcup \dots$ as in Proposition 6.2.5. Moving b a little, we can assume that the representation $b = \sum_{v_i \in B} a_i v_i$ involves finitely many non-zero terms. Let $b = a_0 + \sum_{i=1}^{N} \sum_{u_j \in U} a_{ij} u_j^{p^i}$. If $N > 0$ then subtracting from b elements of the form $a_{ij} u_j^{p^i} - a a_{ij}^{1/p} u_j^{p^{i-1}}$ we can achieve that it is contained in $\mathrm{Span}(1 \sqcup \dots \sqcup U^{p^{N-1}})$. (One has also to check that $|pa_{ij} u_j^{p^i}| < s$, but this is obvious because $|pb| < s$ by Lemma 6.1.2(i)). Iterating the process we achieve that $b = a_0 + \sum_{u_j \in U} a_j u_j$. Since k is algebraically closed, $a_0 = c^p - ac$ for some $c \in k$, and so we can remove a_0 as well. Now, we claim that the absolute value of b cannot be reduced further by adding elements $c^p - ac$. Indeed, if $|b + c^p - ac| < |b|$ then $|b| > s$, so $|c^p - ac| = |b| > s$. Since $s^{\frac{p-1}{p}} \geq |a|$, the latter is possible only when $|c^p| > |ac|$. Therefore $|b + c^p| < |b|$, contradicting the property that $b \in \mathrm{Span}(U)$ is orthogonal to L^p. This settles the Proposition for types 2 and 3.

In the case of 6.2.1(ii), $L = K$ because $\widetilde{K} = \widetilde{k}$ is algebraically closed and $|K^\times| = |k^\times|$ is divisible by Remark 6.2.2. Set $r = \inf |z - k|$, as usually. Replacing b with a sufficiently close element we can achieve that $b \in k[z]$, say $b = \sum_{i=0}^{m} b_i z^i$. Using linear change of the coordinate z we can achieve that $\sum b_i T^i$ is invertible in $k\{|z|^{-1}T\}$, and then $|pb_i z^i| \leq |pb| < s$. Now, if $(p, m) = 1$ and $m > 1$ then Lemma 6.2.9 implies that $|b_m| r^m < s$. Choose $z_0 \in k$ such that $d = b_m(z - z_0)^m$ satisfies $|d| < s$. Subtracting d from b, we decrease the degree of b. If $p|m$ and $\mathrm{char}(k) = p$ then we subtract $c^p - ac$ with $c = b_m^{1/p} z^{m/p}$ from b decreasing the degree of b, and then use another linear coordinate change to restore the condition $|pb_i z^i| < s$. In the mixed characteristic case with $p|m$, $b_m^{1/p}$ does not have to be in k but by 6.2.1(ii) there exists $c_0 \in k$ so that $|c_0^p - b_m| \le |pb_m|$. Since $|pb_m z^m| < s$, for $c = c_0 z^{m/p}$ we have that $|c^p - b_m z^m| < s$. Thus, we can replace b with $b - c^p + ac$ so that the absolute value of the m-th term of b drops below s and then we can safely remove it. All in all, we can move b inside the coset until its degree m drops below two. If $m = 1$ then b is a topological generator of K, and if $m = 0$ then $b \in k$. $\square$

6.3. **One-dimensional analytic fields.** Let $z \in K \setminus k$ be an element. We say that it is an *unramified generator* (resp. *moderately ramified generator*) if $K/\overline{k(z)}$ is a finite unramified extension (resp. finite moderately ramified extension). In this section, generator always means topological generator. The main result of this section states that any one-dimensional analytic k-field possesses an unramified generator.

Theorem 6.3.1. *Let L be a one-dimensional analytic k-field satisfying the conditions of 6.2.1, then:*

(i) L possesses an unramified generator over k;

(ii) if $p > 0$ and $k = k^a$ then any special coset in L contains a moderately ramified generator of L;

(iii) if L is of type 2 or 3 then it is stable.

We start with few simple lemmas. Until the end of Corollary 6.3.4 k is an arbitrary analytic field.

Lemma 6.3.2. *Let $K = \overline{k(z)}$ be a one-dimensional analytic k-field and L be an analytic K-field. Set $r = |z - k^a| > 0$ (computed in $\widehat{K^a}$) and assume that $z' \in L$ satisfies $|z - z'| < \varepsilon r$ for some $\varepsilon < 1$. Then there exists an isometric embedding $\phi : K \hookrightarrow L$ over k such that $\phi(z) = z'$ and $|x - \phi(x)| < \varepsilon |x|$ for any $x \in K^\times$.*

Proof. For any linear polynomial $f(T)$ over k we have that $|f(z) - f(z')| < \varepsilon |f(z)|$. Assume given $x, x', y, y' \in L$ with $|x - x'| < \varepsilon |x|$ and $|y - y'| < \varepsilon |y|$ (this assumption is symmetric in x, x' and y, y', and it forces that $x, x', y, y' \in L^\times$). It is easy to check that $|xy - x'y'| < \varepsilon |xy|$ and $|\frac{x}{y} - \frac{x'}{y'}| < \varepsilon |\frac{x}{y}|$. Hence the above inequality holds for any rational function $f(T) \in k(T)$ (by factoring $f(z)$ in $\widehat{k^a L}$). From this the Lemma follows by continuity. $\square$

Lemma 6.3.3. *If $\alpha, \beta : K \to L$ are embeddings of analytic fields and $\varepsilon < 1$ is a number such that $|\alpha(x) - \beta(x)| < \varepsilon |x|$ for any $x \in K$ then $[L : \alpha(K)] = [L : \beta(K)]$.*

Proof. It suffices to prove that if $[L : \alpha(K)] = n < \infty$ then $[L : \beta(K)] \le n$. Fix a number $\varepsilon < r < 1$ and pick up an r-orthogonal basis $x_1, \dots, x_n$ of L over $\alpha(K)$; we recall (see [BGR, 2.6.1/3]) that this means that for any choice of $a_1, \dots, a_n \in \alpha(K)$ one has that $|\sum a_i x_i| \ge r \max_i(|a_i x_i|)$. Let $x \in L$ be arbitrary. Express it as $x = \sum \alpha(a_i) x_i$ for some elements $a_i \in K$ and set $x' = \sum \beta(a_i) x_i$, then $|x - x'| < \varepsilon \max_i(|a_i||x_i|) \le \frac{\varepsilon}{r} |x|$. In a similar way we can approximate $x - x'$ by an element from the vector space $L' := \sum x_i \beta(K)$, etc., obtaining in the end arbitrary good approximations of x by elements from L'. By completeness, $L' = L$ and obviously $[L' : \beta(K)] \le n$. $\square$

