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Augmented Teichmüller spaces and Orbifolds

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Abstract. We study complex-analytic properties of the augmented Teichmüller spaces $\overline{\mathcal{T}}_{g,n}$ obtained by adding to the classical Teichmüller spaces $\mathcal{T}_{g,n}$ points corresponding to Riemann surfaces with nodal singularities. Unlike $\mathcal{T}_{g,n}$, the space $\overline{\mathcal{T}}_{g,n}$ is not a complex manifold (it is not even locally compact). We prove however that the quotient of the augmented Teichmüller space by any finite index subgroup of the Teichmüller modular group has a canonical structure of a complex orbifold. Using this structure we construct natural maps from $\overline{\mathcal{T}}$ to stacks of admissible coverings of stable Riemann surfaces. This result is important for understanding the cup-product in stringy orbifold cohomology. We also establish some new technical results from the general theory of orbifolds which may be of independent interest.

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Contents

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1. Introduction

1.1. Augmented Teichmüller spaces

The Teichmüller space $\mathfrak{T}_{g,n}$ is the space of pairs $((X, x_1, \ldots, x_n), \phi)$, where $(X, x_1, \ldots, x_n)$ is a compact complex curve (Riemann surface) of genus g with n distinct marked points (which we also call *punctures*) and

$$\phi : (S, p_1, \ldots, p_n) \to (X, x_1, \ldots, x_n)$$

is a *marking* — an isotopy class of orientation preserving diffeomorphisms with a fixed compact oriented surface S with n marked points $p_1, \ldots, p_n \in S$. Lipman Bers in [7] introduced the augmented Teichmüller space $\overline{\mathfrak{T}}_{g,n}$ by adding to $\mathfrak{T}_{g,n}$ points corresponding to Riemann surfaces with nodes. (A marking of a nodal Riemann surface X is an isotopy class of maps $\phi : S \to X$, such that the preimages of nodes are simple closed curves on S, see Definition 5.1.1.)

Let $\Gamma_{g,n} = \pi_0(\mathcal{D}\!\mathit{iff}^+(S, p_1, \ldots, p_n))$ be the Teichmüller modular group, i.e. the group of isotopy classes of orientation preserving diffeomorphisms $(S, p_1, \ldots, p_n)$. (This group is also known as the mapping class group of the n-punctured surface of genus g, cf. [29]). We will frequently denote this group simply by Γ. The modular group Γ naturally acts on $\overline{\mathfrak{T}}_{g,n}$ and the quotient $\Gamma\backslash\overline{\mathfrak{T}}_{g,n}$ is homeomorphic to the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of Riemann surfaces of genus g with n punctures. One of Bers' goals was an attempt to introduce a natural complex structure on $\overline{\mathcal{M}}_{g,n}$ and to prove its projectivity. The existence of a normal complex structure on the quotient $\Gamma\backslash\overline{\mathfrak{T}}_{g,n}$ was announced by Bers [7, 8, 9], but, to the best of our knowledge, no detailed proof of this result had been published.

Unlike the usual Teichmüller space, the space $\overline{\mathfrak{T}}_{g,n}$ is not a manifold (it is not even locally compact). Still, the augmented Teichmüller spaces play an important role in Teichmüller theory (see [3]). In particular they appear in the study of the Weil-Petersson metric on $\mathfrak{T}_{g,n}$ (see [38, 47, 12, 13]). One of the goals of this paper is to understand and study the augmented Teichmüller space from the complex analytic point of view. Our main results suggest that the space $\overline{\mathfrak{T}}_{g,n}$ can be viewed as a certain universal space of coverings of stable Riemann surfaces of genus g ramified in at most n points and from this point of view it should be thought of as a projective system of complex orbifolds.

1.2. Results

Our main result is the following theorem (a combined statement of 6.1.1, 7.2.1 and 7.2.4).

Theorem. *Let G be a finite index subgroup of the Teichmüller modular group $\Gamma_{g,n}$, where (g, n) is in the stable range (i.e. $2g - 2 + n > 0$).*

- (i) *The quotient $G\backslash\overline{\mathfrak{T}}_{g,n}$ has a structure of a complex orbifold such that $G\backslash\mathfrak{T}_{g,n}$ is its open suborbifold. In particular, $G\backslash\overline{\mathfrak{T}}_{g,n}$ is a compact normal complex space.*
- (ii) *For every finite index subgroup $G' \subset G$ there exists a canonical morphism $G'\backslash\overline{\mathfrak{T}} \to G\backslash\overline{\mathfrak{T}}$ of the corresponding orbifolds.*
- (iii) *There exists a finite index subgroup $G' \subset G$ such that the orbifold $G'\backslash\overline{\mathfrak{T}}_{g,n}$ is a manifold (i.e. each point has a trivial stabilizer).*

We will denote the quotient $G\backslash\overline{\mathfrak{T}}_{g,n}$ with this orbifold structure by $[G\backslash\overline{\mathfrak{T}}_{g,n}]$. For $G = \Gamma_{g,n}$ the resulting orbifold $[\Gamma_{g,n}\backslash\overline{\mathfrak{T}}_{g,n}]$ coincides with the Deligne-Mumford moduli stack[1] $\overline{\mathfrak{M}}_{g,n}$ of stable curves of genus g with n punctures. In Section 7.3 we prove that the natural gluing operations on the collection of stacks $\overline{\mathfrak{M}}_{g,n}$ of stable marked curves can be extended to give canonical operations on the collection of orbifolds $[G\backslash\overline{\mathfrak{T}}]$.

[1] We use the fraktur font to distinguish stacks and orbifolds (such as $\overline{\mathfrak{M}}_{g,n}$ and $\mathfrak{Adm}_{g,n,d}$) from underlying coarse spaces denoted by the mathcal font (resp. $\mathcal{M}_{g,n}$ and $\mathcal{A}dm_{g,n,d}$).

As we prove in Section 8.3, the first statement of this theorem also holds for certain finite extensions $\widetilde{G} \to G \subset \Gamma_{g,n}$ of finite index subgroups of $\Gamma_{g,n}$ acting on $\overline{\mathfrak{T}}_{g,n}$ via the homomorphism $\gamma : \widetilde{G} \to \Gamma_{g,n}$. This leads to our second main result—a discovery of a connection between the augmented Teichmüller space $\overline{\mathfrak{T}}_{g,n}$ and the moduli space[2] $\mathfrak{Adm}_{g,n,d}$ of admissible coverings $\pi : \widetilde{X} \to X$ of degree d, where X is a stable complex curve of genus g with n punctures (see e.g. [4, Sect. 4]).

Let S be a compact oriented surface of genus g with n punctures and let $\rho : \widetilde{S} \to S$ be a finite covering unramified outside the punctures. For any stable complex curve X of genus g with n punctures and a marking $\phi : S \to X$ the map ρ induces an admissible covering $\widetilde{X} \to X$. Thus, on the level of points, ρ gives a map

$$v_\rho : \overline{\mathfrak{T}}_{g,n} \to \mathcal{A}dm_{g,n,d} \ . \tag{1}$$

In Section 8 we show that this map can be elevated to a continuous map from $\overline{\mathfrak{T}}_{g,n}$ to the complex orbifold $\mathfrak{Adm}_{g,n,d}$. To do this we first construct a morphism of complex orbifolds $[\widetilde{G}\backslash\overline{\mathfrak{T}}_{g,n}] \to \mathfrak{Adm}_{g,n,d}$, where $\widetilde{G}$ is a finite extension of a finite index subgroup of $\Gamma_{g,n}$. Then we compose this morphism with the canonical map $\overline{\mathfrak{T}}_{g,n} \to [\widetilde{G}\backslash\overline{\mathfrak{T}}_{g,n}]$. (Note that, since $[\widetilde{G}\backslash\overline{\mathfrak{T}}_{g,n}]$ is not a quotient orbifold, the existence of this map is non-obvious. It is constructed in Section 7.2 using parts (ii) and (iii) of the above theorem.)

An important application of this result is a proof given in Section 8.7 of associativity of stringy orbifold cohomology (see below in 1.5).

Since the projection $\left[G\backslash\overline{\mathfrak{T}}_{g,n}\right] \to \overline{\mathfrak{M}}_{g,n}$ is a finite morphism, the complex orbifold $\left[G\backslash\overline{\mathfrak{T}}_{g,n}\right]$ is projective. It is equipped with a tautological family of stable curves $\pi : \mathfrak{X} \to [G\backslash\overline{\mathfrak{T}}_{g,n}]$. Points of $\mathfrak{X}$ can be viewed as stable G-marked curves, where by a G-marking we understand a G-orbit in the set of all markings on a curve. The orbifold $\left[G\backslash\overline{\mathfrak{T}}_{g,n}\right]$ allows to introduce, *a posteriori*, a notion of a family of complex G-marked stable curves. By definition, this is a family of curves induced from the tautological family $\pi : \mathfrak{X} \to \left[G\backslash\overline{\mathfrak{T}}_{g,n}\right]$ via a map $S \to [G\backslash\overline{\mathfrak{T}}_{g,n}]$. For families of smooth curves, this notion coincides with the one given by Grothendieck in [21]. It would be nice to have an *a priori* notion of such a family (defined over $\mathbb{Z}$ or at least over $\mathbb{Q}$) which would identify $\left[G\backslash\overline{\mathfrak{T}}_{g,n}\right]$ with the (analytification of the) corresponding moduli stack.

Stable G-marked curves can be thought of as curves with generalized level structures. Indeed, for certain choices of the group G this notion gives Prym level structures considered by Looijenga in [36]. Let $\widetilde{S} \to S$ be the universal Prym cover of a compact oriented surface S of genus g, i.e. a Galois covering whose Galois group H is the quotient of $\pi_1(S)$ by the normal subgroup generated by the squares of all elements of $\pi_1(S)$.[3] Let $G = \Gamma_{g,\left[\frac{k}{2}\right]}$ be the subgroup of elements of $\Gamma_{g,0}$ whose lifts to $\widetilde{S}$ act on $H_1(\widetilde{S}, \mathbb{Z}/k)$ as

[2] In fact, this is a Deligne-Mumford stack over $\mathbb{Z}[\frac{1}{d!}]$.
[3] Of course H is isomorphic to $H_1(S, \mathbb{Z}/2)$

elements of H. The quotient $\mathcal{M}_{g,\left[\frac{k}{2}\right]} = G\backslash\mathfrak{T}_g$ is the moduli space of smooth curves of genus g with a level-k Prym structure. In [36] Looijenga studied the normalization $\overline{\mathcal{M}}_{g,\left[\frac{k}{2}\right]}$ of the moduli space $\overline{\mathcal{M}}_g$ in the field of meromorphic functions on $\mathcal{M}_{g,\left[\frac{k}{2}\right]}$. The main result of [36] is that, when k is even and $k \geq 6$, the space $\overline{\mathcal{M}}_{g,\left[\frac{k}{2}\right]}$ is smooth.

We use Looijenga's result in our proof of part (iii) of the main theorem. To do this we need to refine it in two ways. First, we generalize Looijenga's theorem to Riemann surfaces with punctures and the corresponding subgroup $G = \Gamma_{g,n,\left[\frac{k}{2}\right]}$ of the modular group $\Gamma_{g,n}$. Second, we show that when k is even and $k \geq 6$ the orbifold $[G\backslash\overline{\mathfrak{T}}_{g,n}]$ is a complex manifold and, therefore, coincides with the Looijenga's space $\overline{\mathcal{M}}_{g,n,\left[\frac{k}{2}\right]}$. This provides a modular description of the space $\overline{\mathcal{M}}_{g,n,\left[\frac{k}{2}\right]}$. We will use this fact in Section 7.2 to construct canonical maps $\pi_G : \overline{\mathfrak{T}}_{g,n} \to [G\backslash\overline{\mathfrak{T}}_{g,n}]$.

The most natural way to introduce an orbifold structure on a topological space is to describe it as the moduli space of some geometric objects. In the lack of a modular description,[4] we had to look for alternative ways. We construct an orbifold structure on $G\backslash\overline{\mathfrak{T}}_{g,n}$ using our formalism of orbifold charts developed in Section 3.

The traditional approach of Satake [44] works only for effective orbifolds and is insufficient for our purposes. We generalize Satake's description of orbifolds in terms of charts and atlases to include non-effective orbifolds. We show that the resulting notion is equivalent by its expressive power to the "modern" approaches to orbifolds based on the language of stacks and étale groupoids.

In particular, our construction in Section 3.2 which associates a stack to an orbifold atlas is analogous to the well-known realization (due to Satake) of an effective C^∞-orbifold as a quotient of a manifold by a compact Lie group. Our result, however, holds also for non-effective and complex orbifolds. Even in the effective C^∞-case our construction is functorial and does not use partitions of unity. It is defined by a universal property which is very convenient when dealing with morphisms from an orbifold.

Our proof of the main theorem uses yet another technical result which may be of independent interest. This is Theorem 4.1.1 on analytification of some algebraic moduli stacks. Namely, we prove that the analytifications of the stacks $\overline{\mathfrak{M}}_{g,n}$ and $\mathfrak{Adm}_{g,n,d}$ represent the corresponding moduli functors in the complex analytic category.

[4] V. Braungardt in his thesis [11] (see also [24]) introduced a concept of a locally complex ringed space and proved that $\overline{\mathfrak{T}}$ can be equipped with such structure which is universal in a certain sense. However, the quotient stacks $G\backslash\overline{\mathfrak{T}}$ produced this way are *non-separated* and therefore are very different from our complex orbifolds $[G\backslash\overline{\mathfrak{T}}]$. We thank the referee for bringing these references to our attention.

1.3. The case $g = 1,\ n = 1$

We will illustrate the main theorem on the simplest interesting case of the one-punctured torus, i.e. when $g = 1$ and $n = 1$.

The moduli space $\mathcal{M}_{1,1}$ of one-dimensional complex tori with one marked point can be viewed as the space of lattices in $\mathbb{R}^2$ up to similarity. Marking of a torus corresponds to a choice of a basis of the lattice. Therefore the Teichmüller space $\mathcal{T}_{1,1}$ is the set of similarity classes of pairs of non-collinear vectors in $\mathbb{R}^2$ which can be identified with the upper half-plane. The boundary of the compactified moduli space $\overline{\mathcal{M}}_{1,1}$ consists of the single point corresponding to the degenerate elliptic curve C (a pinched torus). Therefore, markings of C correspond to isotopy classes of simple closed paths on the standard torus S. So the boundary of $\overline{\mathcal{T}}_{1,1}$ can be identified with $\mathbb{P}^1(\mathbb{Q}) = Q \cup \{\infty\}$ (viewed as the set of pairs of relatively prime integers (p, q) up to a common factor ± 1). The base of the topology of $\overline{\mathcal{T}}_{1,1}$ near a boundary point is given by the collection of open disks tangent to the real line at this point (plus the point itself).

The Teichmüller modular group $\Gamma_{1,1}$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ and the quotient $\Gamma_{1,1}\backslash\overline{\mathcal{T}}_{1,1}$ is the orbifold $\overline{\mathcal{M}}_{1,1}$ whose underlying space is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. The quotient of $\overline{\mathcal{T}}_{1,1}$ by a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is a finite ramified covering of $\mathbb{P}^1(\mathbb{C})$ and, therefore, has a canonical structure of a compact Riemann surface. We will show that analogous results hold for arbitrary g and n.

1.4. Detailed description of the paper

A significant part of the paper deals with general questions of orbifold theory.

In algebraic geometry, the language of stacks is the most adequate for moduli problems. We find it useful for studying orbifolds in other settings as well.

In Section 2 we present two different approaches to defining a 2-category of orbifolds: one based on stacks, the other on groupoids. Whereas the 2-category structure on stacks is standard, the 2-category structure on groupoids is not the one that first comes to mind.

We define an orbifold as a stack of geometric origin, which means that it is equivalent to the stack associated to a separated étale groupoid.[5]

We prove that the functor associating a stack to a groupoid gives an equivalence between 2-categories of separated étale groupoids and orbifolds.

Since Sp may not have arbitrary fiber products, we cannot define representable morphisms for general Sp-stacks and so it is impossible to define Sp-orbifolds simply by modifying the standard definition of algebraic stacks (see [35]). Instead we define an Sp-orbifold as an Sp-stack equivalent to the stack associated to an Sp-groupoid. In the case when Sp is the category of schemes this approach gives separated Deligne-Mumford stacks.

[5]We cannot simply mimic the standard definition of an algebraic stack (see [35]), since the categories of smooth or complex manifolds do not have arbitrary fiber products which are needed to define representable morphisms.

Our treatment of these questions is similar but not identical to the works of Metzler [39], Noohi [42] and Behrend-Xu [6]. The most significant difference between these papers and our approach is that we work in the category of smooth manifolds where general fiber products do not exist. For this reason, we develop the theory so that only fiber products along étale morphisms are used.

In this section we also introduce gerbes over orbifolds which will be used later in Section 8.

In Section 3 we present a more traditional approach to orbifolds. It is based on the notion of (generalized) orbifold charts. For effective orbifolds this approach goes back to the original definition of Satake [44]. We generalize Satake atlases to include non-effective orbifolds. The main result here is a construction of an orbifold from an atlas of generalized orbifold charts.

This gives us a flexibility to use any of the three languages (of orbifold charts, groupoids or stacks) depending on the circumstances. In particular, the orbifold structure on the quotient $G\backslash\overline{\mathfrak{T}}_{g,n}$ will be given using a generalized orbifold atlas.

Throughout the paper we work with the moduli stacks of stable complex curves and admissible coverings in the complex-analytic category. Therefore we need to know that the analytification of the algebraic Deligne-Mumford stacks $\overline{\mathfrak{M}}$ and $\mathfrak{Adm}$ represent the corresponding functors (of families of nodal Riemann surfaces and of families of admissible coverings) in the analytic category. This is proved in Section 4.

To construct an orbifold atlas for the quotient $G\backslash\overline{\mathfrak{T}}_{g,n}$, we start with an orbifold atlas for

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n} = \Gamma_{g,n}\backslash\overline{\mathfrak{T}}_{g,n}$$

and then construct corresponding charts upstairs on $G\backslash\overline{\mathfrak{T}}_{g,n}$. The existence of an orbifold atlas for $\overline{\mathcal{M}}$ follows from the smoothness of the moduli stack $\overline{\mathfrak{M}}$, but to be able to lift the charts to $G\backslash\overline{\mathfrak{T}}_{g,n}$ we need an atlas on $\overline{\mathfrak{M}}$ whose charts satisfy some very special properties. We call such charts *quasiconformal* and prove that there exists a quasiconformal atlas $\overline{\mathfrak{M}}$ in Section 5.

To construct such an atlas we use a version of the Earle-Marden [37] local holomorphic coordinates on the Teichmüller space $\mathfrak{T}_{g,n}$. Let us recall the definition of these coordinates. Start with a maximally degenerate stable Riemann surface X_0 in $\overline{\mathcal{M}}_{g,n}$. This surface is a union of $2g+n-2$ triply punctured spheres glued together along $3g+n-3$ pairs of punctures. For each of $3g+n-3$ nodes $q_i \in X_0$ choose a pair of "coordinate" functions z_i, w_i that identify a neighborhood of q_i with a neighborhood of the node of the curve $V_i = \{(z_i, w_i) | z_i w_i = 0\} \in \mathbb{C}^2$. By replacing V_i with

$$V_{i,t_i} = \{(z_i, w_i) \in \mathbb{C}^2 | z_i w_i = t_i\}$$

we obtain a $3g+n-3$-parameter holomorphic family X_t of nodal Riemann surfaces. This gives a holomorphic map ϕ from a unit polydisk in $\mathbb{C}^{3g+3-n}$ to

$\overline{\mathfrak{M}}$. At every point where ϕ is étale, it defines an orbifold chart of $\overline{\mathfrak{M}}$. However ϕ may not be étale everywhere (see [25] for a counterexample). To circumvent this problem and guarantee existence of étale charts at every point of $\overline{\mathcal{M}}$ we make a very special choice of local coordinates around punctures of X_0 (see Section 5.3.4).

The existence of a quasiconformal atlas on $\overline{\mathfrak{M}}$ reflects two approaches to constructing this space: one based on Teichmüller spaces and another based on the theory of Deligne-Mumford stacks. This indicates that the appearance of quasiconformal charts here may be not coincidental.

In Section 6 we prove our main result—that the quotient of the augmented Teichmüller space $\overline{\mathfrak{T}}_{g,n}$ by a finite index subgroup G of the modular group has a natural structure of a complex orbifold. We do this by constructing an orbifold atlas on $G\backslash\overline{\mathfrak{T}}$ using the existence of a quasiconformal atlas on $\overline{\mathfrak{M}}$ proved in Section 5.

In Section 7 we establish some properties of the orbifold structure on $G\backslash\overline{\mathfrak{T}}$. A special example of G-marked curves give curves with level-ℓ structures. These curves correspond to the subgroup

$$G = \Gamma^{(\ell)} = \mathrm{Ker}(\Gamma \longrightarrow \mathrm{Aut}(H_1(S, \mathbb{Z}/\ell))).$$

Since the orbifold structure on $[G\backslash\overline{\mathfrak{T}}]$ is given by an ad hoc construction and not by a universal property, the existence of the quotient map

$$\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$$

is not guaranteed and requires special attention. This is done in Section 7.2.

First, for each finite index subgroup $G' \subset G$ we define a natural map of orbifolds $[G'\backslash\overline{\mathfrak{T}}] \to [G\backslash\overline{\mathfrak{T}}]$. Then, using a generalization of Looijenga's analysis [36] we prove that for any finite index subgroup G of Γ there exists a finite index subgroup $G' \subset G$ such that $[G'\backslash\overline{\mathfrak{T}}]$ is a complex manifold. Then, the quotient map π_G can be defined as the composition

$$\overline{\mathfrak{T}} \xrightarrow{\pi_{G'}} [G'\backslash\overline{\mathfrak{T}}] \longrightarrow [G\backslash\overline{\mathfrak{T}}].$$

In Section 7.3 we show that the natural gluing operations

$$\overline{\mathfrak{T}}_{g,n} \times \overline{\mathfrak{T}}_{g',n'} \longrightarrow \overline{\mathfrak{T}}_{g+g',n+n'} \quad \text{and} \quad \overline{\mathfrak{T}}_{g,n+2} \longrightarrow \overline{\mathfrak{T}}_{g+1,n}$$

descend to the maps of complex orbifolds when we pass to quotients by finite index subgroups.

As we explained above, given a covering $\rho : \widetilde{S} \to S$ of degree d (where S is, as before, a compact oriented surface of genus g with n punctures), one can naturally assign to each marked stable curve $(X, \phi) \in \overline{\mathfrak{T}}_{g,n}$ an admissible covering $\widetilde{X} \to X$ of degree d.

In Section 8 we elevate the map (1) to a morphism

$$v_\rho : \overline{\mathfrak{T}}_{g,n} \to \mathfrak{Adm}_{g,n,d}$$

of topological stacks (where $\overline{\mathfrak{T}}_{g,n}$ has trivial stabilizers) by composing a complex orbifold morphism $[G\backslash\overline{\mathfrak{T}}] \to \mathfrak{Adm}$ with the quotient map $\overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$.

Here G is a symmetry group of the finite covering $\rho : \widetilde{S} \to S$. It is not a subgroup of the modular group Γ, it acts on $\overline{\mathfrak{T}}$ via a natural homomorphism $\gamma : G \to \Gamma$ whose kernel and the index of the image in Γ are finite. Thus we need to deal with quotients of $\overline{\mathfrak{T}}$ which are slightly more general that the quotients modulo a finite index subgroup of Γ. The quotient $[G\backslash\overline{\mathfrak{T}}]$ is a gerbe over $[\mathrm{Im}(\gamma)\backslash\overline{\mathfrak{T}}]$.

We also prove compatibility of the maps v_ρ with the gluing operations constructed in Section 7.3.

Even though the maps v_ρ are morphisms of topological stacks, we can view them as holomorphic maps by replacing $\overline{\mathfrak{T}}$ with the projective system of complex orbifolds $[G\backslash\overline{\mathfrak{T}}]$.

Finally, in Section 8.7 we show how our results about the spaces $G\backslash\overline{\mathfrak{T}}$ can be used in the study of stringy orbifold cohomology. This was the original motivation for the work presented in this paper and we explain it in a greater detail below.

1.5. Motivation and application: orbifold cup-product

This paper is an offshoot of our project to study generalized multiplicative orbifold cohomology theories [27]. The Chen-Ruan definition of the cup-product in (stringy) orbifold cohomology (see [15] and [19]) uses cohomological correction classes whose construction involves certain equivariant vector bundles on the spaces of admissible coverings $\mathfrak{Adm}_{g,n,d}$. The space $\mathfrak{Adm}_{g,n,d}$ has an open stratum corresponding to non-singular curves and its boundary consists of products of spaces $\mathfrak{Adm}_{g',n',d}$ for $g' \leq g$ and $n' \leq n$. The associativity and commutativity of the orbifold cup-product are derived in [15] and [19] from the fact that the fibers of these bundles on $\mathfrak{Adm}_{g,n,d}$ at certain boundary points are isomorphic. This would be immediate if the spaces $\mathfrak{Adm}$ were connected. However, this is far from being true. For example, components of the open stratum of $\mathfrak{Adm}_{g,n,d}$ correspond to conjugacy classes of actions of the fundamental group of the curve on a d-element set. An attempt to resolve this difficulty brought us to considering augmented Teichmüller spaces.[6]

Let S be a compact oriented surface of genus g with n punctures and let $\rho : \widetilde{S} \to S$ be a finite covering unramified outside of the punctures. For a stable Riemann surface X of genus g with n punctures and a marking $\phi : S \to X$ we obtain an admissible covering of X induced from ρ. This leads to the map (1) of topological orbifolds. Since the augmented Teichmüller space $\overline{\mathfrak{T}}_{g,n}$ is contractible, its image in $\mathfrak{Adm}_{g,n,d}$ is connected. The boundary of $\overline{\mathfrak{T}}_{g,n}$ consists of strata which are products of $\overline{\mathfrak{T}}_{g',n'}$ for smaller values of g' and n' and the maps v_ρ respect the decompositions of boundary strata of $\overline{\mathfrak{T}}$ and $\mathfrak{Adm}$. Therefore this construction allows to move verification of associativity and commutativity of the orbifold cup-product away from a highly disconnected space $\mathfrak{Adm}_{g,n,d}$ to the contractible space $\overline{\mathfrak{T}}_{g,n}$.

[6]The issue of non-connectivity of the spaces of admissible coverings is also addressed in [30], see Lemma 2.30. Unfortunately, we were unable to understand the proof of this lemma.

This is the idea of our approach to the stringy cup-product problem. To implement it, we have to be able to speak about continuous maps from the space $\overline{\mathfrak{T}}_{g,n}$ to $\mathfrak{AD}\mathfrak{m}_{g,n,d}$. However, this task is non-trivial, since the former is a nasty topological space, whereas the latter is the space of $\mathbb{C}$-points of a nice Deligne-Mumford stack. The common ground is found in the 2-category of complex orbifolds and is developed in this paper.

Applications of this construction to generalized stringy orbifold cohomology theories will be described in our forthcoming paper [27].

1.6. Acknowledgments

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2. Generalities on orbifolds

In this section we present our preferred way of working with orbifolds. The language of algebraic stacks has long been the tool of choice for dealing with orbifolds in the context of algebraic geometry. We find it the most appropriate in other categories as well.

In order to obtain different species of the notion of an orbifold (C^∞, complex, algebraic), we have to choose an appropriate basic category Sp of manifolds or spaces and work with stacks over Sp (see Section 2.1). The resulting notion of a Sp-stack is too general to be geometrically meaningful in the same way as the corresponding notion of stack in algebraic geometry is too general. In order to distinguish geometrically meaningful stacks, we restrict our attention to étale groupoids which have already been used for the description of orbifolds.

Since Sp may not have arbitrary fiber products, we cannot define representable morphisms for general Sp-stacks and so it is impossible to define Sp-orbifolds simply by modifying the standard definition of algebraic stacks (see [35]). Instead we define an Sp-orbifold as an Sp-stack equivalent to the stack associated to an Sp-groupoid. In the case when Sp is the category of schemes this approach gives separated Deligne-Mumford stacks.

Both stacks and étale groupoids form a 2-category. Whereas the 2-category structure on stacks is standard, the 2-category structure on groupoids is not the one that first comes to mind. We define an orbifold as a stack of geometric origin, which means that it is equivalent to the stack associated to a separated étale groupoid. We prove further that the functor associating a stack to a groupoid gives an equivalence between 2-categories of separated étale groupoids and orbifolds.

Our treatment is similar but not equivalent to the recent expositions of Metzler [39], Noohi [42] and Behrend-Xu [6]. The most significant difference between these works and our approach is that our basic category of spaces is the category of smooth manifolds in which general fiber products do not exist. For this reason, we develop the theory which uses only fiber products along étale morphisms.

Having in mind the above-mentioned 2-equivalence, our approach to orbifolds via stacks is not so different from the widely accepted approach based on groupoids. We prefer, however, the approach via stacks for various reasons.

In Section 8 we use a related notion of a gerbe which is slightly non-standard. It is presented, along with other miscellanea, in 2.6. Sheaves and vector bundles on orbifolds are defined in 2.7.

2.1. Basic categories of "spaces"

In what follows Sp will denote one of the following categories of "spaces."

(i) The category of Hausdorff topological spaces.
(ii) The category of separated locally ringed topological spaces.
(iii) The category of C^∞-manifolds.
(iv) The category of complex manifolds.
(v) The category of separated complex spaces.
(vi) The category of smooth separated schemes over a field.
(vii) The category of separated schemes over a base scheme.

In each of these categories there exists a notion of an étale morphism. It is a local isomorphism for categories (i)-(v) and an étale morphism of schemes for cases (vi)-(vii).

We consider the category Sp endowed with the topology defined by open covers for Sp of type (i)-(v) and by étale morphisms for cases (vi)-(vii). We could equally consider the étale topology in all cases.

Note that in all our categories of spaces there exist fiber products $X\times_Z Y$ when one of the structure maps $X \to Z,\ Y \to Z$ is étale. Also, the notion of a proper map makes sense for all these categories.

2.2. Groupoids in categories of spaces

2.2.1. Groupoids. A *groupoid* (in category $\mathtt{Set}$) is a small category whose morphisms are invertible. Thus, a groupoid $G_\bullet = (G_0, G_1)$ can be specified by giving a set G_0 of its objects, a set G_1 of its arrows, and operations:

$$\iota : G_0 \to G_1 \text{ (identity)},\ \ s, t : G_1 \to G_0 \text{ (source and target)}$$

and the composition

$$c : G_1 \times_{G_0} G_1 \to G_1$$

satisfying well-known axioms.

Groupoids form a 2-category which we denote $\mathtt{Grp}$. One-morphisms in $\mathtt{Grp}$ are functors between groupoids; 2-morphisms are natural transformations between the corresponding functors. The 2-category $\mathtt{Grp}$ is strict: the composition of functors is strictly associative.

It is sometimes convenient to view groupoids as (very special) simplicial sets. To a groupoid $G_\bullet = (G_0, G_1)$ we assign a simplicial set $(G_0, G_1, G_2, \ldots)$ whose n-simplices are n-tuples of composable arrows in G_1. In this description the source and the target maps $s, t : G_1 \to G_0$ become the face maps d_1 and d_0; the composition of arrows c becomes the face map

$$d_1 : G_2 := G_1 \times_{G_0} G_1 \to G_1.$$

2.2.2. Definition. Let Sp be one of the categories of spaces from Section 2.1.

A *groupoid in*[7] Sp is a pair $G_\bullet = (G_1, G_2)$ of objects of Sp, together with the structure maps ι, s, t, c as in 2.2.1, such that for any $M \in \mathsf{Sp}$ the functor $\mathrm{Hom}(M, \_)$ sends this collection to a groupoid.

(We assume above that the fiber product $G_2 = G_1 \times_{G_0} G_1$ exists in Sp.)

