A SPECTRAL SEQUENCE TO COMPUTE L^2-BETTI NUMBERS OF GROUPS AND GROUPOIDS

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ABSTRACT. We construct a spectral sequence for L^2-type cohomology groups of discrete measured groupoids. Based on the spectral sequence, we prove the Hopf-Singer conjecture for aspherical manifolds with poly-surface fundamental groups. More generally, we obtain a permanence result for the Hopf-Singer conjecture under taking fiber bundles whose base space is an aspherical manifold with poly-surface fundamental group. As further sample applications of the spectral sequence, we obtain new vanishing theorems and explicit computations of L^2-Betti numbers of groups and manifolds and obstructions to the existence of normal subrelations in measured equivalence relations.

1. INTRODUCTION AND STATEMENT OF RESULTS

The aim of this work is to construct, starting with a short exact sequence (later called *strong extension*) of discrete measured groupoids, a spectral sequence for L^2-type cohomology groups. For this, we are using a blend of tools from homological algebra and ergodic theory.

Gaboriau introduced and studied the notion of L^2-Betti numbers of measured equivalence relations [13], which proved to be very fruitful, especially in applications to von Neumann algebras via the work of Popa. Subsequently, more algebraic definitions were developed [23, 28] that build upon Lück's algebraic theory of L^2-Betti numbers [21]. We work with the latter since they are especially suited for our purposes.

Gaboriau's and Lück's works add quite different computational techniques to the theory of L^2-Betti numbers: For instance, Gaboriau's theory allows to exploit the fact that the ergodic dimension of a group might be much smaller than its cohomological dimension; Lück's algebraic theory, on the other hand, allows to use the power of standard homological algebra in computations of L^2-Betti numbers (another algebraic L^2-theory is due to Farber [8]).

The motivation for our spectral sequence was to combine these computational advantages. The reader may wonder whether the generality of the language of groupoids is necessary; we will present computations (Corollaries 1.8, 1.13, 1.15) for groups and manifolds that, in their proofs, make use of measured equivalence relations. The class of measured equivalence relations, however, is not closed under taking quotients, unlike the class of discrete measured groupoids; so it turns out that it is necessary and most natural to work with groupoids in our context.

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We refer the reader for a more detailed review of used tools, concepts, and notation to Section 2.

A central notion is that of a *strongly normal subgroupoid* of a discrete measured groupoid. Building on [9], we see in Section 3 that a strongly normal subgroupoid (Definition 3.5) allows a quotient construction, and every strongly normal subgroupoid appears as the kernel of a quotient map. One calls an ergodic $\mathcal{G}$ a *strong extension* of $\mathcal{S}$ and $\mathcal{Q}$ if $\mathcal{S} \subset \mathcal{G}$ is a strongly normal subgroupoid and $\mathcal{Q}$ the corresponding quotient groupoid. Several examples of strong extensions are discussed in Section 5.

For a discrete measured groupoid $\mathcal{G}$, we define cohomology groups $H^*(\mathcal{G}, M)$ with coefficients in a module M over the groupoid ring of $\mathcal{G}$ (see Section 4). If M is the algebra of affiliated operators $\mathcal{U}(\mathcal{G})$ of the von Neumann algebra of $\mathcal{G}$, then Lück's dimension of the $\mathcal{U}(\mathcal{G})$-module $H^n(\mathcal{G},\mathcal{U}(\mathcal{G}))$ defines the n-th L^2-Betti number $b_n^{(2)}(\mathcal{G})$ of $\mathcal{G}$. This definition of $b_n^{(2)}(\mathcal{G})$ coincides with the definition given by the first author in [23] which itself is an algebraic formulation of Gaboriau's definition [13]. Although it gives the same L^2-Betti numbers, our definition of $H^*(\mathcal{G}, M)$ is more involved: we have to take, in some sense, a huge model of $H^*(\mathcal{G}, M)$ to overcome some serious, technical difficulties.

Theorem 4.7 of Section 4 is our main result: a Grothendieck spectral sequence that computes the cohomology of a strong extension of $\mathcal{S}$ and $\mathcal{Q}$ in terms of the cohomology groups of $\mathcal{S}$ and $\mathcal{Q}$.

We now present applications of the spectral sequence. The proofs of the theorems below are found in Section 7.

1.1. **Applications to ergodic theory.** The following theorem generalizes a corresponding result by Gaboriau [13, Théorème 6.6] for extension of groups where the quotient group is amenable.

Theorem 1.1. *Let*
$$1 \to \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \to 1$$
be a strong extension of discrete measured groupoids where $\mathcal{Q}$ is an infinite amenable discrete measured groupoid. If $b_n^{(2)}(\mathcal{S}) < \infty$, then $b_n^{(2)}(\mathcal{G}) = 0$.

Corollary 1.2. *Let Γ be a countable group with $b_1^{(2)}(\Gamma) \neq 0$. Then there is a free, ergodic Γ-probability space such that the associated orbit equivalence relation $\mathcal{R}$ has a strongly normal subrelation $\mathcal{S} \subset \mathcal{R}$ of infinite index with $b_1^{(2)}(\mathcal{S}) = \infty$.*

Proof of Corollary. Because of $b_1^{(2)}(\Gamma) \neq 0$ the group Γ does not have property (T). By [24, Theorem 1.5] there is an ergodic Γ-probability space (X, μ) that is not *strongly ergodic* (defined in [24, Section 1]). By [24, Theorems 2.1 and 2.3] (the details of the proof are in [18]), there is an amenable equivalence relation $\mathcal{R}_{\text{hyp}}$ and a strong surjection $\theta : X \rtimes \Gamma \to \mathcal{R}_{\text{hyp}}$. The claim now follows for $\mathcal{S} = \ker(\theta)$ from Theorem 1.1 and $b_p^{(2)}(X \rtimes \Gamma) = b_p^{(2)}(\Gamma)$ (see Section 4.5). □

Theorem 1.3. *If an ergodic discrete measured groupoid $\mathcal{G}$ possesses a strongly normal subgroupoid $\mathcal{S}$ such that $b_n^{(2)}(\mathcal{S}) = 0$ for every $n \leq d$, then $b_n^{(2)}(\mathcal{G}) = 0$ for every $n \leq d$.*

The question of (non-)existence of subrelations that are (strongly) normal or split off as a factor is investigated in several works. For example, in [9, Theorem 4.4] it is proved that orbit equivalence relations of free II_1-actions of a lattice in a simple, connected, non-compact

Lie group never have infinite, strongly normal, amenable subrelations. In [1, Corollary 6.2] it is shown that the orbit equivalence relation of a type II_1-action of a non-elementary word-hyperbolic group is irreducible, that is, it never splits off an infinite, discrete measured equivalence relation as a factor.

Corollary 1.4. *Let $\mathcal{G}$ be an infinite, amenable, ergodic discrete measured groupoid. Then $b_p^{(2)}(\mathcal{G}) = 0$ for every $p \geq 0$.*

Proof of Corollary. Consider the principal extension $1 \to \mathcal{G}_{\mathrm{stab}} \to \mathcal{G} \to \mathcal{G}_{\mathrm{rel}} \to 1$ of Subsection 5.3. Assume first that $\mathcal{G}_{\mathrm{stab}}$ is infinite. Since $\mathcal{G}$ is amenable, the stabilizer groups $\mathcal{G}_x = \{\gamma \in \mathcal{G};\ r(\gamma) = s(\gamma) = x\}$ are (infinite) amenable for a.e. $x \in \mathcal{G}^0$. By Lemma 5.1, we have $b_p^{(2)}(\mathcal{G}_{\mathrm{stab}}) = 0$ for every $p \geq 0$. By Theorem 1.3 the assertion follows.

Suppose now that $\mathcal{G}_{\mathrm{stab}}$ is finite. Then $\mathcal{G}_{\mathrm{rel}}$ has to be infinite since $\mathcal{G}$ is so. Hence $\mathcal{G}_{\mathrm{rel}}$ is an infinite hyperfinite equivalence relation. Now the assertion is implied by Lemma 5.1 and Theorem 1.1. □

Theorem 1.5. *Let*

$$1 \to \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \to 1$$

be a strong extension of discrete measured groupoids where $\mathcal{Q}$ is an infinite measured groupoid. Assume that $b_p^{(2)}(\mathcal{S}) = 0$ for all $0 \leq p \leq d$ and $b_{d+1}^{(2)}(\mathcal{S}) < \infty$. Then $b_p^{(2)}(\mathcal{G}) = 0$ for all $0 \leq p \leq d+1$.

As a consequence, we obtain an alternative proof of [3, Théorème 5.4]:

Corollary 1.6 (Bergeron-Gaboriau). *Let Γ be a countable group with $b_1^{(2)}(\Gamma) \neq 0$. Let (X, μ) be an ergodic Γ-probability space. Then either the stabilizer Γ_x is finite for μ-a.e. $x \in X$, or $b_1^{(2)}(\Gamma_x) = \infty$, thus Γ_x is not finitely generated, for μ-a.e. $x \in X$.*

Proof of Corollary. By ergodicity, either Γ_x is finite almost everywhere or infinite almost everywhere. The function $x \mapsto b_1^{(2)}(\Gamma_x)$ is measurable [3, Théorème 5.7]. Thus, by ergodicity again, either $b_1^{(2)}(\Gamma_x) = \infty$ almost everywhere or $b_1^{(2)}(\Gamma_x) < \infty$ almost everywhere. Apply Theorem 1.5 to the principal extension (Subsection 5.3) in combination with Lemma 5.1. Note that the first L^2-Betti number of a finitely generated group is finite. □

Remark 1.7 (Ergodicity hypothesis). We remark that the notion of *strong extension* is based upon ergodicity. Nevertheless, it is possible to drop the assumption on ergodicity in several of the results above. In fact, if $\int_X \mu_x d\mu(x) = \mu$ is the ergodic decomposition of $(\mathcal{G}, \mu)$, then it is possible to show that

$$b_p^{(2)}(\mathcal{G}, \mu) = \int_X b_p^{(2)}(\mathcal{G}_x, \mu_x) d\mu(x).$$

1.2. **Applications to L^2-Betti numbers of groups and manifolds.** Lück [21, Theorem 7.2 on p. 294] proved the following corollary under the additional assumption that the quotient group has a non-torsion element or finite subgroups of arbitrarily high order. Then, using equivalence relations, Gaboriau [13, Théorème 6.8] gave a proof in degree 1, i.e. for the first L^2-Betti number, only assuming the quotient to be infinite. We also mention that [6, Corollary 1] gives a very elementary proof for the degree 1 case if the quotient has a non-torsion element. It is remarked therein that it is *a challenging, and vaguely irritating question to find a purely cohomological proof of Gaboriau's result.* Up to now, there is no proof that does not use measured equivalence relations. The following corollary of Theorem 1.5 generalizes the aforementioned results to all degrees without further assumptions on the quotient.

Corollary 1.8. *Let $\Lambda \subset \Gamma$ be a normal subgroup of infinite index. Suppose that $b_p^{(2)}(\Lambda) = 0$ for $0 \leq p \leq d-1$ and $b_d^{(2)}(\Lambda) < \infty$. Then $b_p^{(2)}(\Gamma) = 0$ for $0 \leq p \leq d$.*

Proof of Corollary. One can find free, ergodic measure-preserving actions of Λ, Γ and $Q = \Gamma/\Lambda$ on probability spaces such that the associated orbit equivalence relations form a strong extension (see Subsection 5.2). Then apply Theorem 1.5 and (4.6) in Section 4. □

The notion of *measure equivalence* was introduced by Gromov and, for the first time, gained prominence in the work of Furman [10, Definition 1.1]:

Definition 1.9. Two countable groups Γ and Λ are called *measure equivalent* if there exists a non-trivial measure space (Ω, μ) on which $\Gamma \times \Lambda$ acts such that the restricted actions of $\Gamma = \Gamma \times \{1\}$ and $\Lambda = \{1\} \times \Lambda$ have measurable fundamental domains $X \subset \Omega$ and $Y \subset \Omega$ with $\mu(X) < \infty$ and $\mu(Y) < \infty$. The space (Ω, μ) is called a *measure coupling* between Γ and Λ.

We denote by $\mathrm{cd}_{\mathbb{C}}(\Gamma)$ the *cohomological dimension* of a group Γ over $\mathbb{C}$, i.e. the projective dimension of $\mathbb{C}$ as a $\mathbb{C}\Gamma$-module. The following theorem is a consequence of the more general Theorem 7.3 and Lemma 7.2.

Theorem 1.10. *Let $1 \to \Lambda \to \Gamma \to Q_0 \to 1$ be a short exact sequence of groups. Suppose that $b_p^{(2)}(\Lambda) = 0$ for $p > m$. Let Q_1 be a group that is measure equivalent to Q_0. Let $n = \mathrm{cd}_{\mathbb{C}}(Q_1)$. Then $b_p^{(2)}(\Gamma) = 0$ for $p > n + m$.*

Remark 1.11. In order to capture the strength of the method of proof that leads to the above theorem, we introduce the concept of *measurable cohomological dimension* in Section 6. It is then clear that the only relevant assumption is that the measurable cohomological dimension of the quotient group is bounded by n. We will prove this fact in Theorem 7.3.

Remark 1.12. A typical situation where the quotient group is measure equivalent to a group with a lower cohomological dimension is the following: Let Γ be a cocompact lattice in $\mathrm{SL}(n, \mathbb{R})$. Then $\mathrm{SL}(n, \mathbb{R})$ endowed with the left multiplication action of Γ and the right multiplication action of $\mathrm{SL}(n, \mathbb{Z})$ is a measure coupling with respect to the Haar measure. The rational cohomological dimension of Γ equals the dimension of the associated symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$; but the rational cohomological dimension of $\mathrm{SL}(n, \mathbb{Z})$ is

$$\mathrm{cd}_{\mathbb{Q}}\big(\mathrm{SL}(n, \mathbb{Z})\big) = \dim\big(\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})\big) - (n-1)$$

by a result of Borel-Serre [5].