Corollary 6.3.4. *A one-dimensional k-field L is finite over any one-dimensional subfield K.*

Proof. We may decrease K, so replace it with a subfield $\overline{k(x)}$ where $x \notin \widehat{k^a}$. Let $y \in L$ be such that L is finite over $L_0 = \overline{k(y)}$ and let L_1 be the separable closure of L_0 in L. Replacing x with some x^{p^n} and decreasing K accordingly we achieve that $x \in L_1$, and then it suffices to show that $[L_1 : K] < \infty$. So, replacing L with L_1 we can assume that L/L_0 is separable and then by Krasner's lemma $L = \overline{l}$ for a finite separable extension $l/k(y)$. Pick up an element $x' \in l$ such that $|x - x'| < |x - k^a|$ and set $K' = \overline{k(x')}$. The above two Lemmas imply that $[L : K] = [L : K']$, and it remains to note that $[L : K'] \leq [l : k(x')] < \infty$ because the degree of an extension can only drop by passing to completions. □

Remark 6.3.5. The caution exercised in the proof of the above Corollary is explained by the following rather surprising fact. One can naturally define the topological transcendence degree of the extensions, but the latter does not have to be additive in towers. In particular, in some cases there exist non-surjective analytic k-endomorphisms of fields $\widehat{k(x)^a}$. However, this non-additivity can only occur in towers containing deeply ramified extensions (i.e. extensions with infinite different). All this is not needed in this paper, so topological transcendence degree will be studied elsewhere.

Proof of Theorem 6.3.1. The case of $p = 0$ is obvious, so we assume that $p > 0$. We start with the type 2 or 3 case. Since $\widetilde{k} = \widetilde{k}^a$ and $|k^\times|$ is divisible, there exists $z \in L$ such that $\widetilde{L}/\widetilde{k}(\widetilde{z})$ is a finite separable extension in the type 2 case and $|L^\times| = |z^{\mathbf{Z}} k^\times|$ in the type 3 case. Note that $L/\overline{k(z)}$ is finite by Corollary 6.3.4. Let K be the unramified closure of $\overline{k(z)}$ in L, then K admits an unramified generator and the extension L/K is finite and immediate. So, L/K is totally wildly ramified and $[L : K] = p^n$. The extension $LK^{\mathrm{mr}}/K^{\mathrm{mr}}$ splits to a tower of extensions of degree p because $G = \mathrm{Gal}(K^{\mathrm{mr}})$ is a pro-p-group (hence any subgroup $H \subset G$ of index p^n possesses a tower of larger subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ with $\#(H_{i+1}/H_i) = p$) and any purely inseparable extension splits into such a tower as well. It follows that there exists a finite moderately ramified extension K'/K such that the extension $K'L/K'$ splits to a tower of p-extensions. In our situation, K' admits a moderately ramified generator and is of type 2 or 3, hence by Propositions 6.2.4 and 6.1.4 K' has no immediate extensions of degree p. Thus $K'L = K'$, and we obtain that $L \subseteq K'$ is moderately ramified over K. So, $L = K$ and we have proved (i). Next we note that for any finite extension L'/L the one-dimensional k-field L' possesses a moderately ramified generator by applying (i) to L'. Propositions 6.2.4 and 6.1.4 imply that L' has no immediate extensions of degree p, hence L is stable. In particular, this gives (iii).

For types 2 and 3 it remains only to prove (ii). Let $b + S_{a,s}(L)$ be a special coset. By Proposition 6.2.4, the value of $|c^p - ac + b|$ accepts its minimum s for some $c \in L$. Set $z = c^p - ac + b$. If $s = |z| \notin |L^\times|^p$ then L is of type 3, $L/\overline{k(z)}$ is Cartesian by (iii) and $|L^\times|/|k(z)^\times|$ is prime to p, hence we obtain that $L/\overline{k(z)}$ is moderately ramified, i.e. z is a moderately ramified generator. If $s \in |L^\times|^p$ but $s \notin |k^\times|$, then necessarily $a = 0$ and L is of type 3. Since $\widetilde{K} = \widetilde{k} = \widetilde{k}^a$, there exists $c \in L$ with $|z - c^p| < |z|$, and we obtain a contradiction with the minimality of $|z|$. It remains to deal with the case when $|z| \in |k^\times|$. Assume first that $a = 0$. Since $k = k^a$ and z is orthogonal to L^p (i.e. $|z + c^p| \geq |z|$ for $c \in L$), the same is true for any element tz with $t \in k = k^p$. Taking t such that $|tz| = 1$ we obtain from orthogonality that $\widetilde{tz} \notin \widetilde{L}^p$. In particular, $\widetilde{L} \neq \widetilde{k}$ and so L is of type 2. The extension $L/\overline{k(z)}$ is Cartesian by (iii), $|L^\times| = |k^\times|$ and $\widetilde{L}$ is separable over $\widetilde{k}(\widetilde{tz})$ and hence over the residue field of $\overline{k(z)}$. Thus, $L/\overline{k(z)}$ is an unramified extension and z is an unramified generator. Finally, we are left with the case when $a = 1$ and $|z| = s = 1$. By our choice of z, the equation $x^p - x + z = 0$ has no solutions in $\widetilde{L}$. Since $\widetilde{k} = \widetilde{k}^a$ we can use Lemma 6.2.6 to prove that there exists an element $\widetilde{c} \in \widetilde{L}$ such that $\widetilde{c}^p - \widetilde{c} + \widetilde{z} \notin \widetilde{L}^p$. Taking a lifting c of $\widetilde{c}$ and replacing z with $c^p - c + z$ we achieve the situation when $\widetilde{z} \notin \widetilde{L}^p$. Then the same argument as above proves that z is an unramified generator.