2.2.3. Definition. Let $G_\bullet$ be a groupoid in Sp.

(i) $G_\bullet$ is called *separated* if the diagonal $(s, t) : G_1 \to G_0 \times G_0$ is proper.
(ii) $G_\bullet$ is called *étale* if the maps $s, t : G_1 \to G_0$ are étale.
(iii) $G_\bullet$ is called a Sp-groupoid if it is étale and separated.

2.2.4. Remark. Separated groupoids are sometimes called *proper groupoids*. We follow Grothendieck's terminology (separatedness = properness of the diagonal).

2.2.5. Example. Let G be a (discrete) group acting on $X \in \mathsf{Sp}$. The *transformation groupoid* $(G\backslash X)_\bullet$ is defined by

$$(G\backslash X)_0 = X, \quad (G\backslash X)_1 = G \times X,$$

where the source map s is the projection $G \times X \to G$ and the target t is the action map $t : G \times X \to X$. If G is finite then $(G\backslash X)_\bullet$ is an Sp-groupoid. If Sp is one of the non-algebraic categories 2.1.(i)-(v), then $(G\backslash X)_\bullet$ is an Sp-groupoid also when the action of G on X is discontinuous.

2.3. Groupoids over categories of spaces. Stacks

By definition, a groupoid $G_\bullet$ in Sp represents a functor from Sp to $\mathtt{Grp}$

$$M \mapsto (\mathrm{Hom}(M, G_0), \mathrm{Hom}(M, G_1)).$$

Since groupoids form a 2-category and not just a category, the notion of a functor to $\mathtt{Grp}$ is too rigid: the most natural constructions produce only pseudofunctors (see 2.3.2) to $\mathtt{Grp}$. This is why we need a relaxed version of the notion of an Sp-groupoid.

The definitions 2.3.1 and 2.3.4 below are special cases of Grothendieck's notions of a fibered category and of a stack, see [45, Ch. 4]. The case when Sp is the category of schemes is described in [35].

2.3.1. Definition. (see [35, Sec. 2]) *A groupoid over*[8] Sp is a category $\mathcal{X}$ endowed with a functor $\pi : \mathcal{X} \to \mathsf{Sp}$ such that

(i) For any $\alpha : U \to V$ in Sp and $x \in \mathcal{X}$ with $\pi(x) = V$ there exists $a : y \to x$ such that $\pi(a) = \alpha$.

[7] In [35] the term *éspace en groupoïdes* is used instead.
[8] *Catégorie fibrée en groupoïdes sur* Sp in the original terminology of [35].

(ii) For any pair of morphisms $a : y \to x$, $b : z \to x$ in $\mathcal{X}$ any map $\gamma : \pi(y) \to \pi(z)$ satisfying $\pi(a) = \pi(b)\gamma$ there exists a unique $c : y \to z$ such that $a = bc$ and $\pi(c) = \gamma$.

Groupoids over Sp form a 2-category denoted $\mathtt{Grp}/\mathsf{Sp}$. A 1-morphism from $\pi : \mathcal{X} \to \mathsf{Sp}$ to $\pi' : \mathcal{X}' \to \mathsf{Sp}$ is a functor $f : \mathcal{X} \to \mathcal{X}'$ strictly commuting with π, π'. A 2-morphism $\theta : f \to g$ between $f, g : \mathcal{X} \to \mathcal{X}'$ is a natural transformation that sends $x \in \mathcal{X}$ to a morphism $\theta(x) : f(x) \to g(x)$ over $\mathrm{id}_{\pi(x)}$ in $\mathcal{X}'$.

2.3.2. Pseudofunctors. Cleavage. Let $\pi : \mathcal{X} \to \mathsf{Sp}$ be a groupoid over Sp. The fibers

$$\mathcal{X}_U := \pi^{-1}(U), \quad \text{for} \quad U \in \mathsf{Sp},$$

are groupoids. For each $\alpha : U \to V$ and for each $x \in \mathcal{X}_V$ choose a lifting $a : \alpha^*(x) \to x$ of α. This choice can be uniquely extended to a functor $\alpha^* : \mathcal{X}_V \to \mathcal{X}_U$. Also, for each pair of composable arrows in Sp, one has a uniquely defined isomorphism $\theta_{\alpha,\beta} : \alpha^*\beta^* \to (\beta\alpha)^*$ These isomorphisms θ satisfy a standard compatibility condition shown on diagram (3).

The above collection $(\mathcal{X}_U, \alpha^*, \theta_{\alpha,\beta})$ defines a *pseudofunctor* $\mathsf{Sp}^{\mathrm{op}} \to \mathtt{Grp}$; it would be a genuine functor if $\theta_{\alpha,\beta}$ were the identity for all α and β.

Vice versa, given a collection of groupoids $\mathcal{X}_U$ for each $U \in \mathsf{Sp}$, together with functors

$$\alpha^* : \mathcal{X}_V \longrightarrow \mathcal{X}_U$$

for each morphism $\alpha : U \to V$ and equivalences

$$\theta_{\alpha,\beta} : \alpha^*\beta^* \to (\beta\alpha)^* \tag{2}$$

for each pair of composable arrows α, β of Sp, such that the diagram

$$\begin{tikzcd}
\alpha^*\beta^*\gamma^* \arrow[r, "\theta_{\alpha,\beta}\cdot\gamma^*"] \arrow[d, "\alpha^*\cdot\theta_{\beta,\gamma}"'] & (\beta\alpha)^*\gamma^* \arrow[d, "\theta_{\beta\alpha,\gamma}"'] \\
\alpha^*(\gamma\beta)^* \arrow[r, "\theta_{\alpha,\gamma\beta}"] & (\gamma\beta\alpha)^*
\end{tikzcd} \tag{3}$$

is commutative for each triple of composable arrows, one can "glue" a groupoid $\pi : \mathcal{X} \to \mathsf{Sp}$ by the formulas

$$\mathrm{Ob}\ \mathcal{X} = \coprod \mathrm{Ob}\ \mathcal{X}_U; \quad \pi(x) = U \Leftrightarrow x \in \mathcal{X}_U;$$
$$\mathrm{Hom}_{\mathcal{X}}(x, y) = \coprod_{\alpha:U\to V} \mathrm{Hom}_{\mathcal{X}_U}(x, \alpha^*(y)).$$

2.3.3. Definition. A choice of functors $a^* : \mathcal{X}_V \to \mathcal{X}_U$ for each $a : U \to V$ in Sp and of compatible equivalences (2) is called *a cleavage* of a groupoid $\pi : \mathcal{X} \to \mathsf{Sp}$ (in SGA1: un clivage). Thus every groupoid over Sp admits a cleavage, and cleaved groupoids over Sp are the same as pseudofunctors $\mathsf{Sp}^{\mathrm{op}} \to \mathtt{Grp}$.

Any groupoid $G_\bullet$ in Sp represents a functor $\mathsf{Sp} \to \mathtt{Grp}$. This, together with the trivial cleavage $\theta_{\alpha,\beta} = \mathrm{id}$, defines a groupoid over Sp. Thus, the notion of groupoid *over* Sp generalizes that of groupoid *in* Sp.

Groupoids over Sp play the role of "presheaves of groupoids" on Sp. Stacks can be viewed as "sheaf of groupoids".

2.3.4. Definition. (see [35], Sect. 2–3) A *stack* (of groupoids) $\mathcal{X}$ over Sp is a groupoid over Sp satisfying the following two conditions.

(i) For any two objects $x, y \in \mathcal{X}_U$ the assignment

$$\alpha : V \longrightarrow U \mapsto \mathrm{Hom}(\alpha^*(x), \alpha^*(y))$$

is a sheaf on $\mathsf{Sp}/U := \{V \to U | V \in \mathsf{Sp}\}$.

(ii) For any covering $\alpha_i : V_i \to U$ in Sp the groupoid $\mathcal{X}_U$ is equivalent to the groupoid $\mathcal{X}(\{V_i\})$ of "local data" whose objects are collections

$$\left(x_i \in \mathcal{X}_{V_i}, \theta_{ij} : x_i|_{V_{ij}} \longrightarrow x_j|_{V_{ij}}\right), \text{ where } V_{ij} := V_i \times_U V_j,$$

with compatible θ_{ij} and whose morphisms are isomorphisms of these collections.

Stacks over Sp form a 2-category $\mathtt{Stacks}/\mathsf{Sp}$ which is a strictly full 2-subcategory of $\mathtt{Grp}/\mathsf{Sp}$.

Let $M \in \mathsf{Sp}$. The functor on Sp represented by M is a stack. It is called the stack represented by M. An 1-morphism $M \to \mathcal{X}$ is given by an object of $\mathcal{X}_M$.

2.3.5. Associated stack. For every $\mathcal{X} \in \mathtt{Grp}/\mathsf{Sp}$ we can associate an Sp-stack $[\mathcal{X}]$ which is constructed in two steps. First one sheafifies all Hom-sets and then "glues" new objects from the local data as in Definition 2.3.4 (see the details in [35, Lemme 3.2]).

The stack associated to an Sp-groupoid $M_\bullet$ will be denoted $[M_\bullet]$. If $M_\bullet = (G\backslash X)_\bullet$, we will write $[M_\bullet] = [G\backslash X]$ rather than $[(G\backslash X)_\bullet]$.

The following lemma gives an explicit description of the groupoid $[M_\bullet]_U$.

2.3.6. Lemma. *Let* $U \in \mathsf{Sp}$.

(i) *Objects of* $[M_\bullet]_U$ *are morphisms* $V_\bullet \to M_\bullet$ *of groupoids where* $\alpha : V \to U$ *is étale surjective in* Sp *and the groupoid* $V_\bullet$ *in* Sp *is defined by the formulas*

$$V_n = V \times_U \ldots \times_U V \ (n+1 \text{ factors}).$$

(ii) *Given two objects,* $\alpha : V_\bullet \to M_\bullet$ *and* $\alpha' : V'_\bullet \to M_\bullet$*, a morphism from* α *to* α' *is a morphism between two functors from* $V_\bullet \times_U V'_\bullet$ *to* $M_\bullet$*.*

This is a direct application of the construction of the associated stack, see the proof of Lemma 3.2 in [35]. □

Proposition 2.3.8 below gives a similar explicit description of the groupoid $\mathrm{Hom}([X_\bullet], [Y_\bullet])$, where $X_\bullet$ and $Y_\bullet$ are Sp-groupoids.

2.3.7. Definition. A map $f : Z_\bullet \to X_\bullet$ is called an *acyclic fibration* if the map $f_0 : Z_0 \to X_0$ is étale surjective and the commutative square

$$\begin{tikzcd} Z_1 \arrow[r, "f_1"] \arrow[d, "(s,t)"'] & X_1 \arrow[d, "(s,t)"'] \\ Z_0 \times Z_0 \arrow[r, "f_0 \times f_0"] & X_0 \times X_0 \end{tikzcd}$$

is Cartesian.[9]

Let $f : Z_\bullet \to X_\bullet$ be an acyclic fibration and let $g : X'_\bullet \to X_\bullet$ be a morphism. Then the "naive" fiber product

$$Z'_\bullet = Z_\bullet \times^{\mathrm{nv}}_{X_\bullet} X'_\bullet$$

defined by

$$Z'_i = Z_i \times_{X_i} X'_i,\ i = 0, 1,$$

gives rise to an acyclic fibration $f' : Z'_\bullet \to X'_\bullet$.

2.3.8. Proposition. *The groupoid* $\mathrm{Hom}([X_\bullet], [Y_\bullet])$ *has the following explicit description.*

(i) *The objects of* $\mathrm{Hom}([X_\bullet], [Y_\bullet])$ *are diagrams of* Sp*-groupoids*

$$X_\bullet \xleftarrow{s} Z_\bullet \xrightarrow{f} Y_\bullet$$

where s is an acyclic fibration.

(ii) *A morphism from* $X_\bullet \xleftarrow{s} Z_\bullet \xrightarrow{f} Y_\bullet$ *to* $X_\bullet \xleftarrow{s'} Z'_\bullet \xrightarrow{f'} Y_\bullet$ *in* $\mathrm{Hom}([X_\bullet], [Y_\bullet])$ *is given by a 2-morphism between the two compositions* $f \circ \mathrm{pr}_1$ *and* $f' \circ \mathrm{pr}_2$ *from* $Z_\bullet \times^{\mathrm{nv}}_{X_\bullet} Z'_\bullet$ *to* $Y_\bullet$.

Proof. By the universality of the associated stack, we need just to describe $\mathrm{Hom}(X_\bullet, [Y_\bullet])$. A map $F : X_\bullet \to [Y_\bullet]$ defines a composition $\hat{F} : X_0 \to [Y_\bullet]$. By Lemma 2.3.6 there exists an étale surjective map $s_0 : Z_0 \to X_0$ and a map $f_0 : Z_0 \to Y_0$ so that the pair (s_0, f_0) presents $\hat{F}$.

Consider the space

$$Z_1 = (Z_0 \times Z_0) \times_{X_0 \times X_0} X_1. \tag{4}$$

This determines an acyclic fibration $s : Z_\bullet \to X_\bullet$. We claim that F canonically determines (and is canonically determined by) a map $f : Z_\bullet \to Y_\bullet$ extending f_0.

The pair (s_0, f_0) gives for each $U \in \mathsf{Sp}$ a functor $F(U)$ which acts on objects by

$$(x : U \to X_0) \longmapsto (U \longleftarrow U \times_{X_0} Z_0 \longrightarrow Z_0 \xrightarrow{f_0} Y_0). \tag{5}$$

Let us describe the action of F on the arrows. To each arrow in $(X_\bullet)_U$ (that is to each map $x : U \to X_1$) F assigns a morphism between two images

[9] Acyclic fibrations are special cases of weak equivalences of groupoids, as defined in [41].

of $sx,\ tx : U \to X_0$ given as in (5). The second part of Lemma 2.3.6 says that this amounts to a map $U \to Z_1$ where Z_1 is defined by (4).

This proves the first part of the proposition. The second part is straightforward. □

2.3.9. Fiber products. Since groupoids over Sp form a 2-category, we will use the following natural 2-categorical fiber product operation.

Definition. The *fiber product* of a diagram of 1-morphisms in $\mathtt{Grp}/\mathsf{Sp}$

$$\mathcal{X} \xrightarrow{f} \mathcal{Z} \xleftarrow{g} \mathcal{Y}$$

is the groupoid in $\mathtt{Grp}/\mathsf{Sp}$ whose objects over $U \in \mathsf{Sp}$ are triples (x, y, θ), where $x \in \mathcal{X}(U),\ y \in \mathcal{Y}(U)$ and $\theta : f(x) \tilde{\to} g(y)$ is an isomorphism; morphisms are compatible pairs of morphisms in $\mathcal{X}$ and $\mathcal{Y}$.

This fiber product has the expected properties.

2.3.10. Lemma. *Let $\mathcal{F}$ be a fiber product of a diagram $\mathcal{X} \to \mathcal{Z} \leftarrow \mathcal{Y}$. Then*

(i) *If $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ are Sp-stacks then $\mathcal{F}$ is as well an Sp-stack.*

(ii) *The associated stack $[\mathcal{F}]$ is a fiber product of the diagram*

$$[\mathcal{X}] \to [\mathcal{Z}] \leftarrow [\mathcal{Y}].$$

Proof. The statement (i) is immediate and (ii) follows from (i). □

2.4. Orbifolds

In this section we define Sp-orbifolds, where Sp is one of the categories of spaces from Section 2.1.

2.4.1. Definition. A stack $\mathcal{X}$ over Sp is called an Sp-*orbifold* if it is equivalent to the stack $[X_\bullet]$ associated to an (étale separated) Sp-groupoid $X_\bullet$.

The full 2-subcategory of Sp-orbifolds in the 2-category of Sp-stacks will be denoted by Sp-$\mathtt{Orbi}$ (or simply $\mathtt{Orbi}$).

Let Sp be the category of schemes over a fixed base scheme. The standard definition of Deligne-Mumford stack requires the diagonal to be quasi-compact (i.e., the preimage under the diagonal map of any quasi-compact open subset is quasi-compact). The stacks having proper diagonal are called *separated stacks.*

Definition of orbifolds as equivalence classes of separated étale groupoids belongs to Moerdijk. We prefer looking at a groupoid as a specific *presentation* of an Sp-orbifold in the sense of the following definition.

2.4.2. Definition. Let $\mathcal{X}$ be an Sp-orbifold. An Sp-groupoid $X_\bullet$ together with a map

$$\alpha : X_\bullet \longrightarrow \mathcal{X}$$

of groupoids over Sp is called a presentation of $\mathcal{X}$ if it induces an equivalence $[X_\bullet] \to \mathcal{X}$.

A presentation $\alpha : X_\bullet \to \mathcal{X}$ of an Sp-orbifold $\mathcal{X}$ is uniquely determined by a morphism $\alpha_0 : X_0 \to \mathcal{X}$ and by an equivalence $\alpha_1 : X_1 \to X_0 \times_{\mathcal{X}} X_0$.

2.5. Representable morphisms

2.5.1. Definition. An Sp-orbifold $\mathcal{X}$ is called *representable* if for every $U \in \mathsf{Sp}$ the groupoid $\mathcal{X}(U)$ is discrete (i.e. the group of automorphisms of every object in $\mathcal{X}(U)$ is trivial).

If Sp is one of the non-algebraic categories 2.1.(i)-(v), then representable orbifolds are functors represented by objects of Sp; representable orbifolds for Sp of type (vi) or (vii) correspond to algebraic spaces.

2.5.2. Definition. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of Sp-orbifolds is called *representable* if for any morphism $a : Y \to \mathcal{Y}$ such that $Y \in \mathsf{Sp}$ and the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y \in \mathsf{Sp}\text{-}\mathtt{Orbi}$ exists, this fiber product is representable.

It is clear that in order to check that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is representable, it is sufficient to prove that for some presentation $Y_\bullet$ of $\mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y_0$ is a representable orbifold.

2.5.3. Proposition. *Let $\mathcal{X}$ be an Sp-orbifold.*

(i) *The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.*
(ii) *Let $f : X \to \mathcal{X}$ be a morphism in $\mathsf{Sp}\text{-}\mathtt{Orbi}$ such that X belongs to Sp. Then f is representable.*

Proof. Choose a presentation $X_\bullet$ of $\mathcal{X}$. Then the fiber product

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} (X_0 \times X_0)$$

is equivalent to the stack associated to

$$X_\bullet \times_{X_\bullet \times X_\bullet} (X_0 \times X_0) = X_0 \times_{X_\bullet} X_0 = X_1.$$

This proves the first statement.

The second statement follows from the equality

$$X \times_{\mathcal{X}} X_0 = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} (X \times X_0).$$

□

2.5.4. Properties of representable morphisms.

Definition. A property (class) P of morphisms in Sp is called *local* if for each Cartesian diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

the following hold:

- $f \in P$ and g is étale implies that $f' \in P$ and
- $f' \in P$ and g is étale surjective implies that $f \in P$.

Let P be a local property of morphisms in Sp. We say that a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of Sp-orbifolds satisfies P if its base change $f' : X_0 \to Y_0$ satisfies P, where $Y_0 \to \mathcal{Y}$ is obtained from a presentation $Y_\bullet$ of $\mathcal{Y}$.

The following classes of morphisms are local: smooth (= submersive), étale, étale surjective, proper, open embedding, and finite (=proper with finite fibers).

Locality of étale surjective morphisms is important for the following description of the 2-category of orbifolds.

2.5.5. Proposition. *The 2-category* Sp-`Orbi` *is equivalent to the 2-category whose objects are* Sp*-groupoids and morphisms are defined as in Proposition 2.3.8.*

Proof. If $X_\bullet$ is a presentation of $\mathcal{X}$, the corresponding map $X_0 \to \mathcal{X}$ is étale surjective since its base change with respect to the morphism $X_0 \to \mathcal{X}$ is $s : X_1 \to X_0$ which is étale and admits a section.

Vice versa, assume $a : Y \to \mathcal{X}$ is étale surjective. Consider $X_0 = Y$, and $X_1 = Y \times_{\mathcal{X}} Y$. The orbifold X_1 is representable and, since it is étale over $X_0 \in \mathsf{Sp}$, it belongs to Sp.[10] Therefore we found a Sp-groupoid $X_\bullet$ presenting $\mathcal{X}$.

Let $X_0 \to \mathcal{X}$ and $Y_0 \to \mathcal{X}$ be étale surjective and let $Z_0 = X_0 \times_{\mathcal{X}} Y_0$. Let $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ be the presentations of $\mathcal{X}$ constructed as above from the maps $X_0 \to \mathcal{X}$, $Y_0 \to \mathcal{X}$, $Z \to \mathcal{X}$. Then the maps $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$ are acyclic fibrations in the sense of 2.3. □

Thus, the 2-category `Orbi` can be defined in terms of of Sp-groupoids. By using language of stacks we do not gain new "expressive power". However, this language has the same advantages in dealing with Sp-orbifolds as it has in the context of algebraic geometry.

Proposition 2.5.5 implies that the 1-category obtained from `Orbi` by identifying isomorphic morphisms, is equivalent to the localization of the category of Sp-orbifolds by the collection of acyclic fibrations. The latter category is what Moerdijk [41] calls the category of orbifolds.

2.6. Some examples and constructions

2.6.1. Points and the coarse space. Let Sp be of one of the non-algebraic categories of spaces 2.1.(i)-(v). For an Sp-orbifold $\mathcal{X}$ define $|\mathcal{X}|$ as the set of connected components of the groupoid $\mathcal{X}(\text{point})$. If $\mathcal{X}$ is represented by an Sp-groupoid $X_\bullet$, one has a natural surjection $X_0 \to |\mathcal{X}|$. The set $|\mathcal{X}|$ endowed with the quotient topology is called *the coarse space of* $\mathcal{X}$. Open subsets of $|\mathcal{X}|$ are in a one-to-one correspondence with (equivalence classes of) open substacks of $\mathcal{X}$.

[10] This is true even in the case when Sp is the category of schemes, see [33, 6.17].

If Sp is of algebraic type (vi) or (vii), the points of $\mathfrak{X}$ are defined as classes of equivalent objects of $\mathfrak{X}(\mathrm{Spec}\ K)$, where K is a field and the equivalence allows to extend the field K.

The set of points $|\mathfrak{X}|$ is endowed with the Zariski topology, whose open sets are defined by points $|\mathfrak{U}|$ of open suborbifolds $\mathfrak{U}$ of $\mathfrak{X}$ (see details in [35]).

If Sp is the category of complex manifolds the coarse space of an Sp-orbifold has a natural structure of a complex space. If Sp is the category of schemes of finite type over a locally Noetherian base, the coarse moduli space is an algebraic space by a result of Keel-Mori, see [31].

2.6.2. Global quotient. Let $X \in \mathsf{Sp}$ and let G be a finite group acting on X. Then the Sp-orbifold $[G\backslash X]$ associated to the transformation groupoid $(G\backslash X)_\bullet$ (see 2.2.5) is called *the global quotient orbifold.*

2.6.3. Change of the base category. Let $F : \mathsf{Sp}_1 \to \mathsf{Sp}_2$ be a functor between two categories of spaces from the list 2.1 that preserves étale morphisms, coverings and proper morphisms, as well as the fiber products. Then the functor F extends to the corresponding categories of Sp-groupoids. Using Proposition 2.3.8 and Proposition 2.5.5, we obtain, up to 2-equivalence, a functor

$$F : \mathsf{Sp}_1\text{-}\mathtt{Orbi} \longrightarrow \mathsf{Sp}_2\text{-}\mathtt{Orbi}.$$

Examples of this construction provide various forgetful functors. A less obvious example is the functor assigning to a scheme of finite type over $\mathbb{C}$ its analytification, see [45, exposé XII].

One of the goals of this paper is to construct maps from the augmented Teichmüller space $\overline{\mathfrak{T}}$ (which is a topological space) to stacks of admissible coverings $\mathfrak{Adm}$ considered either as a Deligne-Mumford stack or as an orbifold in the category of complex spaces (see 2.6.4). The above construction allows one to define the desired map as a 1-morphism in the category of topological orbifolds.

This may seem a weak notion; it is sufficient, however, to be able to pull back a vector bundle on $\mathfrak{Adm}$ to $\overline{\mathfrak{T}}$ (see 2.7).

2.6.4. Moduli stacks. In this paper a few moduli stacks play an important role.

According to [16] and [32] the functor assigning to each scheme S the groupoid of families of stable curves of genus g with n punctures over S, is represented by a smooth projective Deligne-Mumford stack. We denote it $\overline{\mathfrak{M}}_{g,n}$. Its open substack $\mathfrak{M}_{g,n}$ represents the groupoid of smooth families.

The stack of admissible coverings $\mathfrak{Adm}_{g,n,d}$ assigns to a scheme S the groupoid of admissible coverings of degree d of S-families of stable curves of genus g unramified of n points (see [4, Sect. 4]). This is a proper Deligne-Mumford stack having a projective coarse moduli space, see [40]. Similarly, given a finite group H, we denote by $\mathfrak{Adm}_{g,n}(H)$ the stack of admissible H-coverings.

These are algebraic orbifolds (i.e. Sp is the category of schemes) in the sense of our definition. We will prove in Section 4 that the analytifications of the stacks $\mathfrak{M}$, $\overline{\mathfrak{M}}$, $\mathfrak{Adm}$ represent the corresponding groupoids of analytic families (of stable curves or of admissible coverings).

2.6.5. Gerbes.

Definition. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of Sp-orbifolds is called *a gerbe* if

- $f : \mathcal{X} \to \mathcal{Y}$ is surjective.
- $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective.

The first condition means that for any object $y \in \mathcal{Y}_U$ there exists a covering $V \to U$ and an object $x \in \mathcal{X}_V$ such that $f(x)$ is isomorphic to y_V. The second condition means that given a pair of objects $x_1, \; x_2$ in $\mathcal{X}_U$ and an isomorphism $\theta : f(x_1) \to f(x_2)$ in $\mathcal{Y}_U$, there exists a covering $V \to U$ and an isomorphism $\eta : x_{1V} \to x_{2V}$ such that $f(\eta) = \theta$.

A gerbe $f : \mathcal{X} \to \mathcal{Y}$ is called *split* if there exists a morphism $s : \mathcal{Y} \to \mathcal{X}$ such that the composition $f \circ s$ is equivalent to $\mathrm{id}_{\mathcal{Y}}$.

A typical example of a split gerbe is given by a finite group trivially acting on a manifold. Here is a non-split example. Let $\widetilde{G} \to G$ be a surjective homomorphism of finite groups. Let G act on a manifold X. Then the morphism

$$[\widetilde{G}\backslash X] \longrightarrow [G\backslash X]$$

is a gerbe which is not necessarily split.

Note that a base change of a gerbe is a gerbe and that for any gerbe $\mathcal{X} \to \mathcal{Y}$ there exists a covering $\mathcal{Y}' \to \mathcal{Y}$ such that the base change $\mathcal{X}' \to \mathcal{Y}'$ splits. All this immediately follows from the definition.

2.7. Sheaves and vector bundles on orbifolds

A sheaf (or a vector bundle) on an orbifold $\mathcal{X}$ is given by a compatible collection of sheaves (vector bundles) on each étale neighborhood $f : X \to \mathcal{X}$. Here is an appropriate definition.

2.7.1. Definition. A *sheaf* F on an orbifold $\mathcal{X}$ is a collection of the following data:

- Assignment, for each étale morphism $f : X \to \mathcal{X}$, of a sheaf F_f on $X \in \mathsf{Sp}$.
- An isomorphism of sheaves
  $$\theta_{f,g,\phi,\alpha} : \phi^*(F_g) \to F_f$$
  for each quadruple (f, g, ϕ, α), where $f : X \to \mathcal{X}$ and $g : Y \to \mathcal{X}$ are étale morphisms, $\phi : X \to Y$ is a morphism in Sp and $\alpha : f \to g \circ \phi$ a morphism in $\mathcal{X}(X)$.

The isomorphisms θ should be compatible with respect to compositions, i.e. for any morphisms $h : Z \to \mathcal{X}$, $\psi : Y \to Z$, $\beta : g \to h \circ \psi$ we have

$$\theta_1 \circ \phi^*(\theta_2) = \theta_{12},$$

where $\theta_1 = \theta_{f,g,\phi,\alpha}$, $\theta_2 = \theta_{g,h,\psi,\beta}$, $\theta_{12} = \theta_{f,h,\psi\circ\phi,(\beta\phi)\circ\alpha}$ and

$$\beta\phi : g \circ \phi \to h \circ \psi \circ \phi$$

is induced by β.

A vector bundle on orbifolds is defined similarly.

Let $X_\bullet$ be a presentation of $\mathcal{X}$. A sheaf (resp., a vector bundle) F on $\mathcal{X}$ gives a sheaf (a vector bundle) F_0 on X_0 together with an isomorphism $\theta : s^*(F_0) \to t^*(F_0)$ of sheaves on X_1 satisfying the cocycle condition on X_2. It is a standard fact that the above assignment is an equivalence of categories. In particular, if $\mathcal{X} = [G\backslash X]$, where G is a finite group, and $X \in \mathsf{Sp}$, then sheaves (resp., vector bundles) on $\mathcal{X}$ are the same as G-equivariant sheaves (vector bundles) on X.

2.7.2. Inverse image. Given a morphism of orbifolds $f : \mathcal{X} \to \mathcal{Y}$ one can choose presentations $X_\bullet$ and $Y_\bullet$ of $\mathcal{X}$ and $\mathcal{Y}$ so that f lifts to a map

$$f_\bullet : X_\bullet \to Y_\bullet$$

of Sp-groupoids. Then a sheaf (resp., a vector bundle) F on $\mathcal{Y}$ is given by a sheaf (a vector bundle) F_0 on Y_0 together with the descent data (an isomorphism $\theta : s^*(F_0) \to t^*(F_0)$ satisfying the cocycle condition). The inverse image $f_0^*(F_0)$ together with the inverse image descent data define a sheaf (a vector bundle) on X. One can easily check that the result does not depend on the choice of presentations for $\mathcal{X}$ and for $\mathcal{Y}$. This defines the inverse image functor f^*.

3. Satake orbifolds

In this section the category of spaces Sp is either the category of C^∞-manifolds or of complex manifolds.

Originally orbifolds were defined by Satake [44] using the language of orbifold charts. This approach works only for effective orbifolds which is not sufficient for our purposes.

In this section we present a generalization of Satake's description of orbifolds in terms of charts and atlases which also works for non-effective orbifolds and has some other advantages. We will show that this generalized Satake definition of Sp-orbifolds is equivalent to the one based on the language of stacks from Section 2.4. Even though the definition in terms of stacks is more natural, we need to use charts and atlases in Section 6 in order to construct a complex orbifold structure on quotients of the augmented Teichmüller space.

In Section 3.1 we define Satake orbifold atlases. Our definition is more general than the original one given by Satake in [44]. Our atlases, in addition to orbifolds charts, contain information about admissible maps between the charts. This allows us to incorporate non-effective orbifolds. We prove that every Sp-orbifold has such an atlas.

In Section 3.2, conversely, we show that any Satake orbifold specified by a collection of (generalized) orbifold charts and admissible morphisms between them corresponds to an Sp-orbifold. This orbifold is constructed as 2-colimit (in an appropriate sense) of the global quotients defined by the charts.

Our method has several advantages over the standard construction of an equivalence class of groupoids from a Satake orbifold (see e.g. [41]). First, we define the associated Sp-orbifold by a universal property which is very convenient in applications. Second, our procedure works with non-effective orbifolds as well as with effective ones. And third, the same construction works both for C^∞ and complex orbifolds.

3.1. Geographical approach: charts and atlases

Recall the following fact.

3.1.1. Lemma. *Let $X_\bullet$ be an Sp-groupoid. Let $x \in X_0$ and*

$$G = \mathrm{Aut}(x) = \{\gamma \in X_1 | s(\gamma) = t(\gamma) = x\}.$$

For any open neighborhood V of $x \in X_0$ there exists an open neighborhood $U \subset V$ of x so that the restriction of $X_\bullet$ to U is isomorphic to a quotient groupoid $(G\backslash U)_\bullet$.

Proof. See proof of Theorem 4.1 in [41]. $\square$

This lemma implies that any Sp-orbifold can be covered by open suborbifolds of the form $[G\backslash U]$, where G is a finite group.

