# ADMISSIBLE CONSTANTS FOR GENUS 2 CURVES

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Abstract. S.-W. Zhang recently introduced a new adelic invariant $\varphi$ for curves of genus at least 2 over number fields and function fields. We calculate this invariant when the genus is equal to 2.

## 1. Introduction

Let $X$ be a smooth projective geometrically connected curve of genus $g \geq 2$ over a field $k$ which is either a number field or the function field of a curve over a field. Assume that $X$ has semistable reduction over $k$. For each place $v$ of $k$, let $Nv$ be the usual local factor connected with the product formula for $k$.

In a recent paper [11] S.-W. Zhang proves the following theorem:

**Theorem 1.1.** *Let $(\omega, \omega)_a$ be the admissible self-intersection of the relative dualizing sheaf of $X$. Let $\langle \Delta_\xi, \Delta_\xi \rangle$ be the height of the canonical Gross-Schoen cycle on $X^3$. Then the formula:*

$$(\omega, \omega)_a = \frac{2g-2}{2g+1} \left( \langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \varphi(X_v) \log Nv \right)$$

*holds, where the $\varphi(X_v)$ are local invariants associated to $X \otimes k_v$, defined as follows:*

- *if $v$ is a non-archimedean place, then:*

$$\varphi(X_v) = -\frac{1}{4}\delta(X_v) + \frac{1}{4} \int_{R(X_v)} g_v(x,x)((10g+2)\mu_v - \delta_{K_{X_v}}) \,,$$

  *where:*
  - *$\delta(X_v)$ is the number of singular points on the special fiber of $X \otimes k_v$,*
  - *$R(X_v)$ is the reduction graph of $X \otimes k_v$,*
  - *$g_v$ is the Green's function for the admissible metric $\mu_v$ on $R(X_v)$,*
  - *$K_{X_v}$ is the canonical divisor on $R(X_v)$.*

  *In particular, $\varphi(X_v) = 0$ if $X$ has good reduction at $v$;*
- *if $v$ is an archimedean place, then:*

$$\varphi(X_v) = \sum_\ell \frac{2}{\lambda_\ell} \sum_{m,n=1}^{g} \left| \int_{X(\bar{k}_v)} \phi_\ell \omega_m \bar{\omega}_n \right|^2 ,$$

  *where $\phi_\ell$ are the normalized real eigenforms of the Arakelov Laplacian on $X(\bar{k}_v)$ with eigenvalues $\lambda_\ell > 0$, and $(\omega_1, \ldots, \omega_g)$ is an orthonormal basis for the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_{X(\bar{k}_v)} \omega \, \bar{\eta}$ on the space of holomorphic differentials.*

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Apart from giving an explicit connection between the two canonical invariants $(\omega,\omega)_a$ and $\langle\Delta_\xi,\Delta_\xi\rangle$, Zhang's theorem has a possible application to the effective Bogomolov conjecture, *i.e.*, the question of giving effective positive lower bounds for $(\omega,\omega)_a$. Indeed, the height of the canonical Gross-Schoen cycle $\langle\Delta_\xi,\Delta_\xi\rangle$ is known to be non-negative in the case of a function field in characteristic zero, and should be non-negative in general by a standard conjecture of Gillet-Soulé (*op. cit.*, Section 2.4). Further, the invariant $\varphi$ should be non-negative, and Zhang proposes, in the non-archimedean case, an explicit lower bound for it which is positive in the case of non-smooth reduction (*op. cit.*, Conjecture 1.4.2). Note that it is clear from the definition that $\varphi$ is non-negative in the archimedean case; in fact it is positive (*op. cit.*, Remark after Proposition 2.5.3).

Besides $\varphi(X_v)$, Zhang also considers the invariant $\lambda(X_v)$ defined by:

$$\lambda(X_v)=\frac{g-1}{6(2g+1)}\varphi(X_v)+\frac{1}{12}(\varepsilon(X_v)+\delta(X_v))\,,$$

where:

- if $v$ is a non-archimedean place, the invariant $\delta(X_v)$ is as above, and:
$$\varepsilon(X_v)=\int_{R(X_v)}g_v(x,x)((2g-2)\mu_v+\delta_{K_{X_v}})\,,$$
- if $v$ is an archimedean place, then:
$$\delta(X_v)=\delta_F(X_v)-4g\log(2\pi)$$
with $\delta_F(X_v)$ the Faltings delta-invariant of the compact Riemann surface $X(\bar{k}_v)$, and $\varepsilon(X_v)=0$.