Assume now that L is of type 4 in 6.2.1(ii). First, let us prove (ii) assuming (i). In view of (i) and Proposition 6.2.4, we have only to rule out the possibility that a critical coset $b' + S_{a,s}(L)$ contains an element $b \in k$. Since k is algebraically closed, the equation $b = c^p - ac$ has a solution $c \in k$ which implies that already $b + S_{a,s}(k)$ is split. The contradiction shows that (i) implies (ii), so we should only establish (i). Since $\widetilde{L} = \widetilde{k}$ is algebraically closed and $|L^\times| = |k^\times|$ is divisible, L admits only immediate algebraic extensions. In particular, any finite extension of L splits to a tower of p-extensions. Thus we have only to prove that if $K = \overline{k(z)}$ and $[L : K] = p$ then L possesses a generator. Since K has no non-trivial moderately ramified extensions, by Proposition 6.1.4(ii) we can find a special coset $b + S_{a,s}(K)$ as in 6.1.4(i). In particular, the coset $b + S_{a,s}(L)$ is split. By Proposition 6.2.4 we can achieve furthermore that either b is a generator of K or $b \in k$. If $b \in k$ then there exists $\alpha \in L$ with $|\alpha^p - a\alpha + b| < s$ and this easily implies that α approximates a root of $T^p - aT + b = 0$ better than any element from k. The latter would contradict our assumption that L is k-split, hence b is a generator of K.

If $a = 1$ then by Proposition 6.1.4(iii) the polynomial $T^p - aT + b$ has a root $\alpha \in L$ and we claim that α generates L. Indeed, α generates L over K and $b = -\alpha^p + \alpha$ generates K over k. If $a = 0$ we take $\alpha \in L$ with $|\alpha^p + b| < s$ and we will see that this α is a generator. Set $b' = -\alpha^p$ and $K' = \overline{k(b')}$. By the choice of α we have that $|b - b'| < s = \inf|b + K^p|$. We claim that the latter infimum does not exceed $\inf|b - k|$. Indeed, if $|b + c| < s$ for some $c \in k$ then by 6.2.1(ii) there exists $d \in k$ with $|d^p - c|$ arbitrary close to $|pc|$. Since $|pc| = |pb| < s$ by Lemma 6.1.2(i), we can achieve that $|d^p - c| < s$, which implies that $|b + d^p| < s$. The contradiction proves that $|b - b'| < \inf|b + K^p| \le \inf|b - k|$, and the latter infimum equals to $\inf|b - k^a|$ because K is k-split. By Lemmas 6.3.2 and 6.3.3 we obtain that $[L : K'] = [L : K] = p$. It remains to note that $|pb'| < s \le \inf|b - k| = \inf|b' - k| = \inf| - b' - k|$, hence $\alpha = (-b')^{1/p} \notin K'$ by Lemma 6.2.8. Thus, $[\overline{k(\alpha)} : K'] = p$, and therefore $\overline{k(\alpha)} = L$. □

As a corollary we prove a slightly generalized version of the stability theorem of Grauert-Remmert, see [BGR, 5.3.2/1].

Corollary 6.3.6. *If k is a stable analytic field and K is of type 2 or 3 over k then K is stable.*

Proof. Let $z \in K$ be such that either $r := |z|$ is not in $\sqrt{|k^\times|}$ or $r = 1$ and $\widetilde{z} \notin k^a$. By Corollary 6.3.4 K is finite over $\overline{k(z)}$, so it suffices to establish stability of the latter field. Thus we can assume that $K = \overline{k(z)} = \mathrm{Frac}(\widehat{k\{r^{-1}z\}})$. Suppose to the contrary that a finite extension L/K is not Cartesian. Since K^s is dense in K^a, it suffices to consider the case of a separable extension L/K (if $s = \inf|\alpha - K|$

is not achieved for some $\alpha \in K^a$ then there exists $\alpha' \in K^s$ with $|\alpha' - \alpha| < s$, so the separable extension $k(\alpha')/k$ is not Cartesian). In particular, we can assume that L is separable over k. Given a finite extension k'/k, we will use the notation $L' = k'L$ and $K' = k'K$. Note that $K' = \mathrm{Frac}(\widehat{k'\{r^{-1}z\}})$ and one checks straightforwardly that the extension K'/K is Cartesian since k'/k is Cartesian. We claim that for sufficiently large k' the extension L'/K' is Cartesian. Indeed, the extension $\widehat{k^aL}/\widehat{k^aK}$ is Cartesian by Theorem 6.3.1(iii), hence it admits an orthogonal basis $\{a_1, \dots, a_n\} \in \widehat{k^aL}$. Moving a_i's slightly does not spoil orthogonality in the non-Archimedean world, so we can achieve that $a_i \in k^aL$, and then $a_i \in k'L$ for a suitable k'. Clearly, $\{a_i\}$ is an orthogonal basis of L'/K', i.e. this extension is Cartesian. Finally, the extension L'/K splits to a tower of Cartesian extensions $K \subset K' \subset L'$. Hence L'/K is Cartesian and its subextension L/K is Cartesian too. □

Remark 6.3.7. Stability theorem for type 2 fields is the main ingredient in the proof of the Grauert-Remmert finiteness theorem, see [BGR, §6.4]. We saw in this section that stability theorem (both for types 2 and 3) is essentially equivalent to uniformization of one-dimensional analytic fields.

Proof of Theorem 2.1.10. We simply have to combine already proved results. Part (i) of the Theorem is exactly Theorem 6.3.1(i). To prove (ii) we note that by Lemma 6.3.2, if ε is small enough and $|x - x'| \le \varepsilon$ then there exists an analytic k-isomorphism ϕ between the fields $L = \overline{k(x)}$ and $L' = \overline{k(x')}$ such that $|y - \phi(y)| \le |y|/2$ for any $y \in L$. Obviously, $\widetilde{L}$ coincides with $\widetilde{L'}$ as a subfield of $\widetilde{K}$, and also we have that $[K : L] = [K : L']$ by Lemma 6.3.3. So, K/L' is unramified because K/L is unramified. Finally, in (iii) we necessarily have that K is of type 2. Hence $L = \overline{k(x)}$ is stable by Theorem 6.3.1(iii) and it follows that $[K : L] = [\widetilde{K} : \widetilde{L}]$. In particular, K/L is unramified if and only if $\widetilde{K}/\widetilde{L}$ is separable. □

Appendix A. Stable modification and desingularization of surfaces

This appendix is an attempt to systemize known results and methods in the theories of semi-stable curves and desingularization of surfaces. It seems to be impossible to give credits to all mathematicians that have contributed to these theories, but I try to do my best. One of the aims of this systematization is to stress the analogy between the two theories, to compare them and to describe an interplay between them. For the sake of simplicity, all relative curves in this appendix are automatically assumed to have smooth geometrically connected generic fiber.