A pair (U, G) as above is called an orbifold chart of $\mathcal{X}$ (see a more formal definition below). An orbifold chart (U, G) is called *effective* if the action of G on U is effective.

In [44] Satake defined an orbifold (V-manifold in his terminology) as a topological space endowed with an atlas of effective orbifold charts. We will call such objects *effective Satake orbifolds.* Satake proved that every effective Satake orbifold can be presented as a quotient of a manifold by a compact group acting with finite stabilizers. In [41] Moerdijk and Pronk deduce from this that an effective Satake orbifold can be presented by a C^∞-groupoid.

In Section 3.2 we define general (not necessarily effective) Satake orbifolds and construct an orbifold atlas for arbitrary orbifold in the sense of Section 2. We also present a new construction that associates to a general Satake orbifold a C^∞ (or complex) orbifold.

We begin with formal definitions of orbifold charts and atlases.

3.1.2. Abstract orbifold charts.

Definition. An *abstract orbifold chart* is a pair (V, H) where $V \in \mathsf{Sp}$ and H is a finite group acting on V. A *morphism of abstract orbifold charts*

$$f : (V, H) \longrightarrow (V', H')$$

is a pair (f_V, f_H), where $f_V : V \to V'$ is a morphism in Sp and $f_H : H \to H'$ is a group homomorphism, such that

- the map f_V is f_H-equivariant and
- the induced map of orbifolds

$$[H\backslash V] \longrightarrow [H'\backslash V']$$

is an open embedding (see Section 2.5.4).

An abstract orbifold chart (V, H) is called *effective* if H acts effectively on V.

The category of abstract orbifold charts will be denoted by `Charts`.

3.1.3. Remarks.

(i) The second condition in the definition of morphism of charts can be reformulated as follows. After a base change, the map $[H\backslash V] \to [H'\backslash V']$ turns into $[H\backslash(H' \times V)] \to V'$. The latter map is an open embedding if and only if
- the kernel of the map $f_H : H \to H'$ acts freely on V and
- the induced map from the quotient space $H' \times^H V$ to V' is an open embedding.

(ii) If f is a map of abstract orbifold charts, then the map f_V is étale because it is the composition of an open embedding $V \to H' \times V$, the étale morphism

$$H' \times V \to [H\backslash H' \times V]$$

and the open embedding described in the previous remark.

3.1.4. Definition. Let X be a Hausdorff topological space.

An *orbifold chart* of X is a collection $(V, H, \pi : V \to X)$ where $(V, H) \in$ `Charts` and π is a continuous map identifying the quotient V/H with an open subset of X. An orbifold chart (V, H, π) of X is called effective if the abstract orbifold chart (V, H) is effective.

A *morphism*

$$f : (V, H, \pi) \to (V', H', \pi')$$

of orbifold charts of X is a morphism (f_V, f_H) of the abstract orbifold charts satisfying the compatibility $\pi = \pi' \circ f_V$.

The category of orbifold charts of X will be denoted `Charts`$/X$.

Note that a morphism $f : (V, H, \pi) \to (V', H', \pi')$ of effective orbifold charts is uniquely determined by its first component f_V.

Let (V, H, π) be an orbifold chart. Any element $h \in H$ defines *the inner automorphism* h of (V, H, π) by the formulas

$$h_V(x) = h(x),\ h_H(g) = hgh^{-1}.$$

The effective version of these notions is considerably simpler due to the following property of effective orbifold charts.

3.1.5. Lemma. *Let $f, g : (V, H, \pi) \to (V', H', \pi')$ be two injective maps between connected effective orbifold charts. Then there exists $h \in H'$ so that $g = h \circ f$.*

Proof. See Proposition A.1 in Moerdijk-Pronk [41]. □

The following example shows this does not hold in general. Let H act trivially on V and let ϕ be a non-inner automorphism of H. Then the pair (id_V, ϕ) is an automorphism of (V, H, id_V) which cannot be obtained from $(\mathrm{id}_V, \mathrm{id}_H)$ by conjugation.

As a special case of Lemma 3.1.5 we deduce that if a chart (V, H, π) is effective, the semigroup of endomorphisms $\mathrm{End}(V, H, \pi)$ identifies with H.

We start with the (more or less standard) definition of *effective* orbifold atlases.

3.1.6. Definition. An *effective orbifold atlas* of a Hausdorff topological space X is a collection of effective orbifold charts on X covering X, such that for any two charts (V', H', π') and (V'', H'', π'') with $x \in \pi'(V') \cap \pi''(V'')$ there exists a chart (V, H, π) in the collection and a pair of injective morphisms from (V, H, π) to (V', H', π') and (V'', H'', π'') respectively so that $x \in \pi(V)$.

The notion of equivalent atlases and of the maximal atlas in the effective case are defined in a standard way.

3.1.7. Definition. A topological space X with a family of equivalent effective orbifold atlases is called *an effective Satake orbifold.*

Below we present a general definition of (not necessarily effective) a Satake orbifold. To be able to work with non-effective atlases, we will have to specify the admissible morphisms between the orbifold charts explicitly.

Our general definition reduces to 3.1.7 in the effective case (see Remark 3.1.10 below).

The category of the orbifold charts A will satisfy the following properties.

3.1.8. Definition. A category A is called *a chart* category if

- For each $a \in A$ all endomorphisms of a in A are invertible.
- For each $a, b \in A$ the set $\mathrm{Hom}_A(a, b)$ is a (may be, empty) $\mathrm{Aut}(b)$-torsor.

Note that any arrow $f : a \to b$ in a chart category A defines a homomorphism

$$\mathrm{Aut}(f) : \mathrm{Aut}(a) \longrightarrow \mathrm{Aut}(b)$$

uniquely characterized by the property

$$f \circ u = \mathrm{Aut}(f)(u) \circ f \text{ for } u \in \mathrm{Aut}(a).$$

3.1.9. Definition. An *orbifold atlas* of a Hausdorff topological space X consists of

- A chart category A.
- A functor $c : A \to \mathtt{Charts}/X$ which sends $a \in A$ to the chart
$$c(a) = (V_c(a), H_c(a), \pi_c(a)) \in \mathtt{Charts}/X. \quad (6)$$
- A collection of isomorphisms $\iota : \mathrm{Aut}(a) \to H_c(a)$ compatible with the action of both groups on $V_c(a)$ such that $\phi \in \mathrm{Aut}(a)$ induces the inner automorphism of $c(a)$ given by the element $\iota(\phi) \in H_c(a)$.

The above data are assumed to satisfy the following properties.

(i) The images of the charts $c(a)$ cover the whole X.
(ii) For any $x \in X$ belonging to the images of two charts $c(a')$ and $c(a'')$, there exists $a \in A$ with a pair of arrows $a \to a',\ a \to a''$, such that x belongs to the image of $c(a)$.

A *morphism of orbifold atlases* $(A, c) \to (A', c')$ is a fully faithful functor

$$f : A \to A'$$

of the corresponding chart categories together with an isomorphism of functors

$$c \xrightarrow{\simeq} c' \circ f.$$

Two orbifold atlases are called *equivalent* if they can be connected by a sequence of morphisms in the above sense.

We will usually suppress the subscript c in equation (6) and will write simply

$$c(a) = (V(a), H(a), \pi(a)) \text{ or even } c(a) = (V(a), H(a), \pi).$$

3.1.10. Remark. Let X be an effective Satake orbifold defined by a set A of connected effective orbifold charts. Then by 3.1.5 the subcategory of $\mathtt{Charts}/X$ defined by the set A of orbifold charts and all injective morphisms between them, is a chart category. Thus, our definition 3.1.9 reduces to the standard definition 3.1.7 in the effective case.

3.1.11. An orbifold atlas of an orbifold. Proposition. Any Sp-orbifold admits an orbifold atlas in the sense of Definition 3.1.9.

Proof. For an orbifold $\mathcal{X}$, define an atlas category A as follows (see [26, 6.1]). The objects of A are triples $(V, H, \hat{\pi})$, where (V, H) is an abstract orbifold chart and $\hat{\pi} : [H\backslash V] \to \mathcal{X}$ is an open embedding. A morphism

$$f : (V, H, \hat{\pi}) \to (V', H', \hat{\pi}')$$

is a triple (f_V, f_H, θ), where (f_V, f_H) is a morphism of abstract orbifold charts and θ is an isomorphism between $\hat{\pi}$ and $\hat{\pi}' \circ \hat{f}$, where

$$\hat{f} : [H\backslash V] \to [H'\backslash\ V']$$

is the map of orbifolds induced by (f_V, f_H). Let X be the coarse space for $\mathcal{X}$. Define the functor

$$c : A \to \mathtt{Charts}/X$$

by assigning to the triple $(V, H, \hat{\pi}) \in A$ the orbifold chart (V, H, π), where π is the composition of the projection $V \to H\backslash V$ with the map $H\backslash V \to X$ induced by $\hat{\pi}$. Let us check that $c : A \to \mathtt{Charts}/X$ is an orbifold atlas for $\mathcal{X}$. According to Lemma 3.1.1, the conditions (i) and (ii) of Definition 3.1.9 are satisfied.

Now we will show that A is a chart category. Let $a = (V, H, \hat{\pi})$ be an object of A. Since $\hat{\pi} : [H\backslash V] \to \mathcal{X}$ is an open embedding, the category $\mathrm{Hom}([H\backslash V], \mathcal{X})$ is equivalent to the category $\mathrm{Hom}([H\backslash V], [H\backslash V])$. Thus,

$\mathrm{End}(a)$ is isomorphic to the group of automorphisms of the identity functor $[H\backslash V] \to [H\backslash V]$ and by Lemma 2.3.6 we have

$$\mathrm{End}(a) = \mathrm{Aut}(a) = H.$$

A similar argument proves that $\mathrm{Hom}(a, b)$ is an $\mathrm{Aut}(b)$-torsor. □

3.2. An orbifold from a Satake orbifold atlas

In this section we will show that to each Satake orbifold there corresponds an orbifold in the sense of Definition 2.4.1. This correspondence is natural in a sense which we will not try to make precise (because we have not introduced a 2-category structure on Satake orbifolds).

3.2.1. Theorem. *There exists a natural construction which assigns to a Satake orbifold X an Sp-orbifold $[X]$, such that an orbifold chart (V, H, π) of X gives an open embeddings of orbifolds*

$$\hat{\pi} : [H\backslash V] \longrightarrow [X].$$

In particular, the coarse space of the orbifold $[X]$ is homeomorphic to the underlying space of X.

In a certain sense, this result is a converse to Proposition 3.1.11. When X is an effective C^∞ Satake orbifold, it follows from Theorem 4.1 of [41].

We will use Satake orbifolds in the study of quotients of Teichmüller spaces in Section 6. The construction of a complex orbifold from an orbifold atlas of will be used in Section 8 to obtain a map from the augmented Teichmüller space $\overline{\mathfrak{T}}_{g,n}$ to the stack of admissible coverings $\mathfrak{Adm}_{g,n,d}$.

The proof of Theorem 3.2.1 occupies the rest of this section. The idea is very simple. If a manifold X is covered by open subsets $U_\alpha,\ \alpha \in A$, then under some natural assumptions on A, X can be described as the direct limit of the collection U_α. In our situation the realization $[X]$ of a Satake orbifold X will be defined as a (2-) colimit of its orbifold charts. The most difficult part of this project consists of proving that the resulting stack is an orbifold.

3.2.2. Direct limit of stacks. Recall the notion of direct limit of Sp-stacks. Since the stacks form a 2-category, it makes more sense to talk about weak functors into the (2-) category of stacks. Let I be a category. A functor

$$F : I \longrightarrow \mathtt{Stacks/Sp}$$

is defined as a fibered category

$$\pi : \mathcal{F} \longrightarrow I^{\mathrm{op}} \times \mathsf{Sp}$$

such that for each $i \in I$ the fiber

$$\mathcal{F}_i \longrightarrow \mathsf{Sp}$$

is an Sp-stack.

Following [46, VI.6.3] we define $\varinjlim(F)$ as the localization of the total category $\mathcal{F}$ with respect to the morphisms of the type α^* where $\alpha \in \mathrm{Mor}(I)$. The resulting localization is still a category over Sp. As we show below,

$\varinjlim(F)$ is fibered over Sp; we denote the associated Sp-stack by $\mathcal{L}\!\varinjlim(F)$. The category $\mathcal{L}\!\varinjlim(F)$ can be described in terms of pseudofunctors as follows. The composition

$$\Pi = \mathrm{pr}_2 \circ \pi : \mathcal{F} \to \mathsf{Sp}$$

is a fibered category. Choosing a cleavage, we get a pseudofunctor $\mathsf{Sp}^{\mathrm{op}} \to \mathtt{Cat}$. Its composition with the total localization functor $\mathtt{Cat} \to \mathtt{Grp}$ gives a pseudofunctor $\mathsf{Sp}^{\mathrm{op}} \to \mathtt{Grp}$, i.e., a cleaved groupoid over Sp which is denoted by $\varinjlim(F)$.

In general, we have no reason to expect that the direct limit $\mathcal{L}\!\varinjlim(F)$ is an orbifold, even if the fibers $\mathcal{F}_i$ are all Sp-orbifolds. This happens, however, in some cases. Let, for example, $X \in \mathsf{Sp}$ and let a finite group G act on X. These data define an obvious functor

$$\widetilde{X} : BG \to \mathtt{Stacks}/\mathsf{Sp}$$

from the classifying groupoid BG of G to Sp and, therefore, to Sp-stacks. The direct limit $\varinjlim(\widetilde{X})$ is the functor from Sp to groupoids represented by the quotient groupoid $(G\backslash X)_\bullet$; the associated stack $\mathcal{L}\!\varinjlim(\widetilde{X})$ is $[G\backslash X]$.

3.2.3. Constructing orbifold from an atlas. Let X be a Hausdorff topological space and let $c : A \to \mathtt{Charts}/X$ be an orbifold atlas for X, where A is a chart category. The composition

$$\mathcal{V} : A \xrightarrow{c} \mathtt{Charts}/X \xrightarrow{\mathrm{pr}_1} \mathsf{Sp} \longrightarrow \mathtt{Stacks}/\mathsf{Sp} \tag{7}$$

assigns to $a \in A$ the Sp-stack represented by $V(a) \in \mathsf{Sp}$.

We define the *realization* $[X]$ as $\mathcal{L}\!\varinjlim(\mathcal{V})$. This is an Sp-stack which depends on X and on the choice of the atlas $c : A \to \mathtt{Charts}/X$ of X.

We will prove later that $[X]$ is essentially independent of the choice of the atlas.

Let $c : A \to \mathtt{Charts}/X$ be an atlas and let I be a finite subset of $\mathrm{Ob}(A)$. Define A_I as the full subcategory of A which consists of objects $a \in A$ satisfying the condition

$$\mathrm{Hom}(a, i) \neq \emptyset \text{ for each } i \in I.$$

Define X_I as the intersection of the images of the charts corresponding to the elements of I. For each $a \in A_I$ the chart $c(a)$ has its image in X_I. This gives a functor

$$c_I : A_I \to \mathtt{Charts}(X_I).$$

3.2.4. Lemma. *The pair (A_I, c_I) is an orbifold atlas of X_I.*

Proof. The only thing we have to check is that the images of the charts of A_I cover the whole X_I. This follows from property 3.1.9 (ii) of orbifold atlases by induction on the cardinality of I. $\square$

3.2.5. Definition. A collection of arrows $f_i : a_i \to b$ in A with the same target $b \in A$ is called *a covering* if the maps $V(f_i) : V(a_i) \to V(b)$ cover $V(b)$.

3.2.6. Lemma. *Let B be a subset of* $\mathrm{Ob}(A)$ *and let $a \in A$. Assume that*

$$\mathrm{Im}\ c(a) \subset \bigcup_{b\in B} \mathrm{Im}\ c(b).$$

Then the collection of maps $f : x \to a$ from elements x which can be mapped into an element of B is a covering.

Proof. Let $v \in V(a)$ and let $x = \pi(v)$. There exists $b \in B$ such that $x \in \mathrm{Im}\ c(b)$. Then by property 3.1.9 (ii) of orbifold atlases there exists $d \in A$, a pair of maps $\alpha : d \to a$ and $\beta : d \to b$, and $w \in V(d)$ such that $x = \pi(w)$. This implies that the elements v and $V(\alpha)(w)$ belong to the same $H(a)$-orbit. This means that by replacing α with its $H(a)$-conjugate, we can assure that $v = V(\alpha)(w)$. $\square$

Now, we are ready to prove that the realization does not depend on the choice of an atlas.

3.2.7. Proposition. *Let*

$$B \longrightarrow A \xrightarrow{c} \mathtt{Atlas}/X$$

be a morphism of orbifold atlases of X. Suppose, as above, that

$$\mathcal{V} : A \to \mathsf{Sp} \to \mathsf{Stacks}/\mathsf{Sp}$$

sends $a \in A$ to $V(a)$ and let $\mathcal{V}_B$ be the restriction of $\mathcal{V}$ to B.

Then the map of the realizations

$$\varinjlim{\mathcal{L}im}(\mathcal{V}_B) \longrightarrow \varinjlim{\mathcal{L}im}(\mathcal{V})$$

is an equivalence.

Proof. We define a subatlas $\bar{B}$ in A by the formula

$$\bar{B} = \{a \in A | \exists b \in B : \mathrm{Hom}(a, b) \neq \emptyset\}.$$

Since $\bar{B}$ contains the image of B in A, the morphism of atlases is the composition

$$B \to \bar{B} \to A \xrightarrow{c} \mathtt{Atlas}/X.$$

We will prove that the functors

$$B \to \bar{B} \text{ and } \bar{B} \to A$$

induce an equivalence of the corresponding direct limits.

For each $a \in \bar{B}$ choose an arrow $f : a \to b$ with $b \in B$. Given a compatible collection of maps $V(b) \to \mathcal{X}$ for $b \in B$, we will get a collection of maps $V(a) \to \mathcal{X}$ for $a \in \bar{B}$. To prove that it is automatically compatible, we will check that if $g : a \to b'$ is another arrow with $b' \in B$, the compositions

$$V(a) \to V(b) \to \mathcal{X} \text{ and } V(a) \to V(b') \to \mathcal{X}$$

are canonically isomorphic.

We claim that a is covered by the arrows $u : x \to a$ which can be placed in a commutative diagram (8), where $b'' \in B$.

$$\begin{tikzcd}
x \arrow[d, "u"'] \arrow[dr] & \\
a \arrow[d, "f"'] \arrow[dr, "g"] & b'' \arrow[dl] \arrow[d] \\
b & b'
\end{tikzcd} \tag{8}$$

Lemma 3.2.4, for $I = \{b, b'\}$, together with Lemma 3.2.6 applied to the atlas $\bar{B}_I$ and to the subset Ob B_I guarantee that a is covered by arrows $u : x \to a$, such that x can be mapped to an element $b'' \in B$ which, in turn, can be sent to b and to b'. Since A is a chart category, the arrows $b'' \to b$ and $b'' \to b'$ can be chosen so that the diagram (8) becomes commutative.

The equivalence between the compositions

$$V(x) \longrightarrow V(b'') \longrightarrow V(b) \longrightarrow \mathcal{X}$$

and

$$V(x) \longrightarrow V(b'') \longrightarrow V(b') \longrightarrow \mathcal{X}$$

is now immediate. Since the maps $u : x \to a$ cover a, this gives the required equivalence between the compositions

$$V(a) \to V(b) \to \mathcal{X} \quad \text{and} \quad V(a) \to V(b') \to \mathcal{X}\ .$$

Now let us prove a similar statement for the functor $\bar{B} \to A$. Any object $a \in A$ can be covered by objects of $\bar{B}$. Thus, for any stack $\mathcal{X}$ a map $V(a) \to \mathcal{X}$ is uniquely defined by a compatible collection of maps $\alpha_f : V(b_f) \to \mathcal{X}$ for each $f : b_f \to a$ with $b_f \in \bar{B}$. Given a compatible collection of maps $V(b) \to \mathcal{X}$ for $b \in \bar{B}$, the collection of maps $V(a) \to \mathcal{X}$ so defined will be automatically compatible by Lemma 3.2.6. This proves that the functor $\underrightarrow{\mathcal{L}im}\mathcal{V}_{\bar{B}} \to \underrightarrow{\mathcal{L}im}\mathcal{V}$ is an equivalence. $\square$

3.2.8. Corollary. *Assume that a Satake orbifold X admits a global chart, i.e. a chart (V, H, π) with surjective $\pi : V \to X$. Then the realization $[X]$ is naturally equivalent to $[H\backslash V]$.*

Proof. Let $c : A \to \mathtt{Charts}/X$ be an orbifold atlas of X and let $a \in A$ define a global chart $c(a) = (V, H, \pi)$. The embedding $BH \to A$ which sends the unique object of BH to a gives an embedding of orbifold atlases. The realization $\underrightarrow{\mathcal{L}im}\mathcal{V}_{BH}$ is precisely $[H\backslash V]$. $\square$

3.2.9. The induced atlas. Let $c : A \to \mathtt{Charts}/X$ be an orbifold atlas of X and let U be an open subset of X. The *induced orbifold atlas* of U is the functor

$$c_U : A \to \mathtt{Charts}/U$$

defined as the composition of $c : A \to \mathtt{Charts}/X$ with the restriction functor

$$\mathtt{Charts}/X \to \mathtt{Charts}/U$$

which sends a chart (V, H, π) to $(\pi^{-1}(U), H, \pi|_{\pi^{-1}(U)})$.

By definition, a canonical morphism of the realizations $[U] \to [X]$ is defined.

For example, if U is the image in X of a chart $c(a) = (V, H, \pi)$, then by 3.2.8 the realization $[U]$ is equivalent to $[H\backslash V]$.

Later we will need the following explicit description of fiber products in $\mathtt{Charts}/X$.

3.2.10. Lemma. *Let f_1 and f_2 be two morphisms of orbifold charts*

$$f_i : (V_i, H_i, \pi_i) \longrightarrow (V, H, \pi),\ \ i = 1, 2.$$

Define the triple $(V_{12}, H_{12}, \pi_{12})$ by the formulas

$$V_{12} = V_1 \times_V V_2,\ \ H_{12} = H_1 \times_H H_2,\ \ \pi_{12} = \pi_1 \circ \mathrm{pr}_1.$$

Then the projections

$$\mathrm{pr}_i : (V_{12}, H_{12}, \pi_{12}) \to (V_i, H_i, \pi_i),\ \ i = 1, 2,$$

are morphisms of orbifold charts.

Proof. The morphisms $f_i : V_i \to V$ are étale, therefore, the fiber product V_{12} exists in Sp and the projections $\mathrm{pr}_i : V_{12} \to V_i$ are étale. Thus we have to verify that the maps $[H_{12}\backslash V_{12}] \to [H_i\backslash V_i]$ are open embeddings. According to Remark 3.1.3.(i), we need to check two conditions. The first one, that the kernel of the map $\mathrm{pr}_1 : H_{12} \to H_1$ acts freely on V_{12}, immediately follows from the similar property of the map f_2. The second is that the map

$$\alpha : H_1 \times^{H_{12}} V_{12} \to V_1$$

is an open embedding. This map is étale since H_{12} acts freely on $H_1 \times V_{12}$ and pr_1 is étale. Thus, it is enough to check that α is injective.

Assume that we have

$$h_i \in H_1, \quad (x_i, y_i) \in V_{12} \text{ for } i = 1, 2$$

such that $h_1(x_1) = h_2(x_2)$. We need to find $(u, v) \in H_{12}$ such that

$$h_1 = h_2 u \text{ and } (x_2, y_2) = (u, v)(x_1, y_1).$$

Since we must set $u = h_2^{-1} h_1$, we need to show that there exists $v \in H_2$ satisfying the conditions

$$f_2(v) = f_1(h_2^{-1} h_1) \text{ and } y_2 = v y_1. \tag{9}$$

Applying f_1 to the equality $h_1(x_1) = h_2(x_2)$ one obtains

$$f_1(h_1)f_2(y_1) = f_1(h_1)f_1(x_1) = f_1(h_2)f_1(x_2) = f_1(h_2)f_2(y_2).$$

Since f_2 is a morphism of orbifold charts, the above equation implies the existence of $v \in H_2$ which satisfies

$$f_1(h_1) = f_1(h_2)f_2(v), \quad y_2 = vy_1.$$

This is equivalent to equation (9). □

3.2.11. Intersection of charts. Now we will describe an operation that assigns to every pair of objects in the chart category A an orbifold chart. For manifolds it corresponds to the usual operation of intersection of charts.

Proposition-Definition. Let $c : A \to \mathtt{Charts}/X$ be an orbifold atlas. There exists a natural operation that assigns to a pair of objects $a_1, a_2 \in A$ a chart $c(a_1 \cap a_2)$ together with morphisms

$$\mathrm{pr}_i : c(a_1 \cap a_2) \to c(a_i),\ i = 1, 2,$$

which satisfy the following universal property.

For each pair of morphisms

$$\alpha_1 : b \to a_1,\ \alpha_2 : b \to a_2$$

there exists a canonical morphism of charts

$$c(\alpha_1, \alpha_2) : c(b) \to c(a_1 \cap a_2)$$

such that

$$\mathrm{pr}_i \circ c(\alpha_1, \alpha_2) = V(\alpha_i),\ i = 1, 2.$$

We call the chart

$$c(a_1 \cap a_2) = (V(a_1 \cap a_2), H(a_1 \cap a_2), \pi(a_1 \cap a_2))$$

the *intersection* of a_1 and a_2.

This operation resembles a direct product operation, but it is not a direct product. We call it intersection because of the lack of a more appropriate term.

Proof. Let $c_i := c(a_i) = (V_i, H_i, \pi_i),\ i = 1, 2$ be two orbifold charts. We will construct a chart $(V_{12}, H_{12}, \pi_{12})$ together with a pair of maps

$$\mathrm{pr}_i : (V_{12}, H_{12}, \pi_{12}) \longrightarrow (V_i, H_i, \pi_i)$$

satisfying the universal property.

Consider an open subset $U = \pi_1(V_1) \cap \pi_2(V_2)$ of X. We can view it as a Satake orbifold with the induced atlas of orbifold charts, see 3.2.9. The charts (U_i, H_i, π_i), where $U_i = \pi_i^{-1}(U)$, have the same image U in X. Therefore, by

Corollary 3.2.8, the maps $[H_1\backslash U_1] \to [U]$ and $[H_2\backslash U_2] \to [U]$ are equivalences. Consider the 2-fiber product

$$V_{12} = U_1 \times_{[U]} U_2,$$

where the maps $U_i \to [U]$ are defined as the compositions

$$U_i \to [H_i\backslash U_i] \longrightarrow [U].$$

Since V_{12} is a representable Sp-orbifold, we may assume that $V_{12} \in \mathsf{Sp}$.

The group H_2 acts freely on V_{12} with the quotient U_1. Similarly, the group H_1 acts freely on V_{12} with the quotient U_2. These actions commute and define an action of $H_{12} = H_1 \times H_2$ on V_{12}. Thus we have constructed an orbifold chart $(V_{12}, H_{12}, \pi_{12})$ with π_{12} being the composition of the projection to U_1 and π_1. We claim this is the chart we need.

Let $c = (W, H, \rho)$ be a chart. A map from c to $c(a_1 \cap a_2)$ is given by a pair of maps $W \to V_{12}$ and $H \to H_1 \times H_2$. It is uniquely defined by a triple (f_1, f_2, θ) where

$$f_i : c \to (U_i, H_i, \pi_i)$$

are morphisms of charts and $\theta : \psi_1 \to \psi_2$ is an isomorphism between the two compositions

$$\psi_i : W \longrightarrow U_i \longrightarrow [H_i\backslash U_i] \longrightarrow [U], \quad i = 1, 2. \tag{10}$$

Let $\alpha_i : b \to a_i, \quad i = 1, 2$, be two arrows in A. We have a pair of morphisms

$$c(\alpha_i) : c(b) = (W, H, \rho) \to (V_i, H_i, \pi_i).$$

Since $\rho(W) \subseteq U = \pi_1(V_1) \cap \pi_2(V_2)$, the morphisms $c(\alpha_i)$ factor through (U_i, H_i, π_i). By definition of realization, each of the compositions

$$\psi_1 : W \longrightarrow U_1 \longrightarrow [H_1\backslash U_1] \longrightarrow [U] \tag{11}$$

and

$$\psi_2 : W \longrightarrow U_2 \longrightarrow [H_2\backslash U_2] \longrightarrow [U] \tag{12}$$

is canonically isomorphic to the composition

$$W \longrightarrow [\rho(W)] \longrightarrow [U].$$

Therefore, one has a canonical choice of isomorphism between (11) and (12), so that a map $c(b) \to c(a_1 \cap a_2)$ is defined. $\square$

Now we are ready to prove that the realization $[X]$ of a Satake orbifold is an orbifold in the sense of Definition 2.4.1. This will be the last step in the proof of Theorem 3.2.1.

3.2.12. Theorem. *Let X be a Satake orbifold. Then its realization $[X]$ is an orbifold.*

Proof. Let $c : A \to \mathtt{Charts}/X$ be an orbifold atlas of X, $\mathcal{V} : A \to \mathsf{Sp}$ be the obvious functor assigning $V(a)$ to $a \in A$. We wish to present the stack $\underrightarrow{\mathcal{L}im}(\mathcal{V})$ by an Sp-groupoid. The problem here is in the fact that the definition of $\underrightarrow{\mathrm{Lim}}(\mathcal{V})$ includes localization of the total category which may destroy representability.

Fortunately, the intersection operation 3.2.11 allows one to present the localization in a very explicit way.

As it was done in 3.2.2, we will interpret the functor

$$\mathcal{V} : A \longrightarrow \mathsf{Sp},\ a \mapsto V(a)$$

as a category $\mathcal{X}$ fibered over $A^{\mathrm{op}} \times \mathsf{Sp}$. The fibers $\mathcal{X}_{a,M}$ at $(a, M) \in A^{\mathrm{op}} \times \mathsf{Sp}$ are discrete; one has $\mathcal{X}_{a,M} = \mathrm{Hom}(M, V(a))$ for connected $M \in \mathsf{Sp}$.

The category $\mathcal{X}$ considered as a fibered category over Sp comes from a category *in* Sp (which we denote by the same letter) defined as follows

- The objects of $\mathcal{X}$ is $\coprod_{a\in A} V(a)$.
- The morphisms of $\mathcal{X}$ is $\coprod_{\alpha\in \mathrm{Mor}(A)} V(s(\alpha))$
- The map $s\colon \mathrm{Mor}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{X})$ restricted to the α-component is $\mathrm{id}_{V(s(\alpha))}$.
- The map $t : \mathrm{Mor}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{X})$ restricted to the α-component is $V(\alpha)$.

Here, as before, $s\alpha$ and $t\alpha$ denote the source and the target of an arrow α.

Now we will present an étale groupoid $\mathcal{Y}$ in Sp such that the corresponding fibered category over Sp is obtained from $\mathcal{X}$ by the full localization of the fibers. Thus $\mathcal{Y}$ will represent $\underrightarrow{\mathcal{L}im}(\mathcal{V})$.

Define the groupoid $\mathcal{Y}$ as follows.

- $\mathrm{Ob}(\mathcal{Y}) = \mathrm{Ob}(\mathcal{X})$.
- $\mathrm{Mor}(\mathcal{Y}) = \coprod_{a_1,a_2\in A} V(a_1 \cap a_2)$ (we are using here the notation of 3.2.11).
- The maps $s, t : \mathrm{Mor}(\mathcal{Y}) \to \mathrm{Ob}(\mathcal{Y})$ are just the projections $\mathrm{pr}_1,\ \mathrm{pr}_2$ from $V(a_1 \cap a_2)$ to $V(a_1)$ and to $V(a_2)$.

The structure maps s, t are étale. The composition in $\mathcal{Y}$ is given by the canonical projections

$$c(a_1 \cap a_2) \times_{c(a_2)} c(a_2 \cap a_3) \longrightarrow c(a_1 \cap a_3),\ a_1, a_2, a_3 \in A$$

defined as follows.