We present the following sample application of Theorem 1.10, for which we do not know an alternative proof that does not use measured groupoids.

Corollary 1.13. *Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ be infinite amenable groups. Let Γ be an extension of the type*

$$1 \to A_1 * \cdots * A_k \to \Gamma \to B_1 * \cdots * B_l \to 1.$$

Then

$$b_p^{(2)}(\Gamma) = \begin{cases} (k-1)(l-1) & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Corollary. Since $b_p^{(2)}(A_i) = 0$ for every $p \geq 0$ and $i \in \{1, \ldots, k\}$ by [21, Theorem 6.37 on p. 259], we obtain by the Mayer-Vietoris sequence for l^2-homology that $b_p^{(2)}(A_1 * \cdots * A_k) = 0$ whenever $p > 1$.

Since $B_1 * \cdots * B_l$ and the free group F_l of rank l are measure equivalent [14, Section 2.2], it follows that $b_p^{(2)}(\Gamma) = 0$ for $p > 2$ by the previous theorem. Since Γ and $A_1 * \cdots * A_k$ are infinite, we have $b_0^{(2)}(\Gamma) = b_0^{(2)}(A_1 * \cdots * A_k) = 0$, thus, by Corollary 1.8, $b_1^{(2)}(\Gamma) = 0$. Since the Euler characteristic χ is multiplicative for extensions and $\chi(\Gamma) = \sum_{p \geq 0} (-1)^p b_p^{(2)}(\Gamma)$, we obtain that

$$b_2^{(2)}(\Gamma) = \chi(\Gamma) = \chi(A_1 * \cdots * A_k)\chi(B_1 * \cdots * B_l) = (k-1)(l-1). \qquad \square$$

Next we turn our attention to a central conjecture for L^2-Betti numbers, the *Hopf-Singer Conjecture*; it predicts that for a closed aspherical manifold M we have $b_p^{(2)}(\widetilde{M}) = 0$ provided $2p \neq \dim(M)$. For a survey of known results we refer the reader to [21, Chapter 11].

A group is said to be a *poly-surface group*, if it has a series of normal subgroups such that the subquotients are fundamental groups of closed oriented surfaces (see Definition 6.10 for more details). The cohomological dimension of such a group is precisely twice the length of the series and the Euler characteristic is the product of the individual Euler characteristics of the subquotients.

In Section 6, we study the measurable cohomological dimension (which is closely related to Gaboriau's ergodic dimension [13]) in more detail and show, that the measurable cohomological dimension of a poly-surface group is the length of its defining series of normal subgroups, i.e. only half of the expected number.

The following theorem is obtained as Corollary 6.11 in Subsection 6.2.

Theorem 1.14. *The Hopf-Singer conjecture holds for any closed aspherical manifold with poly-surface fundamental group.*

We also obtain a permanence result for the Hopf-Singer conjecture:

Theorem 1.15. *Let M be a closed, aspherical, $2n$-dimensional manifold that satisfies the Hopf-Singer Conjecture, that is, $b_p^{(2)}(\widetilde{M}) = 0$ unless $p = n$. Let L be a closed orientable aspherical manifold of dimension $2m$ with a poly-surface fundamental group. If N is the total space of an orientable fiber bundle over L with fiber M, then*

$$b_p^{(2)}(\widetilde{N}) = \begin{cases} b_n^{(2)}(\widetilde{M}) \cdot \chi(\pi_1(L)) & \text{if } p = m+n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, N satisfies the Hopf-Singer Conjecture, too.

Proof. The manifold N is closed, orientable and aspherical. Let $\Gamma = \pi_1(N)$, and $\Lambda = \pi_1(M)$. From the fiber bundle we get an extension of groups

$$1 \to \Lambda \to \Gamma \to \pi_1(L) \to 1.$$

Now, $\pi_1(L)$ satisfies Singer's condition (see Definition 6.6) by Theorem 6.11. Hence, the group $\pi_1(L)$ has measurable cohomological dimension m. By Theorem 7.3, $b_p^{(2)}(\widetilde{N}) = b_p^{(2)}(\Gamma) = 0$ for $p > m+n$. Thus, by Poincaré duality, $b_p^{(2)}(\widetilde{N}) = 0$ unless $p = m+n$. This yields $b_{m+n}^{(2)}(\widetilde{N}) = \pm\chi(N)$, and from the multiplicativity of χ for fiber bundles the claim follows. $\square$

2. Overview of used concepts and tools

2.1. **Standard Borel and measure spaces.** All measurable spaces in this work are *standard Borel spaces* unless stated otherwise. Maps between standard Borel spaces are measurable unless stated otherwise. Our background references for standard Borel spaces are [7,19]; we recall here some basic notions and facts (see also [15, p. 51/52]).

We use the terms *measurable* and *Borel* interchangeably for maps between or subsets of standard Borel spaces. A measure on a standard Borel space is called *Borel measure*. A *partition* of standard Borel space X is a countable family $(X_i)_{i\in I}$ of pairwise disjoint Borel subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$. A *Borel isomorphism* $f : X \to Y$ between standard Borel spaces is a bijective Borel map. Inverses of Borel isomorphisms are Borel, and Borel subsets of a standard Borel space are again standard Borel.

The following result is a fundamental tool to which we refer throughout the paper as the *selection theorem* (see [19, theorem 18.10 on p. 123]).

Theorem 2.1 (Selection Theorem)**.** *Let $f : X \to Y$ be a Borel map between standard Borel spaces whose fibers are countable. Then $f(X) \subset Y$ is Borel, and there is a partition $X = \bigcup_{i\in\mathbb{N}} X_i$ such that each $f|_{X_i}$ is injective.*

A *measure space* (X, μ) is by definition a standard Borel space X equipped with a Borel measure μ. A *probability space* is a measure space whose measure is a probability measure, that is, has total measure 1. A *measure isomorphism* $f : (X, \mu_X) \to (Y, \mu_Y)$ is a measure-preserving Borel map with the property that there are Borel subsets $A \subset X$ and $B \subset Y$ with $\mu_X(X - A) = \mu_Y(Y - B) = 0$ such that $f|_A$ is a Borel isomorphism $A \to B$. If (X, μ) is continuous, that is, $\mu(\{x\}) = 0$ for every $x \in X$, then there is Borel isomorphism $f : X \to [0, 1]$ with $f_*\mu = \mu \circ f^{-1} = \lambda$. Here, λ denotes the Lebesgue measure.

Another important tool [15, Theorem 3.22 on p. 72] is

Theorem 2.2 (Measure disintegration)**.** *Let (X, μ) and (Y, ν) be probability spaces and $\pi : X \to Y$ a Borel map such that $\pi_*\mu = \nu$. Then there is a map $y \mapsto \mu_y$ that associates to every $y \in Y$ a probability measure μ_y on X such that*

i) *For every Borel subset $A \subset X$, the function $y \mapsto \mu_y(A)$ is Borel.*
ii) *For ν-a.e. $y \in Y$, $\mu_y(\pi^{-1}(y)) = 1$.*
iii)
$$\mu = \int_Y \mu_y d\nu(y).$$

2.2. **Discrete measured groupoids.** The standard reference for measurable groupoids is [2]. The *source and range maps* of a groupoid are denoted by s and r, respectively. We use superscript 0 to denote the *unit space* of a groupoid, as in $\mathcal{G}^0$ for the groupoid $\mathcal{G}$. Our convention is that a *subgroupoid* of a groupoid has the same unit space.

A *discrete measurable groupoid* $\mathcal{G}$ is a groupoid $\mathcal{G}$ equipped with the structure of a standard Borel space such that $\mathcal{G}^0 \subset \mathcal{G}$ is a Borel subset, all the structure maps are Borel, and $s^{-1}(\{x\})$ is countable for all $x \in \mathcal{G}^0$. Let c^s_x and c^r_x denote the counting measures on $s^{-1}(x)$ and $r^{-1}(x)$, respectively.

A *discrete measured groupoid* $(\mathcal{G}, \mu)$ is a discrete measurable groupoid $\mathcal{G}$ together with a Borel measure μ on $\mathcal{G}^0$ such that

$$\int_{\mathcal{G}^0} c^s_x d\mu(x) = \int_{\mathcal{G}^0} c^r_x d\mu(x)$$

as measures on $\mathcal{G}$. This measure on $\mathcal{G}$, which extends the one on $\mathcal{G}^0$, is also denoted by μ.

Moreover, $(\mathcal{G}, \mu)$ is called *infinite* if $s^{-1}(x)$ is infinite for all $x \in \mathcal{G}^0$ in a subset of positive measure.

The discrete measured groupoid $(\mathcal{G}, \mu)$ is said to be *ergodic* if one (thus, all) of the following equivalent conditions hold:

i) Any function $f : \mathcal{G}^0 \to \mathbb{R}$ that is $\mathcal{G}$-*invariant* (that is, $s \circ f = r \circ f$) is μ-a.e. constant.
ii) For any Borel subset $A \subset \mathcal{G}^0$ of positive measure, the so-called *saturation* $A^{\mathcal{G}} := \{x \in \mathcal{G}^0;\ \exists \gamma \in \mathcal{G} : s(\gamma) \in A, r(\gamma) = x\}$ has full measure.
iii) For any two Borel subsets $A, B \subset \mathcal{G}^0$ of positive measure there exists a Borel subset $E \subset \mathcal{G}$ such that $s(E) \subset A$, $r(E) \subset B$ and $\mu(s(E)) > 0$, $\mu(r(E)) > 0$.

Each discrete measured groupoid $(\mathcal{G}, \mu)$ has an *ergodic decomposition*, that is, there is an disintegration map $(\mathcal{G}^0, \mu) \to (X, \nu)$ such that $(\mathcal{G}, \mu_x)$ is for ν-a.e. $x \in X$ ergodic [16, Theorem 6.1]. The measure space (X, ν) is called the *space of ergodic components*.

A Borel map $\phi : (\mathcal{G}, \mu) \to (\mathcal{G}', \nu)$ is called a *homomorphism* if ϕ is a map of groupoids and $f_*\mu = \nu$.

Let (X, μ) be a probability space with a probability measure μ, and let $\Gamma \times X \to X$ be a measure preserving (m.p.) group action. We denote by $(X \rtimes \Gamma, \mu)$ the translation groupoid, i.e. the groupoid with total space $X \times \Gamma$, base space X and where $s = \pi_X$ and $r : X \rtimes \Gamma \to X$ is defined to be the action of Γ on X.

For the definition of an *amenable* or, equivalently, *hyperfinite* discrete measured groupoid we refer to [26, Chapter XIII, §3].

2.3. **Spectral sequences.** Weibel's book [29] is a standard reference for all the homological algebra that we need. We restate [29, Theorem 5.8.3 on p. 158] for the convenience of the reader.

Theorem 2.3 (Grothendieck)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories, such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects. Let $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ be left exact functors, such that that G sends injectives to injectives. Then, given $A \in \mathcal{A}$, there exists a first quadrant spectral sequence:*

$$E_2^{pq} = (R^pF)(R^qG)(A) \Longrightarrow R^{p+q}(FG)(A).$$

Here R^pF, for $p \geq 0$, denotes the p-th right derived functor of F. This very general spectral sequence can be used to construct the classical Hochschild-Serre spectral sequence in group cohomology. The core of our work consists in constructing suitable functors F and G and verifying the above hypothesis in the setting of discrete measured groupoids.

In view of the above assumptions, it is useful to have a criterion that ensures that a functor preserves injective objects:

Lemma 2.4 ([29, Theorem 2.6.1 on p. 50])**.** *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. If G has an exact left-adjoint functor, then it sends injectives to injectives.*

2.4. L^2**-Betti numbers.** The standard reference for L^2-Betti numbers is Lück's book [21]. In this paper we only refer to the homological-algebra-type definition of L^2-Betti numbers in the context of groups [21, Chapter 6] and discrete measured groupoids [23] (see also [27]).

Let (M, τ) be a finite von Neumann algebra with a fixed trace $\tau\colon M \to \mathbb{C}$. Lück introduced a dimension function $L \mapsto \dim_{(M,\tau)}(L)$ for an arbitrary M-module L with very nice properties like additivity for arbitrary modules. If the context is clear, we may also omit the trace τ or the whole subscript in $\dim_{(M,\tau)}$.

If $M = L(\Gamma)$ is the group von Neumann algebra of a group Γ with its standard trace, then the p-th L^2-Betti number of Γ is defined as

$$b_p^{(2)}(\Gamma) := \dim_{L(\Gamma)} \operatorname{Tor}_p^{\mathbb{C}\Gamma}\big(\mathbb{C}, L(\Gamma)\big) = \dim_{L(\Gamma)} H_p\big(\Gamma, L(\Gamma)\big) \in [0,\infty].$$

If $\mathcal{G}$ is a discrete measured groupoid, then one defines in a similar way

$$b_p^{(2)}(\mathcal{G}) := \dim_{L(\mathcal{G})} \operatorname{Tor}_p^{\mathcal{R}(\mathcal{G})}\big(L^\infty(\mathcal{G}^0), L(\mathcal{G})\big) \in [0,\infty].$$

For the definition of $\mathcal{R}(\mathcal{G})$ and $L(\mathcal{G})$ see Section 4.1.

2.5. **Operators affiliated with a finite von Neumann algebra.** Let (M,τ) again be a finite von Neumann algebra with a fixed trace τ. Denote by $L^2(M,\tau)$ the GNS-construction with respect to τ. There is a natural algebra $\mathcal{U}(M,\tau)$ of closable, densely defined and unbounded operators on $L^2(M,\tau)$, which are affiliated with the algebra M. For details, we refer to [25, Chapter IX].