A.1. **Two contexts where semi-stable families of curves appear.** The semi-stable families of curves naturally appear in two different contexts: the context of moduli spaces and the context of desingularization of relative curves. Originally, a systematic study of families of semi-stable curves was motivated by the theory of moduli spaces of curves. The foundational work in that direction is [DM]. This is the work where the stable reduction theorem over a discrete valuation ring R appeared for the first time. It is worth to mention that though it may sound surprisingly today, the theorem was very surprising when discovered [1]. Not only it was

[1] The author is grateful to M. Raynaud for a consultation on this issue.

not expected in the positive/mixed characteristic case, but even in the characteristic zero case its assertion was strikingly new despite the fact that this case follows easily from desingularization of surfaces of characteristic zero proved by Zariski in 1930s. The stable reduction theorem was applied in [DM] to prove that the moduli stack of stable curves is proper, and then the general stable extension theorem (see the Introduction) follows easily, including, as a particular case, the stable reduction theorem over non-discrete valuation rings.

On the other hand, relative semi-stable curves can be considered as a relative analog of the notion of a smooth curve. Indeed, if one starts with a relative curve $\phi : C \to S$ and tries to improve the singularities of ϕ by reasonable (e.g. proper surjective) base changes and modifications of C, then the mildest singularities one can hope to obtain are those of semi-stable curves. (Note that one has to allow base changes which are not modifications in order to obtain relative curves with reduced fibers. For example, to get rid of nilpotents in the central fiber of $\phi : \mathrm{Spec}(\mathbf{Q}[\pi, x, y]/(\pi x - y^2) \to \mathrm{Spec}(\mathbf{Q}[\pi]))$ one has to adjoin $\sqrt{\pi}$ to the base.) Thus, de Jong's semi-stable modification theorem [dJ, 2.4] can be considered as a relative desingularization theorem. Our work in this paper has a clear flavor of desingularization approach, and, as we will see below, our stable modification theorem is an analog of the minimal desingularization of surfaces.

A.2. **Desingularization of surfaces.** There are two main theorems concerning smooth models of surfaces: the minimal model theorem and the minimal desingularization theorem. The first one is an absolute result that is close in nature to the theory of moduli spaces. It states that if k is an algebraically closed field and K is finitely generated over k with $\mathrm{tr.deg.}(K/k) = 2$ and is sufficiently generic (namely, K is not of the form $L(T)$ for a subfield $L \subset K$ containing k), then K admits a minimal k-smooth proper model. The minimal desingularization theorem states that any integral quasi-excellent two-dimensional scheme X admits a minimal desingularization, i.e. a modification $X' \to X$ with regular source such that any other modification $X'' \to X$ with regular source factors through X'). Unlike the minimal model theorem, this second theorem applies to any integral quasi-excellent surface. Moreover, it admits generalizations which treat divisors and finite group actions.

Usually, a proof of the minimal model/desingularization theorem is not direct and goes in three steps: (1) find some regular model/modification, (2) prove that the family of such models/modifications is filtered by domination, (3) given any regular model/modification construct a minimal regular contraction and establish its uniqueness. The last two stages are easy and rather standard. Step (2) follows from the following two facts: (2a) if X is a regular surface then the family of all its modifications that can be obtained by successive blowing up closed points is cofinal in the family of all modifications of X; and (2b) any modification of surfaces $X' \to X$ with regular X and X' can be obtained by successive blowing up closed points. Step (3) is done by successive contraction of exceptional $\mathbf{P}^1$'s (these are $\mathbf{P}^1$'s with self-intersection equal to -1, i.e. with the normal bundle isomorphic to $\mathcal{O}(1)$), and by certain combinatorial computations with the intersection form. The heart of the proof is in the first step which we call desingularization of surfaces. We present two approaches to desingularization of surfaces due to Zariski and Lipman.

Zariski was first to establish desingularization of surfaces over fields of characteristic zero. His approach was to first desingularize a surface X along a valuation ring. Zariski proved the following local uniformization theorem which can be considered as a local (on the Riemann-Zariski space of a variety) solution of the desingularization problem: if X is integral and of finite type over an algebraically closed field k of characteristic zero, $\mathcal{O}$ is a valuation ring of the field of rational functions $k(X)$ with $k \subset \mathcal{O}$ and a k-morphism $\mathrm{Spec}(\mathcal{O}) \to X$ extending the isomorphism of generic points, then there exists a modification $X' \to X$ such that the lifting $\mathrm{Spec}(\mathcal{O}) \to X'$, which exists by the valuative criterion, lands in the smooth locus of X'. Local uniformization implies desingularization of surfaces because one can glue the local solutions using Steps (2a) and (2b) above. Using a much more involved gluing method, Zariski was also able to obtain desingularization of threefolds in characteristic zero via local uniformization.

Lipman proposed in [Lip] another method of desingularization of a quasi-excellent integral two-dimensional scheme X. At the first step, a modification $X' \to X$ is constructed so that X' is normal and has only rational singularities, i.e. singular points that (a posteriori) are resolved by trees of $\mathbf{P}^1$'s (with a negative definite intersection form). One easily sees that X' has rational singularities if and only if the arithmetic genus $p_a(X') = h^1(X', \mathcal{O}_{X'})$ is minimal in the set $\{X_i\}$ of all modifications of X, and the first step is established by proving that arithmetic genus of X_i's is bounded. At the second step, each rational singularity point is resolved by $\mathbf{P}^1$-trees rather explicitly.

A.3. **Desingularization of relative curves.** The theory of semi-stable modifications of relative curves is analogous in many aspects to the theory of desingularization of surfaces. Its two main results are the stable extension theorem (mentioned in the introduction) and the stable modification theorem which are clear analogs of the two main results on desingularization of surfaces. Note also that de Jong's semi-stable modification theorem is an analog of (non-minimal) desingularization of surfaces. Moreover, we will see that the theory of relative curves is slightly easier because some arguments are easier and some results can be proved in a stronger form. In addition, desingularization can often be used to construct semi-stable modifications, while it is much harder (though sometimes possible) to go in the opposite direction.

Localizing the base (in the Riemann-Zariski sense) one obtains a very important particular case of the above theorems: the (semi-)stable reduction theorem. I know two published direct proofs of this theorem: the proof of Bosch-Lütkebohmert in [BL1] and the proof of van der Put in [Put] (other proofs at least use desingularization of surfaces, and we will discuss them in §A.6). The two proofs are close in spirit and have many common features with the method of Lipman. Both are rigid-analytic and apply to a formal curve over any complete valuation ring of height one, and then the algebraic version over any valuation ring of height one is an easy consequence, see [BL1, p. 377]. Similarly to Lipman's method, both proofs run in two stages: first one studies the arithmetic genus of the closed fibers of modifications to prove that there exists an ordinary modification (i.e. a modification whose closed fiber has only ordinary singularities), then ordinary singularities are resolved rather explicitly by trees of $\mathbf{P}^1$'s.