Consider the chart

$$c = (V, H, \pi) = c(a_1 \cap a_2) \times_{c(a_2)} c(a_2 \cap a_3).$$

A map from c to $c(a_1 \cap a_3)$ is uniquely determined by maps $c \to c(a_i),\ i = 1, 2,$ and an isomorphism between the two maps from V to $[\pi_1(V_1) \cap \pi_3(V_3)]$. The maps $c \to c(a_i),\ i = 1, 2,$ are defined by

$$\mathrm{pr}_1 : c(a_1 \cap a_2) \to c(a_1) \text{ and } \mathrm{pr}_2 : c(a_2 \cap a_3) \to c(a_3).$$

To get an isomorphism of the two maps from V we notice that all the realizations involved, $[\pi_i(V_i)]$ and their double and triple intersections, are global quotient orbifolds by 3.2.8 and their inclusions are open embeddings of orbifolds. Since $\pi(V)$ belongs to the triple intersection $\pi_1(V_1)\cap\pi_2(V_2)\cap\pi_3(V_3)$, the isomorphisms between the two maps from V to $[\pi_1(V_1)\cap\pi_2(V_2)]$ and between the two maps from V to $[\pi_2(V_2)\cap\pi_3(V_3)]$ induced by the maps $c\to c(a_1\cap a_2)$ and $c\to c(a_2\cap a_3)$ can be realized as isomorphisms between two pairs of maps from V to $[\pi_1(V_1)\cap\pi_2(V_2)\cap\pi_3(V_3)]$. Their composition, composed with the open embedding of $[\pi_1(V_1)\cap\pi_2(V_2)\cap\pi_3(V_3)]$ into $[\pi_1(V_1)\cap\pi_3(V_3)]$, yields the required datum.

The canonical map $\iota:\mathcal{X}\to\mathcal{Y}$ of Sp-categories is defined as follows. It is identical on the objects. For any morphism $\alpha:a\to b$ in A a canonical map $\iota_\alpha:c(a)\to c(a\cap b)$ corresponds to the pair (id_a,α). This induces a map $V_{\iota_\alpha}:V(a)\to V(a\cap b)$ which assembles into the map $\iota:\mathrm{Mor}(\mathcal{X})\to\mathrm{Mor}(\mathcal{Y})$.

We claim that for connected $M\in\mathsf{Sp}$ the groupoid $\mathcal{Y}(M)$ is the (full) localization of the category $\mathcal{X}(M)$.

One has

$$\mathrm{Ob}\,\mathcal{X}(M)=\mathrm{Ob}\,\mathcal{Y}(M)=\coprod_{a\in A}\mathrm{Hom}(M,V(a))=\{(a,f)|a\in A,f:M\to V(a)\}.$$

Furthermore,

$$\mathrm{Mor}\,\mathcal{X}(M)=\coprod_{\alpha\in\mathrm{Mor}(A)}\mathrm{Hom}(M,V(s\alpha))=\{(\alpha,f)|\alpha\in\mathrm{Mor}(A),f:M\to V(s\alpha)\},$$

where

$$s(\alpha,f)=(s\alpha,f),\ \text{and}\ t(\alpha,f)=(t\alpha,V(\alpha)\circ f:M\to V(s\alpha)\to V(t\alpha)).$$

Similarly,

$$\mathrm{Mor}\,\mathcal{Y}(M)=\coprod_{a,b\in A}\mathrm{Hom}(M,V(a\cap b))=\{(a,b,f)|a,b\in A,f:M\to V(a\cap b)\},$$

where

$$s(a,b,f)=(a,V(\mathrm{pr}_1)\circ f),\ \text{and}\ t(a,b,f)=(b,V(\mathrm{pr}_2)\circ f).\tag{13}$$

In the formula (13) the maps $\mathrm{pr}_1:c(a\cap b)\to c(a)$ and $\mathrm{pr}_2:c(a\cap b)\to c(b)$ are the standard projections.

The functor $\iota:\mathcal{X}(M)\to\mathcal{Y}(M)$ assigns to an arrow (α,f) in $\mathcal{X}(M)$ the arrow $(s\alpha,t\alpha,V(\mathrm{id},\alpha)\circ f)$ where $c(\mathrm{id},\alpha):c(a)\to c(a\cap b)$ is defined by the maps

$$\mathrm{id}:a\to a,\ \text{and}\ \alpha:a\to b$$

in A. A direct calculation shows that

$$(a,b,f)\circ\iota(\mathrm{pr}_1,f)=\iota(\mathrm{pr}_2,f)\tag{14}$$

where, as above, $\mathrm{pr}_1:c(a\cap b)\to c(a),\ \mathrm{pr}_2:c(a\cap b)\to c(b)$ are the standard projections.

Let G be a groupoid and let $F:\mathcal{X}(M)\to G$ be a functor. We claim there exists a unique $\bar{F}:\mathcal{Y}(M)\to G$ such that $F=\bar{F}\circ\iota$. On objects $\bar{F}$ must coincide with F, since $\mathrm{Ob}(\mathcal{X}(M)) = \mathrm{Ob}(\mathcal{Y}(M))$. From (14) it follows how $\bar{F}$ should act on morphisms:

$$\bar{F}(a, b, f) = F(\mathrm{pr}_2, f) \circ F(\mathrm{pr}_1, f)^{-1}. \tag{15}$$

Thus $\bar{F}$ is unique. To prove its existence, we have to check that $\bar{F}$ defined by (15) commutes with compositions. This is a straightforward calculation.

This completes the proof of Theorem 3.2.12. □

Now we can finish the proof of Theorem 3.2.1. Since $[X]$ is represented by the groupoid $\mathcal{Y}$, with

$$\mathrm{Ob}(\mathcal{Y}) = \coprod_{a \in A} V(a),$$

for every $a \in A$ we have an open suborbifold U_a represented by the Sp-groupoid $G_\bullet(a) = (G_0(a), G_1(a))$ with

$$G_0(a) = V(a) \text{ and } G_1(a) = V(a \cap a).$$

Since the set of arrows G_1 can be identified with

$$V(a) \times_{[H_a \backslash V_a]} V(a) = H(a) \times V(a),$$

we see that the open suborbifold can be identified with the global quotient $[H(a) \backslash V(a)]$. □

Note the following important corollaries of this theorem.

3.2.13. Corollary. *Let X be a Satake orbifold and let $[X]$ be its realization. Let $\mathcal{Y}$ be an arbitrary orbifold. Then a map $f : [X] \to \mathcal{Y}$ of orbifolds is determined by the following data:*

- *For each $a \in A$ a map*

$$f_a : [H(a) \backslash V(a)] \to \mathcal{Y}$$

  (where, as usual, $(V(a), H(a), \pi(a)) = c(a)$ is the chart corresponding to a).
- *For each morphism $\alpha : a \to b$ in A a 2-morphism*

$$\theta_\alpha : f_a \to f_b \circ [c(\alpha)]$$

  where for a morphism ϕ of orbifold charts we denote by $[\phi]$ the corresponding map of quotient orbifolds.

These data are required to satisfy obvious compatibility condition for θ_α.

3.2.14. Proposition. *The categories of sheaves (or categories of vector bundles) on a Satake orbifold X and its realization $[X]$ are canonically equivalent.*

Proof. This result follows from our construction of the realization of a Satake orbifold. □

4. Algebraic moduli versus analytic moduli

4.1. Two ways of passing from algebraic to analytic families

The two ways of looking at orbifolds discussed in Sections 2 and 3—as groupoid-valued functors on a certain category of spaces (manifolds) and as geometric objects represented by groupoids in the category of spaces—suggest two possible ways of passing from one category of manifolds to another. We are particularly interested in the passage from the category of schemes (of finite type over $\mathbb{C}$) to the category of analytic spaces.

The first way of passing from schemes to complex spaces is to replace a functor on the category of schemes with a functor on complex spaces. For example, the functor of families of stable curves over schemes becomes the functor of families of stable Riemann surfaces.[11]

The second way is the change of the base category mentioned in 2.6.3. That is for a Deligne-Mumford stack $\mathcal{X}$ represented by a groupoid $X_\bullet$ we can apply the analytification functor which produces a groupoid $X_\bullet^{an}$ representing an orbifold in the analytic category.

Of course, since the first procedure is not even formally defined, we cannot expect that these two processes always give the same result.

However, as we show in this section, for the moduli spaces of stable punctured curves and of admissible coverings the two procedures are equivalent.

Let $\overline{\mathfrak{M}}_{g,n}$ be the stack of stable complex curves of genus g with n punctures and let $\mathfrak{Adm}_{g,n,d}$ (resp. $\mathfrak{Adm}_{g,n}(H)$) be the stack of admissible coverings of degree d (resp. of admissible H-coverings).

These stacks are proper Deligne-Mumford stacks and, therefore, Sp-orbifolds where Sp is the category of schemes.

In this section we prove the following result.

4.1.1. Theorem. *The analytification of the stack $\overline{\mathfrak{M}}_{g,n}$ (resp., of $\mathfrak{Adm}_{g,n,d}$, resp., of $\mathfrak{Adm}_{g,n}(H)$) represents the functor of analytic families of stable curves of genus g with n punctures (resp., of admissible coverings of degree d of curves of genus g unramified outside of n points, resp., of H-admissible coverings).*

Here is a plan of the proof. First, following M. Hakim [22], we define algebraic families whose base is an arbitrary locally ringed topological space. The analytifications of the algebraic moduli stacks automatically represent the corresponding algebraic families with bases in the category of complex-analytic spaces. The rest follows from the fact proved in Section 4.3 that any complex-analytic family of stable complex curves is necessarily projective (and therefore algebraic). This is a generalization of the well-known fact that every compact complex manifold of dimension one is projective. Thus our proof does not work for families of algebraic varieties of dimension higher than one.

[11] This procedure is not defined for an arbitrary Deligne-Mumford stack.

4.2. Analytic families of algebraic curves

When we speak about families of varieties parametrized by an analytic space, we usually mean an analytic family of the corresponding complex analytic spaces. Sometimes it is important to have both “analytic” and “algebraic” directions. This can be done using the notion of a family of schemes parametrized by a ringed topological space introduced by M. Hakim [22] (in a much greater generality).

We present below a definition of a family of objects of a stack $\mathcal{X}$ parametrized by a locally ringed topological space. For the stack $\mathcal{X} = \overline{\mathfrak{M}}_{g,n}$ this gives the notion of an analytic family of algebraic curves. Similarly, for $\mathcal{X} = \mathfrak{Adm}_{g,n,d}$ we get the notion of an analytic family of algebraic admissible coverings.

4.2.1. Definition. Let $(X, \mathcal{O})$ be a locally ringed site and let $\mathcal{X}$ be a stack of groupoids on the category of affine schemes. Let $\mathtt{Pre}\text{-}\mathcal{X}(X, \mathcal{O})$ be the fibered category over X whose fiber over $U \in X$ is the groupoid

$$\mathtt{Pre}-\mathcal{X}(X, \mathcal{O})_U = \mathcal{X}(\mathcal{O}(U)).$$

Denote by $\mathcal{X}(X, \mathcal{O})$ the groupoid of global sections of the stack associated to the fibered category $\mathtt{Pre}\text{-}\mathcal{X}(X, \mathcal{O})$. Objects of $\mathcal{X}(X, \mathcal{O})$ are called *families of objects of $\mathcal{X}$ parametrized by $(X, \mathcal{O}_X)$.*

For $\mathcal{X} = \mathtt{Sch}$ this gives to the notion of a scheme over $(X, \mathcal{O})$ (see [22]); for $\mathcal{X} = \overline{\mathfrak{M}}_{g,n}$ we get the notion of an analytic family of stable algebraic curves, and $\mathcal{X} = \mathfrak{Adm}_{g,n,d}$ we obtain the notion of an analytic family of algebraic admissible coverings.

Thus, by definition, *a scheme over* $(X, \mathcal{O})$ is given by a collection of the following data.

- An open covering $\{U_i\}$ of X;
- A collection of schemes Y_i over Spec $\mathcal{O}(U_i)$;
- A compatible collection of isomorphisms of the pullbacks of Y_i and of Y_j to Spec $\mathcal{O}(U_{ij})$.

A similar description can be given for $\mathcal{X} = \overline{\mathfrak{M}}_{g,n}$ or $\mathfrak{Adm}_{g,n,d}$.

4.2.2. Proposition. *Let $\mathcal{X}$ be an algebraic Deligne-Mumford stack of finite type over $\mathbb{C}$. Then the functor $(X, \mathcal{O}) \mapsto \mathcal{X}(X, \mathcal{O})$ from the category of analytic spaces to the category of groupoids is representable by the analytification of $\mathcal{X}$.*

Proof. The analytification $\mathcal{X}^{an}$ is defined as follows. Let $\mathcal{X}$ be presented by a groupoid $X_\bullet$ where X_i, $i = 0, 1$ are schemes of finite type over $\mathbb{C}$. Then $\mathcal{X}^{an}$ is defined as the stack associated to the groupoid $X_\bullet^{an}$.

The statement of the proposition follows immediately from the following facts.

- A map $\mathrm{Hom}_{\mathtt{LR}}((X, \mathcal{O}), \mathrm{Spec} A) \to \mathrm{Hom}_{\mathtt{COM}}(A, \Gamma(X, \mathcal{O}))$ is a bijection. Here the left-hand side Hom is taken in the category of locally ringed spaces and the right-hand side Hom in the category of commutative rings.

- A map $X \to M$ from an analytic space X to a scheme of locally finite type over $\mathbb{C}$ in the category of locally ringed spaces lifts canonically to a map $X \to M^{an}$ of analytic spaces.

$\square$

Thus, according to Proposition 4.2.2, the complex-analytic stack $\overline{\mathfrak{M}}_{g,n}^{an}$ represents (algebraic) families of stable curves of genus g with n punctures parametrized by complex-analytic spaces. A similar claim is true for $\mathfrak{Adm}_{g,n,d}^{an}$ and $\mathfrak{Adm}_{g,n}(H)^{an}$.

4.3. Analytic families of analytic curves

Here we prove that any analytic family of stable curves (or of stable admissible coverings or of stable admissible H-coverings) is algebraic, i.e. it can be obtained as the analytification of an algebraic family. Together with Proposition 4.2.2 this will give Theorem 4.1.1.

4.3.1. Theorem. *Any analytic family* $(\pi : X \to S, \sigma_1, \ldots, \sigma_n)$ *of stable punctured curves is projective. In particular, it is an analytification of an algebraic family over* S.

Proof. Let ω_π be the relative dualizing sheaf of π. In the analytic category it was defined in [43] as $\pi^!(\mathcal{O}_S)$ where the functor

$$\pi^! = D_X \circ \pi^* \circ D_S$$

is obtained from the inverse image functor by dualization. The morphism π is a locally complete intersection morphism, therefore it follows (see e.g. [23]) that ω_π is an invertible sheaf. It satisfies the base change formula[12] $\omega_{\pi_T} = g^*(\omega_\pi)$ for a Cartesian diagram

$$\begin{tikzcd} X_T \arrow[r, "g"] \arrow[d, "\pi_T"'] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd} \quad . \tag{16}$$

Let $D_i = \sigma_i(S)$ be the divisor in X that corresponds to the ith marked point. We claim that the invertible sheaf

$$L = \left(\omega_\pi \otimes \mathcal{O}_X(-\sum_{i=1}^{n} D_i)\right)^{\otimes 3}$$

[12] A detailed proof of the base change formula for Cohen-Macaulay morphisms of locally Noetherian schemes is given in in [14, Theorem 3.6.1] It is based on a local description of ω_π in terms of Ext functors and on the base change for Ext functors. For local complete intersections it is given by the compatibility lemma [14, Lemma 2.6.2] whose proof remains valid in the analytic setting as well.

gives rise to a finite morphism

$$j : X \to \mathbb{P}((\pi_*(L))^*).$$

Indeed, since the restriction of L to every fiber $X_s,\ s \in S$, is very ample, the map j is well-defined and its restriction to X_s is a closed embedding. Since π is proper, j is also proper and thus is finite. This implies that π is projective.

Let X^{alg} be the scheme over S whose analytification is isomorphic to X. According to the "relative GAGA" (see Theorem VIII.3.5 in [22]) the categories of coherent sheaves on X and on X^{alg} are equivalent.

This implies that the sections $\sigma_i : S \to X$ are algebraic and also that any analytic automorphism of X comes from an automorphism of X^{alg}. This completes the proof. □

Now we can easily obtain a similar result for families of stable admissible coverings.

4.3.2. Theorem. *Any analytic family of stable admissible coverings*

$$(C \to X \to S, \sigma_1, \dots, \sigma_n)$$

is projective. The same is true for families of admissible H-coverings.

Proof. The family $(X \to S, \sigma_1, \dots, \sigma_n)$ is analytification of an algebraic family

$$(X^{alg} \longrightarrow S, \sigma_1, \dots, \sigma_n)$$

by Theorem 4.3.1. Since the covering C of X is given by a coherent sheaf of algebras, the result follows by the "relative GAGA" [22, Theorem VIII.3.5].

Finally, to deal with the case of H-coverings, we notice that the balancedness condition in the definition of admissible H-covering (see Section 4.3.1 of [4]) only involves geometric points. Therefore this condition is the same for analytic and algebraic version. □

5. Teichmüller spaces and quasiconformal charts of $\overline{\mathfrak{M}}$

In the beginning of this section we introduce the Teichmüller spaces $\mathfrak{T}_{g,n}$ and $\overline{\mathfrak{T}}_{g,n}$ and present some facts about them which will be needed later. Then we construct on $\overline{\mathfrak{M}} = [\Gamma_{g,n} \backslash \overline{\mathfrak{T}}_{g,n}]$ an orbifold atlas whose charts satisfy some very special properties. We call such charts *quasiconformal.* Our construction uses a version of the Earle-Marden [37] local holomorphic coordinates on the Teichmüller space $\mathfrak{T}_{g,n}$. In Section 6 we will use this quasiconformal atlas to construct an orbifold atlas on quotient of the augmented Teichmüller space $\overline{\mathfrak{T}}$ by finite-index subgroups of the modular group Γ.

5.1. Teichmüller spaces $\mathcal{T}_{g,n}$ and $\overline{\mathcal{T}}_{g,n}$

Here we recall definitions and standard facts about the Teichmüller spaces $\mathcal{T}_{g,n}$ and $\overline{\mathcal{T}}_{g,n}$.

Let us fix a compact oriented surface S of genus g with n boundary components $L_1, \dots, L_n$ and smooth parametrizations

$$\lambda_i : S^1 \to L_i$$

compatible with the orientation of S. Here and below, we assume that the surface S is of hyperbolic type, i.e. $2g + n - 2 > 0$.

5.1.1. Definition. Let $(X, p_1, \dots, p_n)$ be a stable complex curve X of arithmetic genus g with n punctures $p_i \in X$. A *marking* of the punctured curve $(X, p_1, \dots, p_n)$ is a continuous map

$$\phi : S \to X$$

satisfying the following properties

(i) The preimage $\phi^{-1}(p_i)$ of the ith puncture $p_i \in X$ is the ith boundary component $L_i \subset S$.
(ii) The preimage $\phi^{-1}(q)$ of every node $q \in X$ is a simple closed curve in S.
(iii) The map ϕ induces a homeomorphism

$$\phi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}},$$

where

$$X_{\mathrm{reg}} = X - X_{\mathrm{sing}} - \{p_1, \dots, p_n\},$$

is the complement of the sets of nodes and punctures of X.

Two markings $\phi, \phi' : S \to X$ are called *isotopic* if $\phi' = \phi \circ f$, where f is a diffeomorphism of S, such that

$$f_{L_i} = \mathrm{Id}_{L_i}, \ i = 1, \dots, n, \tag{17}$$

and f is isotopic to the identity in the class of diffeomorphisms satisfying (17).

5.1.2. Definition. A punctured stable curve X with an isotopy class of markings $[\phi]$ is called a *marked curve.*

The set $\overline{\mathcal{T}}_{g,n}$ of isomorphism classes of marked curves of genus g with n punctures is called the *augmented Teichmüller space.*

Remark. Sometimes, when we wish to stress the functorial dependence of the augmented Teichmüller space on S, we will use the notation $\overline{\mathcal{T}}(S)$ instead of $\overline{\mathcal{T}}_{g,n}$. Of course, $\overline{\mathcal{T}}(S)$ depends, up to non-canonical isomorphism, only on the genus of S and on the number of its boundary components.

The points $(X, [\phi])$ of $\overline{\mathcal{T}}_{g,n}$, where X is a non-singular complex curve, form the usual Teichmüller space $\mathcal{T}_{g,n}$.

In order to introduce a topology on $\overline{\mathcal{T}}_{g,n}$ we need the following notion.

5.1.3. Definition. Let $(X, x_1, \dots, x_n, \phi)$ and $(Y, y_1, \dots, y_n, \psi)$ be two marked stable curves. A continuous map $f : X \to Y$ is called a *contraction* if it satisfies the following conditions.

(i) $f(x_i) = y_i$ for $i = 1, \dots, n$.
(ii) f induces a homeomorphism $f^{-1}(Y_{\text{reg}}) \to Y_{\text{reg}}$.
(iii) For every node $y \in Y$ its preimage $f^{-1}(y)$ is either a node of X or a simple closed loop.
(iv) The marking ψ of Y is isotopic to $f \circ \phi$.

For unmarked punctured curves $(X, x_1, \dots, x_n)$ and $(Y, y_1, \dots, y_n)$ a contraction is defined as any continuous map $f : X \to Y$ satisfying conditions (i)—(iii).

The following sets form a basis of the topology of the augmented Teichmüller space. Choose a marked curve

$$(Y, y_1, \dots, y_n, [\psi]) \in \overline{\mathcal{T}}_{g,n},$$

a number $\varepsilon > 0$ and an open subset N of Y containing all the nodes of Y. The neighborhood $\mathcal{U}_{N,\varepsilon} \subset \overline{\mathcal{T}}_{g,n}$ is defined as the set of all

$$(X, x_1, \dots, x_n, [\phi]) \in \overline{\mathcal{T}}_{g,n}$$

for which there exists a contraction $f : X \to Y$ such that the restriction of f to $f^{-1}(Y - \bar{N})$ is $(1+\varepsilon)$-quasiconformal.

5.1.4. Modular group action. Let

$$\Gamma_{g,n} = \pi_0(\mathcal{D}\mathit{iff}^+(S/\partial S))$$

be the Teichmüller *modular group*, i.e. the group of isotopy classes of orientation preserving diffeomorphisms of S identical on the boundary ∂S. (This group is also known as the mapping class group of the n-punctured surface of genus g, cf. [29]). We will usually denote this group by $\Gamma(S)$ or simply by Γ. The modular group Γ naturally acts on $\overline{\mathcal{T}}_{g,n}$ and on $\mathcal{T}_{g,n}$ as follows:

$$[\gamma](X, [\phi]) := (X, [\phi \circ \gamma^{-1}]), \tag{18}$$

where $[\gamma] \in \Gamma$ is a mapping class represented by a diffeomorphism γ and $\phi : S \to X$ is a marking of X.

This action allows the following description of markings of a nodal curve X_0 in terms of markings of nearby smooth curves. Let X be a smooth curve and let X_0 be a nodal curve.

Assume there is a contraction of X to X_0 that contracts several disjoint simple closed curves $C_1, \dots, C_r$ on X.

5.1.5. Proposition. *There is a natural bijection between the set of isotopy classes of markings of the nodal curve X_0 and the set of G-orbits in the set of isotopy classes of markings of X, where G is a subgroup of Γ generated by the Dehn twists around the curves $C_1, \dots, C_r$.*

□

We will use the following classical results about the Teichmüller spaces $\mathcal{T}_{g,n}$ and $\overline{\mathcal{T}}_{g,n}$ and the action of the modular group on them (for details see [3] and references there).

5.1.6. Theorem. (i) *The space $\mathcal{T}_{g,n}$ has a structure of a complex manifold of complex dimension $3g+n-3$ diffeomorphic to an open ball in $\mathbb{R}^{6g+2n-6}$.*
(ii) *The quotient $\Gamma\backslash\mathcal{T}_{g,n}$ is isomorphic, as a complex space, to $\mathcal{M}_{g,n}$.*
(iii) *The quotient space $\Gamma\backslash\overline{\mathcal{T}}_{g,n}$ is homeomorphic to $\overline{\mathcal{M}}_{g,n}$.*

Here and below $\mathcal{M}_{g,n}$ (resp., $\overline{\mathcal{M}}_{g,n}$) denotes the complex space associated to the moduli stack of compact Riemann surfaces of genus g with n marked points (resp., its Deligne-Mumford compactification).

5.2. Complex structure of $\mathcal{T}_{g,n}$

We present below a modular description of the complex space $\mathcal{T}_{g,n}$ (see [21, 17, 18]).

5.2.1. Definition. Let B be a complex space. A *family of smooth curves* of genus g with n punctures over the base B is a flat proper morphism of complex spaces $\pi : C \to B$ with n sections $\sigma_i : B \to C$, such that fibers of π are complex curves of genus g and the images of the sections $\sigma_i,\ i = 1, \dots, n$, are pairwise disjoint.

To each $b \in B$ we assign a set

$$P_b = \pi_0(\mathcal{D}\mathit{iff}^+(S, C_b)),$$

where $C_b = \pi^{-1}(b)$. Since π is topologically a locally trivial fibration, these sets assemble into a covering

$$p : P \longrightarrow B$$

with fibers P_b.

5.2.2. Definition. A *marking* of a family of smooth curves $\pi : C \to B$ is a section of the associated covering

$$p : P(\pi) \longrightarrow B.$$

If $G \subset \Gamma$ is a subgroup of the modular group, a section of the covering

$$p_G : G\backslash P(\pi) \longrightarrow B$$

is called a *G-marking* of the family π.

The following result proved in [17, 18] generalizes the theorem of Grothendieck on modular description of the Teichmüller space $\mathcal{T}_g = \mathcal{T}_{g,0}$.

5.2.3. Theorem. *For $2g + n > 2$, the functor*

$$B \mapsto F(B),$$

where $F(B)$ is the set of isomorphism classes of marked curves of genus g with n punctures over B, is representable by a complex manifold. The representing object is isomorphic to the Teichmüller space $\mathcal{T}_{g,n}$.

5.3. Quasiconformal atlas for $\overline{\mathfrak{M}}$

In this section we prove the existence of an atlas on $\overline{\mathcal{M}}$ with especially nice orbifold charts. These charts, which we call *quasiconformal*, satisfy a collection of properties described in 5.3.1. Our approach is based on the plumbing construction of Earle and Marden [37]. This construction produces a family of stable curves over a polydisk starting with a maximally degenerate curve X_0 and a collection of local coordinates near the nodes of X_0. This family of curves is not everywhere locally universal, i.e. it does not necessarily give an orbifold chart for the moduli space $\overline{\mathcal{M}}$ (see a counterexample in [25]).

However, as we show in this section, open subsets of those coordinate polydisks which do form an orbifold chart cover the whole moduli space and therefore give an orbifold atlas with required properties.

To prove that the charts obtained from the plumbing construction cover the whole moduli space, we proceed as follows. First, for each stable curve X we describe very special plumbing data $(X_0,\ z_i)$, where X_0 is a maximally degenerate stable curve of genus g with n punctures (all such curves have $m = 3g + n - 3$ nodes) and $z_1, \dots, z_{2m}$ are local parameters near the nodes of X_0.

This data gives rise to a family of curves

$$\pi : \mathfrak{X} \longrightarrow U \tag{19}$$

whose base U is an neighborhood of the origin in $\mathbb{C}^{3g+n-3}$. This family, which we construct in 5.3.4, contains X and has the property that the geodesics (in the hyperbolic metric) which cut X into a union of "pairs of pants" in local coordinates z_i have equations $|z_i| = s_i$.

The family (19) is induced from the universal family over the moduli stack $\overline{\mathfrak{M}}$ via a map $U \to \overline{\mathfrak{M}}$ which gives rise to an orbifold chart

$$\hat{\beta} : [A\backslash U] \to \overline{\mathfrak{M}}.$$

To prove étalness of $\hat{\beta}$ we first show in 5.3.5 that, when X is non-singular, the restriction of the family (22) to a certain subspace of U of *real* dimension m is the Fenchel-Nielsen family (see Section 5.3.6). This gives étalness in the non-singular case and in 5.3.7 we deduce from it the general case.

5.3.1. Quasiconformal charts on $\overline{\mathfrak{M}}$. We start with a definition of quasiconformal charts.

Let U be an open subset of $\mathbb{C}^m$ with an action of a finite group A. Let

$$\hat{\beta} : [A\backslash U] \to \overline{\mathfrak{M}}$$

be an open embedding and let

$$\beta : U \to \overline{\mathcal{M}}$$

be the corresponding map to the coarse moduli space. Denote by

$$\pi : \mathfrak{X} \to U$$

the family of nodal curves on U induced by β and let U_0 be the smooth locus of π:

$$U_0 = U \times_{\overline{\mathfrak{M}}} \mathfrak{M}.$$

The complement $U - U_0$ will be called the singular locus (of π). For $t \in U$ we denote by X_t the fiber $\pi^{-1}(t)$.

The construction of 3.1.11 provides $\overline{\mathcal{M}}$, the coarse moduli space of the smooth complex orbifold $\overline{\mathfrak{M}}$, with an orbifold atlas. Below, in our construction of an orbifold atlas for $G\backslash\overline{\mathfrak{T}}_{g,n}$, we will need charts satisfying some nice properties. We call such charts *quasiconformal.*

Definition. An orbifold chart (U, A, β) of the complex orbifold $\overline{\mathfrak{M}}$ is called *quasiconformal* if it satisfies the following conditions (QC1)–(QC6).

(QC1) The manifold U is analytically equivalent to a contractible neighborhood of 0 in $\mathbb{C}^m$, so that the singular locus $U - U_0$ corresponds to the union of (some) coordinate hyperplanes. In particular, if $U \neq U_0$, the intersection of the components of the singular locus is stable under the A-action. We assume that there exists a point $z \in U$ fixed by A. If $U \neq U_0$, we assume that z lies in the intersection of the components of the singular locus.

(QC2) For every $t \in U$ there exists an open neighborhood U^t of t in U and a *quasiconformal contraction* — a continuous map

$$c^t : \mathcal{X}^t \to X_t,$$

where $\mathcal{X}^t$ is the restriction of $\mathcal{X}$ to U^t, such that for every fiber X_s, $s \in U^t$, the restriction

$$c^t_s = c^t|_{X_s} : X_s \longrightarrow X_t$$

is a contraction (see Definition 5.1.3). In addition, the map c^t is *quasiconformal* in the following sense.

Let $\phi_t : S \to X_t$ be a marking; choose a neighborhood N of the nodes of X_t and $\varepsilon > 0$. Then there exists a small neighborhood U^δ of t in U^t such that for any $s \in U^\delta$ and for any marking $\phi_s : S \to X_s$ for which ϕ_t is isotopic $c^t_s \circ \phi_s$, the contraction $c^t_s : X_s \to X_t$ is $(1+\varepsilon)$-quasiconformal outside the preimage of $\overline{N}$.

(QC3) For every $t \in U$ there exist neighborhoods $\mathcal{O}_i \ni x_i$ of the nodes x_i, $i = 1, \dots, r$, of the curve X_t such that

(a) The maps

$$c^t : \mathcal{X}^t \longrightarrow X_t \text{ and } \pi^t : \mathcal{X}^t \longrightarrow U^t$$

define an analytic isomorphism

$$(c^t)^{-1}(X_t - \bigcup \mathcal{O}_i) \longrightarrow U^t \times (X_t - \bigcup \mathcal{O}_i). \tag{20}$$

(b) For every $i = 1, \dots, r$, the map

$$(c^t)^{-1}(\mathcal{O}_i) \longrightarrow U^t \tag{21}$$

is analytically isomorphic to the standard projection

$$P_i \to D^m$$

from

$$P_i = \{(u, v, t_1, \dots, t_m) \in D^2 \times D^m | uv = t_i\}$$

to the standard polydisk $D^m \subset \mathbb{C}^m$.

(QC4) For any $s \in U^t,\ u \in U^s \cap U^t \cap U_0$ there exists a homeomorphism

$$\theta : X_u \to X_u$$

isotopic to the identity, such that

$$c_u^t \circ \theta = c_s^t \circ c_u^s$$

(QC5) One has $U = U^z$.