There is also the notion of dimension $\dim_{\mathcal{U}(M,\tau)}(M) \in [0,\infty]$ for an arbitrary $\mathcal{U}(M,\tau)$-module M [21, Chapter 8; 22]. As it turns out, a useful dimension function is obtained by restricting the module structure to M and taking the dimension of the $\mathcal{U}(M,\tau)$-module as an M-module. Again, we may omit τ or the whole subscript in $\dim_{\mathcal{U}(M,\tau)}$ if the context is clear.

This algebra has been studied in connection with L^2-invariants in [22, 28]. It shares very nice ring-theoretic properties, which we want to summarize in the sequel:

i) $\mathcal{U}(M,\tau)$ is *von Neumann regular*, i.e. all modules are flat,
ii) $\mathcal{U}(M,\tau)$ is left and right *self-injective*, i.e. $\mathcal{U}(M,\tau)$ is injective as left and as right module over itself,
iii) $\iota\colon M \to \mathcal{U}(M,\tau)$ is a flat ring extension, i.e. $L \mapsto \mathcal{U}(M,\tau) \otimes_M L$ is exact,
iv) $\dim_{\mathcal{U}(M,\tau)} \mathcal{U}(M,\tau) \otimes_M L = \dim_{(M,\tau)} L$, for every M-module L,
v) $\dim_{\mathcal{U}(M,\tau)} \hom_{\mathcal{U}(M,\tau)}(L, \mathcal{U}(M,\tau)) = \dim_{\mathcal{U}(M,\tau)} L$, for every $\mathcal{U}(M,\tau)$-module L.
vi) If $\dim_{\mathcal{U}(M,\tau)} L = 0$, then $\hom_{\mathcal{U}(M,\tau)}(L, \mathcal{U}(M,\tau)) = 0$.

For proofs, we refer to [22, 28] and the references therein.

2.6. **Localization and completion.** Let (M,τ) be as above. Let $\mathcal{A}$ be an abelian category with a faithful functor $F\colon \mathcal{A} \to \operatorname{Mod}(M)$. Assume that the functor F preserves limits and co-limits. In the category of M-modules, the sub-category of zero-dimensional modules is a Serre sub-category. Moreover, the full sub-category $\mathcal{S} \subset \mathcal{A}$, given by those modules, that map to zero-dimensional modules, forms a Serre sub-category as well.

Example 2.5. The example to have in mind, is the category of $L^\infty(X) \rtimes \Gamma$-modules with its forgetful functor to $L^\infty(X)$-modules.

Some of our computations will be carried out in the *localized* category $\mathcal{A}/\mathcal{S}$, which we also denote by $\mathcal{A}_{\mathrm{loc}}$. This category is abelian and its properties, for the examples of special interest, were studied in [27], where the second author showed that it has enough projective objects and naturally embeds in $\mathcal{A}$ as the sub-category of those modules, which are *complete* with respect to the rank metric, see [27] for details. We recall some of the results in Section 4.

3. Quotients and Normality

In this section, we provide the technical underpinning of the concept of (strong) extension of discrete measurable groupoids.

3.1. **Ergodic discrete measurable groupoids.**

Lemma 3.1. *Let $(\mathcal{G},\mu)$ be an ergodic discrete measured groupoid with atom-free μ, that is, $\mu(\{x\})=0$ for every $x\in\mathcal{G}^0$. Let $A,B\subset\mathcal{G}^0$ be Borel subsets of positive measure. Then*

$$\mu\big(s^{-1}(A)\cap r^{-1}(B)\big)=\infty.$$

Proof. We start with a general observation about ergodic groupoids: The function $\mathcal{G}^0\to\mathbb{Z}\cup\{\infty\}, x\mapsto \#r^{-1}(x)=\#s^{-1}(x)$ is a.e. constant because of ergodicity. If $\mathcal{G}^0$ is atom-free, then $\#s^{-1}(x)=\infty$ for a.e. $x\in\mathcal{G}^0$ since otherwise there existed $n\geq 1$ and $A\subset\mathcal{G}^0$ with $\mu(A)<1/n$ and $\#s^{-1}(x)\leq n$ for $x\in A$, implying $\mu(A^{\mathcal{G}})<1$.

Again by ergodicity, we can pick a set $E\subset\mathcal{G}$ with $A':=s(E)\subset A$ and $B':=r(E)\subset B$ such that $\mu(A'),\mu(B')>0$. By the selection theorem we can assume that there is such E with injective $s|_E$ and $r|_E$. Denote by $f:B'\to E$ the inverse of $r:E\to B'$. Consider the Borel map

$$\phi: s^{-1}(A')\cap r^{-1}(B')\to s^{-1}(A')\cap r^{-1}(A')=\mathcal{G}_{A'}, \phi(\gamma)=f(r(\gamma))^{-1}\circ\gamma.$$

Notice that the restricted groupoid $\mathcal{G}_{A'}:=s^{-1}(A')\cap r^{-1}(A')$ is ergodic if $\mathcal{G}$ is so.

Since ϕ is a map over A' with respect to the source maps and fiberwise bijective, we obtain, with the general observation above, that

$$\mu\big(s^{-1}(A)\cap r^{-1}(B)\big)\geq\mu\big(s^{-1}(A')\cap r^{-1}(B')\big)=\mu\big(\mathcal{G}_{A'}\big)=\infty. \qquad\square$$

The following lemma is certainly well known but we failed to find a reference.

Lemma 3.2. *Let $(\mathcal{G},\mu)$ be an ergodic discrete measured groupoid. Then there is a countable set I and a measure isomorphism $\phi:\mathcal{G}^0\times I\to\mathcal{G}$ such that $s\circ\phi=\mathrm{pr}^0_{\mathcal{G}}$, where $\mathrm{pr}^0_{\mathcal{G}}:\mathcal{G}^0\times I\to\mathcal{G}^0$ is the projection and $\mathcal{G}^0\times I$ is endowed with the product of μ and the counting measure on I. Further, for every $i\in I$, the map $\mathcal{G}^0\to\mathcal{G}^0, x\mapsto r(\phi(x,i))$, is a measure isomorphism.*

Proof. By ergodicity, $(\mathcal{G}^0,\mu)$ is either discrete (thus $\mathcal{G}$ is finite) or atom-free. We leave the easy proof of the first case to the reader and proceed to the atom-free case. The following auxiliary fact is needed for the proof.

Claim: Let $F\subset\mathcal{G}$ be a Borel subset of finite measure. Let $E\subset\mathcal{G}\backslash F$ be a Borel subset on which r and s are injective, then there is a Borel subset $D\subset\mathcal{G}\backslash F$ containing E such that r,s are injective on D and $s(D)=\mathcal{G}^0$ up to null sets. Note that by $\mathcal{G}$-invariance of μ, this also implies that $r(D)=\mathcal{G}^0$ up to null sets.

The set $\mathcal{B}$ of Borel sets D such that $E\subset D\subset\mathcal{G}\backslash F$ and $r|_D,s|_D$ are injective contains E and is partially ordered as follows: $D_1\leq D_2$ if and only if $D_1\subset D_2$ up to null sets. Let $\mathcal{T}\subset\mathcal{B}$ be a totally ordered subset. To apply Zorn's lemma later on, we show that $\mathcal{T}$ has an upper bound in $\mathcal{B}$. Let $r=\sup_{D\in\mathcal{T}}\mu(s(D))\in(0,1]$. Pick a countable family $\{D_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$ with $D_n\leq D_{n+1}$ and $\mu(s(D_n))\to r$. Let $D=\bigcup D_n$. Upon subtracting a suitable null set from D we can ensure that for all $x,y\in D$ there is $n_0\in\mathbb{N}$ with $x,y\in D_{n_0}$. Thus $r|_D,s|_D$ are injective, and D is an upper bound of $\mathcal{T}$. By Zorn's lemma the set $\mathcal{B}$ possesses a maximal element $D_{max}\in\mathcal{B}$. Suppose $\mu(s(D_{max}))<1$, thus $\mu(s(D_{max}))<1$. Let $A\subset\mathcal{G}^0\backslash s(D_{max})$ and $B\subset\mathcal{G}^0\backslash r(D_{max})$ be subsets of positive measure. Since $\mu(F)<\infty$ by assumption, Lemma 3.1 implies that $\mu(s^{-1}(A) \cap r^{-1}(B) \cap (\mathcal{G} \backslash F)) = \infty$. In particular, there is $E \subset \mathcal{G}$ of positive measure such that $s(E) \subset A$ and $r(E) \subset B$. Once more using the selection theorem, we can assume that there is such E with r, s being injective on E. Since $E \cup D_{max}$ would contradict maximality of D_{max}, it follows that $\mu(s(D_{max})) = \mu(r(D_{max})) = 1$.

Let us continue with the proof of the lemma. Using the selection theorem, pick a countable Borel partition $\{E_n\}_{n\in\mathbb{N}}$ of $\mathcal{G}$ such that r, s are injective on each E_n. We construct inductively Borel subsets $D_n \subset \mathcal{G}$ such that

i) $E_n \subset \bigcup_{i=1}^{n} D_i$ for every $n \geq 1$, and the union is disjoint up to null sets,
ii) r and s are injective on D_n for every $n \geq 1$,
iii) $r(D_n) = s(D_n) = \mathcal{G}^0$ up to null sets.

According to the claim above there is such D_1, and for $n > 1$ there is a Borel subset $D_n \subset \mathcal{G} \backslash \bigcup_{i=1}^{n-1} D_i$ containing $E_n \backslash \bigcup_{i=1}^{n-1} D_i$ such that r, s are injective on D_n and $s(D_n) = \mathcal{G}^0$ up to null sets. Since $E_n \subset \bigcup_{i\geq 1} D_i$ for every $n \in \mathbb{N}$, we have $\mathcal{G} = \bigcup_{i\geq 1} D_i$. The (inverse of the) desired isomorphism is now given as

$$\mathcal{G} \to \mathcal{G}^0 \times \mathbb{N}, \gamma \mapsto (s(\gamma), n) \text{ if } \gamma \in D_n. \qquad \square$$

Definition 3.3. Let $\mathcal{S} \subset \mathcal{G}$ be a subgroupoid of a discrete measured groupoid $\mathcal{G}$. Let $A \subset \mathcal{G}^0$ be a Borel subset. Let $\mathrm{End}_{\mathcal{S}}(\mathcal{G})|_A$ denote the set of all Borel maps $\phi : A \to \mathcal{G}$ such that

i) $s(\phi(x)) = x$ for a.e. $x \in A$,
ii) $\gamma \in \mathcal{S} \Leftrightarrow \phi(r(\gamma))\gamma\phi(s(\gamma))^{-1} \in \mathcal{S}$ for every $\gamma \in \mathcal{G}$ with $r(\gamma) \in A$ and $s(\gamma) \in A$.

Let $\mathrm{Aut}_{\mathcal{S}}(\mathcal{G})|_A \subset \mathrm{End}_{\mathcal{S}}(\mathcal{G})|_A$ be the subset consisting of those ϕ such that $x \to r(\phi(x))$ is a.e. injective. An element in $\mathrm{Aut}_{\mathcal{S}}(\mathcal{G})|_A$ is called a *local section of* $\mathcal{G}$. If $A = \mathcal{G}^0$, we will drop the subscript A in the notation. If $\mathcal{S} = \mathcal{G}$, the condition ii) is void, and we will drop the subscript $\mathcal{S}$ in the notation.

Definition 3.4. Let $\phi : A \to \mathcal{G}$ and $\psi : B \to \mathcal{G}$ be local sections. The *composition* $\phi \circ \psi$ is the local section defined by

$$\psi^{-1}\big(r(B) \cap A\big) \to \mathcal{G},\ x \mapsto \phi\big(r(\psi(x))\big) \circ \psi(x).$$

3.2. **Definition of strong extensions.** The following notion is a generalization of (strong) normality for equivalence relations [9, Theorem 2.2 and Definition 2.14].

Definition 3.5. We call a subgroupoid $\mathcal{S} \subset \mathcal{G}$ of a discrete measured groupoid *(strongly) normal* if there is a countable family $\{\phi_n\}$ (in $\mathrm{Aut}_{\mathcal{S}}(\mathcal{G})$) in $\mathrm{End}_{\mathcal{S}}(\mathcal{G})$ such that for a.e. $\gamma \in \mathcal{G}$ there exists exactly one ϕ_n in the family with $\phi_n(r(\gamma))\gamma \in \mathcal{S}$. We say that $\{\phi_n\}$ is a family of *(strongly) normal choice functions*.

Remark 3.6. Note that for a translation groupoid $X \rtimes G$ and a subgroup $H \subset G$, the groupoid $X \rtimes H \subset X \rtimes G$ is strongly normal if and only if H is normal in G. A countable family of strongly normal choice functions is given by the automorphisms of X which are induced by any complete set of representatives of the cosets G/H.

The previous definition makes sure that all the relevant choices can be made in a measurable way.

Definition 3.7. Let $\theta : (\mathcal{G}, \mu) \to (\mathcal{Q}, \nu)$ be a homomorphism of discrete measured groupoids. Let $\mathcal{S} = \theta^{-1}(\mathcal{Q}^0)$ be the kernel of θ, and let $\mu = \int_{\mathcal{Q}^0} \nu_y d\nu(y)$ be the measure disintegration with respect to θ. Then

i) θ is a *surjection* if for a.e. $x \in \mathcal{G}^0$ and a.e. $\gamma \in \mathcal{Q}$ with $s(\gamma) = \theta(x)$ there exists $g \in \mathcal{G}$ with $\theta(g) = \gamma$ (*pointwise surjectivity*),
ii) θ is a *strong surjection* if θ is a surjection and $(\mathcal{S}|_{\theta^{-1}(y)}, \nu_y)$ is ergodic for a.e. $y \in \mathcal{Q}^0$.