The new proof of the stable reduction theorem presented in this paper is a close analog of Zariski's approach. The main ingredient of the proof is uniformization of one-dimensional valued fields in Theorem 2.1.8. Local uniformization of a relative curve along a valuation was easily deduced in Proposition 4.3.3. The latter statement is a clear analog of Zariski's local uniformization, and similarly to local uniformization, which is still unknown in positive characteristic and large dimensions, the case of positive characteristic was much more difficult. Indeed, the main effort in the proof of 2.1.8 was in struggling with the effects of wild ramification, in particular in controlling extensions with defect in §6.1. Gluing local desingularizations to a global one is, again, rather similar to the surface case described in A.2. Analogs of Steps (2a) and (3) from A.2 are Propositions 4.3.1 and 4.4.6. The only subtle point is that we managed to avoid factorization of modifications from Step (2b); see §4.5 for details.

Finally, we would like to say few words about the history of the closely related and nearly equivalent problems of uniformization of one-dimensional (analytic) valued fields and local description (or uniformization) of non-Archimedean curves. Despite the fact that the formulation of Theorem 2.1.8 seems to be new, it was clear for experts that similar statements can be deduced from the stable reduction theorem. For example, Berkovich deduced a local description of analytic curves from the stable reduction theorem, see [Ber2, 3.6.1]. As for direct valuation-theoretic proofs of uniformization of one-dimensional valued fields, the author knows about the following works: in the analytic case stability theorem of Grauert-Remmert covered type 2 case (and, probably, the case of type 3 fields was known to experts), and M. Matignon established the case of analytic type 4 fields in an unpublished work; in the general case F.-V. Kuhlmann proved generalized stability theorem for types 2 and 3 in [Kuh1] and uniformization of type 4 fields will be worked out in [Kuh2]. It seems that in all these works the argument is more computational than ours because one studies p-extensions of valued fields by use of Kummer and Artin-Shreier theories, while we struggled with defect by use of Proposition 6.1.4 which covered all cases in a uniform manner. Also, it seems that currently only our method covers some cases when the ground field is not algebraically closed (see 6.2.1). We note also that probably a classical valuation-theoretic work [Epp] of Epp was initially motivated by a hope to obtain uniformization of valued fields algebraically and to then deduce the stable reduction theorem.

A.4. **Comparison of the two theories.** In the two previous sections we described two parallel desingularization theories. We summarize the analogies between them in the following table.

Surfaces:	Relative curves:
Modification of the surface	Alteration of the base and modification of the curve
Desingularization of surfaces	Semi-stable modification
Minimal model theorem	Stable extension theorem
Minimal desingularization theorem	Stable modification theorem
No analog (no localization on the base)	Stable reduction theorem
Local uniformization of surfaces	Uniformization of one-dimensional valued fields
Surfaces with rational singularities	Ordinary relative curves
Lipman's method	The methods of Bosch-Lütkebohmert's and van der Put
Zariski's method	The method of this paper

A.5. **The link between the two theories.** The two parallel theories we have described meet together in the following very important particular case. If $X \to S$ is a relative curve and the base S is a curve then X is a surface. In particular, one can wonder what is the connection between the desingularizations of X of two kinds when S is a regular curve. Until the end of this section we assume that S is a local regular curve, that is $S = \mathrm{Spec}(R)$ for a discrete valuation ring R.

The minimal surface desingularization $X_{\mathrm{sm}} \to X$ does not need to be semi-stable over S because its special fiber can be non-reduced. In such situations, one is guaranteed that a non-trivial alteration of the base (i.e. replacing of R with its integral closure in a finite separable extension K'/K of its field of fractions) is required in order to construct a semi-stable modification. Conversely, a semi-stable modification which involves a non-trivial alteration of the base does not help (at least at first glance) to desingularize X. If a semi-stable modification is possible already over S then X_{sm} and the stable modification X_{st} are tightly connected (e.g. X_{sm} is semi-stable) and one can easily use either of them to construct another one. Thus, one can expect that the Galois group G_K of K is essentially responsible for the gap between the two theories, and, indeed, we will see that a good control on the Galois group sometimes makes it possible to pass from desingularizations to semi-stable models and vice versa.

If the residue field $k = R/m_R$ is algebraically closed of characteristic zero then the Galois group G_K has a simple structure because it coincides with the tame inertia group. In this case the link between the two theories is so tight that it even extends to higher dimensions. Given X_{sm} one can easily predict what is the minimal extension K' over which stable modification exists (K'/K is totally ramified and its degree is the minimal common multiple of the multiplicities of the irreducible components in the special fiber of X_{sm}). Moreover, the same argument was used in [KKMS] to deduce a higher dimensional semi-stable reduction theorem from the desingularization theorem of Hironaka. In opposite direction, de Jong and Abramovich proved in [AdJ] that the quotient $X_{\mathrm{st}}/G_{K'/K}$ has very mild toric singularities which can be easily resolved (thus, giving a link from X_{st} to X_{sm}). Moreover, their argument applies to any base S, so they deduce weak desingularization of higher dimensional algebraic varieties in characteristic zero.

The situation with k of positive characteristic is far more complicated. No general way is known to go in the difficult direction $X_{\mathrm{st}} \mapsto X_{\mathrm{sm}}$ even when S is a curve. The main problem here is to control the properties of the quotient by a wildly ramified Galois group. The easier link $X_{\mathrm{sm}} \mapsto X_{\mathrm{st}}$ can be established at least in the case of curves. The main idea here is to control the Galois group through its action on the l-adic cohomology group $H^1(X_\eta, \mathbf{Q}_l)$ or another invariant of close nature (e.g. Jacobian's l-torsion). In particular, it turns out that the Galois group of K' acts unipotently on $H^1(X_\eta, \mathbf{Q}_l)$ (via the embedding $G_{K'} \hookrightarrow G_K$) if and only if X_η admits a stable model after the base change corresponding to the extension K'/K. This underlies the proofs of the stable reduction theorem by Deligne-Mumford (using Grothendieck's semi-stable reduction of abelian varieties), Artin-Winters and Saito.

A.6. **Proofs of the stable reduction theorem.** The stable reduction theorem is a fundamental result which has been proved in many ways, though no easy self-contained proof is known. The author knows about six published proofs of the stable reduction theorem [DM], [AW], [Gi], [BL1], [Put] and [Sa], and a new proof is presented in the paper. It seems natural to systemize different proofs and we try to do this below.