(QC6) For a node x of X_z let D_x be the space

$$D_x = \{t \in U | (c_t^z)^{-1}(x) \text{ is a point }\}.$$

Then D_x is a component of the singular locus and every component of the singular locus is obtained in this way.

Remarks.

1. Note that if the condition (QC2) is valid for some marking ϕ_t of X_t, then it is valid for all markings of X_t. Also, since $\mathfrak{M} = [\Gamma \backslash \mathfrak{T}]$, the condition (QC2) is empty for $t \in U_0$.

2. Existence of continuous contractions c^t in (QC2) is not a very restrictive condition. What makes it non-trivial is the requirement that c^t is quasiconformal.

3. The property (QC3) means that the family of curves over U is constant outside neighborhoods of the nodes and is equivalent to the family given by the plumbing construction (see 5.3.3) in the neighborhoods of the nodes.

4. The property (QC6) identifies the set of components of the singular locus with the set of nodes of X_z. A marking $\phi : S \to X_z$ of X_z allows to identify the fundamental group of U_0 with the subgroup of the modular group Γ generated by the Dehn twists around $\phi^{-1}(x)$, where x runs through the nodes of X_z.

5. Below we will construct a collection of quasiconformal charts for $\overline{\mathfrak{M}}$ using a plumbing construction and will prove that they give an orbifold atlas of $\overline{\mathfrak{M}}$. This means that, in a certain sense, all sufficiently small orbifold charts of $\overline{\mathfrak{M}}$ are quasiconformal.

The notion of a quasiconformal chart serves a bridge between the Teichmüller and the stack-theoretic approach to the description of the moduli space of stable curves.

5.3.2. Theorem. *The moduli stack $\overline{\mathfrak{M}}$ of stable curves admits an orbifold atlas of quasiconformal charts.*

Proof of this theorem occupies the rest of this subsection (5.3.3—5.3.8).

5.3.3. Plumbing construction. Fix a maximally degenerate stable curve X_0 of genus g with n punctures, i.e. X_0 has a maximal possible number of nodes

$$m = 3g + n - 3.$$

Let $x_1, \dots, x_m \in X_0$ be the nodes of X_0. For each node $x_i \in X_0$ fix an open neighborhood $x_i \in V_i \subset X_0$, such that these sets V_i are pairwise disjoint, do not contain punctures and each V_i is a union of two subsets

$$V_i = U_i \cup U_{i+m}, \ i = 1, \dots, m,$$

meeting at the point x_i and homeomorphic to the open unit disk $D \subset \mathbb{C}$. Finally, let

$$z_k : D \to X_0, \ k = 1, \dots, 2m,$$

be holomorphic maps such that z_k gives a homeomorphism between D and $z_k(D) = U_k$ and

$$z_i(0) = z_{i+m}(0) = x_i, \ i = 1, \dots, m.$$

Using the choices of the curve X_0 and of $2m$ local coordinate functions z_i, we will construct a family $\mathcal{X}$ of stable punctured curves over the polydisk D^m as follows.

Take an open subset $\mathcal{Y} \subset X_0 \times D^m$ given by

$$\mathcal{Y} = X_0 \times D^m - \bigcup_{i=1}^{m} W_i,$$

where

$$W_i = \{(x, t_1, \dots, t_m) \in X_0 \times D^m \mid x = z_i(z) \text{ or } x = z_{i+m}(z) \text{ for } |z| \le |t_i|\}$$

and

$$P_i = \{(u, v, t_1, \dots, t_m) \in D^2 \times D^m \mid uv = t_i\} \ .$$

We glue the manifolds $\mathcal{Y}$ and P_i using the equivalence relation generated by the following conditions.

- The point of $\mathcal{Y}$ with coordinates $(z_i(z), t_1, \dots, t_m)$ is equivalent to the point of P_i with coordinates $(z, t_i/z, t_1, \dots, t_m)$
- The point of $\mathcal{Y}$ with coordinates $(z_{i+m}(z), t_1, \dots, t_m)$ is equivalent to the point of P_i with coordinates $(t_i/z, z, t_1, \dots, t_m)$.

One easily sees that the quotient of $\mathcal{Y} \sqcup P_1 \sqcup \ldots \sqcup P_m$ by the equivalence relation described above is Hausdorff; it is, therefore, a complex manifold which we denote by $\mathcal{X}$. It is fibered over D^m; its fiber X_t over $t = (t_1, \dots, t_m)$ is obtained from the original nodal curve X_0 by "holomorphic plumbing" which replaces a neighborhood of the node x_i, for which $t_i \neq 0$, locally parametrized by a neighborhood of the node of the curve $uv = 0$, with a piece of the smooth curve $uv = t_i$.

The fiber X_t of the above family is smooth if and only if all the coordinates of t are nonzero.

Introduce the following notation

$$D_0 = D - \{0\}, \ B_0 = (D_0)^m \text{ and } \ B = D^m$$

and let

$$\pi : \mathcal{X} \longrightarrow B \tag{22}$$

be the family of curves constructed above. The restriction of π to B_0 gives the family

$$\pi_0 : \mathcal{X}_0 \to B_0$$

of smooth curves.

According to the results of Section 4, the stack $\overline{\mathfrak{M}}$ represents complex families of nodal curves with punctures. Thus, the family (22) defines a map

$$\hat{\beta} : B \to \overline{\mathfrak{M}}.$$

As was shown in [25], the map $\hat{\beta}$ is not necessarily étale. We will show however, that for any stable punctured curve X there exists a choice of a maximally degenerated curve X_0, together with a choice of local coordinates near the nodes so that, for some point $t \in B$, the map $\hat{\beta}$ is étale at t and $\beta(t)$ is presented by X.

For the point and the plumbing data chosen as above, consider the group $A = \mathrm{Aut}(\hat{\beta}(t))$. According to Lemma 3.1.1, there exists a contractible neighborhood U such that $(U, A, \hat{\beta})$ gives an open embedding $[A \backslash U] \to \overline{\mathfrak{M}}$. We also assume that U does not intersect coordinate hyperplanes which do not contain t. The singular locus of (U, A, β) is the union of coordinate hyperplanes containing t. The collection of quasiconformal contractions is given by the standard contraction of the family

$$\{(z, w, t) \in \mathbb{C}^3 |\ |z| \leq 1,\ |w| \leq 1,\ |t| \leq 1,\ zw = t\}$$

over the closed disk $|t| \leq 1$ to the fiber at $t = 0$.

5.3.4. Construction of the family. Let $(X, x_1, \ldots, x_n)$ be a punctured curve with r nodes.

We endow the complement

$$X_{\mathrm{reg}} = X - \{\text{nodes and punctures}\}$$

with the canonical complete hyperbolic metric. Choose a maximal collection of simple disjoint geodesics

$$C_i, i = 1, \ldots, m - r$$

on X such that their complement

$$X_{\mathrm{reg}} - \bigcup_i C_i$$

is a disjoint union of pairs of pants P_j, $j = 1, \ldots, 2g - 2 + n$.

Note that each geodesic C_i has a natural (angular) parametrization. To each boundary component of each pair of pants P_j we glue a punctured disk, so that the angular parametrizations on the common circle coincide. As a result, we get an embedding of each pair of pants P_j into a triply punctured sphere S_j; each punctured disk glued to a pair of pants P_j defines an open embedding $z : D_0 \to S_j$ which is *almost* the local coordinate near the puncture we need.

Here is the reason we will have to make a small adjustment to the embeddings $z : D_0 \to S_j$. If $w : D_0 \to S_k$ is the other local coordinate corresponding to the same geodesic C_i, the gluing formula is $zw = 1$, whereas we were supposed to get $zw = t$ with $|t| < 1$.

The lemma below claims that each open embedding $z_i : D_0 \to S_j$ can be extended to an open embedding $Z_i : D_0' \to S_j$ of a greater punctured disk. Then we can substitute z_i with $Z_i(1+\varepsilon)$ so that the geodesic C_i will be given by the equation $|z| = \frac{1}{1+\varepsilon}$ and the images of the unit disks will still have no intersection.

Lemma. *Let X be a bordered Riemann surface and C be its boundary component endowed with the intrinsic metric. Glue a unit disk D to X so that the common boundary component acquires the same angular coordinate from X and from D. Let $\widehat{X}$ be the resulting Riemann surface. Then the map $D \to \widehat{X}$ extends to an open embedding $D' \to \widehat{X}$ of a strictly greater disk $D' \supsetneq D$ having the same center.*

Proof. The claim is clear if X is a half-annulus $A = \{z |\ 1 \le |z| < c\}$. Then $\widehat{X}$ identifies with the disk $\{z |\ |z| < c\}$ strictly containing the unit disk.

Now, if X is arbitrary, let

$$X^d = X \cup_C \bar{X},$$

be the double of X with respect to C, where $\bar{X}$ is the antiholomorphic copy of X. Then the Nielsen extension of $\bar{X}$ at C embeds into X^d and has form $A \cup \bar{X}$ where A is a half-annulus having C as the boundary and embedded into X. This gives the required extension. □

The r nodes of the original curve X identify some pairs of punctures of $\coprod_j S_j$. This gives a maximally degenerated curve X_0 having r "original" nodes and $m - r$ new nodes, endowed with local coordinates

$$z_1, \dots, z_{m-r}, w_1, \dots, w_{m-r}$$

near the $m - r$ new nodes. We can choose the $2r$ coordinates near r "original" nodes in an arbitrary way. The curve X is obtained from X_0 by the plumbing construction with parameters $t = (t_1, \dots, t_m)$ where the geodesic C_i in the corresponding pair of local coordinates is given by the equations $|z| = \sqrt{t_i}$, $|w| = \sqrt{t_i}$, and $t_i = 0$ for $i > m - r$.

5.3.5. The case of a smooth curve. Assume that X has no nodes. We assume $X = X_t$ for some $t \in B_0$. The family $\pi_0 : \mathfrak{X}_0 \to B_0$ of Riemann surfaces defines a map $T\beta : T_t B_0 \to T_{\beta(t)}\mathfrak{M}$ of complex vector spaces. We want to prove that this map is an isomorphism if π is the family constructed in 5.3.4.

The tangent space $T_{\beta(t)}\mathfrak{M}$ identifies with the cohomology $H^1(X_t, T)$ where T is the sheaf of vector fields vanishing at the punctures. The image of a vector $v \in T_t(B) = \mathbb{C}^m$ is described by an explicit Čech 1-cocycle. Thus the problem reduces to proving that some Čech 1-cocycles are not coboundaries. This is, however, difficult to calculate explicitly, and this is not true for a general choice of local coordinates—see a counterexample in [25].

5.3.6. The Fenchel-Nielsen family. Recall the construction of the Fenchel-Nielsen coordinates on the Teichmüller space. As above, we have chosen a maximal collection of free loops on the basic surface S. For each $(X, \phi) \in \mathcal{T}_{g,n}$ a collection of geodesics is therefore defined. Their lengths give a (real-analytic) map

$$L : \mathcal{T}_{g,n} \longrightarrow \mathbb{R}_+^m \tag{23}$$

(Fenchel-Nielsen length coordinates). Fix $l = (l_1, \ldots, l_m) \in \mathbb{R}_+^m$. The preimage $L^{-1}(l)$ is a $\mathbb{R}^m$-torsor with the action of the i-th component of $\mathbb{R}$ given by cutting of a Riemann surface along the i-th geodesic, twisting the boundary components one with respect to the other, and gluing them back.

The map L has a section which allows one to define what is classically known as Fenchel-Nielsen coordinates. This coordinate system consists of m length coordinates (23) and m angular Fenchel-Nielsen coordinates $\theta_1, \ldots, \theta_m$, chosen so that the shift by 2π along each coordinate corresponds to the Dehn twist. In what follows we will use modified angular coordinates $\tau_i = \frac{l_i}{2\pi i}\theta_i$.

For a fixed value $l \in \mathbb{R}_+^m$ the Riemann surfaces from $L^{-1}(l)$ can be organized in a family with the base $\mathbb{R}^m$ — this family is sometimes called the Fenchel-Nielsen deformation.

Kodaira-Spencer theory [34] provides for any $X \in L^{-1}(l)$ an $\mathbb{R}$-linear map $\mathbb{R}^m \to H^1(X, T)$, T being the sheaf of vector fields vanishing at the punctures of X.[13]

We denote the images of the coordinate vectors by $\frac{\partial}{\partial \tau_i} \in H^1(X, T)$.

Lemma. *The vectors* $\frac{\partial}{\partial \tau_i}$, $i = 1, \ldots, m$, *form a basis of* $H^1(X, T)$ *over* $\mathbb{C}$.

Proof. This follows from the Wolpert's formula [28, 8.3]

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{m} d\tau_i \wedge dl_i$$

for the Weil-Petersson form on the Teichmüller space. Since ω_{WP} is nondegenerate, $d\tau_i$, and therefore $\frac{\partial}{\partial \tau_i}$ are linearly independent. $\square$

Now we can explain what is special about our choice of local coordinates.

The pullback of the family $\pi_0 : \mathfrak{X}_0 \to B_0$ along the map

$$u : \mathbb{R}^m \longrightarrow B_0$$

defined by the formula $u(x_1, \ldots, x_m) = (t_1 e^{2\pi i x_1}, \ldots, t_m e^{2\pi i x_m})$, is the Fenchel-Nielsen family.

Consider the diagram of maps of tangent spaces ("chain rule")

$$\mathbb{R}^m \xrightarrow{Tu} \mathbb{C}^m \xrightarrow{T\beta} T_{\beta(t)}\mathfrak{M} = H^1(X, T).$$

The composition $T\beta \circ Tu$ sends the standard basis $\{e_i\}$ of $\mathbb{R}^m$ into a $\mathbb{C}$-basis $\{\frac{\partial}{\partial \tau_i}\}$ of $T_{\beta(t)}\mathfrak{M}$. Since the map Tu also sends the basis of $\mathbb{R}^m$ into a basis of $\mathbb{C}^m$, the map $T\beta$ is an isomorphism.

[13]Fortunately, Kodaira and Spencer developed their theory for C^∞ families of complex manifolds!

This proves that in the case of X smooth the special chart we have defined in 5.3.4 is étale at $t \in B_0$ for which $X = X_t$.

The case of nodal curves is considered below.

5.3.7. Proof of the étalness for nodal curves. Let X be a nodal punctured curve and let $\pi : \mathcal{X} \to B$ be the family of curves built by the plumbing construction with the special choice of the local coordinates as in 5.3.4. Assume that $X = X_t$ for $t \in \bar{B}$. We have to check that the map of the tangent spaces

$$T(\beta) : T_t B \longrightarrow T_{\beta(t)} \overline{\mathfrak{M}}$$

is an isomorphism. Since the dimensions of the vector spaces coincide, it is sufficient to prove the injectivity. Let $t = (t_1, \dots, t_m)$ and let

$$v = (v_1, \dots, v_m) \in T_t B$$

belong to the kernel of $T(\beta)$. The target of $T(\beta)$ is the collection of deformations of X_t over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. Triviality of such a deformation means in particular that all nodes of X_t are preserved under the deformation; in other words, one has

$$t_i = 0 \Longrightarrow v_i = 0.$$

Assume for simplicity that $t_1 = \dots = t_k = 0 = \tau_1 = \dots = \tau_k$ and $t_i \neq 0$ for $i > k$. The normalization X_t^{nor} of X_t is a smooth curve with $n + 2k$ punctures (all preimages of the nodes become punctures). Let X_0' be obtained from X_0 by ungluing the first k nodes and turning them into $2k$ punctures.[14] Let $B' = D^{n-k}$ and let

$$\pi' : \mathcal{X}' \to B'$$

be the family of curves obtained from X_0' by the plumbing construction with the same special choice of the local coordinates near the punctures as specified in 5.3.4. Then X_t^{nor} appears in this family as the fiber of π' at $t' = (t_{k+1}, \dots, t_n) \in B'$. Therefore, the tangent vector

$$v' = (v_{k+1}, \dots, v_n) \in T_{t'} B'$$

belongs to the kernel of the map

$$T(\beta') : T_t' B' \to T_{\beta'(t')} \overline{\mathfrak{M}},$$

where $\beta' : B' \to \overline{\mathfrak{M}}_{g,n+2k}$ is the map inducing the family π'.

Since we have already proved the étalness for the smooth curve X^{nor}, it follows that it also holds for X.

[14] To unglue a single node q we choose a neighborhood $U \ni q$ which does not contain other nodes, normalize U and paste the result back. We assign labels $n+1, n+2, \dots, n+2k$ to the new $2k$ punctures in an arbitrary way.

5.3.8. The charts form an atlas. First of all, organize the charts (U, A, β) constructed above into a category as is explained in 3.1.11. This gives a category $\mathcal{Q}$ whose objects are triples $(U, A, \hat{\beta})$, where $\hat{\beta} : [A\backslash U] \to \overline{\mathfrak{M}}$ is an open embedding and whose morphisms consist of morphisms of such charts, together with a 2-isomorphism between their maps to $\overline{\mathfrak{M}}$.

Note that, according to our choice, each chart (U, A) satisfies the following property: A has a fixed point in U. This implies, in particular, that all maps of charts defined by arrows of $\mathcal{Q}$, are injective.

Let us show that the the category $\mathcal{Q}$ together with the obvious functor

$$c : \mathcal{Q} \to \texttt{Charts}/\overline{\mathcal{M}},$$

defines an orbifold atlas. The only thing to check is the condition (ii) in the definition of orbifold atlas 3.1.9. Let

$$[A_i\backslash U_i] \to \overline{\mathfrak{M}},\ i = 1, 2,$$

be two orbifold charts having a common point $x \in \overline{\mathcal{M}}$ in the image. We can assume that U_i is small enough so that x_i is the only preimage of x in it. In this case the groups A_1 and A_2 can be identified with $A = \mathrm{Aut}(x)$. We will use this identification. Consider

$$W = [A\backslash U_1] \times_{\overline{\mathfrak{M}}} U_2.$$

The induced map $W \to U_2$ is an open embedding, equivariant with respect to the action of A. This defines an abstract orbifold chart (W, A) together with open embeddings $[A\backslash W] \to [A\backslash U_1]$ and $[A\backslash W] \to [A\backslash U_2]$. Since W is an open subset of U_2, the chart (W, A) belongs to our collection.

The atlas of quasiconformal charts for $\overline{\mathfrak{M}}$ is constructed.

6. Augmented Teichmüller spaces from the complex-analytic point of view

In this section we study complex-analytic properties of Bers' augmented Teichmüller spaces $\overline{\mathcal{T}}_{g,n}$. The space $\overline{\mathcal{T}}_{g,n}$ is obtained by adding to the classical Teichmüller space $\mathcal{T}_{g,n}$ points corresponding to Riemann surfaces with nodal singularities. Unlike $\mathcal{T}_{g,n}$, the space $\overline{\mathcal{T}}_{g,n}$ is not a complex manifold (it is not even locally compact). However, as we show in this section, the quotient of $\overline{\mathcal{T}}_{g,n}$ by any finite index subgroup G of the Teichmüller modular group $\Gamma_{g,n}$ is a normal complex space. More precisely, we prove (see Theorem 6.1.1) that $G\backslash\overline{\mathcal{T}}_{g,n}$ has a canonical structure of a complex orbifold.

6.1. Complex structure on $G\backslash\overline{\mathcal{T}}_{g,n}$: markings

Let $2g+n > 2$ and G be a finite index subgroup of the corresponding modular group Γ. Two markings ϕ, ϕ' of a nodal curve X are called G-equivalent if there exists $g \in G$ such that ϕ' is isotopic to $g(\phi)$. Points of the quotient space $G\backslash\overline{\mathcal{T}}_{g,n}$ are pairs (X, ϕ), where X is a stable curve of genus g with n punctures and ϕ is a G-equivalence class of markings (a G-marking).

We are going to construct an orbifold atlas for the quotient $G\backslash\overline{\mathcal{T}}_{g,n}$.

Shortly, the idea is the following. We start with a quasiconformal orbifold atlas $\mathfrak{Q}$ atlas for the moduli stack $\overline{\mathfrak{M}}$ of stable curves (see Section 5.3).

Then, for each chart $(U, A, \beta) \in \mathfrak{Q}$, endowed with an additional datum (a marking of the singular fiber) we construct a chart (V, H, α) for $G\backslash\overline{\mathfrak{T}}_{g,n}$ making the diagram

$$\begin{tikzcd}
V \arrow[rrr, "\alpha"] \arrow[d] & & & G\backslash\overline{\mathfrak{T}}_{g,n} \arrow[d] \\
U \arrow[r] & {[A\backslash U]} \arrow[r, "\hat{\beta}"] & \overline{\mathfrak{M}} \arrow[r] & \overline{\mathcal{M}}
\end{tikzcd}$$

commutative. Here $\hat{\beta}$ is the embedding of stacks determined by β.

Finally some work is needed to get everything arranged into an orbifold atlas and to prove various compatibilities.

In this subsection we present the construction of a chart (V, H, α) of $G\backslash\overline{\mathfrak{T}}_{g,n}$ based on a choice of $(U, A, \beta) \in \mathfrak{Q}$ and on a choice of a marking of the special fiber of the family defined by U. We show that these charts can be arranged into an orbifold atlas $\mathcal{A} \to \mathtt{Charts}(G\backslash\overline{\mathfrak{T}})$.

As a result of the construction of the atlas, we get a natural complex orbifold structure on the quotient $G\backslash\overline{\mathfrak{T}}_{g,n}$. We denote the obtained orbifold by $[G\backslash\overline{\mathfrak{T}}_{g,n}]$. It is connected to other spaces and orbifolds as shown in the diagram (24) below. These connections are described in the following theorem whose proof occupies Sections 6.1–7.2.

6.1.1. Theorem. *Let G be a finite index subgroup of the Teichmüller modular group $\Gamma = \Gamma_{g,n}$. Then the quotient space $G\backslash\overline{\mathfrak{T}}$ is the coarse space of a naturally defined complex orbifold $[G\backslash\overline{\mathfrak{T}}]$ so that the quotient orbifold $[G\backslash\mathfrak{T}]$ becomes its open substack.*

The quotient map $\overline{\mathfrak{T}} \to G\backslash\overline{\mathfrak{T}}$ factors through a map

$$\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}];$$

the composition $[G\backslash\mathfrak{T}] \to G\backslash\overline{\mathfrak{T}}$ factors through $[G\backslash\overline{\mathfrak{T}}] \to G\backslash\overline{\mathfrak{T}}$ and the composition $[G\backslash\mathfrak{T}] \to \overline{\mathfrak{M}}$ factors through a canonically defined morphism $[G\backslash\overline{\mathfrak{T}}] \to \overline{\mathfrak{M}}$ (see the dashed arrows in the diagram (24) below).

In particular, the quotient $G\backslash\overline{\mathfrak{T}}$ has a natural structure of a normal complex space extending that on $G\backslash\mathfrak{T}$.

$$
\begin{tikzcd}
\mathcal{T} \arrow[rr] \arrow[d] & & \overline{\mathcal{T}} \arrow[dl, dashed, "\pi_G"'] \arrow[dd] \\
{[G\backslash\mathcal{T}]} \arrow[r, dashed] \arrow[d] & {[G\backslash\overline{\mathcal{T}}]} \arrow[dr, dashed] \arrow[dd, dashed] & \\
G\backslash\mathcal{T} \arrow[rr] \arrow[d] & & G\backslash\overline{\mathcal{T}} \arrow[d] \\
\mathfrak{M} \arrow[r] & \overline{\mathfrak{M}} \arrow[r] & \overline{\mathcal{M}}
\end{tikzcd}
\tag{24}
$$

6.1.2. Space of markings of fibers. Let (U, A, β) be a quasiconformal chart in $\mathcal{Q}$. In what follows we adopt the notations of 5.3.1 where the notion of quasiconformal chart is discussed.

A collection of contractions $c_s^t : X_s \to X_t$ allows one to transfer markings from X_s to X_t. We will show in 6.1.4 below that, even though we do not fix the contractions but only require their existence, the transfer of markings in a quasiconformal chart is defined uniquely.

Fix a quasiconformal contraction

$$c^t : \mathcal{X}^t \to X_t \tag{25}$$

and a marking

$$\phi : S \to X_t.$$

We say that a marking ϕ_s of $X_s,\ s \in U^t$, is *consistent* with the given marking $\phi : S \to X_t$ via c^t if the marking $c_s^t \circ \phi_s$ of X_t is equivalent to ϕ.

Fix t and $\phi : S \to X_t$ as above. For $s \in U_0^t = U_0 \cap U^t$ denote by P_s the set of all markings of X_s and by Q_s the subset of markings in P_s consistent with ϕ. The sets P_s and Q_s combine into coverings of U_0^t,

$$p : P \longrightarrow U_0^t \ \text{ and } \ q : Q \longrightarrow U_0^t,$$

so that $P_s = p^{-1}(s),\ Q_s = q^{-1}(s)$.

The coverings p and q are torsors over U_0^t respectively for the groups Γ and Γ_0, the free abelian subgroup of Γ generated by the Dehn twists around the curves $\phi^{-1}(x_i)$, where $x_i,\ i = 1, \ldots, r$ are the nodes of X_t.

The covering q is a universal covering of U_0^t and p can be recovered from it as follows:

$$P = \Gamma \times^{\Gamma_0} Q. \tag{26}$$

The roles of the coverings p and q is explained by the following.

Lemma. *Let $\pi' : \mathcal{X}' \to Y$ be the family of curves induced from $\pi : \mathcal{X} \to U_0^t$ via a map $Y \to U_0^t$. Then markings of π' correspond to sections of the covering $P' \to Y$ induced from p. The sections of the covering $Q' \to Y$ induced from q correspond to the markings of π' consistent with ϕ.*

$\square$

One has a sequence of canonical maps $Q \to P \xrightarrow{\alpha} \mathcal{T}$.

Note that Q is a connected component of P; its choice depends on the choice of the marking ϕ. If $\gamma \in \Gamma$ then the marking $\phi' = \phi \circ \gamma$ corresponds to the component $Q' = \gamma(Q)$ of P. This gives the following geometric way of marking a curve X_t.

6.1.3. Corollary. *For a fixed quasiconformal contraction (25) $c^t : \mathcal{X}^t \to X_t$, there is a natural one-to-one correspondence between markings of X_t and components of P.*

$\square$

We claim that this correspondence is independent of the choice of a quasiconformal contraction. To justify this we will present an independent characterization of a marking defined by the choice of a component in P. We proceed as follows.

The point $t \in U^t$ admits a basis of neighborhoods U^δ, such that (U^δ, A^t), where $A^t = \mathrm{Stab}_A(t))$, is a subchart of (U, A) satisfying the conditions (QC1)–(QC6) in 3.1.9.

Let P^δ and Q^δ be the spaces defined as above with U^δ instead of U. Since U_0^t and $U_0^\delta = U_0^t \cap U^\delta$ have the same fundamental groups, each component of P contains precisely one component of P^δ. Denote by $\overline{Q^\delta}$ the closure of Q^δ in the augmented Teichmüller space $\overline{\mathcal{T}}$.

6.1.4. Proposition. *In the above notation, ϕ is the only marking of X_t for which (X_t, ϕ) belongs to the intersection $\bigcap_\delta \overline{Q^\delta}$.*

The proposition immediately implies that the notion of consistency of markings, defined in 6.1.2 with the help of contraction, is in fact independent of the choice of contraction.

Proof of the proposition. First of all, $(X_t, \phi) \in \overline{Q^\delta}$ for each δ since any neighborhood of (X_t, ϕ) contains Q^δ for U^δ small enough.

Assume $(X_t, \phi') \in \bigcap_\delta \overline{Q^\delta}$. If (X_t, ϕ) and (X_t, ϕ') represent different points of $\overline{\mathcal{T}}$, they have disjoint neighborhoods. On the other hand, by (QC2) there exist U^δ such that Q^δ belongs to both of them.

Thus, choosing a component Q of P, we reconstruct the transfer of markings from X_s to X_t for each $s \in U_0^t$. The property (QC4) implies that the transfer is uniquely defined also for any $s \in U^t$. $\square$

From now on we will keep the notation of 6.1.2 for $t = z$. Thus, we have $U^t = U,\ U_0^t = U_0$, and $p : P \to U_0,\ q : Q \to U_0$.

Note the following consequence of the above discussion.

6.1.5. Corollary. *There is a one-to-one correspondence between the markings of X_z and the connected components of P.*

The following description of the covering space P is very useful.

6.1.6. Lemma. *We have the isomorphism*

$$P = U_0 \times_{\mathfrak{M}} \mathfrak{T},$$

where the fiber product is taken in the 2-category of complex orbifolds.

Proof. Let f be the map $P \to U_0 \times_{\mathfrak{M}} \mathfrak{T}$ given by the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\alpha_p"] \arrow[d, "p"'] & \mathfrak{T} \arrow[d] \\ U_0 \arrow[r, "\beta"] & \mathfrak{M} \end{tikzcd}$$

Since $p : P \to U_0$ and $U_0 \times_{\mathfrak{M}} \mathfrak{T} \to U_0$ are coverings and f is a morphism of coverings over U_0, to prove that f is an isomorphism, it is sufficient to compare the action of the fundamental group of U_0 on the fibers. After identification of $\pi_1(U_0)$ with Γ_0 both fibers can be identified with Γ and the action of $\pi_1(U_0)$ with the left action of $\Gamma_0 \subset \Gamma$ on Γ. ☐

As a result, the space P acquires an action of the group A commuting with the action of Γ.

6.2. Complex structure on $G\backslash\overline{\mathfrak{T}}_{g,n}$: charts

Let G be a finite index subgroup of the modular group Γ.

Fix a quasiconformal orbifold chart (U, A, β) of $\overline{\mathfrak{M}}$. Fix a marking

$$\phi : S \to X_z$$

(recall that this is equivalent to fixing a connected component Q of P). We will assign to the pair (chart, marking) an orbifold chart (V, H, α) of the quotient $G\backslash\overline{\mathfrak{T}}$.

The marking ϕ determines the spaces $Q \subset P$, the isomorphism $\pi_1(U_0) \simeq \Gamma_0$ and the presentation $P = \Gamma \times^{\Gamma_0} Q$.

The moduli stack $\overline{\mathfrak{M}}$ contains as an open substack the stack $\mathfrak{M}$ of non-singular curves. The triple $(U_0, A, \beta|_{U_0})$ is, of course, a chart for $\mathfrak{M}$.

6.2.1. A big commutative diagram. As a first step in the construction of our orbifold chart, we have to describe the spaces and the arrows of the diagram (29) below.

The quotient $G\backslash P$ can be described by the bijection i

$$G\backslash P = G\backslash(\Gamma \times^{\Gamma_0} Q) \xleftarrow{\ i\ } \coprod_{\gamma \in G\backslash\Gamma/\Gamma_0} (\gamma^{-1}G\gamma \cap \Gamma_0)\backslash Q \ , \tag{27}$$

where γ runs through a set of representatives of double cosets $G\backslash\Gamma/\Gamma_0$ and $i=\{i_\gamma\}$ is the collection of maps

$$i_\gamma:(\gamma^{-1}G\gamma\cap\Gamma_0)\backslash Q\longrightarrow G\backslash(\Gamma\times^{\Gamma_0}Q),\ [x]\mapsto[\gamma x].$$

Recall that Γ_0 is the free abelian group generated by the Dehn twists $D_i,\ i=1,\dots,r$, around the curves $\phi^{-1}(x_i)$ of S, where $x_1,\dots,x_r$ are the nodes of X_z. Let

$$k_i=\min\{k|D_i^k\in G\},\ \text{for } i=1,\dots,r.$$

Denote by Γ_0' the subgroup of $G\cap\Gamma_0$ generated by $D_1^{k_1},\dots,D_r^{k_r}$. Let $Y=\Gamma_0'\backslash Q$. The natural map

$$Y=\Gamma_0'\backslash Q\to U_0=\Gamma_0\backslash Q$$

is a covering with the Galois group $\Gamma_0/\Gamma_0'=\mathbb{Z}/\mathbb{Z}_{k_1}\times\ldots\times\mathbb{Z}/\mathbb{Z}_{k_r}$.