The following lemma is standard and proved in the context of equivalence relations in [11, Lemma 2.1]. We present a proof of the groupoid version here so that the reader can see the validity of Remark 3.9.

Lemma 3.8. *Let $\mathcal{G}$ be an ergodic discrete measured groupoid. Let $E, F \subset \mathcal{G}^0$ be Borel subsets with the same measure. Then there is $\phi \in \operatorname{Aut}(\mathcal{G})$ such that $r(\phi(E)) \subset F$ and $r \circ \phi : E \to F$ is a measure isomorphism.*

Proof. Choose a measure isomorphism $\mathcal{G}^0 \times I \to \mathcal{G}$, $(x, i) \to \phi_i(x)$ as in Lemma 3.2. For any Borel subsets $A, B \subset \mathcal{G}^0$ define

$$c(A, B) := \sup_{n \in \mathbb{N}} \mu\big(r(\phi_n(A)) \cap B\big).$$

By ergodicity, $c(A, B) > 0$ whenever $\mu(A) > 0$ and $\mu(B) > 0$. Let $E_0 = E$ and $F_0 = F$. By induction we define Borel sets $E_n, F_n \subset \mathcal{G}^0$ and elements $\psi_n \in \{\phi_i;\ i \in \mathbb{N}\}$ as follows: Given E_n and F_n let $i \in \mathbb{N}$ be the minimal number such that

$$\mu(r(\phi_i(E_n)) \cap F_n) \geq c(E_n, F_n)/2,$$

and let $\psi_n = \phi_i$ and $E_{n+1} = E_n \backslash (r \circ \phi_n)^{-1}(F_n)$, $F_{n+1} = F_n \backslash (r \circ \psi_n)(E_n)$. Let $E_\infty = \bigcap_n E_n$ and $F_\infty = \bigcap_n F_n$. For every $n \in \mathbb{N}$, the sets E_n and F_n have the same measure, thus $\mu(E_\infty) = \mu(F_\infty)$. Further, we have $\mu(E_\infty) = \mu(F_\infty) = 0$ since otherwise $c(E_n, F_n) \geq c := c(E_\infty, F_\infty) > 0$ for all n which contradicts the choice of ψ_n at the stage where $\mu(E_n \backslash E_{n+1}) < c/2$. So $(E'_n)_{n \in \mathbb{N}}$ and $(F'_n)_{n \in \mathbb{N}}$ defined by $E'_n = E_n \backslash E_{n+1}$ and $F'_n = F_n \backslash F_{n+1}$ are partitions of E and F, respectively. Now we can define $\phi : \mathcal{G}^0 \to \mathcal{G}$ as follows:

$$\phi(x) = \begin{cases} \mathrm{id}_x & \text{if } x \notin E, \\ \psi_n(x) & \text{if } x \in E'_n. \end{cases} \qquad \square$$

Remark 3.9. After fixing a measure isomorphism $\mathcal{G}^0 \times I \to \mathcal{G}$, the construction of $\phi \in \operatorname{Aut}(\mathcal{G})$ in the previous lemma involves no other choices. This fact is important in a situation where one wants to conclude that ϕ depends on the input data μ, E, F in a measurable way.

The next lemma turns out to be crucial for the construction of our spectral sequence in Section 4.

Lemma 3.10. *Let $\theta : (\mathcal{G}, \mu) \to (\mathcal{Q}, \nu)$ be a strong surjection of ergodic, discrete measured groupoids. For every Borel subset $A \subset \mathcal{Q}^0$ and every $\phi \in \operatorname{Aut}(\mathcal{Q})|_A$ there is a lift $\psi \in \operatorname{Aut}(\mathcal{G})|_{\theta^{-1}(A) \cap \mathcal{G}^0}$, that is, $\theta(\psi(x)) = \phi(\theta(x))$ for a.e. $x \in \theta^{-1}(A) \cap \mathcal{G}^0$.*

Proof. Let $\mathcal{S}$ be the kernel of θ. If the cardinality of $\mathcal{G}^0$ is finite, Lemma 3.10 becomes very easy and is left to the reader. We assume in the sequel that $\mathcal{G}^0$ is infinite. Thus there is a Borel isomorphism $\mathcal{G}^0 \cong [0, 1]$. Let us fix one. Let $\mu = \int_{\mathcal{Q}^0} \mu_y d\nu(y)$ be the disintegration of $(\mathcal{G}^0, \mu)$ with respect to θ. By assumption, $\mathcal{S}|_{\theta^{-1}(y)}$ is ergodic for a.e. $y \in \mathcal{Q}^0$.

First of all, we reproduce an argument in [15, Proof of Theorem 3.18 on p. 70]. The function $p : x \mapsto \mu_{\theta(x)}(\{x\})$ on $\mathcal{G}^0$ is measurable and $\mathcal{G}$-invariant. By ergodicity we have $p(x) = c$ for μ-a.e. $x \in \mathcal{G}^0$ and a constant $c \geq 0$. If $c > 0$, then $\theta : \mathcal{G}^0 \to \mathcal{Q}^0$ has finite fibers ν-a.e., and the lemma is easily proved using the selection theorem. We leave that to the reader and restrict ourselves to the more complicated case that $c = 0$, that is, μ_y is continuous for ν-a.e. $y \in \mathcal{Q}^0$.

Consider the following Borel maps

$$f : \mathcal{G}^0 \to \mathcal{Q}^0 \times [0,1], \quad f(x) = (\theta(x), \mu_{\theta(x)}([0,x])),$$
$$g : \mathcal{Q}^0 \times [0,1] \to \mathcal{G}^0, \quad (y,t) \mapsto \min\{x \in \mathcal{G}^0; \mu_y([0,x]) \geq t\}.$$

One sees that $g \circ f = \mathrm{id}$. In particular, f is injective. Further we have for a Borel subset $A \subset \mathcal{Q}^0$

$$\begin{aligned}\mu\big(f^{-1}(A \times [0,t])\big) &= \int_A \mu_y\big(g(A \times [0,t])\big) d\nu(y)\\ &= \int_A \mu_y\big(g(\{y\} \times [0,t])\big) d\nu(y)\\ &= \int_A \mu_y\big([0, g(y,t)]\big) d\nu(y) = t \cdot \nu(A).\end{aligned}$$

Thus $f_*\mu = \nu \times \lambda$, where λ denotes the Lebesgue measure. Hence $\mathrm{im}(f)$ has full measure, and f is a measure isomorphism $(\mathcal{G}^0, \mu) \to (\mathcal{Q}^0 \times [0,1], \nu \times \lambda)$.

Since θ is pointwise surjective, there is a, at least set-theoretic, lift ψ' of ϕ, and one uses Lemma 3.2 to obtain a measurable such $\psi' \in \mathrm{End}(\mathcal{G})|_{\theta^{-1}(A)\cap\mathcal{G}^0}$. By the selection theorem there is a partition

$$\theta^{-1}(A) \cap \mathcal{G}^0 = \bigcup_{k\geq 1} D_k \text{ with } \psi'|_{D_k} \in \mathrm{Aut}(\mathcal{G})|_{D_k} \text{ for any } k \geq 1.$$

Let $D'_k := r(\psi'(D_k))$. In the following, we neglect null sets. By the uniqueness of disintegration we obtain that

$$\mu_{r(\phi(y))}(D'_k) = \mu_y(D_k) \text{ for every } y \in \mathcal{Q}^0.$$

The problem is that the sets $\{D'_k\}_{k\geq 1}$ could have overlaps, thus causing ψ to be non-injective. The idea is to make these sets fiberwise disjoint by an application of Lemma 3.8 to $(\mathcal{S}, \mu_y)$ for $y \in C$. To make this precise, consider for $y \in \mathcal{Q}^0$ the unique minimal sequence $0 = m(y,0) < m(y,1) < m(y,2) < \ldots$ of numbers in $[0,1]$ with the property

$$\mu_{r(\phi(y))}(D'_k) = \mu_{r(\phi(y))}\big(g(\mathcal{Q}^0 \times [m(y,k-1), m(y,k)])\big).$$

Let $\tau_k^y : D'_k \to \mathcal{S}$ be the local section of $(\mathcal{S}, \mu_{r(\phi(y))})$ constructed in Lemma 3.8 with the property that

$$r \circ \tau_k^y : D'_k \to g\big(\mathcal{Q}^0 \times [m(y,k-1), m(y,k)]\big)$$

is a $\mu_{r(\phi(y))}$-measure isomorphism. We define a local section ψ on $\theta^{-1}(A)$ by

$$\psi(x) = \tau_k^{\theta(x)} \circ \psi'(x) \text{ for } x \in D_k.$$

It follows from the explicit construction of τ_k^y (see Remark 3.9) that ψ is measurable. It is clear that ψ is still a lift of ϕ since $\tau_k^{\theta(x)}$ is (fiberwise) a local section of $\mathcal{S} = \ker(\theta)$. One easily verifies that $\mu(r(\psi(B))) = \mu(B)$ for all Borel subsets $B \subset \theta^{-1}(A)$, thus $\psi : \theta^{-1}(A) \to \mathcal{G}$ lies indeed in $\mathrm{Aut}(\mathcal{G})|_{\theta^{-1}(A)\cap\mathcal{G}^0}$. □

3.3. **Strongly normal subgroupoids vs. strong quotients.**

Theorem 3.11. *Let $\theta : \mathcal{G} \to \mathcal{Q}$ be a strong surjection of ergodic, discrete measured groupoids. Then the kernel $\ker(\theta) = \theta^{-1}(\mathcal{Q}^0)$ is a strongly normal subgroupoid of $\mathcal{G}$.*

Proof. Consider a measure isomorphism $\phi : \mathcal{Q}^0 \times I \to \mathcal{Q}$ as in Theorem 3.2. According to Lemma 3.10 we can lift each $\phi_i := \phi|_{\mathcal{Q}^0 \times \{i\}} \in \mathrm{Aut}(\mathcal{Q})$ to $\psi_i \in \mathrm{Aut}(\mathcal{G})$. Then $\{\psi_i\}_{i \in I}$ is a family of strongly normal choice functions for $\ker(\theta)$. □

The following theorem is a straightforward generalization of [9, Theorem 2.2], where $\mathcal{S}$ and $\mathcal{G}$ are assumed to be equivalence relations, to groupoids. One does the same constructions as in *loc. cit.* line by line, but for groupoids.

Theorem 3.12 (Quotient construction)**.** *Let $\mathcal{S} \subset \mathcal{G}$ be a strongly normal subgroupoid of an ergodic discrete measured groupoid. Then there is a strong surjection $\theta : \mathcal{G} \to \mathcal{Q}$ onto an ergodic discrete measured groupoid, called the* quotient of $\mathcal{G}$ by $\mathcal{S}$*, such that*

i) $\ker(\theta) := \theta^{-1}(\mathcal{Q}^0) = \mathcal{S}$,
ii) for a.e. $x \in \mathcal{G}^0$ and every $\gamma \in \mathcal{Q}$ with $\theta(x) = s(\gamma)$ there exists $g \in \mathcal{G}$ such $\theta(g) = \gamma$,
iii) for any ergodic discrete measured groupoid $\mathcal{Q}'$ and any homomorphism $\theta' : \mathcal{G} \to \mathcal{Q}'$ with $\mathcal{S} \subset \ker\theta'$ there is a m.p. homomorphism $\kappa : \mathcal{Q} \to \mathcal{Q}'$ such that $\kappa \circ \theta = \theta'$.

Definition 3.13. With the setting of the previous theorem, we say that $(\mathcal{G}, \mu)$ is a *strong extension* of $(\mathcal{S}, \mu)$ by $(\mathcal{Q}, \nu)$, and we indicate this, similarly to groups, by writing:

$$1 \to (\mathcal{S}, \mu) \longrightarrow (\mathcal{G}, \mu) \longrightarrow (\mathcal{Q}, \nu) \to 1$$

We record the following lemma for later reference.

Lemma 3.14. *We retain the notation of the preceding Theorem 3.12. For every local section $\psi : A \to \mathcal{G}$ there is a countable Borel partition $A = \bigcup_{n \geq 1} A_n$, and local sections $q_n \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{G})|_{A_n}$ and $s_n \in \mathrm{Aut}(\mathcal{S})$ such that $\psi|_{A_n} = q_n \circ s_n$*

Proof. Let $\{\phi_n\}_{n \geq 1}$ be a family of strongly normal choice functions. Define

$$A_n := \{a \in A;\ (\phi_n \circ \psi)(a) \in \mathcal{S}\}.$$

Then $s_n := \phi_n \circ \psi|_{A_n} \in \mathrm{End}(\mathcal{S})|_{A_n}$. Upon further partitioning each A_n, we can assume that $s_n \in \mathrm{Aut}(\mathcal{S})$. For $q_n = \phi_n^{-1}$ the assertion follows. □

Remark 3.15. Let $\mathcal{S} \subset \mathcal{G}$ be strongly normal with quotient $\mathcal{G} \xrightarrow{\theta} \mathcal{Q}$ as in Definition 3.13. A local section $\phi \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{G})$ is the lift of some $\phi' \in \mathrm{Aut}(\mathcal{Q})$ by the following argument: The map $\theta \circ \phi : \mathcal{G}^0 \to \mathcal{Q}$ is $\mathcal{S}$-invariant. The restrictions of $\mathcal{S}$ to fibers $\theta^{-1}(q)$ for $q \in \mathcal{Q}^0$ are ergodic since θ is a strong surjection. Thus, $\theta \circ \phi$ descends to a map $\mathcal{Q}^0 \to \mathcal{Q}$ of which ϕ is a lift.