All proofs are naturally divided to three types. The proof of Gieseker in [Gi] is the only proof of the first type. It is based on the geometric invariant theory. One constructs moduli spaces of stable curves by global projective methods then the stable reduction theorem is obtained as a by-product.

Three direct proofs perform the main work in the framework of non-Archimedean analytic geometry. They apply to any complete valuation ring of height one and construct a semi-stable modification similarly to desingularization of a surface. The proofs of Bosch-Lütkebohmert, [BL1], and van der Put, [Put], are close to Lipman's desingularization of surfaces. Arithmetic genus plays an important role in these proofs. Our proof is an analog of Zariski's desingularization of surfaces. It is of rather valuation-theoretic nature, and the arithmetic genus (and sheaves $R^1f_*(\mathcal{O}_X)$) shows up only when we want to contract a semi-stable modification to the stable one.

The proofs of the third type apply to discrete valuation rings. One uses desingularization of surfaces as a (non-trivial!) starting point. If it is known that $X' = X \times_S S'$ admits a stable modification, where $S' = \mathrm{Spec}(R')$ and R' is the integral closure of R in a finite separable extension K'/K, then such a desingularization of X' is a required semi-stable modification. The extension K'/K is specified via the action of the Galois group G_K on an appropriate invariant of X. It is the group of l-torsion points, unless $l = 2$, of the relative generalized Jacobian $J_{X/S}$ in the Deligne-Mumford-Grothendieck or Artin-Winters approaches (K' is chosen so that it splits the l-torsion of the Jacobian), or the étale cohomology group $H^1(X_{\overline{\eta}}, \mathbf{Q}_l)$ in Saito's approach (K' is chosen so that $G_{K'}$ acts unipotently on this cohomology group).

Appendix B. Curves over separably closed fields

The material of this section is rather standard, so we give sketched proofs only. We assume that k is a separably closed field and $S = \mathrm{Spec}(k)$. Let C be a proper connected geometrically reduced S-curve and $\pi : \widetilde{C} \to C$ be its normalization. For any point $x \in C$ we define a number $g(x)$ as follows: if $x \in C^0$, where C^0 is the set of generic points of C, then $g(x)$ is the geometric genus $h^1(\mathcal{O})$ of its irreducible component; if $x \in C \setminus C^0$ then $g(x)$ is the dimension of the k-vector space $\mathcal{O}_{\widetilde{C},\widetilde{x}}/\mathcal{O}_{C,x}$, where $\widetilde{x} = \pi^{-1}(x)$. In particular, $g(x) = 0$ if and only if either x is a regular closed point or x is the generic point of a rational irreducible component.

A point $x \in C$ is called an *ordinary n-fold point* if the completed local ring $\widehat{\mathcal{O}}_{C,x}$ is isomorphic to $k[[T_1, \dots, T_n]]/(\{T_iT_j\}_{i\neq j})$, and for $n = 2$, x is called an *ordinary double point*. An ordinary n-fold point x is a k-point and $\widetilde{x} = \{\widetilde{x}_1, \dots, \widetilde{x}_n\}$ where each $\widetilde{x}_i$ is a smooth k-point. Furthermore, Zariski locally at x the curve C is obtained from $\widetilde{C}$ by gluing the points $\widetilde{x}_i$ to a single point, i.e. $\mathcal{O}_{C,x}$ is the subring of $\mathcal{O}_{\widetilde{C},\widetilde{x}}$ whose elements satisfy $f(\widetilde{x}_1) = \cdots = f(\widetilde{x}_n)$. It follows easily from this description that if x is the only non-regular point in C then C is the pushout of the diagram $\widetilde{C} \leftarrow \widetilde{x} \to x$. Alternatively, one can describe ordinary points as follows: any k-point x satisfies $g(x) \geq |\widetilde{x}| - 1$ (where $|\widetilde{x}|$ is the cardinality of $\widetilde{x}$) and the equality holds if and only if x is ordinary. Indeed, $\mathcal{O}_{\widetilde{C},\widetilde{x}}/\mathcal{O}_{C,x}$ admits a natural surjective k-linear map ϕ onto $k(\widetilde{x})/k(x)$ (where $k(\widetilde{x})$ is the quotient of the semi-local ring $\mathcal{O}_{\widetilde{C},\widetilde{x}}$ by its radical), hence the inequality is satisfied. If x is ordinary then ϕ is bijective and $k(\widetilde{x})\widetilde{\to}k^n$, hence the exact equality holds. Conversely, if $g(x) = |\widetilde{x}| - 1$ then $k(\widetilde{x})/k(x)$ is of dimension at most $|\widetilde{x}| - 1$, hence $\widetilde{x}$ consists of k-points. By the pushout property, the normalization $\widetilde{C} \to C$ factors through a curve C' obtained from $\widetilde{C}$ by gluing all points of $\widetilde{x}$ to a single point x'. We proved that $g(x') = |\widetilde{x}| - 1$, hence $g(x') = g(x)$ and it follows that the subrings $\mathcal{O}_{C',x'}$ and $\mathcal{O}_{C,x}$ of $\mathcal{O}_{\widetilde{C},\widetilde{x}}$ coincide. Since x' is ordinary, x is ordinary too. Note also that this characterization of ordinary points implies that a k-point x is ordinary if and only if its preimage $x^a \in C \otimes_k k^a$ is ordinary. (For the sake of completeness we remark also that a k-point x is ordinary if and only if C is semi-normal at x, i.e. $\mathrm{Spec}(\mathcal{O}_{C,x})$ does not admit non-trivial bijective finite modifications.)

We say that a curve C is *ordinary* (resp. *semi-stable*), if all its non-smooth points are ordinary (resp. ordinary double points). We claim that C is ordinary if and only if $C^a = C \otimes_k k^a$ is ordinary. The direct implication is obvious and to prove the converse one it suffices to show that if the preimage $x^a \in C^a$ of a point x is an ordinary singular point then x is such a point too. Note that $g(x^a) \geq g(x)$ and the equality holds if and only if $\widetilde{C} \otimes_k k^a$ is normal over x. Also, $|\widetilde{x}|$ is the number of valuation rings of $k(C)$ centered on $\mathcal{O}_{C,x}$ and any such ring extends uniquely through the purely inseparable extension k^a/k. Hence $|\widetilde{x^a}| = |\widetilde{x}|$ and we obtain that $|\widetilde{x}| - 1 = |\widetilde{x^a}| - 1 = g(x^a) \geq g(x)$, and hence $\widetilde{x}$ is ordinary. A multipointed curve (C, D) is called *ordinary* (resp. *semi-stable*) if C is ordinary (resp. semi-stable) and D is a union of smooth k-points. As usual, $p_a = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C) = h^1(\mathcal{O}_C)$ denotes the arithmetic genus of C.