We define $Z=(G\cap\Gamma_0)\backslash Q$. This is the component of $G\backslash P$ corresponding to $\gamma=1$. The natural projection $Y\to Z$ gives a map

$$u:Y\to G\backslash P$$

commuting with the projections of Y and of $G\backslash P$ to U_0.

The projection $G\backslash P\to U_0$ is a finite map. Now let V be the normalization of U in the field of meromorphic functions of Y. The variety V is a smooth; it looks locally like a polydisk ramified over the components of the singular locus with the ramification degree $k_1,\dots,k_r$. We denote by $\varkappa$ both the projection $V\to U$ and its restriction to the smooth part $Y\to U_0$. Let

$$\pi':\mathfrak{X}'\longrightarrow V \tag{28}$$

be the family of curves induced from the family $\pi:\mathfrak{X}\to U$ via $\varkappa$.

The manifold V is the "space component" of the orbifold chart we are building.

Let

$$\alpha:G\backslash P\to[G\backslash\mathfrak{T}]$$

be the map induced by $\alpha_p:P\to\mathfrak{T}$. Now we will extend the composition

$$Y\xrightarrow{u}G\backslash P\xrightarrow{\alpha}[G\backslash\mathfrak{T}]$$

to get the dashed map $\alpha:V\to G\backslash\overline{\mathfrak{T}}$ in the following big commutative diagram (do not pay attention to the other dashed arrows for the time being).

$$
\begin{tikzcd}
 & & P \arrow[r, "\alpha_p"] \arrow[d] & \mathcal{T} \arrow[rr] \arrow[d] & & \overline{\mathcal{T}} \arrow[dl, dashed, "\pi_G"'] \arrow[dd] \\
 & Y \arrow[r, "u"] \arrow[dl] \arrow[dd, "\varkappa"'] & G\backslash P \arrow[r, "\alpha"] \arrow[dd] & {[G\backslash\mathcal{T}]} \arrow[r, dashed] \arrow[dd] \arrow[drr] & {[G\backslash\overline{\mathcal{T}}]} \arrow[dr, dashed] \arrow[ddd, dashed] & \\
V \arrow[rrrrr, dashed, "\alpha"] \arrow[dd, "\varkappa"'] & & & & & G\backslash\overline{\mathcal{T}} \arrow[dd] \\
 & U_0 \arrow[r, "="] \arrow[dl] & U_0 \arrow[r, "\hat{\beta}"] & \mathfrak{M} \arrow[dr] & & \\
U \arrow[rrrr, "\hat{\beta}"] & & & & \overline{\mathfrak{M}} \arrow[r] & \overline{\mathcal{M}}
\end{tikzcd}
\tag{29}
$$

Here $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,n}$ is the complex orbifold associated to the smooth algebraic stack of moduli of stable curves of genus g with n punctures. According to Theorem 4.1.1, $\overline{\mathfrak{M}}$ represents complex-analytic families of stable curves of genus g with n punctures. The family of curves $\pi : \mathcal{X} \to U$ defines therefore a map $\hat{\beta} : U \to \overline{\mathfrak{M}}$. The map from $\mathfrak{M}$ to $\overline{\mathfrak{M}}$ is the obvious open embedding. The complex space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}$ is the coarse moduli space for $\overline{\mathfrak{M}}$ and the horizontal map $\overline{\mathfrak{M}} \to \overline{\mathcal{M}}$ is obvious. The map from $[G\backslash\mathcal{T}]$ to $G\backslash\overline{\mathcal{T}}$ is the composition of the projection $[G\backslash\mathcal{T}] \to G\backslash\mathcal{T}$ to the "naïve quotient" space and of the embedding $G\backslash\mathcal{T} \to G\backslash\overline{\mathcal{T}}$. Finally, the projection $\overline{\mathcal{T}} \to \overline{\mathcal{M}}$ forgetting the marking is continuous and factors through the quotient $G\backslash\overline{\mathcal{T}}$.

The family of curves $\pi' : \mathcal{X}' \to V$ restricted to Y, admits a canonical G-marking induced via u from the canonical G-marking on the family based on $G\backslash P$.

Choose a point $x \in V$ and let $t = \varkappa(x) \in U$. Since G has finite index in Γ and since the quotient $G\backslash\overline{\mathcal{T}}$ is Hausdorff, there exist a neighborhood N of the nodes of X_t and a positive ε such that the standard neighborhoods $\mathcal{U}_{N,\varepsilon}(X_t, \phi)$ have no intersection for different G-markings ϕ.

Choose a neighborhood U^t and a contraction $c^t : \mathcal{X}^t \to X_t$. There exists a neighborhood U^δ of t in U^t such that c^t_s is $(1+\varepsilon)$-quasiconformal outside N for all $s \in U^\delta$. Define $\mathcal{O}$ as the component of $\varkappa^{-1}(U^\delta)$ containing x.

For $y \in \mathcal{O} \cap Y$ let $\alpha(y) = (X_s, \psi)$ where we denote $s = \varkappa(y)$. The G-marking $c^t_s \circ \psi$ of X_t does not depend on y. This G-marking defines the image of $x \in V$ in $G\backslash\overline{\mathcal{T}}$ and thus gives the required dashed map

$$\alpha : V \longrightarrow G\backslash\overline{\mathcal{T}}_{g,n} \tag{30}$$

which is automatically continuous. In Proposition 6.2.2 below we will prove that α is open.

To get an orbifold chart (V, H, α) we have to specify the group H.

Recall that $Y = \varkappa^{-1}(U_0)$ and the image of the map $u : Y \to G\backslash P$ is a component Z of $G\backslash P$. By Lemma 6.1.6 the groups A and Γ act on P and the actions commute. Thus, A acts on the quotient $G\backslash P$ as well. Let

$$A_Z = \{a \in A : a(Z) = Z\} \tag{31}$$

be the stabilizer of the component Z.

We define H as the group of pairs $(\widetilde{a}, a)$, where $\widetilde{\alpha} : Y \to Y$ and $a : Z \to Z$ are automorphisms with $a \in A_Z$ such that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\widetilde{a}"'] & Z \arrow[d, "a"] \\ Y \arrow[r] & Z \end{tikzcd}$$

is commutative.

Another description of the groups A_Z and H is given in 6.2.5.

The action of H on Y extends to V, since V is the normalization of U in the field of meromorphic functions on Y, see [20, 7.3]. Now we will prove that (V, H, α) is an orbifold chart for $G\backslash\overline{\mathfrak{T}}$.

6.2.2. Proposition. *The map*

$$\alpha : V \longrightarrow G\backslash\overline{\mathfrak{T}}$$

is open.

Proof. We will prove that the image $\alpha(V)$ is open in $G\backslash\overline{\mathfrak{T}}$. Since we can replace the chart (U, A, β) with a smaller quasiconformal chart, this will prove that α carries any open set to an open set.

Let $x \in V$ and let $\alpha(x)$ be presented by a marked curve (X, ϕ). We have to prove that there is a pair $(N, \varepsilon > 0)$ where N is a neighborhood of the nodes of X, so that the standard neighborhood $\mathcal{U}_{N,\varepsilon}(X, \phi)$ of (X, ϕ) in $G\backslash\overline{\mathfrak{T}}$ lies in $\alpha(V)$.

Let $t = \varkappa(x)$. By (QC2) there exists an open neighborhood U^t of t in U and a contraction $c^t : \mathfrak{X}^t \to X_t$. The map $\beta : U \to \overline{\mathcal{M}}$ is open; thus, a pair (N, ε) can be chosen so that $\mathcal{U}_{N,\varepsilon}(X, \phi)$ lies in $\pi^{-1}(\beta(U^t))$, where $\pi : G\backslash\overline{\mathfrak{T}} \to \overline{\mathcal{M}} = \Gamma\backslash\overline{\mathfrak{T}}$ is the standard projection.

Since G has finite index in Γ, we can also assume that the neighborhoods $\mathcal{U}_{N,\varepsilon}(X, \phi)$ and $\mathcal{U}_{N,\varepsilon}(X, \phi')$ have no intersection if ϕ and ϕ' define different G-markings.

We claim that $\alpha(V)$ contains the neighborhood $\mathcal{U}_{N,\varepsilon}(X, \phi)$.

In fact, let $(X', \phi') \in \mathcal{U}_{N,\varepsilon}(X, \phi)$. By construction, there exists $s \in U^t$ with $\beta(s)$ represented by X'. Let U^δ be a small neighborhood of s contained in $U^s \cap U^t$ and consider $U_0^\delta = U^\delta \cap U_0$. Choose a point $u \in U_0^\delta$, lift it to a point $y \in Y$ and consider $\alpha(y) = (X'', \phi'')$. The contraction of ϕ'' to X

gives ϕ, therefore, the contraction of ϕ'' to X' gives ϕ' up to an element of the group Γ_0. This means that we can find another lift y_1 of u to Y with $\alpha(y_1) = (X'', \phi_1'')$ such that the contraction of ϕ_1'' to X' will be ϕ'.

Let Y^δ be the component of $\varkappa^{-1}(U_0^\delta)$ containing y_1. Then the intersection $\bar{Y^\delta} \cap \varkappa^{-1}(s)$ consists of one point z such that $\alpha(z) = (X', \phi')$. $\square$

6.2.3. Lemma. *The homomorphism $H \to A_Z$ is surjective with the kernel*

$$K = \mathrm{Aut}_Z(Y).$$

Proof. We have to verify that any automorphism $a : Z \to Z$ from A_Z lifts to an automorphism of Y. We have the following picture. Three spaces, U_0, Z and Y have a common universal covering Q. The fundamental group of U_0 is Γ_0, and the coverings Y and Z correspond to the subgroups Γ_0' and $G \cap \Gamma_0$ of Γ_0.

Let $a \in A_Z \subset A$. Since A is abelian, its action on U_0 induces an action on Γ_0. The action of A on U_0 comes from an action on U, therefore its action on Γ_0 must be a signed permutation of the Dehn twists $D_1, \ldots, D_r$ which generate Γ_0.

If an element $a \in A$ belongs to A_Z then $G \cap \Gamma_0$ is an a-invariant subgroup of Γ_0. This implies that Γ_0' is also a-invariant due to the specific form of the action of A on Γ_0.

Note that the kernel K of the epimorphism $H \to A_Z$ identifies with $(G \cap \Gamma_0)/\Gamma_0'$. $\square$

6.2.4. Theorem. *Let $x_1, x_2 \in V$, then $\alpha(x_1) = \alpha(x_2)$ if and only if $x_2 \in Hx_1$.*

Proof. Let (X_i, ϕ_i), $i = 1, 2$, be the marked curves representing the points $\alpha(x_i) \in G\backslash\overline{\mathfrak{T}}$.

The equality $\alpha(x_1) = \alpha(x_2)$ gives an isomorphism of G-marked curves, that is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi_1"] \arrow[d, "g"'] & X_1 \arrow[d, "\theta"] \\ S \arrow[r, "\phi_2"] & X_2 \end{tikzcd} \tag{32}$$

for some $g \in G$.

We will show that for any two open sets $U_1 \ni x_1$ and $U_2 \ni x_2$ in V the intersection $U_2 \cap (H \cdot U_1)$ is nonempty. Since H is a finite group, this will imply that $x_2 \in Hx_1$.

From now on we fix x_i and U_i, $i = 1, 2$ as above.

Since the map $\alpha : V \to G\backslash\overline{\mathfrak{T}}$ is open by Proposition 6.2.2, we may assume that $\alpha(U_1) = \alpha(U_2)$. Choose a point $x_1' \in U_1$ which corresponds to a smooth curve (X_1', ϕ_1'). Since the images of U_i under α coincide, there exists $x_2' \in U_2$ having the same image. The corresponding G-marked curve (X_2', ϕ_2') is, obviously, smooth as well. Moreover, there exist $g' \in G$ and an isomorphism $\theta' : X_1' \to X_2'$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\phi_1'"] \arrow[d, "g'"'] & X_1' \arrow[d, "\theta'"] \\ S \arrow[r, "\phi_2'"] & X_2' \end{tikzcd} \tag{33}$$

is commutative.

The images of x_1' and x_2' in Z have the same image in $G\backslash\mathcal{T}$. Therefore, $x_2' = ax_1'$ for some $a \in A_Z$. By Lemma 6.2.3 we can lift the element a to $h \in H$ acting on Y. Therefore x_2' and $h(x_1')$ have the same image in Z. This implies that there exists an element h' in the kernel K of the epimorphism $H \to A_Z$ such that $x_2' = h'hx_1'$. This concludes the proof of the theorem. $\square$

6.2.5. Second description of the groups A_Z and H. Here we will present yet another interpretation of the groups A_Z and H which appear in the description of the orbifold charts (V, H, α). This description will be needed in Section 8 where we use a slightly more general quotient of $\overline{\mathcal{T}}$ than the one described here.

Let A_Q be the group of pairs $(\widetilde{a}, a)$, where $a \in A$ and $\widetilde{a} : Q \to Q$ satisfies the condition $q \circ \widetilde{a} = a \circ q$, where $q : Q \to U_0$.

The natural map $[A_Q\backslash Q] \to [A\backslash U_0]$ is an equivalence. Thus, the map $Q \to \mathcal{T}$ induces a map of the quotients $[A_Q\backslash Q] \to [\Gamma\backslash\mathcal{T}]$. By Lemma 2.3.6 this gives rise to a homomorphism $\iota : A_Q \to \Gamma$. This homomorphism is uniquely determined by the requirement of commutativity of the diagram

$$\begin{tikzcd} Q \arrow[r, "\alpha_Q"] \arrow[d, "a"'] & \mathcal{T} \arrow[d, "\iota(a)"'] \\ Q \arrow[r, "\alpha_Q"] & \mathcal{T} \end{tikzcd}$$

Consider $A_{Q,G} = A_Q \times_\Gamma G$. We claim that the image of the composition

$$A_{Q,G} \to A_Q \to A$$

is precisely A_Z, so that we have got a morphism of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \Gamma_0 \cap G \arrow[r] \arrow[d] & A_{Q,G} \arrow[r] \arrow[d] & A_Z \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \Gamma_0 \arrow[r] & A_Q \arrow[r] & A \arrow[r] & 1 \end{tikzcd} . \tag{34}$$

In fact, since $A_{Q,G}$ acts on Q, the quotient $A'_Z = A_{Q,G}/(\Gamma_0 \cap G)$ acts on $Z = (\Gamma_0 \cap G)\backslash Q$ and is a subgroup of A. Thus $A'_Z \subseteq A_Z$. Since the composition

$$[A'_Z\backslash Z] \to [A_Z\backslash Z] \to [G\backslash \mathfrak{T}]$$

is an open embedding, one should necessarily have $A'_Z = A_Z$.

As it was explained in the proof of 6.2.3, the group Γ'_0 is normal in $A_{Q,G}$. Passing to the quotient by Γ'_0 in the upper line of (34), one gets the short exact sequence

$$1 \longrightarrow (\Gamma_0 \cap G)/\Gamma'_0 \longrightarrow A_{Q,G}/\Gamma'_0 \longrightarrow A_Z \longrightarrow 1$$

which identifies with the sequence

$$1 \longrightarrow K \longrightarrow H \longrightarrow A_Z \longrightarrow 1 \tag{35}$$

defined by Lemma 6.2.3.

Recall that $Y = \Gamma'_0\backslash Q$. Thus, one has an open embedding

$$[H\backslash Y] = [A_{Q,G}\backslash Q] = [A_Z\backslash Z] \longrightarrow [G\backslash \mathfrak{T}]. \tag{36}$$

6.3. Orbifold atlas for $G\backslash\overline{\mathfrak{T}}$

We now have a sufficient supply of orbifold charts for constructing an orbifold atlas for $G\backslash\overline{\mathfrak{T}}$. In order to arrange the constructed orbifold charts into an atlas, we have to present a chart category $\mathcal{A}$ and a functor $c : \mathcal{A} \to \texttt{Charts}/(G\backslash\overline{\mathfrak{T}})$ satisfying the properties of 3.1.9.

The chart (V, H, α) constructed above depends on the following choices.

1) A chart (U, A, β) in $\mathfrak{Q}$. The singular locus $U - U_0$, where $U_0 = \mathfrak{M} \times_{\overline{\mathfrak{M}}} U$, is a normal crossing divisor. The group A acts on U with a fixed point z (belonging to the intersection of the components)
2) A marking ϕ of the curve X_z (or, what is equivalent, a choice of a component of P).

The chart category $\mathcal{A}$ will be constructed simultaneously with a functor $p : \mathcal{A} \to \mathfrak{Q}$, so that if $c(p(a))$ is the chart (U, A, β) of $\overline{\mathfrak{M}}$, $c(a)$ is the chart (V, H, α) constructed in 6.1.

Recall some notation from Section 6.2. The chart V contains a dense open H-equivariant subset $Y = \Gamma'_0\backslash Q$ giving an open embedding

$$\hat{\alpha} : [H\backslash Y] \longrightarrow [G\backslash \mathfrak{T}]. \tag{37}$$

We define $\mathcal{A}$ as the category whose objects are open embeddings (37), where Y and H are obtained from an orbifold chart (V, H, α) described above. A morphism in $\mathcal{A}$ from $(Y_1, H_1, \hat{\alpha}_1)$ to $(Y_2, H_2, \hat{\alpha}_2)$ is defined as a morphism of abstract orbifold charts

$$(f_Y, f_H) : (Y_1, H_1) \longrightarrow (Y_2, H_2)$$

together with a 2-morphism

$$\theta_Y : \hat{\alpha}_2 \circ f_Y \simeq \hat{\alpha}_1.$$

Note that any object $(Y, H, \hat{\alpha})$ of $\mathcal{A}$ gives rise to a 2-commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d] & {[G\backslash\mathfrak{T}]} \arrow[d] \\ U \arrow[r, "\beta"] & \overline{\mathfrak{M}} \end{tikzcd} \tag{38}$$

so that the assignment $(Y, H, \hat{\alpha}) \mapsto (U, A, \hat{\beta})$ defines a functor $\mathcal{A} \to \mathfrak{Q}$. In fact, $Y = \Gamma_0'\backslash Q$ and the morphism $\hat{\alpha} : [H\backslash Y] \to [G\backslash\mathfrak{T}]$ can be realized by a pair of morphisms $(q : Q \to \mathfrak{T}, A_{Q,G} \to G)$. The pair $(q : Q \to \mathfrak{T},\ A_Q \to \Gamma)$ is compatible it on one side and with $\hat{\beta} : [A\backslash U] \to \overline{\mathfrak{M}}$ on the other side.

Furthermore, any map

$$\eta : (Y_1, H_1, \hat{\alpha}_1) \longrightarrow (Y_2, H_2, \hat{\alpha}_2)$$

in $\mathcal{A}$ lifts to a map $\eta : Q_1 \to Q_2$ which by Proposition 2.3.8 defines a unique diagram

$$\begin{tikzcd} Q_1 \arrow[r, "q_1"] \arrow[d, "\eta"'] & \mathfrak{T} \arrow[d, "g"] \\ Q_2 \arrow[r, "q_2"] & \mathfrak{T} \end{tikzcd} \tag{39}$$

where $g \in G$. This diagram induces a morphism of orbifold charts

$$\bar{\eta} : (U_{01}, A_1, \hat{\beta}_1) \longrightarrow (U_{02}, A_2, \hat{\beta}_2)$$

compatible with η.

Note that since U_{0i} is the smooth locus of U_i, any map $(U_1, A_1, \hat{\beta}_1) \to (U_2, A_2, \hat{\beta}_2)$ in $\mathfrak{Q}$ carries U_{01} to U_{02}, so that $\bar{f}$ extends uniquely to a morphism $(U_1, A_1, \hat{\beta}_1) \to (U_2, A_2, \hat{\beta}_2)$.

Let us prove that the above morphisms in $\mathcal{A}$ give an orbifold atlas for $G\backslash\overline{\mathfrak{T}}$. First of all, the category $\mathcal{A}$ is a chart category since it is a full subcategory of the chart category defined by the orbifold $[G\backslash\mathfrak{T}]$ via 3.1.11.

Let us show that an arrow of $\mathcal{A}$ defined as above, gives a morphism of the corresponding orbifold charts. First, $\eta_Y : Y_1 \to Y_2$ uniquely defines a map $\eta_V : V_1 \to V_2$ since V_i can be identified as the normalization of U_i in the field of meromorphic functions on Y_i. The map of abstract orbifold charts $(V_1, H_1) \to (V_2, H_2)$ is automatically defined since $\mathcal{A}$ is a chart category. To check that we have a map of orbifold charts over $G\backslash\overline{\mathfrak{T}}$, we have to check that the map $\eta_V : V_1 \to V_2$ is compatible with the projections $\alpha_i : V_i \to G\backslash\overline{\mathfrak{T}}$. This is enough to check on the dense subset Y_1 of V_1 where compatibility follows from the definition.

The required collection of isomorphisms $\iota : \mathrm{Aut}(a) \to H(a),\ a \in \mathcal{A}$, comes from the construction of $\mathcal{A}$ as a full subcategory of the chart category for $[G\backslash\mathfrak{T}]$.

Let us check that the images of the charts (V, H, α) cover the whole space $G\backslash\overline{\mathfrak{T}}$. Let (X, ψ), where X is a curve and ψ is a G-marking of X, represent a point of $G\backslash\overline{\mathfrak{T}}$. Choose a quasiconformal chart (U, A, β) of $\overline{\mathfrak{M}}$ containing $x \in U$ with $\beta(x) = X$. A choice of representative for the G-marking ψ defines a marking $\phi : S \to X_z$ and therefore a chart (V, H, α) for $G\backslash\overline{\mathfrak{T}}$. If $y \in V$ is a lifting of x, its image $\alpha(y)$ is a pair (X, ψ'). The G-markings ψ and ψ' define the same G-marking ϕ on X_z. Therefore, they differ by an element $\gamma \in \Gamma_0$. Since Γ_0 acts on V,[15] the point $\gamma(y)$ has the required image in $G\backslash\overline{\mathfrak{T}}$.

The last thing to be checked is the condition (ii) of definition 3.1.9.

Let $x \in G\backslash\overline{\mathfrak{T}}$ belong to the images of the orbifold charts $(V_i, H_i, \alpha_i),\ i = 1, 2$. Then the image y of x in $\overline{\mathcal{M}}$ is covered by $(U_i, A_i, \beta_i),\ i = 1, 2$. If $y_i \in U_i$ are preimages of $y \in \overline{\mathcal{M}}$, we can assume as in 5.3.8 that there exists an isomorphism of charts $\eta : (U_1, A_1, \beta_1) \to (U_2, A_2, \beta_2)$ sending y_1 to y_2.

Let (X_1, ϕ_1) and (X_2, ϕ_2) represent the curves at the points y_1 and y_2 of U_1 and U_2. Since both marked curves represent the same point $x \in G\backslash\overline{\mathfrak{T}}$, there exist an isomorphism $\theta : X_1 \to X_2$ and an element $g \in G$ making the diagram (32) commutative.

The isomorphism $\eta : U_1 \to U_2$ commutes with the maps $\hat{\beta}_i : U_i \to \overline{\mathfrak{M}}$. Therefore, η induces an isomorphism

$$\eta : U_{01} \longrightarrow U_{02} \tag{40}$$

The isomorphism (40) induces an isomorphism of the corresponding fundamental groups so that the Dehn twists defined by the nodes of X_1 map (up to sign) to the Dehn twists defined by the corresponding nodes of X_2. In particular, this implies that the numbers $k_1, \ldots, k_r$ defining the coverings Y_1 and Y_2, coincide.

Also, an isomorphism $\eta : Q_1 \to Q_2$ of the covering spaces of U_{0i} is induced so that the diagram

$$\begin{tikzcd} Q_1 \arrow[r] \arrow[d, "\eta"'] & \mathfrak{T} \arrow[d, "g"] \\ Q_2 \arrow[r] & \mathfrak{T} \end{tikzcd}$$

is commutative. This defines an isomorphism of the factors

$$Y_1 = \Gamma_0'\backslash Q_1 \longrightarrow \Gamma_0'\backslash Q_2 = Y_2.$$

This induces an isomorphism of V_i since V_i is the normalization of U_i in the field of meromorphic functions of Y_i.

[15] via Γ_0/Γ_0'

7. Properties of orbifolds $G\backslash\overline{\mathfrak{T}}$

In this section we establish some properties of the orbifold structure on $G\backslash\overline{\mathfrak{T}}$ introduced in the previous section. We start with showing that for the subgroup

$$\Gamma^{(\ell)} = \mathrm{Ker}(\Gamma \longrightarrow \mathrm{Aut}(H_1(S, \mathbb{Z}/\ell))).$$

the orbifold $[\Gamma^{(\ell)}\backslash\overline{\mathfrak{T}}]$ corresponds to the moduli stack of curves with level-ℓ structures. Besides of providing an interesting example, this fact will be used in 7.2.4 to construct the canonical map

$$\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$$

for an arbitrary finite-index subgroup $G \subset \Gamma$.

We also construct here gluing operations on the orbifolds $[G\backslash\overline{\mathfrak{T}}]$ which are induced by gluing operations for bordered surfaces.

7.1. Example: level-ℓ curves

Let $\ell > 2$ be a natural number. Define

$$\Gamma^{(\ell)} = \mathrm{Ker}(\Gamma \longrightarrow \mathrm{Aut}(H_1(S, \mathbb{Z}/\ell))).$$

$\Gamma^{(\ell)}$-marking on a smooth curve X is the same as a level-ℓ structure on X.

Let $(X, \phi : S \to X)$ represent a $\Gamma^{(\ell)}$-marked nodal Riemann surface. Choose a quasiconformal neighborhood (U, A, β) of $X \in \overline{\mathfrak{M}}$ and construct a corresponding chart (V, H, α) for $[\Gamma^{(\ell)}\backslash\overline{\mathfrak{T}}]$ as in 6.2.1. We assume that $\alpha(s) = (X, \phi)$ for some $s \in V$.

Recall that the group H is an extension

$$1 \longrightarrow K \longrightarrow H \longrightarrow A_Z \longrightarrow 1,$$

where A_Z is the subgroup of the automorphism group A of $X \in \overline{\mathfrak{M}}$ stabilizing the component Z.

7.1.1. Proposition. *For $G = \Gamma^{(\ell)}$ one has $A_Z = 1$.*

Proof. Let $h \in H$. We will check that the image $a \in A_Z$ of h induces a trivial action on the homology $H_1(X, \mathbb{Z}/\ell)$. This will imply that $a = 1$ since the automorphism group A of X acts faithfully on $H_1(X, \mathbb{Z}/\ell)$. Since the map $H \to A_Z$ is surjective, this will imply our claim.

Recall that $Y = \varkappa^{-1}(U_0) \subset V$. Choose $x \in Y$, $y = h(x)$ and let (X_x, ϕ_x), (X_y, ϕ_y) be the corresponding G-marked Riemann surfaces. The map α from diagram (29) induces an open embedding

$$[H\backslash Y] \longrightarrow [G\backslash\mathfrak{T}].$$

This gives rise to the following commutative diagram

$$\begin{tikzcd} H_1(X_x, \mathbb{Z}/\ell) \arrow[d, "a"'] & H_1(S, \mathbb{Z}/\ell) \arrow[l, "H_1(\phi_x)"'] \arrow[d, equal] \\ H_1(X_y, \mathbb{Z}/\ell) & H_1(S, \mathbb{Z}/\ell) \arrow[l, "H_1(\phi_y)"'] \end{tikzcd} \tag{41}$$

On the other hand, the family of curves with the base V defines a morphism $V \to U \to \overline{\mathfrak{M}}$. The element $h \in H$ induces an automorphism $a \in A$ of the family. This implies the commutativity of the diagram

$$\begin{tikzcd} H_1(X,\mathbb{Z}/\ell) \arrow[d, "a"'] & H_1(X_x,\mathbb{Z}/\ell) \arrow[l, "v_x"'] \arrow[d, "a"'] \\ H_1(X,\mathbb{Z}/\ell) & H_1(X_y,\mathbb{Z}/\ell) \arrow[l, "v_y"'] \end{tikzcd} , \tag{42}$$

where the horizontal arrows are the vanishing cycles maps.

We will show later that

$$v_x \cdot H_1(\phi_x) = v_y \cdot H_1(\phi_y) : H_1(S,\mathbb{Z}/\ell) \longrightarrow H_1(X,\mathbb{Z}/\ell). \tag{43}$$

Then comparing the diagrams (41) and (42) we see that

$$a : H_1(X,\mathbb{Z}/\ell) \to H_1(X,\mathbb{Z}/\ell)$$

is the identity, which yields the claim.

Let us now explain (43).

Let $\pi : \mathcal{X} \to V$ be the family of curves described in (28) (where the notation $\mathcal{X}'$ was used instead of $\mathcal{X}$). One has $X = \pi^{-1}(s),\ X_x = \pi^{-1}(x),\ X_y = \pi^{-1}(y)$. Let $j_s,\ j_x,\ j_y$ be the respective embeddings of $X,\ X_x,\ X_y$ into $\mathcal{X}$. The space $\mathcal{X}$ contracts to X, so j_s is a homotopy equivalence. Consider the diagram

$$\begin{tikzcd} H_1(S,\mathbb{Z}/\ell) \arrow[r, "H_1(\phi_y)"] \arrow[d, "H_1(\phi_x)"'] & H_1(X_y,\mathbb{Z}/\ell) \arrow[d, "H_1(j_y)"'] & \\ H_1(X_x,\mathbb{Z}/\ell) \arrow[r, "H_1(j_x)"] & H_1(\mathcal{X},\mathbb{Z}/\ell) & H_1(X,\mathbb{Z}/\ell) \arrow[l, "H_1(j_s)"', "\sim"] \end{tikzcd} . \tag{44}$$

The vanishing cycle homomorphism is the composition $j_s^{-1} j_x$; therefore, the compatibility (43) is equivalent to the commutativity of the diagram (44).

Finally, commutativity of (44) can be shown as follows. The restriction of the family $\pi : \mathcal{X} \to V$ to Y is locally trivial; thus, the assignment

$$x \in Y \mapsto H_1(X_x,\mathbb{Z}/\ell)$$

is a local system on Y. The maps

$$H_1(S,\mathbb{Z}/\ell) \xrightarrow{H_1(\phi_x)} H_1(X_x,\mathbb{Z}/\ell) \xrightarrow{H_1(j_x)} H_1(\mathcal{X},\mathbb{Z}/\ell)$$

give rise to a map of constant local systems $H_1(S,\mathbb{Z}/\ell) \to H_1(\mathcal{X},\mathbb{Z}/\ell)$ which therefore does not depend on $x \in Y$.

$\square$

7.2. Functoriality with respect to G

In this section we will prove that the orbifold structure on spaces $G\backslash\overline{\mathfrak{T}}$ is natural with respect to a subgroup G of the modular group Γ. Then we will use this fact to produce in 7.2.4 the map $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$ from the big commutative diagram (29).

7.2.1. A canonical map $[G_1\backslash\overline{\mathfrak{T}}] \to [G_2\backslash\overline{\mathfrak{T}}]$. Let $G_1 \subset G_2$ be two finite index subgroups of Γ. Then a canonical map

$$[G_1\backslash\overline{\mathfrak{T}}] \longrightarrow [G_2\backslash\overline{\mathfrak{T}}] \tag{45}$$

can be constructed as follows. Starting from an orbifold chart $(U, A, \beta) \in \mathcal{Q}$ of $\overline{\mathfrak{M}}$, we get as in 6.2.1 the charts $(V_i, H_i, \alpha_i),\ i = 1, 2$, and a compatible pair of maps $V_1 \to V_2,\ H_1 \to H_2$. Thus, a map of charts $(V_1, H_1, \alpha_1) \to (V_2, H_2, \alpha_2)$ is canonically defined, giving finally a map of orbifolds (45).

The group homomorphism $H_1 \to H_2$ appears in the commutative diagram whose construction is obvious.

$$\begin{tikzcd}
1 \arrow[r] & K_1 \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & A_{Z_1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_2 \arrow[r] & H_2 \arrow[r] & A_{Z_2} \arrow[r] & 1
\end{tikzcd}$$

The map $A_{Z_1} \to A_{Z_2}$ is injective. This implies that if, for instance, $G_2 = \Gamma^{(l)},\ l \geq 3$, then $A_{Z_1} = 1$.