Remark 3.16. Consider Gaboriau's extension (5.1) of Subsection 5.2

$$1 \to (Z \rtimes \Lambda, \mu \times \nu) \longrightarrow (Z \rtimes \Gamma, \mu \times \nu) \longrightarrow (Y \rtimes Q, \nu) \to 1$$

associated to a group extension $1 \to \Lambda \to \Gamma \xrightarrow{p} Q \to 1$. If the action of Γ on $Z = X \times Y$ is not ergodic, then this is not a strong extension in the sense of Definition 3.13. However, the conclusion from the combination of Lemma 3.14 and Remark 3.15 is still true: For every local section $\psi : A \to Z \rtimes \Gamma$ there is a countable Borel partition $A = \bigcup_{n \geq 1} A_n$, local sections $q_n : A_n \to Z \rtimes \Gamma$ that are lifts of local sections of $Y \rtimes Q$, and local sections $s_n : A_n \to Z \rtimes \Lambda$ such that $\psi|_{A_n} = q_n \circ s_n$.

For that, note that a local section $\psi : A \to Z \rtimes \Gamma$ is essentially given by a map $A \to \Gamma$ that we denote by the same name. There are a countable Borel partition $Y = \bigcup_{n\geq 1} Y_n$ and elements $q_n \in Q$ for every $n \geq 1$ such that $p(\psi(a)) = q_n$ if $a \in A_n = A \cap X \times Y_n$. Choose lifts $\gamma_n \in \Gamma$ for each $q_n \in Q$. Then the desired $q_n : A_n \to \Gamma$ and $s_n : A_n \to \Lambda$ are defined by $q_n(a) = \gamma_n$ and $s_n(a) = \psi(a)q_n(a)^{-1}$.

4. Construction of the spectral sequence

4.1. $\mathcal{R}(\mathcal{G})$**-modules.** Let $(\mathcal{G}, \mu)$ be a discrete measured groupoid with source and range maps $s, r\colon \mathcal{G} \to \mathcal{G}^0$. Denote by $L(\mathcal{G}, \mu)$ the associated von Neumann algebra and by $\mathcal{U}(\mathcal{G}, \mu)$ the algebra of operators, which are affiliated with the finite von Neumann algebra $L(\mathcal{G}, \mu)$. Recall that a local section of $(\mathcal{G}, \mu)$ is a Borel section $\phi\colon A \to \mathcal{G}$ of s such that the restriction of r to $\phi(A)$ is injective.

Let $\mathcal{R}(\mathcal{G}, \mu)$ be the convolution algebra of complex valued Borel functions on $\mathcal{G}$, which can be written as *finite* sums of products of essentially bounded complex-valued Borel functions on $\mathcal{G}^0$ and characteristic functions $\chi_{\phi(A)}$ of graphs of local sections $\phi : A \to \mathcal{G}$. Clearly, $\mathcal{R}(\mathcal{G}, \mu) \subset L(\mathcal{G}, \mu)$ is a sub-ring.

Remark 4.1. We consider the case $(\mathcal{G}, \mu) = (X \rtimes \Gamma, \mu)$ of a translation groupoid. The *crossed product ring* $L^\infty(X) \rtimes \Gamma$ is the free $L^\infty(X)$-module with basis Γ. Its multiplication is determined by the rule $\gamma f = f(\gamma^{-1}\_)\gamma$ for $\gamma \in \Gamma, f \in L^\infty(X)$. The crossed product ring embeds as a rank-dense subring into $\mathcal{R}(X \rtimes \Gamma, \mu)$ via $\sum f_g g \mapsto f \in L^\infty(X \rtimes \Gamma)$ with $f(x, g^{-1}) = f_g(x)$ (compare [27, Proposition 4.1]).

The following lemma is a useful characterization of modules over $\mathcal{R}(\mathcal{G}, \mu)$, which should be seen in analogy to group rings.

Lemma 4.2. *Let M be an abelian group. To give a $\mathcal{R}(\mathcal{G}, \mu)$-module structure on M is the same as to give a $L^\infty(\mathcal{G}^0)$-module structure and compatible partial isomorphism as follows: For each local section ϕ of $(\mathcal{G}, \mu)$, there exists an isomorphism*

$$\hat{\phi}\colon \chi_A M \to \chi_{r(\phi(A))} M$$

that is compatible with composition, restriction and orthogonal sum.

Proof. Denote by $\overline{\mathcal{R}}(\mathcal{G}, \mu)$ the universal $L^\infty(\mathcal{G}^0)$-algebra, generated by symbols $\overline{\phi}$ for every local section ϕ of $\mathcal{G}$, subject to relations implementing compatibility with restriction, composition and orthogonal sum. Clearly, the natural map $\sigma\colon \overline{\mathcal{R}}(\mathcal{G}, \mu) \to \mathcal{R}(\mathcal{G}, \mu)$ is surjective and M is a $\overline{\mathcal{R}}(\mathcal{G}, \mu)$-module. The proof is finished by showing that σ is injective and hence an isomorphism.

Let $h = \sum f_i \overline{\phi}_i$ be a finite sum in $\overline{\mathcal{R}}(\mathcal{G}, \mu)$. It follows from the compatibility with respect to composition and restriction that every element in $\overline{\mathcal{R}}(\mathcal{G}, \mu)$ can be written in this form. Assume that $\sum f_i \phi_i = 0$. In order to show injectivity, we have to prove $h = 0$. We may assume that the support of f_i is equal to the domain of ϕ_i. If we partition $\mathcal{G}^0$ into sets $X_1, \ldots, X_k$, on which

(i) the domains of the local sections $\phi_i|_{X_j}$ are either X_j or empty, and
(ii) the local sections $\phi_i|_{X_j}$ are either a.e. equal or a.e. different,

we can restrict our attention to one set X_j at a time. Indeed, since $\overline{\phi_i}$ is the orthogonal sum of its restrictions to the X_j, it suffices to show $\sum_i f_i|_{X_j} \overline{\phi}_i|_{X_j} = 0$. Without loss of generality, we can now assume that the local sections $\phi_i|_{X_j}$ are a.e. different from each other. Clearly, $\sum_i f_i|_{X_j}\phi_i|_{X_j} = 0$ is only possible if $f_i = 0$, since the $\phi_i|_{X_j}$ have disjoint support as characteristic functions on $\mathcal{G}$. This finishes the proof. □

Remark 4.3. The abelian group $L^\infty(\mathcal{G}^0)$ becomes an $\mathcal{R}(\mathcal{G},\mu)$-module via

$$\hat{\phi} : L^\infty(A) \to L^\infty(r(\phi(A))),\ f \mapsto \big(x \mapsto f((r\circ\phi)^{-1}(x))\big)$$

for a local section $\phi : A \to \mathcal{G}$.

4.2. **Completion and localization of modules.** Every $L^\infty(\mathcal{G}^0)$-module M carries a canonical metric (*rank metric*), which is induced by the so-called *rank*, i.e.

$$d(\xi,\eta) = \inf\{\mu(A^c) \mid A \subset \mathcal{G}^0 \text{ Borel}, \chi_A\xi = \chi_A\eta\}, \quad \forall \xi,\eta \in M.$$

It was shown in [27, Lemma 4.4] that the completion of a $\mathcal{R}(\mathcal{G},\mu)$-module with respect to the underlying $L^\infty(\mathcal{G}^0)$-module carries a natural $\mathcal{R}(\mathcal{G},\mu)$-module structure and that the associated completion functor is exact [27, Lemma 2.6]. Moreover, the category of complete $\mathcal{R}(\mathcal{G},\mu)$-modules was shown to be abelian with enough projective objects [27, Theorem 2.7].

We denote by $\mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{comp}}$ the full subcategory of $\mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))$, formed by complete $\mathcal{R}(\mathcal{G},\mu)$-modules and denote by

$$c\colon\ \mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu)) \to \mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{comp}}$$

the completion functor.

Projectives in $\mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{comp}}$ are obtained by completing free modules. The following lemma shows that there are also enough injective objects.

Lemma 4.4. *The abelian category* $\mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{comp}}$ *has enough injective objects.*

Proof. Clearly, the abelian category $\mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))$ has enough injective objects. Let Z be an injective object in the abelian category $\mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))$. It is sufficient to construct an complete injective object Z', which contains Z as a submodule. Let α be an ordinal with uncountable cofinality, i.e. there exists no countable cofinal subset. For every $\beta \le \alpha$, we define Z_β by the following transfinite inductive procedure. We set $Z_0 = Z$ and $Z_{\beta+1}$ to be some injective $\mathcal{R}(\mathcal{G},\mu)$-module, containing the completion of Z_β as a sub-module. If β is a limit ordinal, we set $Z_\beta = \cup_{\beta'<\beta} Z_{\beta'}$. Every Cauchy sequence in Z_α is contained in some Z_β for $\beta < \alpha$ and thus its limit exists in $Z_{\beta+1} \subset Z_\alpha$. Hence Z_α is complete. Moreover, choosing α big enough, Z_α is injective as a $\mathcal{R}(\mathcal{G},\mu)$-module, since injectivity can be tested on the *set* of sub-modules of the trivial module. We can define $Z' = Z_\alpha$. □

$\mathcal{R}(\mathcal{G},\mu)$-modules, which are zero-dimensional as $L^\infty(\mathcal{G}^0)$-modules, complete to the zero module. Indeed, this is a reformulation of the local criterion of zero-dimensionality, that can be found in [23, Theorem 2.4]. Hence, there exists a natural exact functor of abelian categories

$$c\colon\ \mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{loc}} \to \mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{comp}}, \tag{4.1}$$

which is an equivalence of abelian categories. Indeed, the inverse is the restriction of the natural quotient functor. In the sequel we concentrate on complete $\mathcal{R}(\mathcal{G},\mu)$-modules. The completion of $L^\infty(\mathcal{G}^0)$ identifies naturally with the algebra of measurable functions on $\mathcal{G}^0$, which we denote by $\mathcal{M}(\mathcal{G}^0)$. Note that $\mathcal{U}(\mathcal{G},\mu)$ is complete, whereas $L(\mathcal{G},\mu)$ is not necessarily.

4.3. **Some derived functors.** In this subsection, we introduce some derived functors, which are appropriate to make our approach to a spectral sequence work. They are defined on the localized category; and thus we are heavily using its nice homological properties which were established in preceding section and [27].

Definition 4.5. Let M be a complete left $\mathcal{R}(\mathcal{G},\mu)$-module. We define:

$$H_*(\mathcal{G},M)=\underline{\operatorname{Tor}}_*^{\mathcal{R}(\mathcal{G},\mu)}\left(\mathcal{M}(\mathcal{G}^o),M\right), \quad \text{and} \quad H^*(\mathcal{G},M)=\underline{\operatorname{Ext}}^*_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),M),$$

to be the *complete* $\mathcal{G}$-homology and $\mathcal{G}$-cohomology of the $\mathcal{R}(\mathcal{G},\mu)$-module M.

Here, $\underline{\operatorname{Tor}}_*^{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),?)$ and $\underline{\operatorname{Ext}}^*_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),?)$ denote the derived functors of the functors

$$M\mapsto \mathcal{M}(\mathcal{G}^o)\otimes_{\mathcal{R}(\mathcal{G},\mu)} M, \quad \text{and} \quad M\mapsto \hom_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),M),$$

from the abelian category $\operatorname{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\rm comp}$ to abelian groups. Following the arguments in [29, Chapter 2.7], we see that the bi-functors $\underline{\operatorname{Tor}}_*^{\mathcal{R}(\mathcal{G},\mu)}(?,?)$ and $\underline{\operatorname{Ext}}^*_{\mathcal{R}(\mathcal{G},\mu)}(?,?)$ are balanced, as are their classical counterparts.

4.4. **Spectral sequence.** The following theorem enables us to construct a spectral sequence for the cohomology which we just defined.

Theorem 4.6. *Let*

$$1\to(\mathcal{S},\mu')\to(\mathcal{G},\mu)\to(\mathcal{Q},\nu)\to 1$$

be a strong extension of discrete measured groupoids, i.e. $(\mathcal{G},\mu)\to(\mathcal{Q},\nu)$ *is a strong surjection with kernel* $(\mathcal{S},\mu')$. *Let* M *be a complete left-module over* $\mathcal{R}(\mathcal{G},\mu)$. *The abelian group*

$$\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0),M)$$

is naturally a complete left module over $\mathcal{R}(\mathcal{Q},\nu)$, *and there is a natural isomorphism of abelian groups*

$$\hom_{\mathcal{R}(\mathcal{Q},\nu)}\left(\mathcal{M}(\mathcal{Q}^0),\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0),M)\right)=\hom_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),M).$$

Proof. Since $\mathcal{M}(\mathcal{Q}^0)\subset\mathcal{M}(\mathcal{G}^0)$, there is a natural $L^\infty(\mathcal{Q}^0)$-module structure on

$$\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0),M).$$

It remains to provide partial isomorphisms as described above. Let ϕ be a local section of $\mathcal{Q}$. Since $(\mathcal{G},\mu)\to(\mathcal{Q},\nu)$ is a strong surjection, there exists a section ϕ' of $\mathcal{G}$ which lifts ϕ by Lemma 3.10. For $f\in\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0),M)$, we define

$$(\phi\rhd f)(g)=\phi' f(\phi'^{-1}g), \quad \forall g\in\mathcal{M}(\mathcal{G}^0).$$

This is well-defined and, together with the aforementioned $\mathcal{M}(\mathcal{Q}^0)$-module structure, it defines a left-module structure of $\mathcal{R}(\mathcal{Q},\nu)$. Moreover, $\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0),M)$ is easily seen to be complete. We now aim to show that

$$\hom_{\mathcal{R}(\mathcal{Q},\nu)}\left(\mathcal{M}(\mathcal{Q}^0),\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0),M)\right)=\hom_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),M).$$

An element in $f\in\hom_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0),M)$ is described by the image of $1\in\mathcal{M}(\mathcal{G}^0)$. For this, note that all $L^\infty(\mathcal{Q}^0)$-module maps are automatically contractive and hence continuous in the rank metric, see [27, Lemma 2.3].