Lemma B.1. *The equality $p_a(C) = 1 - |C^0| + \sum_{x \in C} g(x)$ holds.*

Proof. The normalization morphism π is affine, hence we have an isomorphism $H^i(C, \pi_*\mathcal{O}_{\widetilde{C}})\widetilde{\to}H^i(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$. Also, the sheaf $\mathcal{F} = \pi_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C$ is a skyscraper because π is an isomorphism over non-closed points. As a consequence, we obtain the following exact sequence

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_{\widetilde{C}}) \to H^0(\mathcal{F}) \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_{\widetilde{C}}) \to 0.$$

Since $k = k^s$ and C is geometrically reduced, k is algebraically closed in the function field of each irreducible component of C. In particular, $H^0(\mathcal{O}_{\widetilde{C}})$ is the direct sum of $|C^0|$ copies of k. The Lemma follows now by computing the dimensions of the cohomology groups. □

The Lemma has the following Corollary which will serve us in applications. Let Z be a connected proper semi-stable k-curve. If $p_a(Z) = 0$ then we say that Z is a $\mathbf{P}^1_k$-*tree*. This condition is equivalent to requiring that the irreducible components of Z are isomorphic to $\mathbf{P}^1_k$, and the incidence graph of Z is a tree (vertex per generic point and edge per double point). If C is a curve then by the boundary $\partial(Z) = \partial_C(Z)$ of a closed subscheme Z we mean the intersection of Z with the Zariski closure of its complement.

Corollary B.2. *Let $f : C' \to C$ be a morphism of proper geometrically reduced k-curves such that $p_a(C') = p_a(C)$. Assume that $x \in C$ is a point such that $Z = f^{-1}(x)$ is a connected curve and $f : C' \setminus Z \to C \setminus x$ is an isomorphism. Let us assume also that C' is semi-stable at all points of Z. Then x is an ordinary n-fold point if and only if Z is a $\mathbf{P}^1_k$-tree and $\partial(Z)$ contains exactly n points.*

Proof. Apply the genus formula of Lemma B.1 to compute $p_a(C')$ and $p_a(C)$. Since the genera are equal and $C'\setminus Z\widetilde{\to}C\setminus\{x\}$, we obtain that $g(x)=\sum_{z\in Z}g_{C'}(z)-|Z^0|$. Set $n=|\partial_{C'}(Z)|$, then the number of double points on Z equals to $n+l$ where l is the number of non-boundary double points of Z. If Γ is the incidence graph of Z then $h^1(\Gamma)=l-|Z^0|+1$. So, $g(x)=\sum_{z\in Z}g_{C'}(z)-|Z^0|=n+l+\sum_{z\in Z^0}g_{C'}(z)-|Z^0|=n-1+h^1(\Gamma)+\sum_{z\in Z^0}g_{C'}(z)\geq n-1$, and the equality holds if and only if $h^1(\Gamma)=\sum_{z\in Z^0}g_{C'}(z)=0$. The latter means that Γ is a tree and the genera are all zero, i.e. Z is a $\mathbf{P}^1_k$-tree. Thus, $g(x)=n-1$ if and only if Z is a $\mathbf{P}^1_k$-tree. It only remains to show that $|\widetilde{x}|$ equals to $n=|\partial_{C'}(Z)|$. We have an obvious embedding of normalizations $\widetilde{C}\hookrightarrow\widetilde{C}'$, hence a morphism $h:\widetilde{C}\to C'$ arises. Clearly, h maps $\widetilde{x}$ surjectively onto $|\partial_{C'}(Z)|$, hence it is enough to show that h is injective on $\widetilde{x}$. If $y\in C'$ is the image of two different points of $\widetilde{x}$ then y is necessarily a double point of C' and hence no component of Z can pass through y. But then $\{y\}$ is a connected component of Z, that contradicts Z being a connected curve. □

In the sequel, by $(f,f_D):(C',D')\to(C,D)$ we denote a proper surjective morphism of connected geometrically reduced multipointed k-curves. We say that an irreducible component Z of C' is *exceptional* if Z lies in the semi-stable locus of (C',D'), is isomorphic to $\mathbf{P}^1_k$, is contracted in C, and contains at most two points x from $D'\cup C'_{\rm sing}$ (i.e. either x is on the divisor or x is an ordinary double point). Note that the pushout C'' of the diagram $\overline{C'\setminus Z}\leftarrow\partial_{C'}(Z)\to\mathrm{Spec}(k)$ is also the pushout of the diagram $C'\leftarrow Z\to\mathrm{Spec}(k)$. Let D'' be the image of D' in C'', then $D'\widetilde{\to}D''$, f factors through C'' and the image of Z is a point contained in the semi-stable locus of (C'',D''). We say that (C'',D'') is obtained from (C',D') by *contracting* Z. Contracting exceptional components successively, we construct a surjective proper morphism $(\overline{C},\overline{D})\to(C,D)$ which has no exceptional components. We call such $(\overline{C},\overline{D})$ a *stable blow down of* (f,f_D).

Lemma B.3. *A stable blow down of (f,f_D) is unique up to unique isomorphism.*

Proof. Let $Y\subset C'$ be a $\mathbf{P}^1_k$-tree lying in the semi-stable locus of (C',D'). We say that Y is exceptional if it is contracted to a point of C and $|Y\cap D'|+|\partial_{C'}(Y)|\leq 2$. We claim that given two exceptional $\mathbf{P}^1_k$-trees Y_1 and Y_2 with non-empty intersection, their union is an exceptional $\mathbf{P}^1_k$-tree too. Indeed, if none of them is contained in the other then there exists a double point x such that one irreducible component through x is in Y_1 but not in Y_2, and the other one is in Y_2 but not in Y_1. So, x is contained in $\partial_{C'}(Y_i)$ for $i=1,2$ but not in $\partial_{C'}(Y_1\cup Y_2)$ and therefore $|\partial_{C'}(Y_1\cup Y_2)|\leq|\partial_{C'}(Y_1)|+|\partial_{C'}(Y_2)|-2$ and the claim follows. Thus, C' contains disjoint exceptional $\mathbf{P}^1_k$-trees $Y_1,\dots,Y_n$ such that any exceptional $\mathbf{P}^1_k$-tree is contained in some Y_i. Now it is clear that the stable blow down of f is determined up to an isomorphism by the property that it contracts each Y_i to a point. Indeed, by Corollary B.2 any semi-stable blow down $C'\to\overline{C}'$ contracts few disjoint exceptional $\mathbf{P}^1_k$-trees, hence the stable blow down $C'\to\overline{C}$ factors through $\overline{C}'$. □

For the sake of completeness, we note that analogous stabilization lemma holds in the absolute situation (i.e. we contract a multipointed k-curve (C',D') without specified base curve (C,D)) when one of the following cases holds: $p_a(C')\geq 2$, or $p_a(C')=1$ and $|D'|\geq 1$, or $p_a(C')=0$ and $|D'|\geq 3$. Note also that one can similarly construct an ordinary blow down of C', but it is not unique in general. In particular, there is no minimal ordinary modification of relative curves.