The significance of this invariant is that if $\deg\det R\pi_*\omega$ denotes the (non-normalized) geometric or Faltings height of $X$ one has a simple expression:

$$\deg\det R\pi_*\omega=\frac{g-1}{6(2g+1)}\langle\Delta_\xi,\Delta_\xi\rangle+\sum_v\lambda(X_v)\log Nv$$

for $\deg\det R\pi_*\omega$, as follows from the Noether formula:

$$12\deg\det R\pi_*\omega=(\omega,\omega)_a+\sum_v(\varepsilon(X_v)+\delta(X_v))\log Nv\,.$$

Now assume that $X$ has genus $g=2$. Our purpose is to calculate the invariants $\varphi(X_v)$ and $\lambda(X_v)$ explicitly. For the $\lambda$-invariant we obtain:

- if $v$ is non-archimedean, then:
$$10\lambda(X_v)=\delta_0(X_v)+2\delta_1(X_v)\,,$$
where $\delta_0(X_v)$ is the number of non-separating nodes and $\delta_1(X_v)$ is the number of separating nodes in the special fiber of $X\otimes k_v$;
- if $v$ is archimedean, then:
$$10\lambda(X_v)=-20\log(2\pi)-\log\|\Delta_2\|(X_v)\,,$$
where $\|\Delta_2\|(X_v)$ is the normalized modular discriminant of the compact Riemann surface $X(\bar{k}_v)$ (see below).

Thus, the $\lambda(X_v)$ are precisely the well-known local invariants corresponding to the discriminant modular form of weight 10 [6] [9] [10]. In particular we have:

$$\deg\det R\pi_*\omega=\sum_v\lambda(X_v)\log Nv$$

and we recover the fact that the height of the canonical Gross-Schoen cycle vanishes for $X$.

## 2. The non-archimedean case

Let $k$ be a complete discretely valued field. Let $X$ be a smooth projective geometrically connected curve of genus 2 over $k$. Assume that $X$ has semistable reduction over $k$. In this section we give the invariants $\varphi(X)$ and $\lambda(X)$ of $X$.

The proof of our result is based on the classification of the semistable fiber types in genus 2 and consists of a case-by-case analysis. The notation we employ for the various fiber types is as in [8]. We remark that there are no restrictions on the residue characteristic of $k$.

**Theorem 2.1.** *The invariant $\varphi(X)$ is given by the following table, depending on the type of the special fiber of the regular minimal model of $X$:*

| *Type* | $\delta_0$ | $\delta_1$ | $\varepsilon$ | $\varphi$ |
|---|---|---|---|---|
| $I$ | 0 | 0 | 0 | 0 |
| $II(a)$ | 0 | $a$ | $a$ | $a$ |
| $III(a)$ | $a$ | 0 | $\frac{1}{6}a$ | $\frac{1}{12}a$ |
| $IV(a,b)$ | $b$ | $a$ | $a+\frac{1}{6}b$ | $a+\frac{1}{12}b$ |
| $V(a,b)$ | $a+b$ | 0 | $\frac{1}{6}(a+b)$ | $\frac{1}{12}(a+b)$ |
| $VI(a,b,c)$ | $b+c$ | $a$ | $a+\frac{1}{6}(b+c)$ | $a+\frac{1}{12}(b+c)$ |
| $VII(a,b,c)$ | $a+b+c$ | 0 | $\frac{1}{6}(a+b+c)+\frac{1}{6}\frac{abc}{ab+bc+ca}$ | $\frac{1}{12}(a+b+c)-\frac{5}{12}\frac{abc}{ab+bc+ca}$ |