Let ϕ be a local section of $(\mathcal{G},\mu)$. We set $\hat\phi=\chi_{\phi(A)}\in\mathcal{R}(\mathcal{G},\mu)$. The image of 1 under f satisfies

$$\hat\phi f(1)=f(\chi_{r(\phi(A))})=\chi_{r(\phi(A))}f(1).$$

Conversely, every such element gives rise to a module homomorphism.

An element in $\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0), M)$ is described as an element in M, satisfying the above relation for local sections of $(\mathcal{S},\mu')$. Having this description, it is obvious that

$$\hom_{\mathcal{R}(\mathcal{Q},\nu)}\left(\mathcal{M}(\mathcal{Q}^0), \hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{G}^0), M)\right)$$

identifies with $\hom_{\mathcal{R}(\mathcal{G},\mu)}(\mathcal{M}(\mathcal{G}^0), M)$ since we can write each local section of $(\mathcal{G},\mu)$ as a countable orthogonal sum of local sections, which are products of lifts of a local section of $(\mathcal{Q},\nu)$ and a local section of $(\mathcal{S},\mu')$ by Lemma 3.14 and Remark 3.15. This finishes the proof, since M is complete and the orthogonal decomposition gives rise to a rank convergent sum M. □

The decomposition of functors which was established in the preceding theorem yields a Grothendieck spectral sequence by Theorem 2.3:

Theorem 4.7. *Let*

$$1 \to (\mathcal{S},\mu') \to (\mathcal{G},\mu) \to (\mathcal{Q},\nu) \to 1$$

be a strong extension of discrete measured groupoids. Let M be a complete left $\mathcal{R}(\mathcal{G},\mu)$-module. Then there is a first quadrant spectral sequence with E_2-term

$$\begin{aligned} E_2^{pq} &= \underline{\mathrm{Ext}}^p_{\mathcal{R}(\mathcal{Q},\nu)}\left(\mathcal{M}(\mathcal{Q}^0), \underline{\mathrm{Ext}}^q_{\mathcal{R}(\mathcal{S},\mu')}\left(\mathcal{M}(\mathcal{G}^0), M\right)\right) \\ &= H^p(\mathcal{Q}, H^q(\mathcal{S}, M)) \end{aligned} \tag{4.2}$$

converging to $\underline{\mathrm{Ext}}^{p+q}_{\mathcal{R}(\mathcal{G},\mu)}\left(\mathcal{M}(\mathcal{G}^0), M\right) = H^{p+q}(\mathcal{G}, M)$. For $M = \mathcal{U}(\mathcal{G},\mu)$, which is a $\mathcal{R}(\mathcal{G},\mu)$-$\mathcal{U}(\mathcal{G},\mu)$-bimodule, we obtain a spectral sequence of $\mathcal{U}(\mathcal{G},\mu)$-modules.

Proof. The only claim that remains to be verified is that $\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{R}^0), ?)$ sends injective objects to injective objects. However, this follows from Lemma 2.4 if we can provide an exact left adjoint functor. Clearly, a similar argument as above shows that

$$M \mapsto c\left(\mathcal{M}(\mathcal{R}^0) \otimes_{\mathcal{M}(\mathcal{Q}^0)} M\right)$$

is left adjoint to $\hom_{\mathcal{R}(\mathcal{S},\mu')}(\mathcal{M}(\mathcal{R}^0), ?)$. Moreover, it is the composition of a flat ring extension (note that $\mathcal{M}(\mathcal{Q}^0)$ is von Neumann regular) and the exact completion functor, and thus exact. This finishes the proof. □

Remark 4.8. For later reference we note that the E_1 is of the spectral sequence is given by

$$E_1^{p,q} = \hom_{\mathcal{R}(\mathcal{Q},\nu)}\left(P_p, \underline{\mathrm{Ext}}^q_{\mathcal{R}(\mathcal{S},\mu')}\left(\mathcal{M}(\mathcal{G}^0), M\right)\right)$$

for a projective resolution P_* of $\mathcal{M}(\mathcal{Q}^0)$ in $\mathrm{Mod}(\mathcal{R}(\mathcal{Q},\nu))_{\mathrm{comp}}$.

Remark 4.9. The conclusion of Theorem 4.7 also holds true for Gaboriau's extension (5.1) even if the group Γ does not act ergodically and so Gaboriau's extension is not a strong extension in the sense of Definition 3.13: For that, note that the conclusion of Lemma 3.10, which is used in Theorem 4.6, is very easy to see for Gaboriau's extension (compare the argument in Remark 3.16). Furthermore, one replace the application of Lemma 3.14 and Remark 3.15 in the proof of Theorem 4.6 by Remark 3.16. All other steps in the proofs of Theorems 4.6 and 4.7 stay the same in the case of Gaboriau's extension.

4.5. **Identification of L^2-Betti numbers.** In order to apply the spectral sequence from Section 4 to the computation of L^2-Betti numbers of groups, we need to study the special case of a m.p. group action more closely. Let Γ be a discrete group, and let (X,μ) be a standard Borel probability space on which Γ acts by m.p. Borel isomorphisms. Note that we do *not* impose any conditions like ergodicity or freeness of the action.

In many situations, invariants of a m.p. action of a discrete group are actually invariants of the group itself. In the sequel we want to identify the dimensions of the homological and cohomological invariants for $(X \rtimes \Gamma, \mu)$, which we just introduced, with ordinary L^2-Betti numbers of Γ.

First of all, for any discrete measured groupoid $(\mathcal{G},\mu)$, there is a natural transformation

$$\operatorname{Tor}_*^{\mathcal{R}(\mathcal{G},\mu)}(?, L(\mathcal{G},\mu)) \to \underline{\operatorname{Tor}}_*^{\mathcal{R}(\mathcal{G},\mu)}(?,\mathcal{U}(\mathcal{G},\mu)), \tag{4.3}$$

consisting of dimension isomorphisms. Indeed, it is induced by the natural map

$$? \otimes_{\mathcal{R}(\mathcal{G},\mu)} L(\mathcal{G},\mu) \to c(?) \otimes_{\mathcal{R}(\mathcal{G},\mu)} \mathcal{U}(\mathcal{G},\mu).$$

For a proof that this is a dimension isomorphism, we refer to the proof of [27, Proposition 4.7]. If follows from [27, Lemma 1.1] that the induced map on derived functors are dimension isomorphisms as well.

Secondly, there is an isomorphism of right $\mathcal{U}(\mathcal{G},\mu)$-modules

$$\hom_{\mathcal{U}(\mathcal{G},\mu)}\left(\underline{\operatorname{Tor}}_*^{\mathcal{R}(\mathcal{G},\mu)}(?,\mathcal{U}(\mathcal{G},\mu)),\mathcal{U}(\mathcal{G},\mu)\right) \cong \underline{\operatorname{Ext}}^*_{\mathcal{R}(\mathcal{G},\mu)}(?,\mathcal{U}(\mathcal{G},\mu)),$$

since $\mathcal{U}(\mathcal{G},\mu)$ is self-injective. Moreover, since dualizing is dimension preserving, we get:

$$\dim \underline{\operatorname{Tor}}_*^{\mathcal{R}(\mathcal{G},\mu)}(?,\mathcal{U}(\mathcal{G},\mu)) = \dim \underline{\operatorname{Ext}}^*_{\mathcal{R}(\mathcal{G},\mu)}(?,\mathcal{U}(\mathcal{G},\mu)). \tag{4.4}$$

In [23] the first author observed that due to dimension flatness of the ring extensions

$$\mathbb{C}\Gamma \subset \mathcal{R}(X \rtimes \Gamma), \quad \text{and} \quad L(\Gamma) \subset L(X \rtimes \Gamma),$$

there are natural dimension isomorphisms as follows:

$$\operatorname{Tor}_*^{\mathcal{R}(\mathcal{G},\mu)}(L^\infty(\mathcal{G}^0), L(\mathcal{G},\mu)) \cong H_*(\Gamma, L(X \rtimes \Gamma)) \cong H_*(\Gamma, L(\Gamma)) \otimes_{L\Gamma} L(X \rtimes \Gamma). \tag{4.5}$$

Thus, combining the dimension isomorphisms (4.3) for $L^\infty(\mathcal{G}^0)$, and (4.4), and (4.5), we get:

$$\begin{aligned} b_*^{(2)}(\Gamma) &= \dim_{L(X\rtimes\Gamma)} H_*(X \rtimes \Gamma, L(X \rtimes \Gamma)) \\ &= \dim_{L(X\rtimes\Gamma)} H^*(X \rtimes \Gamma, \mathcal{U}(X \rtimes \Gamma)) \end{aligned} \tag{4.6}$$

Remark 4.10. Note that $H^n(\mathcal{G},\mathcal{U}(\mathcal{G},\mu))$ is the $\mathcal{U}(\mathcal{G},\mu)$-dual of $H_n(\mathcal{G},\mathcal{U}(\mathcal{G},\mu))$ and thus has the pleasant feature that it vanishes as soon as its dimension is zero. This follows from the results in [28].

5. Examples of groupoid extensions

5.1. **Group extensions.** Let Γ be a group and let $\Lambda \subset \Gamma$ be a normal subgroup. Let (X,μ) be a standard Borel space with a probability measure μ, on which Γ acts by m.p. Borel isomorphisms. For example, one can take any measure on $\{0,1\}$ and consider the infinite product $\{0,1\}^\Gamma$ on which Γ acts by shifts.

Clearly, the translation groupoid $(X \rtimes \Gamma, \mu)$ is a discrete measured groupoid and $(X \rtimes \Lambda, \mu)$ is a strongly normal subgroupoid. If $(X \rtimes \Gamma, \mu)$ is ergodic, then let $\mathcal{Q}^X_{\Lambda\subset\Gamma}$ denote the quotient groupoid, which exists by Theorem 3.12.

5.2. **Gaboriau's extension.** Let $1 \to \Lambda \to \Gamma \xrightarrow{p} Q \to 1$ be a short exact sequence of groups. We describe a special case of 5.1 that Gaboriau used to prove vanishing results for L^2-Betti numbers of groups [13].

Let (X, μ) be a Γ-probability space and (Y, ν) be a Q-probability space. Let $Z = X \times Y$ be the product of probability spaces. The group Γ acts on Y via p and on Z by the diagonal action. Then $(Z \rtimes \Lambda, \mu \times \nu)$ is a strongly normal subgroupoid of $(Z \rtimes \Gamma, \mu \times \nu)$. We refer to

$$1 \to (Z \rtimes \Lambda, \mu \times \nu) \longrightarrow (Z \rtimes \Gamma, \mu \times \nu) \longrightarrow (Y \rtimes Q, \nu) \to 1 \tag{5.1}$$

as *Gaboriau's extension*; it is a strong extension provided Γ acts ergodically on Z. For every ergodic Q-probability space (Y, ν) we can find an ergodic Γ-probability space (X, μ) such that Γ acts ergodically on $(Z, \mu \times \nu)$: Take, for example, the infinite product $\{0,1\}^\Gamma$ with the equidistribution on $\{0, 1\}$ and Γ acting by the shift (*Bernoulli action*). Since this action is mixing, the diagonal Γ-action on Z is still ergodic.

5.3. **The principal extension.** Let $(\mathcal{G}, \mu)$ be an ergodic discrete measured groupoid. We denote by

$$\mathcal{G}_{\mathrm{stab}} := \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}$$

the *stabilizer groupoid*. We call $\mathcal{G}_x := \{\gamma \in \mathcal{G}; r(\gamma) = s(\gamma) = x\}$ the *isotropy group*, or the *stabilizer*, of $x \in \mathcal{G}^0$. $(\mathcal{G}_{\mathrm{stab}}, \mu)$ is a strongly normal subgroupoid of $(\mathcal{G}, \mu)$. Indeed, the Borel map $(r \times s)\colon (r \times s)^{-1}(\Delta(\mathcal{G}^0)) \to \mathcal{G}^0$ has countable fibers. Hence, we can use the selection theorem to find choice functions. We denote by $(\mathcal{G}_{\mathrm{rel}}, \mu)$ the quotient groupoid and call it the *groupoid of the associated equivalence relation*. Obviously, $(\mathcal{G}_{\mathrm{rel}}, \mu)$ has the same unit space $\mathcal{G}^0$. The strong extension

$$0 \to (\mathcal{G}_{\mathrm{stab}}, \mu) \to (\mathcal{G}, \mu) \to (\mathcal{G}_{\mathrm{rel}}, \mu) \to 0$$

is called the *principal extension*, associated to the discrete measured groupoid $(\mathcal{G}, \mu)$.

The discrete measured groupoid $\mathcal{G}_{\mathrm{stab}}$ can be best viewed as a *direct integral* or *Borel field* of discrete groups. All functorial constructions can be carried out fiberwise. The following lemma is therefore no surprise and we omit its proof, which needs a little technical detour.

Lemma 5.1. *With the notation of the previous example, for every $p \geq 0$ we have*

$$b_p^{(2)}(\mathcal{G}_{\mathrm{stab}}) = \int_{\mathcal{G}^0} b_p^{(2)}(\mathcal{G}_x) d\mu(x).$$

6. Measurable cohomological dimension

We are now coming to a more conceptional study of the cohomological properties of a discrete measured groupoid. The main application of the results in this section is the proof of the Hopf-Singer Conjecture for poly-surface groups.