References

[AdJ] Abramovich, D.; de Jong, A.J.: *Smoothness, semistability, and toroidal geometry.*, J. Alg. Geom. **6** (1997), no. 4, 789–801.

[AK] Abramovich, D.; Karu, K.: *Weak semistable reduction in characteristic 0*, Inv. Math. **139** (2000), 241–273.

[AO] Abramovich, D.; Oort, F.: *Alterations and resolution of singularities*, Resolution of singularities (Obergurgl, 1997), 39–108, Progr. Math., 181, Birkhauser, Basel, 2000.

[AW] Artin, M.; Winters, G.: *Degenerate fibres and stable reduction of curves*, Topology **10** (1971), 373–383.

[Ber1] Berkovich, V.: *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, 1990.

[Ber2] Berkovich, V.: *étale cohomology for non-Archimedean analytic spaces*, Publ. Math. IHES **78** (1993), 5–161.

[Ber3] Berkovich, V.: *Smooth p-adic analytic spaces are locally contractible*, Inv. Math. **137** (1999), 1–84.

[BGR] Bosch, S.; Güntzer, U.; Remmert, R.: *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*, Springer, Berlin-Heidelberg-New York, 1984.

[BL1] Bosch, S.; Lütkebohmert, W.: *Stable reduction and uniformization of abelian varieties. I.*, Math. Ann. **270** (1985), no. 3, 349–379.

[BL2] Bosch, S.; Lütkebohmert, W.: *Formal and rigid geometry. II. Flattening techniques.*, Math. Ann. **296** (1993), no. 3, 403–429.

[BLR] Bosch, S.; Lütkebohmert, W.; Raynaud, M.: *Formal and rigid geometry. IV. The reduced fibre theorem.*, Inv. Math. **119** (1995), no. 2, 361–398.

[Bou] Bourbaki, N.: *Algèbre commutative*, Hermann, Paris, 1961.

[Con1] Conrad, B.: *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257.

[Con2] Conrad, B.: *Grothendieck duality and base change*. Lecture Notes in Mathematics, 1750. Springer-Verlag, Berlin, 2000. vi+296 pp.

[Del] Deligne, P.: *Le lemme de Gabber*, Asterisque **127** (1985), 131–150.

[DM] Deligne, P.; Mumford, D.: *The irreducibility of the space of curves of given genus*, IHES Publ. Math. **36** (1969), 75–109.

[Duc] Ducros, A: *Toute forme modérément ramiée d'un polydisque ouvert est triviale*, preprint.

[dJ] de Jong, A.J. *Families of curves and alterations.*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 599–621.

[EGA] Dieudonné, J.; Grothendieck, A.: *Éléments de géométrie algébrique*, Publ. Math. IHES, **4, 8, 11, 17, 20, 24, 28, 32**, (1960-7).

[EGA I] Dieudonné, J.; Grothendieck, A.: *'Eléments de géométrie algébrique, I: Le langage des schemas*, second edition, Springer, Berlin, 1971.

[Epp] Epp, H.: *Eliminating wild ramification*, Inv. Math. **19** (1973), 235–249.

[FK] Fujiwara, K; Kato, F.: *Rigid geometry and applications*, Moduli spaces and arithmetic geometry, 327–386, Adv. Stud. Pure Math., 45, Math. Soc. Japan, Tokyo, 2006.

[Gi] Gieseker, D.: *Lectures on moduli of curves*, Published for the Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1982.

[GR] Gabber, O.; Ramero, L.: *Almost ring theory*, Lecture Notes in Mathematics, 1800. Springer-Verlag, Berlin, 2003, vi+307 pp.

[Har] Hartshorne, R.: *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[Hub] Huber, R.: *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Aspects of Mathematics, Vol. 30, Vieweg, 1996.

[KKMS] Kempf, G.; Knudsen, F.; Mumford, D.; Saint-Donat, B.: *Toroidal embeddings*, Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973. viii+209 pp.

[Kuh1] Kuhlmann, F.-V.: *Elimination of Ramification I: The Generalized Stability Theorem*, to appear in Trans. Amer. Math. Soc.

[Kuh2] Kuhlmann, F.-V.: *Elimination of Ramification II: Henselian Rationality of Valued Function Fields*, in preparation.

[Lip] Lipman, J.: *Desingularization of two-dimensional schemes*, Ann. Math., **107** (1978), no. 1, 151–207.

[Mat] Matsumura, H.: *Commutative ring theory*, Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.

[Put] van der Put, M.: *Stable reductions of algebraic curves*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no. 4, 461–478.

[RG] Raynaud, M.; Gruson, L.: *Critères de platitude et de projectivité*, Inv. Math. **13** (1971), 1–89.

[Sa] Saito, T.: *Vanishing cycles and geometry of curves over a discrete valuation ring*, Amer. J. Math. **109** (1987), no. 6, 1043–1085.

[Tem1] Temkin, M.: *On local properties of non-Archimedean analytic spaces.*, Math. Ann. **318** (2000), no. 3, 585–607.

[Tem2] Temkin, M.: *On local properties of non-Archimedean analytic spaces II*, Isr. J. of Math. **140** (2004), 1–27.

[Tem3] Temkin, M.: *Relative Riemann-Zariski spaces*, preprint, arXiv:[0804.2843], to appear in Isr. J. of Math.

[Tem4] Temkin, M.: *Inseparable local uniformization*, preprint, arXiv:[0804.1554].

[TT] Thomason, R.; Trobaugh, T.: *Higher algebraic K-theory of schemes and of derived categories*, in The Grothendieck Festschrift, Vol. III, 247–436, Progress in Mathematics vol. 88, Birkhauser, Boston, 1990.

[Zar] Zariski, O.: *Local uniformization on algebraic varieties*, Ann. of Math., **41,** (1940), 852–896.

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
E-mail address: `temkin@math.ias.edu`