Note that the map (45) is seldom étale.

The following result generalizes [36, Proposition 3].

7.2.2. Proposition. *For each positive integer k there exists a finite index subgroup $\Gamma_{(k)}$ of the modular group Γ satisfying the following property. For each collection $D_1, \dots, D_m$ of independent Dehn twists the intersection of $\Gamma_{(k)}$ with the group generated by $D_1, \dots, D_m$ is generated by some powers $D_1^{k_1}, \dots, D_m^{k_m}$ where all k_i are divisible by k.*

Proof. The case $n = 0$ follows from Looijenga's result [36, Proposition 3] Here is the definition of $\Gamma_{(k)}$ for $n = 0$. Let $\widetilde{S} \to S$ be a universal Prym cover, i.e. a Galois cover with $\mathrm{Gal}(\widetilde{S}/S) = H^1(S, \mathbb{Z}/2)$ considered as the quotient of $\pi_1(S)$ by the normal subgroup generated by the squares of the elements.

Without loss of generality we can assume that k is even and $k \geq 6$. The group $\Gamma_{(k)}$ is then the group of $\gamma \in \Gamma$ whose (arbitrary) lift $\widetilde{\gamma} : \widetilde{S} \to \widetilde{S}$ acts on $H^1(\widetilde{S}, \mathbb{Z}/k)$ as an element of $\mathrm{Gal}(\widetilde{S}/S)$. By [36, Proposition 3] the group $\Gamma_{(k)}$ satisfies the requirements of the proposition: its intersection with a group generated by $D_1, \dots, D_m$ is the group generated by $D_1^{k_1}, \dots, D_m^{k_m}$ where $k_i = k$ if D_i disconnects S, and $k_i = 2k$ otherwise.

The general case will be reduced to the case $n = 0$.

Let S be a compact oriented surface of genus g with n boundary components. We define a new surface T as the result of gluing S to $-S$ along the boundary in an obvious way. Thus T has no boundary and it is of genus $2g+n-1$. Any diffeomorphism ϕ of S preserving the boundary defines a diffeomorphism $\Delta(\phi)$ of T acting as ϕ on both S and $-S$. This construction preserves isotopy, and, therefore, induces a homomorphism of the modular groups

$$\Delta : \Gamma_S \longrightarrow \Gamma_T \tag{46}$$

of S and of T respectively. Lemma 7.2.3 below claims that Δ is injective. Then we define the subgroup $\Gamma_{(k),S}$ of Γ_S as $\Delta^{-1}(\Gamma_{(k),T})$.

If $D_1,\dots,D_m$ are independent Dehn twists in Γ_S, one has $2m$ independent Dehn twists $D_i^{\pm}$, $i=1,\dots,m$ in T defined by the corresponding circles in S and in $-S$. By [36, Proposition 3] an element $\prod(D_i^+)^{k_i^+}\prod(D_i^-)^{k_i^-}$ belongs to $\Gamma_{(k),T}$ if and only if $k_i^{\pm}$ are divisible by k or by $2k$, depending on i.[16] Since $\Delta(\prod D_i^{k_i}) = \prod(D_i^+)^{k_i}\prod(D_i^-)^{k_i}$, we get the required property. $\square$

Now we will prove injectivity of Δ.

7.2.3. Lemma. *The map $\Delta : \Gamma_S \to \Gamma_T$ defined in (46) is injective.*

Proof. Denote $\pi = \pi_1(S)$, $\Pi = \pi_1(T)$. We choose as the base point for both S and T a boundary point of S. The embedding $S \to T$ admits an obvious section which identifies $-S$ with S. Thus, the embedding $i : \pi \to \Pi$ of fundamental groups induced by the embedding $S \to T$ splits by a projection $\rho : \Pi \to \pi$.

The modular groups Γ_S and Γ_T act by outer automorphisms on the corresponding fundamental groups π and Π; the canonical maps

$$\alpha_S : \Gamma_S \to \mathrm{Out}(\pi), \text{ and } \alpha_T : \Gamma_T \to \mathrm{Out}(\Pi)$$

are well-known to be injective.

Define a map (this is not a group homomorphism!)

$$\nabla : \mathrm{Aut}(\Pi) \to \mathrm{Aut}(\pi)$$

by the formula $\nabla(\phi) = \rho\circ\phi\circ i$. One has for $\phi \in \mathrm{Aut}(\Pi)$, $g \in \Pi$,

$$\nabla(\mathrm{ad}(g)\circ\phi) = \rho\circ\mathrm{ad}(g)\circ\phi\circ i = \mathrm{ad}(\rho(g))\circ\nabla(\phi).$$

Thus ∇ induces a map $\nabla : \mathrm{Out}(\Pi) \to \mathrm{Out}(\pi)$. We claim that the diagram

$$\begin{tikzcd} \Gamma_S \arrow[r, "\Delta"] \arrow[d] & \Gamma_T \arrow[d] \\ \mathrm{Out}(\pi) & \mathrm{Out}(\Pi) \arrow[l, "\nabla"'] \end{tikzcd}$$

is commutative. In fact, if $\gamma : S \to S$ defines an element of Γ_S then $\alpha_S(\gamma)$ sends a loop $u \in \pi$ to $\gamma(u)$.[17] On the other hand, $\nabla\circ\alpha_T(\Delta(\phi)$ sends $u \in \pi$ to $\rho(\Delta(\phi)(u)) = \phi(u)$. Thus Δ is injective since α_S is injective. $\square$

[16] More precisely, this is k if D_i^+ disconnects T, and $2k$ if it does not.

[17] α_S can be actually defined as a homomorphism to $\mathrm{Aut}(\pi)$ since we chose the base point preserved by any γ.

7.2.4. Construction of the map $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$. Now we will use the reasoning of [10, Corollary 2.10] to prove that for any finite index subgroup $G \subset \Gamma$ there exists a smaller finite index subgroup $H \subset G \subset \Gamma$ such that $[H\backslash\overline{\mathfrak{T}}]$ is a manifold. This will allow to construct the canonical map $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$ from the big commutative diagram (29).

Let G be any finite index subgroup of the modular group Γ. The intersection $G' = G \cap \Gamma^{(l)}$ has also finite index. Let

$$G'' = \bigcap_{g\in\Gamma} gG'g^{-1}.$$

This is a normal subgroup of Γ contained in G and having a finite index. Since there are only finitely many Dehn twists in Γ up to conjugation, there exists k such that for each Dehn twist D one has $D^k \in G''$. Finally, consider the subgroup $G''' = G'' \cap \Gamma_{(k)}$. We claim that the quotient $[G'''\backslash\overline{\mathfrak{T}}]$ is a complex manifold.

Look at a chart (V, H, α) constructed as in in 6.2.1 for the quotient $G'''\backslash\overline{\mathfrak{T}}$. Recall that the group H appears as the extension of A_Z with the quotient $K = (G''' \cap \Gamma_0)/\Gamma_0'$, see the notation of 6.2.1–6.2.3. The group A_Z is trivial since $G''' \subseteq \Gamma^{(l)}$, see 7.2.1. Let us show the group K is trivial. Let $\gamma = \prod_{i=1}^r D_i^{d_i} \in G''' \cap \Gamma_{00}$. Then $\gamma \in \Gamma_{(k)}$ since $G''' \subseteq \Gamma_{(k)}$. Therefore, d_i are all divisible by k and $D_i^{d_i} \in \Gamma_{(k)}$. By the choice of k $D_i^{d_i} \in G''$ as well, so they belong to Γ_0'.

Thus, we see that $H = 1$. Therefore, $[G'''\backslash\overline{\mathfrak{T}}]$ is a complex manifold.

Assume now that $H_1,\ H_2$ are two finite index subgroups of $G \subset \Gamma$ such that $[H_i\backslash\overline{\mathfrak{T}}]$ are manifolds for $i = 1, 2$. Then the intersection $H_1 \cap H_2$ contains as well a finite index subgroup H_3 such that $[H_3\backslash\overline{\mathfrak{T}}]$ is a manifold. This proves that the compositions $\overline{\mathfrak{T}} \to [H_i\backslash\overline{\mathfrak{T}}] \to [G\backslash\overline{\mathfrak{T}}]$ coincide.

We can now define the map $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$ as the composition

$$\overline{\mathfrak{T}} \longrightarrow [H\backslash\overline{\mathfrak{T}}] \longrightarrow [G\backslash\overline{\mathfrak{T}}],$$

where H is any finite index subgroup of G such that the quotient $[H\backslash\overline{\mathfrak{T}}]$ is a manifold.

7.3. Gluing operations

For Riemann surfaces with parametrized boundary components, as well for stable curves with punctures, there exist natural gluing operations which correspond to compositions in a *modular operad*. Given two surfaces (resp., stable curves) $S_i,\ i = 1, 2$ of genus g_i with n_i parametrized boundary components (resp., with n_i punctures), one can glue them along ath component (resp., puncture) of S_1 and bth component (resp., puncture) of S_2 to get a surface (resp., a stable curve) of genus $g_1 + g_2$ with $n_1 + n_2 - 2$ boundary components (resp., punctures). Similarly we can glue two boundary components (resp., punctures) of S_1 and produce a surface (resp., a stable curve) of genus $g_1 + 1$ and $n_1 - 2$ boundary components (resp., punctures).

These gluing operations on surfaces and on stable curves are compatible in the sense that for two marked stable curves $\phi_1 : S_1 \to X_1$ and $\phi_2 : S_2 \to X_2$ one can define a new marked curve $\phi : S \to X$, where S is obtained by gluing S_1 and S_2 and X is obtained by gluing X_1 and X_2.

All this is almost obvious. Note, however, that gluing stable curves is canonical in the best possible way — it defines the maps of the corresponding moduli stacks (as described in a more detail in 7.3.3 below). To be justify our suggestion to interpret augmented Teichmüller spaces as projective limits of complex orbifolds $[G\backslash\overline{\mathcal{T}}]$ we have to show that the gluing operations for the augmented Teichmüller spaces descend to well-defined operations on complex orbifolds $[G\backslash\overline{\mathcal{T}}]$. This is done in the current subsection.

Below we describe gluing operations for different types of objects: first for surfaces with boundary in 7.3.1, then for augmented Teichmüller spaces—on the level of points—in 7.3.2. After that in 7.3.3 we recall the gluing operations for the stacks of stable curves, and in the last two subsections, 7.3.4 and 7.3.5, we describe the gluing operations on the level of complex orbifolds — quotients of the augmented Teichmüller spaces. Note that the description in 7.3.1 and 7.3.3 contain nothing new and the construction in 7.3.2 is fairly obvious.

7.3.1. Gluing bordered surfaces. In what follows we denote by $\mathcal{S}_{g,n}$ the groupoid whose objects are oriented surfaces of genus g with n labeled boundary components together with a parametrization of each component. The morphisms are diffeomorphisms preserving the parametrization of the boundary components, up to isotopy.[18] In particular, for $S \in \mathcal{S}_{g,n}$ the modular group of S is just $\Gamma(S) := \mathrm{Aut}_{\mathcal{S}_{g,n}}(S)$.

The following gluing operations are defined.

- Gluing two bordered surfaces: given $S_1 \in \mathcal{S}_{g_1,n_1}$ and $S_2 \in \mathcal{S}_{g_2,n_2}$, a choice of a pair of boundary components in S_1 and in S_2 defines

$$S_1 \circ S_2 \in \mathcal{S}_{g_1+g_2,n_1+n_2-2}.$$

- Gluing two boundary components: given $S \in \mathcal{S}_{g,n}$, a choice of a pair of boundary components defines a new surface $\bar{S} \in \mathcal{S}_{g+1,n-2}$.

The gluing operations are functorial; in particular, for $S = S_1 \circ S_2$ one has natural group homomorphisms $\Gamma_i \to \Gamma$ with $\Gamma_i = \Gamma(S_i),\ \Gamma = \Gamma(S)$.

The operations described above satisfy standard axioms saying that the collection

$$g, n \mapsto \mathcal{S}_{g,n}$$

gives a *modular operad* in the 2-category of groupoids.

[18]A version of this groupoid with non-numbered boundary components is called *extended Teichmüller groupoid* in [5].

7.3.2. Gluing augmented Teichmüller spaces. It is convenient to consider the augmented Teichmüller spaces as a collection of functors

$$\mathcal{S}_{g,n} \longrightarrow \mathtt{Top}$$

to topological spaces. The action of the modular groups on $\mathcal{T}(S)$ is built in in this approach. The gluing operations described above extend to the following maps connecting different $\overline{\mathcal{T}}(S)$:

$$\overline{\mathcal{T}}(S_1) \times \overline{\mathcal{T}}(S_2) \longrightarrow \overline{\mathcal{T}}(S_1 \circ S_2). \tag{47}$$

$$\overline{\mathcal{T}}(S) \longrightarrow \overline{\mathcal{T}}(\bar{S}). \tag{48}$$

The result of gluing $(X_i, \phi_i : S_i \to X_i)$ gives the pair

$$(X, \phi : S_1 \circ S_2 \to X),$$

where $X = X_1 \vee X_2$ is obtained by gluing X_i along the corresponding punctures, with $\phi = \phi_1 \vee \phi_2$ defined by ϕ_1 and ϕ_2 in an obvious way.

The second operation is defined similarly.

7.3.3. Gluing stable curves. The famous modular operad

$$(g, n) \mapsto \overline{\mathfrak{M}}_{g,n}$$

of moduli of stable curves is a close relative of the above.[19] In order to define the gluing operations

$$\overline{\mathfrak{M}}_{g_1,n_1} \times \overline{\mathfrak{M}}_{g_2,n_2} \longrightarrow \overline{\mathfrak{M}}_{g_1+g_2,n_1+n_2-2},$$

one has to be able to glue two families of punctured stable curves

$$\mathcal{X}_i \to V,\ i = 1, 2,$$

of types (g_i, n_i) along a chosen pair of punctures

$$s_1 : V \to \mathcal{X}_1,\ s_2 : V \to \mathcal{X}_2.$$

This is much easier than one could have imagined: the result is given by the colimit of the diagram

$$\mathcal{X}_1 \longleftarrow V \longrightarrow \mathcal{X}_2$$

defined by the choice of the punctures. The existence of such (very special) colimit is easily verified.

The second type gluing operation

$$\overline{\mathfrak{M}}_{g,n} \longrightarrow \overline{\mathfrak{M}}_{g+1,n-2}$$

is defined similarly. Let $\mathcal{X} \to V$ be a family of punctured stable curves of type (g, n) and let $s_{1,2} : V \to \mathcal{X}$ be a pair of punctures. Then the corresponding family $\bar{\mathcal{X}} \to V$ of type $(g + 1, n - 2)$ is defined by the coequalizer of the pair (s_1, s_2).

[19]Since $\overline{\mathfrak{M}}_{g,n} = [\Gamma(S)\backslash\overline{\mathcal{T}}(S)]$ for $S \in \mathcal{S}_{g,n}$

7.3.4. Gluing quotients of $\overline{\mathfrak{T}}_{g,n}$. Disconnected case. The gluing operations on augmented Teichmüller spaces described in 7.3.2 are just continuous maps of topological spaces. In this section we will show that they can be lifted to the level of orbifold maps for corresponding quotient orbifolds $[G\backslash\overline{\mathfrak{T}}]$.

Let us consider first the operation that corresponds to gluing two different surfaces.

Proposition. *Consider two surfaces $S_i \in \mathcal{S}_{g_i,n_i}$, $i = 1, 2$. Set as above*

$$S = S_1 \circ S_2,\ \overline{\mathfrak{T}}_i = \overline{\mathfrak{T}}(S_i),\ \Gamma_i = \Gamma(S_i),\ \Gamma = \Gamma(S).$$

Let $G \subset \Gamma$ be a finite index subgroup. Set $G_i = \Gamma_i \times_\Gamma G$, $i = 1, 2$. Then there exists a natural map of complex orbifolds

$$[G_1\backslash\overline{\mathfrak{T}}_1] \times [G_2\backslash\overline{\mathfrak{T}}_2] \longrightarrow [G\backslash\overline{\mathfrak{T}}] \tag{49}$$

which is compatible with the gluing operation (47) of topological spaces.

Proof. Choose an arbitrary pair of marked Riemann surfaces

$$((X_1, \phi_1), (X_2, \phi_2)) \in \overline{\mathfrak{T}}_1 \times \overline{\mathfrak{T}}_2.$$

By 5.3, there exists a quasiconformal orbifold chart (U, A, β) for $\overline{\mathfrak{M}}_{g,n}$, where $(g, n) = (g_1 + g_2, n_1 + n_2 - 2)$, with $X_z = X_1 \vee X_2$, $A = \mathrm{Aut}(X_z)$ and a pair of quasiconformal charts (U_i, A_i, β_i), $i = 1, 2$, for $\overline{\mathfrak{M}}_{g_i,n_i}$, with $X_i = X_{z_i}$, $A_i = \mathrm{Aut}(X_i)$, such that $U_1 \times U_2$ belongs to the preimage of U under the gluing map (47). This induces maps

$$f_U : U_1 \times U_2 \longrightarrow U, \text{ and } f_A : A_1 \times A_2 \longrightarrow A.$$

Since by (2.6.1) open substacks of a stack correspond to open subsets of its coarse space, we obtain the following 2-commutative diagram of stacks

$$\begin{tikzcd} {[A_1\backslash U_1] \times [A_2\backslash U_2]} \arrow[r, "(f_U, f_A)"] \arrow[d] & {[A\backslash U]} \arrow[d] \\ \overline{\mathfrak{M}}_{g_1,n_1} \times \overline{\mathfrak{M}}_{g_2,n_2} \arrow[r] & \overline{\mathfrak{M}}_{g,n} \end{tikzcd} \tag{50}$$

where the vertical arrows are embeddings of open substacks.

Recall that each node of $X_1 \vee X_2$ defines a component of the singular locus $U - U_0$ of U; in particular, the node x obtained by gluing the punctures of X_1 and X_2, defines a component D_x. The image of f_U lies in D_x and, moreover, f_U is an open embedding of $U_1 \times U_2$ to D_x.

Let $x_1, \ldots, x_r$ be the nodes of X_1, and $x_{r+1}, \ldots, x_{r+s}$ be the nodes of X_2. Then the nodes of $X_1 \vee X_2$ are

$$x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}, x.$$

The corresponding circles in $S_1 \circ S_2$ consist of the circles in S_1 of the form $C_i = \phi_1^{-1}(x_i)$, $i = 1, \ldots, r$, the circles in S_2 of the form $C_i = \phi_2^{-1}(x_i)$, $i = r + 1, \ldots, r + s$, and the common boundary component of S_1 and S_2.

Let D_i be the Dehn twists around $C_i,\ i=1,\dots,r$, in Γ_1, and around $C_i,\ i=r+1,\dots,r+s$, in Γ_2, let $\bar{D}_i$ be their images in Γ. Let

$$k_i=\begin{cases}\min\{d|(D_i)^d\in G_1\} & \text{for } i=1,\dots,r,\\ \min\{d|(D_i)^d\in G_2\} & \text{for } i=r+1,\dots,r+s.\end{cases}$$

By the choice of G_i the same values k_i define the ramification indices of V over U around the corresponding components of the singular locus. This implies that the fiber product $(U_1\times U_2)\times_U V$ is isomorphic to $V_1\times V_2$. Choose a morphism $f_V: V_1\times V_2\to V$ so that the diagram

$$\begin{tikzcd} V_1\times V_2 \arrow[r, "f_V"] \arrow[d] & V \arrow[d] \\ U_1\times U_2 \arrow[r, "f_U"] & U \end{tikzcd}$$

is Cartesian. Let us show that f_V in the diagram above can be chosen to be compatible with the maps

$$\alpha: V\longrightarrow G\backslash\overline{\mathfrak{T}},\ \alpha_i: V_i\longrightarrow G_i\backslash\overline{\mathfrak{T}}_i\ (i=1,2),$$

together with the operations

$$\overline{\mathfrak{T}}_1\times\overline{\mathfrak{T}}_2\longrightarrow\overline{\mathfrak{T}}$$

defined in 7.3.2. Let $v_i,\ i=1,2$, be the (only) preimages of $z_i\in U_i$ in V_i. Any choice of f_V sends the pair $(v_1,v_2)\in V_1\times V_2$ to the only preimage $v\in V$ of $z\in U$. Both $\alpha(f_V(v_1,v_2))$ and the result of gluing $\alpha_i(v_i)$ give the element $(X_1\vee X_2,\phi_1\vee\phi_2)\in G\backslash\overline{\mathfrak{T}}$. If now $(y_1,y_2)\in Y_1\times Y_2\subset V_1\times V_2$ with $\alpha_i(y_i)=(X_i',\phi_i')$, the image $\alpha(f_V(y_1,y_2))$ has form $(X_1'\vee X_2',\phi')$ where the G-markings ϕ' and $\phi_1'\vee\phi_2'$ are both consistent with $\phi_1\vee\phi_2$, that is differ by an element $\gamma\in\Gamma_{01}\times\Gamma_{02}$, where, as in 6.1.2, Γ_{01} and Γ_{02} are generated by the Dehn twists $D_1,\dots,D_r$ and $D_{r+1},\dots,D_{r+1}$. The element γ is unique modulo the intersection $(\Gamma_{01}\times\Gamma_{02})\cap(G_1\times G_2)$. The dependence of γ on the choice of the point (y_1,y_2) is continuous; therefore, γ is constant. Replacing now f_V to its composition with γ, we get a new f_V with the required property.

This allows one to lift the maps f_U and f_A to maps

$$f_V: V_1\times V_2\longrightarrow V,\ f_H: H_1\times H_2\longrightarrow H$$

connecting the orbifold charts of $[G_i\backslash\overline{\mathfrak{T}}_i]$ and $[G\backslash\overline{\mathfrak{T}}]$ and giving rise to a 2-commutative diagram

$$\begin{tikzcd} {[H_1\backslash V_1]\times[H_2\backslash V_2]} \arrow[r] \arrow[d] & {[H\backslash V]} \arrow[d] \\ {[G_1\backslash\overline{\mathfrak{T}}_1]\times[G_2\backslash\overline{\mathfrak{T}}_2]} \arrow[r] & {[G\backslash\overline{\mathfrak{T}}]} \end{tikzcd}\ . \tag{51}$$

The collections $(V_1 \times V_2, H_1 \times H_2, \alpha_1 \times \alpha_2)$ form an orbifold atlas for the product $[G_1 \backslash \overline{\mathfrak{T}}_2] \times [G_2 \backslash \overline{\mathfrak{T}}_2]$. The diagram (51) gives, in particular, a collection of maps

$$[H_1 \backslash V_1] \times [H_2 \backslash V_2] \longrightarrow [G \backslash \overline{\mathfrak{T}}].$$

Any morphism of charts

$$(V_1 \times V_2, H_1 \times H_2, \alpha_1 \times \alpha_2) \longrightarrow (V_1' \times V_2', H_1' \times H_2', \alpha_1' \times \alpha_2')$$

can be uniquely completed to a 2-commutative diagram

$$\begin{tikzcd} {[H_1 \backslash V_1] \times [H_2 \backslash V_2]} \arrow[r, "\hat{f}_V"] \arrow[d] & {[H \backslash V]} \arrow[d] \\ {[H_1' \backslash V_1'] \times [H_2' \backslash V_2']} \arrow[r, "\hat{f}_{V'}"] & {[H' \backslash V']} \end{tikzcd} . \qquad (52)$$

This gives the required map (49). □

7.3.5. Gluing quotients of $\overline{\mathfrak{T}}_{g,n}$. Connected case. Now we will describe gluing operation of the second type which corresponds to gluing two boundary components of the same surface.

Proposition. *Let $S \in \mathcal{S}_{g,n}$ and let $\bar{S} \in \mathcal{S}_{g+1,n-2}$ be obtained from S by gluing two chosen boundary components. Let $\Gamma = \Gamma(S)$, $\bar{\Gamma} = \Gamma(\bar{S})$. One has a group homomorphism $\Gamma \to \bar{\Gamma}$. Choose a finite index subgroup $\bar{G}$ of $\bar{\Gamma}$ and let $G = \Gamma \times_{\bar{\Gamma}} \bar{G}$. Then there exists a natural map of complex orbifolds*

$$[G \backslash \overline{\mathfrak{T}}(S)] \longrightarrow [\bar{G} \backslash \overline{\mathfrak{T}}(\bar{S})] \qquad (53)$$

compatible with the continuous map (48) of topological spaces.

Proof. Let $(X, \phi) \in \overline{\mathfrak{T}}(S)$ and let $(\bar{X}, \bar{\phi})$ be the corresponding point in $\overline{\mathfrak{T}}(\bar{S})$. By 5.3, there exists a quasiconformal orbifold chart $(\bar{U}, \bar{A}, \bar{\beta})$ for $\overline{\mathfrak{M}}_{g+1,n-2}$ with the exceptional curve $X_{\bar{z}} = \bar{X}$, $\bar{A} = \mathrm{Aut}(\bar{X})$ and a quasiconformal chart (U, A, β) for $\overline{\mathfrak{M}}_{g,n}$, with the exceptional curve $X_z = X$, $A = \mathrm{Aut}(X)$, such that U is contained in the preimage of $\bar{U}$ under the gluing map (48). This induces a pair of maps

$$f_U : U \longrightarrow \bar{U}, \ f_A : A \longrightarrow \bar{A},$$

and gives the following 2-commutative diagram of stacks

$$\begin{tikzcd} {[A \backslash U]} \arrow[r, "(f_U, f_A)"] \arrow[d] & {[\bar{A} \backslash \bar{U}]} \arrow[d] \\ \overline{\mathfrak{M}}_{g,n} \arrow[r] & \overline{\mathfrak{M}}_{g+1,n-2} \end{tikzcd} \qquad (54)$$

where the vertical arrows are open embeddings of stacks which are defined by (2.6.1).

Similarly to 7.3.4, the maps $f_U,\ f_A$ can be lifted to maps $f_V : V \to \bar{V}$ and $f_H : H \to \bar{H}$ defining the maps of orbifolds

$$V \longrightarrow [H\backslash V] \longrightarrow [\bar{H}\backslash\bar{V}] \longrightarrow [\bar{G}\backslash\overline{\mathfrak{T}}_{\bar{S}}].$$

Here (V, H, α) is a chart of $[G\backslash\overline{\mathfrak{T}}(S)]$ and $(\bar{V}, \bar{H}, \bar{\alpha})$ is the corresponding chart of $[\bar{G}\backslash\overline{\mathfrak{T}}(\bar{S})]$. A morphism

$$(V, H, \alpha) \longrightarrow (V', H', \alpha')$$

between the charts defines a canonical 2-morphism connecting $V \to [\bar{G}\backslash\overline{\mathfrak{T}}_{\bar{S}}]$ with $V' \to [\bar{G}\backslash\overline{\mathfrak{T}}_{\bar{S}}]$. This defines (53). $\square$

8. Augmented Teichmüller spaces and admissible coverings

Let S be a compact oriented surface S of genus g with n boundary components. Fix an unramified covering $\rho : \widetilde{S} \to S$ of degree d. To each marked stable curve $(X, \phi) \in \overline{\mathfrak{T}}(S)$ a very natural construction (described below in 8.1) assigns an admissible covering $\phi_*(\rho) : \widetilde{X} \to X$. The goal of this section is to show that this leads to a continuous map

$$v_\rho : \overline{\mathfrak{T}}(S) \to \mathfrak{Adm}_{g,n,d}.$$

The morphism $v_\rho : \overline{\mathfrak{T}}(S) \to \mathfrak{Adm}_{g,n,d}$ is defined as the composition

$$\overline{\mathfrak{T}}(S) \xrightarrow{\pi_{\widetilde{G}}} [\widetilde{G}\backslash\overline{\mathfrak{T}}] \xrightarrow{v_\rho^{\widetilde{G}}} \mathfrak{Adm}_{g,n,d}$$

where $\widetilde{G}$ is a group defined in 8.3 below and $v_\rho^{\widetilde{G}}$ is a morphism of complex orbifolds. Thus, using our interpretation of $\overline{\mathfrak{T}}$ as a projective system of complex orbifolds, v_ρ may be viewed as a projective system of morphisms of complex orbifolds.

The group $\widetilde{G}$ consists of pairs $(\widetilde{\gamma}, \gamma)$ where $\gamma \in \Gamma$ and $\widetilde{\gamma}$ is a lifting of γ to $\widetilde{S}$. It is not a subgroup of Γ since such lifting is not unique. Instead, one has a group homomorphism $\widetilde{G} \to \Gamma$ whose kernel is finite and whose image G has finite index in Γ. Thus, our standard definition of the quotients $[G\backslash\overline{\mathfrak{T}}]$ and of the canonical maps π_G given in 6.3, 7.2.4 does not meet our needs; the quotient $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$ and the canonical map $\pi_{\widetilde{G}}$ are defined in 8.4. The resulting orbifold is a gerbe over the quotient $[G\backslash\overline{\mathfrak{T}}]$ which is of the type we studied in Section 6. The orbifold charts for $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$ have form $(V, \widetilde{H}, \alpha)$ where $(V, H, \alpha) \in \mathcal{A}$ is a chart for $[G\backslash\overline{\mathfrak{T}}]$ and $\widetilde{H}$ is a group endowed with a surjective map to H.

The definition of the morphism $v_\rho^{\widetilde{G}} : [\widetilde{G}\backslash\overline{\mathfrak{T}}] \to \mathfrak{Adm}_{g,n,d}$ amounts to the construction of a compatible collection of admissible coverings for the families of curves corresponding to each orbifold chart of $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$.

There is an equivariant version of the construction: if $\rho : \widetilde{S} \to S$ is an H-covering where H is a finite group, a continuous map $v_{\rho,H} : \overline{\mathfrak{T}}(S) \to \mathfrak{Adm}_{g,n}(H)$ is defined. This is done in 8.5.

The morphisms v_ρ and $v_{\rho,H}$ have some important factorization properties with respect to gluing bordered surfaces, see 8.6. The factorization properties of the maps v_ρ and $v_{\rho,H}$ follow from the comparison of the corresponding admissible coverings for the orbifold charts of $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$.

The maps $v_\rho : \overline{\mathfrak{T}}(S) \to \mathfrak{Adm}_{g,n,d}$ are of ultimate importance in the construction of correction classes for the definition of stringy cohomology, see [27] and Section 8.7.

8.1. Pointwise construction

Fix a bordered surface S of genus g with n boundary components and a finite covering $\rho : \widetilde{S} \to S$.

Let $(X, \phi : S \to X)$ be a point of $\overline{\mathfrak{T}}(S)$. Using the marking $\phi : S \to X$ one can push the covering $\rho : \widetilde{S} \to S$ forward (see 8.1.1) to get an admissible covering $\phi_*(\rho) : \widetilde{X} \to X$.

In the case $\rho : \widetilde{S} \to S$ is an H-covering where H is a finite group, the map $\phi_*(\rho) : \widetilde{X} \to X$ acquires an action of H which is automatically balanced as we show in Lemma 8.1.2 below.

Thus, the map $\phi_*(\rho) : \widetilde{X} \to X$ becomes an admissible H-covering in the sense of Definition 4.3.1 of [4].

8.1.1. Pushforward of ρ. Here is the construction of $\phi_*(\rho)$. Outside the nodes and the punctures of X the covering $\phi_*(\rho)$ is the pullback of ρ via ϕ^{-1} with the complex structure on $\widetilde{X}$ induced from X. By passing to the normalization we get a ramified covering β of the normalization X^{nor} of X. Let p_1 and p_2 be two points of X^{nor} that correspond to a node p of X. The fibers of β at p_1 and p_2 are canonically identified with the orbits of monodromy of ρ around the loop $\phi^{-1}(p)$. Thus we obtain an admissible covering $\phi_*(\rho) : \widetilde{X} \to X$.

Assume now that ρ is an H-covering where H is a finite group. The group H in this case acts upon the map $\phi_*(\rho) : \widetilde{X} \to X$.