6.1. **Definition of the measurable cohomological dimension.** Let $(\mathcal{G}, \mu)$ be a discrete measured groupoid and $\mathrm{Mod}(\mathcal{R}(\mathcal{G}, \mu))_{\mathrm{comp}}$ the category of complete $\mathcal{R}(\mathcal{G}, \mu)$-modules. We define the *measurable cohomological dimension* $\mathrm{mcd}_{\mathbb{C}}(\mathcal{G}, \mu)$ to be the length of the shortest resolution of $\mathcal{M}(\mathcal{G}^0)$ by projective objects in $\mathrm{Mod}(\mathcal{R}(\mathcal{G}, \mu))_{\mathrm{comp}}$. The following result is classical [29, Lemma 4.1.6 on p. 93]:

Theorem 6.1. *Let $(\mathcal{G}, \mu)$ be a discrete measured groupoid. The following statements are equivalent:*

i) $\mathrm{mcd}_{\mathbb{C}}(\mathcal{G}, \mu) \leq n$.

ii) For all $M \in \mathrm{Mod}(\mathcal{R}(\mathcal{G},\mu))_{\mathrm{comp}}$ *and* $m > n$*, one has*

$$\underline{\mathrm{Ext}}^m_{\mathcal{R}(\mathcal{G},\mu)}\big(\mathcal{M}(\mathcal{G}^0), M\big) = 0.$$

The spectral sequence of Theorem 4.7 immediately yields a product estimate as follows:

Corollary 6.2 (of Theorem 4.7 and Remark 4.9)**.** *Let*

$$1 \to (\mathcal{S},\mu') \to (\mathcal{G},\mu) \to (\mathcal{Q},\nu) \to 1$$

be a strong extension of discrete measured groupoids or a Gaboriau's extension (5.1)*. Then,*

$$\mathrm{mcd}_{\mathbb{C}}(\mathcal{G},\mu) \le \mathrm{mcd}_{\mathbb{C}}(\mathcal{S},\mu') + \mathrm{mcd}_{\mathbb{C}}(\mathcal{Q},\nu).$$

Turning to the interesting case of translation groupoids, we make the following definition:

Definition 6.3. Let Γ be a discrete group. We set:

$$\mathrm{mcd}_{\mathbb{C}}(\Gamma) = \min\{\mathrm{mcd}_{\mathbb{C}}(X \rtimes \Gamma,\mu) \mid \Gamma \curvearrowright (X,\mu)\},$$

where we take the minimum over all m.p. actions of Γ on a standard probability space (X,μ). We call $\mathrm{mcd}_{\mathbb{C}}(\Gamma)$ the *measurable cohomological dimension* of Γ.

We need the following lemma.

Lemma 6.4. *Let* Γ *be a discrete group and let* $\Gamma \curvearrowright (X,\mu)$ *and* $\Gamma \curvearrowright (Y,\nu)$ *measure preserving actions as above. Let* $f\colon Y \to X$ *be a measure preserving* Γ*-equivariant map. Then,*

$$\mathrm{mcd}_{\mathbb{C}}(Y \rtimes \Gamma,\nu) \le \mathrm{mcd}_{\mathbb{C}}(X \rtimes \Gamma,\mu).$$

Proof. Clearly, there is a Γ-equivariant ring homomorphism $f^*\colon \mathcal{M}(X) \to \mathcal{M}(Y)$. Let $P_* \to \mathcal{M}(X)$ be a projective resolution in the category $\mathrm{Mod}(\mathcal{R}(X \rtimes \Gamma,\mu))_{\mathrm{comp}}$. Since $\mathcal{M}(X)$ is von Neumann regular, $\mathcal{M}(Y)$ is a flat $\mathcal{M}(X)$-module. Hence,

$$(\mathcal{M}(Y) \rtimes \Gamma) \otimes_{\mathcal{M}(X)\rtimes\Gamma} P_* = \mathcal{M}(Y) \otimes_{\mathcal{M}(X)} P_* \to \mathcal{M}(Y) \otimes_{\mathcal{M}(X)} \mathcal{M}(X) = \mathcal{M}(Y)$$

is a resolution. Setting $Q_* = (\mathcal{M}(Y) \rtimes \Gamma) \otimes_{\mathcal{M}(X)\rtimes\Gamma} P_*$ and using exactness of completion (with respect to the $\mathcal{M}(Y)$-module structure), we obtain a resolution $Q'_* \to \mathcal{M}(Y)$ by complete modules. The modules Q'_* are projective in $\mathrm{Mod}(\mathcal{R}(Y \rtimes \Gamma,\nu))_{\mathrm{comp}}$ [27, Theorem 2.7 (3)]. ☐

Proposition 6.5. *Let* $1 \to \Lambda \to \Gamma \to Q \to 1$ *be an extensions of groups. Then*

$$\mathrm{mcd}_{\mathbb{C}}(\Gamma) \le \mathrm{mcd}_{\mathbb{C}}(\Lambda) + \mathrm{mcd}_{\mathbb{C}}(Q).$$

Proof. Let $\Lambda \curvearrowright (X',\mu')$ and $Q \curvearrowright (Y,\nu)$ be measurable actions as above that achieve the measurable cohomological dimension. Let X be the coinduction of X', i.e. the Γ-space $X = \mathrm{map}_{\Lambda}(\Gamma, X')$, on which $\gamma \in \Gamma$ acts from the left by composition with the Γ-map $r_{\gamma^{-1}}\colon \Gamma \to \Gamma,\ \gamma_0 \mapsto \gamma_0\gamma^{-1}$. By choosing a set theoretic section $s\colon \Gamma/\Lambda \to \Gamma$ of the projection with $s(1) = 1$, we obtain a bijection $X \xrightarrow{\cong} \prod_{\Gamma/\Lambda} X'$. We endow X with the structure of a standard probability space (X,μ) by pulling back the product measure. This structure does not depend on the choice of s; the measure μ is Γ-invariant [14, 3.4]. We have $\mathrm{mcd}_{\mathbb{C}}(X \rtimes \Lambda,\mu) \le \mathrm{mcd}_{\mathbb{C}}(X' \rtimes \Lambda,\mu')$ by the previous lemma. Setting $Z = X \times Y$ as in Gaboriau's extension (see Subsection 5.2) yields a translation groupoid $(Z \rtimes \Gamma, \mu \times \nu)$ and an extension of discrete measured groupoids as follows:

$$1 \to (Z \rtimes \Lambda, \mu \times \nu) \longrightarrow (Z \rtimes \Gamma, \mu \times \nu) \longrightarrow (Y \rtimes Q, \nu) \to 1$$

Again by the previous lemma, $\mathrm{mcd}_{\mathbb{C}}(Z \rtimes \Lambda, \mu \times \nu) \leq \mathrm{mcd}_{\mathbb{C}}(X \rtimes \Lambda, \mu)$. By Corollary 6.2 we obtain that

$$\mathrm{mcd}_{\mathbb{C}}(\Gamma) \leq \mathrm{mcd}_{\mathbb{C}}(X' \rtimes \Lambda, \mu') + \mathrm{mcd}_{\mathbb{C}}(Y \rtimes Q, \nu).$$

Hence we completed the proof. □

The precise relationship between the measurable cohomological dimension to Gaboriau's ergodic dimension [13] is not clear at present. Certainly, the measurable cohomological dimension is smaller or equal to the ergodic dimension; however, the reverse inequality seems to be more difficult to establish.

6.2. **The Singer condition.** Striving for a conceptional explanation of the Hopf-Singer conjecture in terms of ergodic theory, we make the following definition. Recall that a *Poincaré duality group of dimension* n is a group Γ of cohomological dimension n such that

$$H^p(\Gamma, \mathbb{Z}\Gamma) \cong \begin{cases} \mathbb{Z}^\epsilon & \text{if } p = n, \\ 0 & \text{if } p \neq n. \end{cases}$$

Here $\mathbb{Z}^\epsilon = \mathbb{Z}$ as an abelian group. The group $H^p(\Gamma, \mathbb{Z}\Gamma)$ carries a natural right Γ-module structure, and $\mathbb{Z}^\epsilon$ denotes the right Γ-module structure on $\mathbb{Z}$ that makes the above isomorphism Γ-equivariant. If $\mathbb{Z}^\epsilon = \mathbb{Z}$ is the trivial Γ-module, then Γ is called *orientable*.

Definition 6.6. Let Γ be a Poincaré duality group of dimension $2n$. We say that Γ satisfies *Singer's condition* if

$$\mathrm{mcd}_{\mathbb{C}}(\Gamma) \leq n.$$

Remark 6.7. The definition could be phrased in a straightforward way for proper and cocompact Γ-CW-complexes, satisfying equivariant Poincaré duality. We leave this to the reader.

Note that Singer's condition and (4.6) imply that all L^2-Betti numbers above the middle dimension vanish. Hence, by Poincaré duality, the only possible non-trivial L^2-Betti number is in the middle dimension. Note that this is also true in the non-orientable case since every non-orientable Poincare duality group has an orientable subgroup of index 2. It is thus of some interest to understand the class of groups which satisfy Singer's condition. Fundamental groups of closed surfaces with genus ≥ 2 are measure equivalent to the free group of rank 2 [14]. The free abelian groups $\mathbb{Z}^n$ are measure equivalent to $\mathbb{Z}$ [14]. Because of Lemma 7.2 we thus obtain the first interesting examples of groups satisfying Singer's condition:

Theorem 6.8. *Let S_g be the closed orientable surface of genus $g \geq 1$. Then $\pi_1(S_g)$ satisfies Singer's condition.*

Since Poincare duality groups are closed under extensions [4, Satz 2.5], Proposition 6.5 yields:

Theorem 6.9. *Let Λ be a Poincaré duality group of dimension $2n$ and Q be a Poincaré duality group of dimension $2m$. Let $1 \to \Lambda \to \Gamma \to Q \to 1$ be an extension of groups. Then Γ is a Poincaré duality group of dimension $2(n+m)$, and it satisfies Singer's condition if Λ and Q satisfy Singer's condition.*

It is obvious that this provides a powerful tool for the construction of groups which satisfy Singer's condition. Indeed, free products of amenable groups and surface groups (in fact all groups measure equivalent to free groups) can be used as basic building blocks. A class of groups which has been studied to some extend and fits into our framework is given by the following definition:

Definition 6.10. A group Γ is said to be a *poly-surface group*, if there exists a series of normal subgroups

$$\{e\} = \Gamma_0 \lhd \Gamma_1 \lhd \cdots \lhd \Gamma_n = \Gamma,$$

such that Γ_k/Γ_{k-1} is a surface group, i.e. the fundamental group of a closed, orientable surface of genus ≥ 2, for all $1 \leq k \leq n$.

It was proved in [17] that every poly-surface group admits a subgroup of finite index, which is the fundamental group of a closed, orientable, aspherical manifold. Hence, the following corollary is of importance.

Corollary 6.11. *Poly-surface groups satisfy Singer's condition. In particular, the Hopf-Singer conjecture holds for closed aspherical manifolds with poly-surface fundamental group.*

7. Proofs of applications

7.1. **Proof of Theorem 1.1.** We give the proof only in the case where $\mathcal{Q} = \mathcal{R}_{\text{hyp}}$ is an infinite amenable equivalence relation. (The more general case of infinite amenable measured groupoids is more tedious but analogous.) There exists an increasing chain of finite sub-relations $\mathcal{R}^n_{\text{hyp}} \subset \mathcal{R}_{\text{hyp}}$, which are isomorphic to the groupoid $(X \rtimes S_n)_{\text{rel}}$, where S_n permutes a partition of a continuous Borel probability space (X, μ) into n sets of equal measure. Moreover, we find these sub-relations such that

$$\mathcal{R}_{\text{hyp}} = \bigcup_{n=1}^{\infty} \mathcal{R}^n_{\text{hyp}}.$$

In the extension

$$1 \to \mathcal{S} \to \mathcal{R} \xrightarrow{\phi} \mathcal{R}_{\text{hyp}} \to 1$$

we can take inverse images $\mathcal{R}^n = \phi^{-1}(\mathcal{R}^n_{\text{hyp}})$ and note that

$$n \cdot b_p^{(2)}(\mathcal{R}^n) = b_p^{(2)}(\mathcal{S}).$$

Indeed, this follows from standard homological arguments since $\mathcal{R}(\mathcal{R}^n, \mu)$ is just an $n \times n$-matrix algebra over $\mathcal{R}(\mathcal{S}, \mu)$. Since $\mathcal{R} = \cup_{n=1}^{\infty} \mathcal{R}^n$, we conclude that

$$b_p^{(2)}(\mathcal{R}) \leq \liminf_{n\to\infty} b_p^{(2)}(\mathcal{R}^n) = \liminf_{n\to\infty} n^{-1} \cdot b_p^{(2)}(\mathcal{S}).$$

Provided the first inequality holds, this would finish the proof since $b_p^{(2)}(\mathcal{S}) < \infty$ implies now $b_p^{(2)}(\mathcal{R}) = 0$. The first inequality follows from the following three facts:

(i) $\mathcal{R}(\mathcal{R}^n, \mu) \subset \mathcal{R}(\mathcal{R}, \mu)$ is a dimension flat ring extension since $\cup_n \mathcal{R}(\mathcal{R}^n, \mu)$ is flat over $\mathcal{R}(\mathcal{R}^n, \mu)$ and rank-dense in $\mathcal{R}(\mathcal{R}, \mu)$,

(ii)
$$L^\infty(\mathcal{R}^0) \otimes_{\mathcal{R}(\mathcal{R},\mu)} ? = c\left(\operatorname{colim} L^\infty(\mathcal{R}^0) \otimes_{\mathcal{R}(\mathcal{R}^n,\mu)} ?\right)$$
for the same reason that $\cup_n \mathcal{R}(\mathcal{R}^n, \mu)$ is rank-dense in $\mathcal{R}(\mathcal{R}, \mu)$, and

(iii)
$$\dim \operatorname{colim} M_n \leq \liminf_{n\to\infty} \dim M_n$$
by [21, Theorem 6.13 (2) on p. 243].