*For $\lambda(X)$ the formula:*

$$10\lambda(X) = \delta_0(X) + 2\delta_1(X)$$

*holds.*

Let us indicate how the theorem is proved. Let $r$ be the effective resistance function on the reduction graph $R(X)$ of $X$, extended bilinearly to a pairing on $\mathrm{Div}(R(X))$. By Corollary 2.4 of [2] the formula:

$$\varphi(X) = -\frac{1}{4}(\delta_0(X) + \delta_1(X)) - \frac{3}{8}r(K,K) + 2\varepsilon(X)$$

holds, where $K$ is the canonical divisor on $R(X)$. The invariant $r(K,K)$ is calculated by viewing $R(X)$ as an electrical circuit. The invariant $\varepsilon$ is calculated on the basis of explicit expressions for the admissible measure and admissible Green's function; see [7] and [8] for such computations. The results we find are as follows:

| *Type* | $\delta_0$ | $\delta_1$ | $r(K,K)$ | $\varepsilon$ |
|---|---|---|---|---|
| $I$ | 0 | 0 | 0 | 0 |
| $II(a)$ | 0 | $a$ | $2a$ | $a$ |
| $III(a)$ | $a$ | 0 | 0 | $\frac{1}{6}a$ |
| $IV(a,b)$ | $b$ | $a$ | $2a$ | $a+\frac{1}{6}b$ |
| $V(a,b)$ | $a+b$ | 0 | 0 | $\frac{1}{6}(a+b)$ |
| $VI(a,b,c)$ | $b+c$ | $a$ | $2a$ | $a+\frac{1}{6}(b+c)$ |
| $VII(a,b,c)$ | $a+b+c$ | 0 | $2\frac{abc}{ab+bc+ca}$ | $\frac{1}{6}(a+b+c)+\frac{1}{6}\frac{abc}{ab+bc+ca}$ |

The values of $\varphi$ follow.

The formula for $\lambda(X)$ is verified for each case separately.

## 3. The archimedean case

Let $X$ be a compact and connected Riemann surface of genus 2. In this section we calculate the invariants $\varphi(X)$ and $\lambda(X)$ of $X$. Let $\mathrm{Pic}(X)$ be the Picard variety of $X$, and for each integer $d$ denote by $\mathrm{Pic}^d(X)$ the component of $\mathrm{Pic}(X)$ of degree $d$. We have a canonical theta divisor $\Theta$ on $\mathrm{Pic}^1(X)$, and a standard hermitian metric $\|\cdot\|$ on the line bundle $\mathcal{O}(\Theta)$ on $\mathrm{Pic}^1(X)$. Let $\nu$ be its curvature form. We have:

$$\int_{\mathrm{Pic}^1(X)} \nu^2 = \Theta.\Theta = 2\,.$$

Let $K$ be a canonical divisor on $X$, and let $\mathbf{P}$ be the set of 10 points $P$ of $\mathrm{Pic}^1(X)-\Theta$ such that $2P \equiv K$. Denote by $\|\theta\|$ the norm of the canonical section $\theta$ of $\mathcal{O}(\Theta)$. We let:

$$\|\Delta_2\|(X) = 2^{-12} \prod_{P\in\mathbf{P}} \|\theta\|^2(P)\,,$$

the normalized modular discriminant of $X$, and we let $\|H\|(X)$ be the invariant of $X$ defined by:

$$\log\|H\|(X) = \frac{1}{2}\int_{\mathrm{Pic}^1(X)} \log\|\theta\|\,\nu^2\,.$$

These two invariants were introduced in [1].

**Theorem 3.1.** *For the $\varphi$-invariant and the $\lambda$-invariant of $X$, the formulas:*

$$\varphi(X) = -\frac{1}{2}\log\|\Delta_2\|(X) + 10\log\|H\|(X)$$

*and*

$$10\lambda(X) = -20\log(2\pi) - \log\|\Delta_2\|(X)$$

*hold.*

The key to the proof is the following lemma. Let $\Phi$ be the map:

$$X^2 \to \mathrm{Pic}^1(X)\,, \qquad (x,y)\mapsto [2x-y]\,.$$

**Lemma 3.2.** *The map $\Phi$ is finite flat of degree* 8.

*Proof.* Let $y \mapsto y'$ be the hyperelliptic involution of $X$. We have a commutative diagram:

$$\begin{array}{ccc} X^2 & \xrightarrow{\Phi} & \mathrm{Pic}^1(X) \\ \downarrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ X^2 & \xrightarrow{\Phi^\vee} & \mathrm{Pic}^3(X) \end{array}$$

where $\alpha$ and $\beta$ are isomorphisms, with:

$$\begin{array}{ccc} \alpha\colon X^2 \to X^2\,, & \Phi^\vee\colon X^2 \to \mathrm{Pic}^3(X)\,, & \beta\colon \mathrm{Pic}^3(X) \to \mathrm{Pic}^1(X)\,, \\ (x,y) \mapsto (x,y')\,, & (x,y) \mapsto [2x+y]\,, & [D] \mapsto [D-K]\,. \end{array}$$

It suffices to prove that $\Phi^\vee$ is finite flat of degree 8. Let $p\colon X^{(3)} \to \mathrm{Pic}^3(X)$ be the natural map; then $p$ is a $\mathbf{P}^1$-bundle over $\mathrm{Pic}^3(X)$, and $\Phi^\vee$ has a natural injective lift to $X^{(3)}$. A point $D$ on $X^{(3)}$ is in the image of this lift if and only if $D$, when seen as an effective divisor on $X$, contains a point which is ramified for the morphism $X \to \mathbf{P}^1$ determined by the fiber $|D|$ of $p$ in which $D$ lies. Since every morphism $X \to \mathbf{P}^1$ associated to a $D$ on $X^{(3)}$ is ramified, the map $\Phi^\vee$ is surjective. As every morphism $X \to \mathbf{P}^1$ associated to a $D$ on $X^{(3)}$ has only finitely many ramification points, the map $\Phi^\vee$ is quasi-finite, hence finite since $\Phi^\vee$ is proper. As $X^2$ and $\mathrm{Pic}^3(X)$ are smooth and the fibers of $\Phi^\vee$ are equidimensional, the map $\Phi^\vee$ is flat. By Riemann-Hurwitz the generic $X \to \mathbf{P}^1$ associated to a $D$ on $X^{(3)}$ has 8 simple ramification points. It follows that the degree of $\Phi^\vee$ is 8. □

Let $G\colon X^2 \to \mathbf{R}$ be the Arakelov-Green's function of $X$, and let $\Delta$ be the diagonal divisor on $X^2$. We have a canonical hermitian metric on the line bundle $\mathcal{O}(\Delta)$ on $X^2$ by putting $\|1\|(x,y) = G(x,y)$, where 1 is the canonical section of $\mathcal{O}(\Delta)$. Denote by $h_\Delta$ the curvature form of $\mathcal{O}(\Delta)$. We have:

$$\int_{X^2} h_\Delta^2 = \Delta.\Delta = -2\,.$$

Restricting $\mathcal{O}(\Delta)$ to a fiber of any of the two natural projections of $X^2$ onto $X$ and taking the curvature form we obtain the Arakelov $(1,1)$-form $\mu$ on $X$. We have $\int_X \mu = 1$ and:

$$\int_X \log G(x,y)\,\mu(x) = 0$$

for each $y$ on $X$. Let $(\omega_1, \omega_2)$ be an orthonormal basis of $\mathrm{H}^0(X, \omega_X)$, the space of holomorphic differentials on $X$. We can write explicitly:

$$h_\Delta(x,y) = \mu(x) + \mu(y) - i\sum_{k=1}^{2}(\omega_k(x)\bar{\omega}_k(y) + \omega_k(y)\bar{\omega}_k(x))$$

and:

$$\mu(x) = \frac{i}{4}\sum_{k=1}^{2}\omega_k(x)\bar{\omega}_k(x)\,.$$

By [11, Proposition 2.5.3] we have:

$$\varphi(X) = \int_{X^2} \log G\, h_\Delta^2\,.$$

We compute the integral using our results from [4] and [5]. Let $W$ be the divisor of Weierstrass points on $X$, and let $p_1\colon X^2 \to X$ be the projection onto the first coordinate. The divisor $W$ is reduced effective of degree 6. According to [3, p. 31] there exists a canonical isomorphism:

$$\sigma \colon \Phi^* \mathcal{O}(\Theta) \xrightarrow{\cong} \mathcal{O}(2\Delta + p_1^* W)$$

of line bundles on $X^2$, identifying the canonical sections on both sides. In [4, Proposition 2.1] we proved that this isomorphism has a constant norm over $X^2$. Thus, the curvature forms on both sides are equal:

$$\Phi^* \nu = 2h_\Delta + 6\mu(x) \quad \text{on } X^2 \, .$$

Squaring both sides of this identity we get:

$$h_\Delta^2 = \frac{1}{4} \Phi^*(\nu^2) - 6 h_\Delta \mu(x) \, ,$$

since $\mu(x)^2 = 0$. Denote by $S(X)$ the norm of $\sigma$. Then we have:

$$2 \log G(x,y) + \sum_w \log G(x,w) = \log \|\theta\|(2x - y) + \log S(X)$$

for generic $(x,y) \in X^2$, where $w$ runs through the Weierstrass points of $X$. By fixing $y$ and integrating against $\mu(x)$ on $X$ we find that:

$$\log S(X) = - \int_X \log \|\theta\|(2x-y) \, \mu(x) \, .$$

By integrating against $h_\Delta^2$ on $X^2$ we obtain:

$$2\varphi(X) + \sum_w \int_{X^2} \log G(x,w) \, h_\Delta^2 = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x-y) \, h_\Delta^2 \, .$$

As we have:

$$h_\Delta^2 = 2\mu(x)\mu(y) - \sum_{k,l=1}^{2} (\omega_k(x)\bar{\omega}_l(x)\bar{\omega}_k(y)\omega_l(y) + \bar{\omega}_k(x)\omega_l(x)\omega_k(y)\bar{\omega}_l(y))$$

it follows that:

$$\int_{X^2} \log G(x,w) \, h_\Delta^2 = 0$$

for each $w$ in $W$ and hence we simply have:

$$2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x-y) \, h_\Delta^2 \, .$$

Using our earlier expression for $h_\Delta^2$ this becomes:

$$2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x-y) \left( \frac{1}{4} \Phi^*(\nu^2) - 6 h_\Delta \mu(x) \right) \, .$$

It is easily verified that $h_\Delta \mu(x) = h_\Delta \mu(y) = \mu(x)\mu(y)$ and hence:

$$\int_{X^2} \log \|\theta\|(2x-y) \, h_\Delta \mu(x) = \int_{X^2} \log \|\theta\|(2x-y) \, \mu(x)\mu(y) = -\log S(X) \, .$$

From Lemma 3.2 it follows that:

$$\int_{X^2} \log \|\theta\|(2x-y) \, \Phi^*(\nu^2) = 8 \int_{\mathrm{Pic}^1(X)} \log \|\theta\| \, \nu^2 = 16 \log \|H\|(X) \, .$$

All in all we find:

$$\varphi(X) = 2 \log S(X) + 2 \log \|H\|(X) \, .$$

Let $\delta_F(X)$ be the Faltings delta-invariant of $X$. According to [5, Corollary 1.7] the formula:

$$\log S(X) = -16\log(2\pi) - \frac{5}{4}\log\|\Delta_2\|(X) - \delta_F(X)$$

holds, and in turn, according to [1, Proposition 4] we have:

$$\delta_F(X) = -16\log(2\pi) - \log\|\Delta_2\|(X) - 4\log\|H\|(X)\,.$$

The formula:

$$\varphi(X) = -\frac{1}{2}\log\|\Delta_2\|(X) + 10\log\|H\|(X)$$

follows.

By definition we have:

$$\lambda(X) = \frac{1}{30}\varphi(X) + \frac{1}{12}\delta_F(X) - \frac{2}{3}\log(2\pi)$$

so we obtain:

$$10\lambda(X) = -20\log(2\pi) - \log\|\Delta_2\|(X)$$

by using [1, Proposition 4] once more.

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