By (i), (ii), and exactness of completion and directed co-limits,
$$\begin{aligned} H_k(\mathcal{R},\mathcal{U}(\mathcal{R},\mu)) &= \operatorname{colim} H_k(\mathcal{R}^n,\mathcal{U}(\mathcal{R}^n,\mu)) \\ &= \operatorname{colim} \big(H_k(\mathcal{R}^n,\mathcal{U}(\mathcal{R}^n,\mu)) \otimes_{\mathcal{U}(\mathcal{R}^n,\mu)} \mathcal{U}(\mathcal{R},\mu)\big) . \end{aligned}$$
This implies the claim since, using (iii), we get:
$$\begin{aligned} b_k^{(2)}(\mathcal{R}^n) &= \dim_{\mathcal{U}(\mathcal{R}^n,\mu)} H_k(\mathcal{R}^n,\mathcal{U}(\mathcal{R}^n,\mu)) \\ & \dim_{\mathcal{U}(\mathcal{R},\mu)} H_k(\mathcal{R}^n,\mathcal{U}(\mathcal{R}^n,\mu)) \otimes_{\mathcal{U}(\mathcal{R}^n,\mu)} \mathcal{U}(\mathcal{R},\mu) \end{aligned}$$

7.2. **Proof of Theorem 1.3 and 1.5.** Consider the strong extension
$$1 \to (\mathcal{S},\mu) \to (\mathcal{G},\mu) \xrightarrow{p} (\mathcal{Q},\nu) \to 1.$$
The E_1-term of the corresponding spectral sequence for is
$$E_1^{p,q} = \hom_{\mathcal{R}(\mathcal{Q},\nu)}\big(F_q(\mathcal{Q}), H^q(\mathcal{S};\mathcal{U}(\mathcal{G},\mu))\big),$$
where $F_*(\mathcal{Q})$ is a projective resolution of $\mathcal{M}(\mathcal{Q}^0)$ in $\operatorname{Mod}(\mathcal{R}(\mathcal{Q},\nu))_{\text{comp}}$.

Thus, using Remark 4.10, $E_1^{p,q} = 0$ for $0 \leq q \leq d$. It follows that $E_r^{p,q} = 0$ for $0 \leq q \leq d$ and $1 \leq r \leq \infty$. This implies that $E_r^{0,d+1} \cong E_\infty^{0,d+1}$ for $r \geq 2$ and $H^{d+1}(\mathcal{G},\mathcal{U}(\mathcal{G},\mu)) \cong E_\infty^{0,d+1}$. Thus,

$$b_p^{(2)}(\mathcal{G}) = 0 \text{ for } 0 \leq p \leq d. \tag{7.1}$$

and

$$\begin{aligned} b_{d+1}^{(2)}(\mathcal{G}) &= \dim_{\mathcal{U}(\mathcal{G},\mu)} H^0\big(\mathcal{Q}; H^{d+1}(\mathcal{S};\mathcal{U}(\mathcal{G},\mu))\big) \\ &= \dim_{\mathcal{U}(\mathcal{G},\mu)} \hom_{\mathcal{R}(\mathcal{Q},\nu)}\big(\mathcal{M}(\mathcal{Q}^0), H^{d+1}(\mathcal{S};\mathcal{U}(\mathcal{G},\mu))\big). \end{aligned} \tag{7.2}$$

This completes the proof of Theorem 1.3. For the proof of Theorem 1.5, it remains to show that $b_{d+1}^{(2)}(\mathcal{G}) = 0$ provided $b_{d+1}^{(2)}(\mathcal{S}) < \infty$.

Let us assume in addition that $(\mathcal{Q},\nu)$ is an infinite equivalence relation. Since $(\mathcal{Q},\nu)$ is infinite, there exists an infinite, amenable subrelation $\mathcal{R}_{\text{hyp}} \subset \mathcal{Q}$ (see [12, Proof of Proposition III.3; 13, Proof of Théorème 6.8]). By the same argument as for (7.2) applied to $1 \to \mathcal{S} \to p^{-1}(\mathcal{R}_{\text{hyp}}) \to \mathcal{R}_{\text{hyp}} \to 1$ we get that

$$b_{d+1}^{(2)}\big(p^{-1}(\mathcal{R}_{\text{hyp}})\big) = \dim_{\mathcal{U}(\mathcal{G},\mu)} \hom_{\mathcal{R}(\mathcal{R}_{\text{hyp}},\nu)} \big(\mathcal{M}(\mathcal{Q}^0), H^{d+1}(\mathcal{S};\mathcal{U}(\mathcal{G},\mu))\big) . \tag{7.3}$$

Clearly, Equation 7.3 implies $b_{d+1}^{(2)}\big(p^{-1}(\mathcal{R}_{\text{hyp}})\big) \leq b_{d+1}^{(2)}(\mathcal{S})$, but this holds also for the inverse image of any finite index sub-relation $\mathcal{R}'_{\text{hyp}} \subset \mathcal{R}_{\text{hyp}}$. Hence, since such sub-relations exist for any given index n,
$$n \cdot b_{d+1}^{(2)}(p^{-1}(\mathcal{R}_{\text{hyp}})) = b_{d+1}^{(2)}(p^{-1}(\mathcal{R}'_{\text{hyp}})) \leq b_{d+1}^{(2)}(\mathcal{S})$$
for any n, and so $b_{d+1}^{(2)}(p^{-1}(\mathcal{R}_{\text{hyp}})) = 0$ as $b_{d+1}^{(2)}(\mathcal{S}) < \infty$. Since the natural map
$$\hom_{\mathcal{R}(\mathcal{Q},\nu)} \big(\mathcal{M}(\mathcal{Q}^0), H^{d+1}(\mathcal{S};\mathcal{U}(\mathcal{G},\mu)\big) \to \hom_{\mathcal{R}(\mathcal{R}_{\text{hyp}},\mu)} \big(\mathcal{M}(\mathcal{Q}^0), H^{d+1}(\mathcal{S};\mathcal{U}(\mathcal{G},\mu)\big) ,$$
is obviously injective, $b_{d+1}^{(2)}(\mathcal{G}) = 0$ follows now from (7.1) and (7.2).

In the course of the proof, we assumed that $(\mathcal{Q}, \nu)$ was an equivalence relation. Again, the more general case of infinite discrete measured groupoids is a bit more tedious but analogous.

7.3. **Proof of Theorem 1.10.** Let us recall the definition of orbit equivalence.

Definition 7.1. We say that two countable groups Γ and Λ are *orbit equivalent* if there is a probability space (X, μ) and essentially free μ-preserving actions of Γ and Λ such that for μ-a.e. $x \in X$ we have $\Gamma x = \Lambda x$. Furthermore, Γ and Λ are *weakly orbit equivalent* if there is a probability space (X, μ), essentially free μ-preserving actions of Γ and Λ on X, and Borel subsets $A \subset X$ and $B \subset X$ such that $\Gamma A = X$ and $\Lambda B = X$ up to null sets and such that for μ-a.e. $x \in X$ we have $\Gamma x \cap A = \Lambda x \cap B$.

Lemma 7.2. *If the groups Q_0 and Q_1 are measure equivalent, then* $\operatorname{mcd}_{\mathbb{C}}(Q_0) \leq \operatorname{cd}_{\mathbb{C}}(Q_1)$.

Proof. Furman [10] showed that groups are measure equivalent if and only if they are weakly orbit equivalent.

Let us assume first that Q_0 and Q_1 are orbit equivalent. Let (X, μ) be a probability space equipped with m.p. free actions of Q_0 and Q_1 such that the actions have the same orbit equivalence relation, which coincides with the translation groupoid. By [10, Lemma 2.2] one can assume that both actions are ergodic. So we have an identification of the corresponding groupoid rings $\mathcal{R}(X \rtimes Q_0) \cong \mathcal{R}(X \rtimes Q_1)$. Consider the following commutative diagram of functors:

$$\begin{array}{ccccc}
\operatorname{Mod}(\mathbb{C}Q_1) & \longrightarrow & \operatorname{Mod}(L^\infty(X) \rtimes Q_1) & \longrightarrow & \operatorname{Mod}(L^\infty(X) \rtimes Q_1)_{\mathrm{comp}} \\
 & & & & \downarrow \cong \\
 & & & & \operatorname{Mod}(\mathcal{R}(X \rtimes Q_1))_{\mathrm{comp}} \\
 & & & & \downarrow \cong \\
 & \searrow & & \longrightarrow & \operatorname{Mod}(\mathcal{R}(X \rtimes Q_0))_{\mathrm{comp}}
\end{array}$$

The horizontal arrows either denote completion or ring extension. The vertical arrows are natural isomorphisms of abelian categories. Concerning the upper vertical arrow, we use that $L^\infty(X) \rtimes Q_1$ is dense in $\mathcal{R}(X \rtimes Q_1)$, thus leading to an identification of complete $L^\infty(X) \rtimes Q_1$- with complete $\mathcal{R}(X \rtimes Q_1)$-modules (see [27, Lemma 4.4]). The first horizontal arrow is an exact functor because of $L^\infty(X) \rtimes Q_1 \otimes_{\mathbb{C}Q_1} M \cong L^\infty(X) \otimes_{\mathbb{C}} M$. The completion functors are exact and preserve projectives by [27, Lemma 2.6 and Theorem 2.7]. Starting with a projective $\mathbb{C}Q_1$-resolution of $\mathbb{C}$ of length n, one obtains by following the arrows a projective resolution P_* of $\mathcal{M}(X)$ in $\operatorname{Mod}(\mathcal{R}(X \rtimes Q_0))_{\mathrm{comp}}$ of length n.

The general case where Q_0 and Q_1 are weak orbit equivalent demands only small modifications. For the upcoming discussion of full idempotents and Morita equivalence we recommend Lam's book [20, Section 18]. Recall that an idempotent p in a ring R is called *full* if the elements rpr' with $r, r' \in R$ generate R additively.

Let $A \subset X$ and $B \subset X$ be Borel subsets of positive measure such that for a.e. $x \in X$ we have $Q_0 x \cap A = Q_1 x \cap B$. In particular, this gives an identification of the restricted translation groupoids

$$\mathcal{R}(X \rtimes Q_0)|_A \cong \mathcal{R}(X \rtimes Q_1)|_B.$$

The characteristic functions χ_A and χ_B are *full* idempotents in the groupoid rings $\mathcal{R}(X \rtimes Q_0)$ and $\mathcal{R}(X \rtimes Q_1)$, respectively. This is easily concluded from ergodicity and Lemma 3.8. Hence

$\mathcal{R}(X \rtimes Q_0)$ and

$$\chi_A \mathcal{R}(X \rtimes Q_0)\chi_A = \mathcal{R}(X \rtimes Q_0|_A)$$

are Morita equivalent, i.e. their module categories are equivalent as abelian categories. Explicitly, the Morita equivalence is given by $M \mapsto \chi_A M$. From this, it is obvious that the Morita equivalence restricts to the subcategories of complete modules. The same argument holds for Q_1 in place of Q_0. After replacing the lower vertical arrow in the diagram above by the equivalence of abelian categories

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{R}(X \rtimes Q_1))_{\mathrm{comp}} & \xrightarrow{\simeq} & \mathrm{Mod}(\mathcal{R}(X \rtimes Q_1)|_B)_{\mathrm{comp}} \\ \Big\downarrow & & \Big\downarrow \simeq \\ \mathrm{Mod}(\mathcal{R}(X \rtimes Q_0))_{\mathrm{comp}} & \xleftarrow[\simeq]{} & \mathrm{Mod}(\mathcal{R}(X \rtimes Q_0)|_A)_{\mathrm{comp}} \end{array}$$

we can run the same kind of argument as before. This finishes the proof of the lemma. □

We now turn to the prove of Theorem 1.10 in the form needed for the application towards Theorem 1.15.

Theorem 7.3. *Let $1 \to \Lambda \to \Gamma \to Q \to 1$ be a short exact sequence of groups. Suppose that $b_p^{(2)}(\Lambda) = 0$ for $p > m$ and let $n = \mathrm{mcd}_{\mathbb{C}}(Q)$. Then $b_p^{(2)}(\Gamma) = 0$ for $p > n + m$.*

Proof. Let $Q \curvearrowright (X, \nu)$ realize $\mathrm{mcd}_{\mathbb{C}}(Q)$. Consider Gaboriau's extension (see Subsection 5.2) of the form

$$1 \to (Z \rtimes \Lambda, \mu \times \nu) \longrightarrow (Z \rtimes \Gamma, \mu \times \nu) \longrightarrow (X \rtimes Q, \nu) \to 1.$$

The E^1-term of the corresponding spectral sequence is

$$E_1^{p,q} = \hom_{\mathcal{R}(X \rtimes Q)}\big(P_p, H^q(\mathcal{R}(Z \rtimes \Lambda); \mathcal{U}(Z \rtimes \Gamma))\big), \tag{7.4}$$

where P_* is a projective resolution of $\mathcal{M}(X)$ of length n in the category $\mathrm{Mod}(\mathcal{R}(X \rtimes Q))_{\mathrm{comp}}$ (see Remark 4.8). Hence $E_1^{p,q} = 0$ whenever $p > n$. Moreover, $E_1^{p,q} = 0$ if $q > m$, since $H^q(\mathcal{R}(Z \rtimes \Lambda); \mathcal{U}(Z \rtimes \Gamma)) = 0$ for $q > m$, by Remark 4.10 and Equation (4.6) . This yields the theorem since $E_\infty^{p,q} = 0$ for all $p + q > n + m$, and thus $b_i^{(2)}(\Gamma) = b_i^{(2)}(Z \rtimes \Gamma) = 0$ for $i > n + m$. □

Clearly, Theorem 7.3 and Lemma 7.2 imply Theorem 1.10.

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