8.1.2. Lemma. *The action of H on $\phi_*(\rho) : \widetilde{X} \to X$ is balanced.*

Proof. Let $y \in \widetilde{X}$ be a node over $x \in X$ and let $h \in H_y$ stabilize y. Let $\widetilde{D}_+ \vee \widetilde{D}_-$ be a small neighborhood of y consisting of a pair of unit disks glued at y and let $D_+ \vee D_-$ be the corresponding neighborhood of $x \in X$. An element $h \in H_y$ acts on $\widetilde{D}_+$ and on $\widetilde{D}_-$ by multiplication by primitive n-th roots of unity, $\zeta_\pm$. Balancedness condition means that $\zeta_+\zeta_- = 1$. One can read out the values of $\zeta_\pm$ from the action of h on the nearby fiber of $\phi_*(\rho)$ at $x_\pm \in D_\pm$. Let $C = \phi^{-1}(x)$ and $\widetilde{C}$ be the component of $\rho^{-1}(C)$ corresponding to y. The annulus $\phi^{-1}(D_+ \vee D_-)$ in S admits an involution identifying the fibers at x_+ and x_-; the corresponding involution identifying D_+ and D_- is antiholomorphic. Therefore, ζ_+ and ζ_- are complex conjugate. $\square$

8.2. Modular group and some other automorphism groups

The classical Dehn-Nielsen-Baer theorem states that the modular group $\Gamma(S)$ embeds into the outer automorphism group $\mathrm{Out}(\pi_1(S))$. The latter group has an especially nice interpretation in terms of the fundamental groupoid $\Pi(S)$. In this subsection we present a groupoid interpretation for the modular group Γ and for some of its relatives.

8.2.1. Fundamental groupoid and the modular group. Recall that for a topological space X its fundamental groupoid $\Pi(X)$ has the points of X as the objects, and the homotopy classes of paths connecting the points as the arrows. We will be especially interested in $\Pi = \Pi(S)$ where S is a fixed oriented surface with boundary.

For (any) groupoid Π let $\mathrm{Seq}(\Pi)$ denote the groupoid of self-equivalences of Π and let $\mathrm{Aut}(\Pi)$ denote the corresponding group of isomorphism classes of objects of $\mathrm{Seq}(\Pi)$.

For a connected groupoid Π the group $\mathrm{Aut}(\Pi)$ is nothing but $\mathrm{Out}(\pi)$ where π is the automorphism group of an object of Π. Thus, for $\Pi = \Pi(S)$ the natural homomorphism from the modular group Γ to $\mathrm{Aut}(\Pi)$ is injective.

8.2.2. Variations. More generally, for a pair of groupoids Π_1, Π_2 we denote by $\mathrm{Eq}(\Pi_1, \Pi_2)$ the groupoid of equivalences $f : \Pi_1 \to \Pi_2$, so that $\mathrm{Eq}(\Pi, \Pi) = \mathrm{Seq}(\Pi)$. We write $\mathrm{Iso}(\Pi_1, \Pi_2)$ for the set of isomorphism classes of objects of Eq.

For a pair of functors $j_{1,2} : \Pi_{1,2} \to \Pi$ a groupoid $\mathrm{Eq}(j_1, j_2)$ has as objects pairs of equivalences,

$$f : \Pi_1 \longrightarrow \Pi_2, \quad g : \Pi \longrightarrow \Pi,$$

together with an isomorphism $\theta : g \circ j_1 \simeq j_2 \circ f$. Similarly to the above, $\mathrm{Iso}(j_1, j_2)$ is the set of isomorphism classes of objects of $\mathrm{Eq}(j_1, j_2)$. As a special case we get a groupoid $\mathrm{Seq}(j)$ and a group $\mathrm{Aut}(j)$.

8.2.3. Variations with coverings. Let X be a topological space with the fundamental groupoid Π. A covering $\rho : \widetilde{X} \to X$ can be described by a functor $\Sigma : \Pi \to \mathtt{Set}$ given by $\Sigma(x) = \rho^{-1}(x)$. This is a "basepoint-free" version of the usual description of a covering by the action of the fundamental group of X on a fiber.

We can define now more groupoids similarly to 8.2.2. Thus given

$$\Sigma_i : \Pi_i \to \mathtt{Set}, \ i = 1, 2,$$

one defines $\mathrm{Eq}((\Pi_1; \Sigma_1), (\Pi_2; \Sigma_2))$ as the groupoid whose objects are pairs (f, ϕ) where $f : \Pi_1 \to \Pi_2$ is an equivalence and $\phi : \Sigma_1 \to f^*(\Sigma_2)$ is an isomorphism. Similarly, for a pair of functors $j_{1,2} : \Pi_{1,2} \to \Pi$ and $\Sigma : \Pi \to \mathtt{Set}$ one defines $\mathrm{Eq}(j_1, j_2; \Sigma)$ to be the groupoid whose objects are quadruples (f, g, θ, ϕ) where

$$f : \Pi_1 \longrightarrow \Pi_2, \quad g : \Pi \longrightarrow \Pi, \quad \theta : g \circ j_1 \simeq j_2 \circ f, \quad \phi : \Sigma \simeq g^*\Sigma.$$

Isomorphism classes of objects of $\mathrm{Eq}(j_1, j_2; \Sigma)$ are denoted by $\mathrm{Iso}(j_1, j_2; \Sigma)$. The notations

$$\mathrm{Eq}((\Pi_1; \Sigma_1), (\Pi_2; \Sigma_2)), \ \mathrm{Iso}((\Pi_1; \Sigma_1), (\Pi_2; \Sigma_2)), \ \mathrm{Seq}(\Pi; \Sigma), \ \mathrm{Aut}(\Pi; \Sigma)$$

are self-evident.

The above defined groups and sets are connected by a bunch of forgetful maps which are all seen in the following commutative diagram corresponding to a pair $j_{1,2} : \Pi_{1,2} \to \Pi$ and to a functor $\Sigma : \Pi \to \mathtt{Set}$

$$\begin{tikzcd} \mathrm{Iso}((\Pi_1; j_1^*\Sigma), (\Pi_2; j_2^*\Sigma)) \arrow[d] & \mathrm{Iso}(j_1, j_2; \Sigma) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Aut}(\Pi; \Sigma) \arrow[d] \\ \mathrm{Iso}(\Pi_1, \Pi_2) & \mathrm{Iso}(j_1, j_2) \arrow[l] \arrow[r] & \mathrm{Aut}(\Pi) \end{tikzcd} . \tag{55}$$

Note that the right-hand side square of the diagram is Cartesian.

8.3. Choice of the group

In this subsection we present the group $\widetilde{G}$ which will appear in the decomposition

$$\overline{\mathfrak{T}}(S) \xrightarrow{\pi_{\widetilde{G}}} [\widetilde{G}\backslash\overline{\mathfrak{T}}] \xrightarrow{v_\rho} \mathfrak{Adm}_{g,n,d}.$$

The group $\widetilde{G}$ will be chosen as a certain subgroup of $\widetilde{\Gamma}$ which is defined as the group of pairs $(\widetilde{\gamma}, \gamma)$ where $\gamma \in \Gamma$ and $\widetilde{\gamma}$ is a lifting of γ to $\widetilde{S}$.

In the notation of 8.2.3 $\widetilde{\Gamma}$ is just the fiber product $\Gamma \times_{\mathrm{Aut}(\Pi)} \mathrm{Aut}(\Pi; \Sigma)$ where Π is the fundamental groupoid of S and Σ is defined by ρ.

Let C be a circle in S. We denote by ρ_C the pullback

$$\rho_C : C \times_S \widetilde{S} \to C,$$

and by ρ_C^k the pullback of ρ_C with respect to the k-sheeted covering $C \to C$. The covering ρ_C is determined up to isomorphism, by a monodromy operator acting on a d-element set; the covering ρ_C^k corresponds to the k-th power of this operator.

8.3.1. Proposition. *There exists a subgroup $\widetilde{G}$ of $\widetilde{\Gamma}$ satisfying the following properties.*

- *The kernel of the map $\widetilde{G} \to \widetilde{\Gamma} \to \Gamma$ is finite.*
- *The image G of the map $\widetilde{G} \to \widetilde{\Gamma} \to \Gamma$ has finite index.*
- *For any circle C in S with the Dehn twist $D \in \Gamma$, if for some k $D^k \in G$, then ρ_C^k is trivial.*

Proof. The kernel of the map $\widetilde{\Gamma} \to \Gamma$ identifies with $\mathrm{Aut}(\widetilde{S}/S)$; it is, therefore, finite. Thus, the first property of $\widetilde{G}$ is automatically fulfilled for any subgroup $\widetilde{G}$ of $\widetilde{\Gamma}$. Let us show that the image $\bar{\Gamma}$ of the map $\widetilde{\Gamma} \to \Gamma$ has finite index in Γ.

The covering $\rho : \widetilde{S} \to S$ is uniquely defined by the action of the fundamental group $\pi_1(S, s)$ at a point $s \in S$ on the finite set $\Sigma = \rho^{-1}(s)$. Since $\pi_1(S, s)$ is finitely generated, there are finite number of isomorphism classes of such coverings. An element $g \in \Gamma$ belongs to $\bar{\Gamma}$ if and only if the inverse image $g^*(\widetilde{S})$ is isomorphic to $\widetilde{S}$. Thus, Γ acts on a finite set (the set of isomorphism classes of coverings of degree d) and $\bar{\Gamma}$ is the stabilizer of one of its elements.

We will now prove that there is a finite index subgroup $\widetilde{G}$ of $\widetilde{\Gamma}$ satisfying the third property. Then the second property will be automatically fulfilled for $\widetilde{G}$.

The group Γ has a finite number of orbits on the set of (free homotopy classes of) non-trivial circles in S. Since $\bar{\Gamma}$ has finite index in Γ, it has as well a finite number of orbits. This implies that there exists an integer K such that for each non-trivial circle C one has $\rho_C^K = \mathrm{id}$. By Proposition 7.2.2 one can choose a finite index subgroup G in $\bar{\Gamma}$ such that for each non-trivial circle C in S the corresponding Dehn twist $D \in \Gamma$ satisfies the condition

$$D^k \in G \Longrightarrow k \text{ is divisible by } K.$$

We can now define $\widetilde{G} = G \times_\Gamma \widetilde{\Gamma}$. Clearly, the map $\widetilde{G} \to G$ is surjective. □

8.4. The quotient $[\widetilde{G} \backslash \overline{\mathfrak{T}}]$.

In this subsection we construct an orbifold atlas for the quotient of $\overline{\mathfrak{T}}$ modulo the group $\widetilde{G}$. The orbifold so defined is endowed with a canonical projection $\pi_{\widetilde{G}} : \overline{\mathfrak{T}} \to [\widetilde{G} \backslash \overline{\mathfrak{T}}]$.

Recall that our construction of the quotient $[G \backslash \overline{\mathfrak{T}}]$ described in Section 6 is valid only for finite index subgroups of Γ. We lack a general construction of the quotient modulo a group $\widetilde{G}$ acting on $\overline{\mathfrak{T}}$ via $f : \widetilde{G} \to \Gamma$ such that $\mathrm{Ker}(f)$ and $[\Gamma : \mathrm{Im}(f)]$ are finite. Our construction is specifically tailored for the groups $\widetilde{G}$ described in 8.3.

The orbifold atlas for the quotient $[\widetilde{G} \backslash \overline{\mathfrak{T}}]$ is a slight modification of the atlas for $[G \backslash \overline{\mathfrak{T}}]$ where G is the image of $\widetilde{G}$ in Γ. For each orbifold chart $(V, H, \alpha) \in \mathcal{A}$ we construct a group epimorphism $\widetilde{H} \to H$ which will give rise to a chart $(V, \widetilde{H}, \alpha)$ for the quotient modulo $\widetilde{G}$. Here is how to get $\widetilde{H}$.

8.4.1. Construction of the chart $(V, \widetilde{H}, \alpha)$. Recall 6.2.5 that the group H of symmetries of an orbifold chart (V, H, α) appears as the quotient

$$H = A_{Q,G} / \Gamma'_0 \tag{56}$$

where $A_{Q,G} = A_Q \times_\Gamma G$ and $\Gamma'_0 = \langle D_1^{k_1}, \dots, D_r^{k_r} \rangle$ is generated by appropriate powers of the Dehn twists D_i around the circles $C_i = \phi^{-1}(x_i)$ which are the preimages in S of the nodes of X_z. Define

$$A_{Q,\widetilde{G}} = A_Q \times_\Gamma \widetilde{G}. \tag{57}$$

One has a natural projection $A_{Q,\widetilde{G}} \to A_{Q,G}$. We claim that the subgroup Γ'_0 of $A_{Q,G}$ canonically lifts to $A_{Q,\widetilde{G}}$.

Let $\widetilde{C}_{ij}$ be the components of $\rho^{-1}(C_i)$ and let d_{ij} denote the degree of $\widetilde{C}_{ij}$ over C_i. By the choice of $\widetilde{G}$, k_i is divisible by all d_{ij}. Therefore, $D_i^{k_i}$ can be lifted to $\prod_j D_{ij}^{\frac{k_i}{d_{ij}}}$, where D_{ij} denotes the Dehn twist around $\widetilde{C}_{ij}$.

The image of Γ'_0 in $A_{Q,\widetilde{G}}$ will be denoted $\widetilde{\Gamma}'_0$.

Define now $\widetilde{H}=A_{Q,\widetilde{G}}/\widetilde{\Gamma}'_0$.

The formula (56) immediately gives a canonical surjection $\widetilde{H}\to H$ with the kernel isomorphic to $\mathrm{Ker}(\widetilde{\Gamma}\to\Gamma)=\mathrm{Aut}(\widetilde{S}/S)$.

The group $\widetilde{H}$ acting on V via H, we have got a (highly non-effective) orbifold chart $(V,\widetilde{H},\alpha)$ of $G\backslash\overline{\mathfrak{T}}=\widetilde{G}\backslash\overline{\mathfrak{T}}$.

Recall that V contains an open dense H-equivariant subset $Y=\Gamma'_0\backslash Q$. The group $\widetilde{H}$ acts on Y via H.

Lemma. *The map $\alpha:Y\to G\backslash\mathfrak{T}$ defines an orbifold chart $(Y,\widetilde{H},\alpha)$ for the quotient $[\widetilde{G}\backslash\mathfrak{T}]$.*

Proof. We have to check that the map $\alpha:[\widetilde{H}\backslash Y]\to[\widetilde{G}\backslash\mathfrak{T}]$ is an open embedding. Making the base change with respect to the map $\mathfrak{T}\to[\widetilde{G}\backslash\mathfrak{T}]$, we get the map

$$f:[A_{Q,\widetilde{G}}\backslash\widetilde{G}\times Q]\longrightarrow\mathfrak{T}, \tag{58}$$

where the group $A_{Q,\widetilde{G}}$ acts on the product $\widetilde{G}\times Q$ by $a(g,q)=(ga^{-1},aq)$. [20] We have to check that (58) is an open embedding. Since the map $[H\backslash Y]\to[G\backslash\mathfrak{T}]$ (see (36)) is an open embedding, the base change with respect to $\mathfrak{T}\to[G\backslash\mathfrak{T}]$ gives an open embedding

$$[A_{Q,G}\backslash G\times Q]\longrightarrow\mathfrak{T}$$

which is equivalent to the map f in (58). $\square$

We will show now how to organize the charts $(\widetilde{H},V,\alpha)$ into an atlas.

8.4.2. Construction of an atlas for $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$. The atlas category $\widetilde{\mathcal{A}}$ for $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$ is defined very similarly to the definition of $\mathcal{A}$, see 6.3.

The category $\widetilde{\mathcal{A}}$ is a full subcategory of the chart category corresponding to the orbifold $[\widetilde{G}\backslash\mathfrak{T}]$ via 3.1.11. Its objects are the triples $(Y,\widetilde{H},\hat{\alpha})$ coming from the charts $(V,\widetilde{H},\alpha)$.[21]

Note that, similarly to 6.3, every object in $\widetilde{\mathcal{A}}$ defines canonically a commutative diagram

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d] & U \arrow[d, "\beta"] \\ {[\widetilde{G}\backslash\mathfrak{T}]} \arrow[r] & {[G\backslash\mathfrak{T}]} \arrow[r] & \overline{\mathfrak{M}} \end{tikzcd}, \tag{59}$$

giving rise to the functors $\widetilde{\mathcal{A}}\to\mathcal{A}\to\mathcal{Q}$.

The functor $c:\widetilde{\mathcal{A}}\to\mathtt{Charts}/(\widetilde{G}\backslash\overline{\mathfrak{T}})$ assigns a chart $(V,\widetilde{H},\alpha)$ to a triple $(Y,\widetilde{H},\hat{\alpha})$.

[20] Here we identify an element $a\in A_{Q,\widetilde{G}}$ with its image in $\widetilde{G}$.

[21] So these are basically the same objects as in $\mathcal{A}$

The required collection of isomorphisms $\iota : \mathrm{Aut}(a) \to \widetilde{H}(a)$, $a \in \widetilde{\mathcal{A}}$, comes from the construction of $\widetilde{\mathcal{A}}$ as a full subcategory of the chart category for $[\widetilde{G}\backslash\mathfrak{T}]$.

Verification of the axioms of Definition 3.1.9 is immediate.

Thus, we proved the following result.

Proposition. *The category $\widetilde{\mathcal{A}}$ defined above, together with the charts $(V, \widetilde{H}, \alpha)$, gives an orbifold atlas for the quotient $\widetilde{G}\backslash\overline{\mathfrak{T}}$. The realization of the atlas denoted as $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$ contains the quotient $[\widetilde{G}\backslash\mathfrak{T}]$ as an open dense suborbifold.*

□

8.4.3. The canonical projection $\pi_{\widetilde{G}} : \overline{\mathfrak{T}} \to [\widetilde{G}\backslash\overline{\mathfrak{T}}]$. In Section 6 the canonical projection $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$ was constructed in the case G is a finite index subgroup of Γ. The idea was to find a smaller group H in G so that the quotient $[H\backslash\overline{\mathfrak{T}}]$ is a complex manifold, and to present the quotient map as the composition

$$\overline{\mathfrak{T}} \longrightarrow [H\backslash\overline{\mathfrak{T}}] \longrightarrow [G\backslash\overline{\mathfrak{T}}].$$

This approach will not work for the quotient modulo $\widetilde{G}$ since the action of $\widetilde{G}$ on $\overline{\mathfrak{T}}$ is not effective. To construct the canonical projection

$$\pi_{\widetilde{G}} : \overline{\mathfrak{T}} \to [\widetilde{G}\backslash\overline{\mathfrak{T}}] \tag{60}$$

we will use the already constructed map $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$.

The canonical map $[\widetilde{G}\backslash\overline{\mathfrak{T}}] \to [G\backslash\overline{\mathfrak{T}}]$ is a gerbe. Its base change with respect to the map $\pi_G : \overline{\mathfrak{T}} \to [G\backslash\overline{\mathfrak{T}}]$ gives a gerbe $\widetilde{\overline{\mathfrak{T}}} \to \overline{\mathfrak{T}}$. Since $\overline{\mathfrak{T}}$ is contractible, the gerbe $\widetilde{\overline{\mathfrak{T}}} \to \overline{\mathfrak{T}}$ is trivial, i.e. is isomorphic to the gerbe $\mathrm{Aut}(\widetilde{S}/S) \times \overline{\mathfrak{T}} \rightrightarrows \overline{\mathfrak{T}}$.

Fortunately, we can point out to a canonical trivialization of this gerbe. In fact, the base change of this gerbe with respect to the embedding $\mathfrak{T} \to \overline{\mathfrak{T}}$ gives a gerbe $\widetilde{\mathfrak{T}} \to \mathfrak{T}$ which is canonically trivialized by the fact that

$$\widetilde{\mathfrak{T}} = \mathfrak{T} \times_{[G\backslash\mathfrak{T}]} [\widetilde{G}\backslash\mathfrak{T}].$$

This trivialization defines a unique trivialization of the gerbe $\widetilde{\overline{\mathfrak{T}}} \to \overline{\mathfrak{T}}$. In particular, we have a canonical splitting $s : \overline{\mathfrak{T}} \to \widetilde{\overline{\mathfrak{T}}}$ ("zero section").

Now we can define the map $\pi_{\widetilde{G}}$ as the composition

$$\overline{\mathfrak{T}} \xrightarrow{s} \widetilde{\overline{\mathfrak{T}}} \longrightarrow [\widetilde{G}\backslash\overline{\mathfrak{T}}].$$

8.5. Construction of the map $v_\rho : \overline{\mathfrak{T}} \to \mathfrak{AOm}_{g,n,d}$

The canonical map $[\widetilde{G}\backslash\overline{\mathfrak{T}}] \to [G\backslash\overline{\mathfrak{T}}]$ gives rise to a family of marked nodal curves over $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$. In order to obtain a morphism of orbifolds

$$v_\rho^{\widetilde{G}} : [\widetilde{G}\backslash\overline{\mathfrak{T}}] \longrightarrow \mathfrak{AOm}_{g,n,d}, \tag{61}$$

we will construct below an admissible covering of this family corresponding to $\rho : \widetilde{S} \to S$. Then, composing $v_\rho^{\widetilde{G}}$ with the canonical projection $\pi_{\widetilde{G}}$ constructed in 8.4.3, we will finally produce the desired map

$$v_\rho : \overline{\mathcal{T}} \longrightarrow \mathfrak{Adm}_{g,n,d}.$$

8.5.1. Admissible covering of $\mathcal{X}_V \to V$. Let $(V, \widetilde{H}, \alpha) \in \widetilde{\mathcal{A}}$ be the orbifold chart for $[\widetilde{G}\backslash\overline{\mathcal{T}}]$ corresponding to a chart $(U, A, \beta) \in \mathcal{Q}$ and to a marking ϕ of the fiber X_z of the universal family $\pi : \mathcal{X} \to U$ at $z \in U$.

We intend to construct an admissible covering of the induced family $\pi_V : \mathcal{X}_V \to V$ corresponding to $\rho : \widetilde{S} \to S$.

Choose a contraction $c : \mathcal{X} \to X_z$ (the result will not depend on the choice). This induces a contraction $c : \mathcal{X}_V \to X_z$.

Let $x_1, \ldots, x_r$ be the nodes of X_z. Choose small neighborhoods $\mathcal{O}_i$ of x_i in X_z as in (QC3). The manifold $\mathcal{X}_V$ is covered by the following open subsets.

1. $\mathcal{Y} = c^{-1}(X_z - \{x_1, \ldots, x_r\})$.
2. $\mathcal{P}_i = c^{-1}(\mathcal{O}_i)$.

The sets $\mathcal{P}_i$ are disjoint; one has

$$\mathcal{Y} \cap \mathcal{P}_i = c^{-1}(\mathcal{O}_i - \{x_i\}).$$

A fiber of $\mathcal{P}_i$ at x looks as follows: if x does not belong to the i-th component of the singular locus, it is a small annulus around the circle $c_x^{-1}(x_i)$. Otherwise it is a standard neighborhood of the node $zw = 0$.

An admissible covering of $\mathcal{X}_V$ is uniquely described by admissible coverings on $\mathcal{Y}$ and on $\mathcal{P}_i$ together with isomorphisms on the intersections $\mathcal{Y} \cap \mathcal{P}_i$.

$\mathcal{Y}$ is a family of (non-compact) Riemann surfaces on V. Admissible covering of $\mathcal{Y}$ is the same as a unramified covering; it is defined uniquely up to unique isomorphism by a unramified covering of $X_z - \{x_1, \ldots, x_r\}$.

In particular, the marking $\phi : S \to X_z$ uniquely defines a unramified covering on $\mathcal{Y}$.

We denote $\mathcal{C}_\mathcal{Y} \to \mathcal{Y} \to V$ the resulting admissible covering.

The intersection of each $\mathcal{P}_i$ with $\mathcal{Y}$ is (homotopically) a union of two annuli. The induced unramified coverings on these annuli are determined by the restriction of $\rho : \widetilde{S} \to S$ to the circle $C_i = \phi^{-1}(x_i) \subset S$. The latter is a degree-d covering $\widetilde{C}_i = \coprod_j \widetilde{C}_{ij} \to C_i$, see 8.4.1.

Note that the constructed covering $\mathcal{C}_\mathcal{Y} \to \mathcal{Y}$ is endowed with a canonical isomorphism of the restriction $\mathcal{C}_\mathcal{Y}|_{\mathcal{Y}\cap\mathcal{P}_i} \to \mathcal{Y} \cap \mathcal{P}_i$ with the one defined by $\widetilde{C}_i$.

Proposition. *The covering $\mathcal{C}_\mathcal{Y} \to \mathcal{Y} \to V$ extends uniquely up to unique isomorphism to an admissible covering $\mathcal{C}_V \to \mathcal{X}_V \to V$.*

Proof. We have to construct admissible coverings $\mathcal{C}_i$ of $\mathcal{P}_i \to V$ endowed with isomorphisms of the restrictions $\mathcal{C}_i|_{\mathcal{Y}\cap\mathcal{P}_i} \to \mathcal{Y} \cap \mathcal{P}_i$ with the coverings defined by $\widetilde{C}_i$. This will allow to canonically glue the coverings into an admissible covering of $\mathcal{X}_V \to V$.

Recall that by the choice of G (see 8.3) k_i are divisible by the degrees d_{ij} of the components $\widetilde{C}_{ij}$ of $\widetilde{C}_i$ over C_i.

We are now looking for an admissible covering $\mathcal{C}_i$ of $\mathcal{P}_i \to V$ inducing $\widetilde{C}_i$ on both components of the intersection $\mathcal{Y} \cap \mathcal{P}_i$.

Let $V_{0i} = V - D_{x_i}$ be the collection of points which do not belong to the i-th component of the singular locus, see condition (QC6) of the quasi-conformal charts.

Let $\mathcal{P}_i^0$ be the preimage of V_{0i} in $\mathcal{P}_i$. Since $\mathcal{P}_i^0$ is smooth, the restriction of an admissible covering to $\mathcal{P}_i^0$ is unramified; it is therefore determined by the action of the fundamental group of $\mathcal{P}_i^0$ on a typical fiber of ρ.

The fundamental group of $\mathcal{P}_i^0$ is the free abelian group generated by two loops:

(Lp1) around the annulus in any fiber of $\pi : \mathcal{P}_i^0 \to V_{0i}$, and
(Lp2) around the singular locus of V_{0i}.

The first loop is homotopic to each one of the components of $\mathcal{Y} \cap \mathcal{P}_i$. The second loop is contractible in $\mathcal{Y}$.

Thus, the covering $\mathcal{C}_{\mathcal{Y}}$ of $\mathcal{Y}$ uniquely extends to a non-ramified covering $\mathcal{C}_i^0$ of the open part $\mathcal{P}_i^0$ of $\mathcal{P}_i$ so that its restriction to (Lp1) canonically identifies with $\widetilde{C}_i$, whereas the restriction to (Lp2) is trivial.

Now we have to show that the covering $\mathcal{C}_i^0$ of $\mathcal{P}_i^0$ uniquely extends to an admissible covering $\mathcal{C}_i$ of the family $\mathcal{P}_i \to V$.

Let us start with the uniqueness. The admissible covering $\mathcal{C}_i$ of $\mathcal{P}_i$, if it exists, is normal[22] and finite over $\mathcal{P}_i$. It can therefore be described as the normalization of $\mathcal{P}_i$ in the field of meromorphic functions on $\mathcal{C}_i^0$. This gives the uniqueness.

To prove the existence, note that the projection $\pi : \mathcal{P}_i \to V$ is analytically isomorphic by (QC3)(b) to the standard projection of the space

$$P_i = \{(u, v, t_1, \ldots, t_m) \in D^2 \times D^m | uv = t_i^{k_i}\}$$

to D^m. Here D is the standard polydisk and k_i is defined by the condition $k_i = \min\{k | D_i^k \in G\}$ where D_i is the Dehn twist around C_i.

The generators of the fundamental group are now presented by the loops

(Lp1) $\theta \mapsto (u \exp(2\pi i\theta), v \exp(-2\pi i\theta), t_1, \ldots, t_m)$.
(Lp2) $\theta \mapsto (u, v, t_1, \ldots, t_i \exp(2\pi i\theta), \ldots, t_m)$.

We have to present an admissible covering C_i of P_i which induces $\widetilde{C}_i$ on (Lp1) and a trivial covering on (Lp2).

The covering $\widetilde{C}_i$ of the circle C_i is uniquely determined by the degrees d_{ij} of each component. We know that d_{ij} divides k_i for each j. Thus, it is sufficient to present for each divisor d of k_i an admissible covering of $P_i \to D^m$ of degree d, such that the monodromy around (Lp1) acts transitively on the generic fiber of the covering, whereas the monodromy around (Lp2) acts trivially on it.

[22] $k[x, y, t]/(xy - t^r)$ is normal by Serre's criterion $R_1 + S_2$

Consider

$$\widetilde{P}_i = \{(\widetilde{u}, \widetilde{v}, t_1, \ldots, t_m) \in D^2 \times V | \widetilde{u}\widetilde{v} = t_i^{\frac{k_i}{d}}\},$$

and define the map $\widetilde{P}_i \to P_i$ by $u = \widetilde{u}^d,\ v = \widetilde{v}^d$. This gives the required admissible covering.

As we have already mentioned, the admissible coverings $\mathcal{C}_{\mathcal{Y}} \to \mathcal{Y} \to V$ and $\mathcal{C}_i \to \mathcal{P}_i \to V$ glue uniquely to get an admissible covering of the family $\mathcal{X}_V \to V$. □

The resulting admissible covering will be denoted as

$$\mathcal{C}_V \longrightarrow \mathcal{X}_V \longrightarrow V.$$

8.5.2. Theorem. *The admissible coverings $\mathcal{C}_V \to \mathcal{X}_V \to V$ constructed above canonically glue into an admissible covering of the universal curve $\mathcal{X}$ of $[\widetilde{G}\backslash\overline{\mathcal{T}}]$.*

Proof. To get an admissible covering over the whole quotient $[\widetilde{G}\backslash\overline{\mathcal{T}}]$, we have to construct a canonical isomorphism

$$\mathcal{C}_{V_1} \longrightarrow u^*\mathcal{C}_{V_2}$$

for each morphism $u : a_1 \to a_2$ in $\widetilde{\mathcal{A}}$, where $c(a_i) = (V_i, \widetilde{H}_i, \alpha_i)$.

Let $a_i = (Y_i, \widetilde{H}_i, \hat{\alpha}_i)$. A morphism $u : a_1 \to a_2$ is given by a triple

$$u_Y : Y_1 \longrightarrow Y_2, \quad u_H : \widetilde{H}_1 \longrightarrow \widetilde{H}_2, \quad \theta : \hat{\alpha}_1 \longrightarrow \hat{\alpha}_2 \circ \hat{u},$$

where $\hat{u} : [\widetilde{H}_1\backslash Y_1] \to [\widetilde{H}_2\backslash Y_2]$ is induced by (u_Y, u_H).

The admissible coverings $\mathcal{C}_{V_i},\ i = 1, 2$, are uniquely determined by their restrictions $\mathcal{C}_{Y_i},\ i = 1, 2$, to Y_i. Thus, it is enough to present a canonical isomorphism

$$\mathcal{C}_{Y_1} \longrightarrow u_Y^*(\mathcal{C}_{Y_2}) \tag{62}$$

of coverings of Y_1. Since u_Y is always injective, we can consider separately two cases: u is an embedding and u is an isomorphism. The case when u is an embedding is obvious. Let us assume now that u is an isomorphism.

Lift a map u_Y to a map $u_Q : Q_1 \to Q_2$ of the universal coverings. The obvious equivalences

$$[A_{Q_i,\widetilde{G}}\backslash Q_i] \longrightarrow [\widetilde{H}_i\backslash Y_i]$$

of the orbifolds allow one to translate a morphism u into a pair of commutative diagrams

$$\begin{array}{ccc} Q_1 & \longrightarrow & \mathcal{T} \\ \Big\downarrow{\scriptstyle u_Q} & & \Big\downarrow{\scriptstyle g} \\ Q_2 & \longrightarrow & \mathcal{T} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_{Q_1,\widetilde{G}} & \longrightarrow & \widetilde{G} \\ \Big\downarrow{\scriptstyle \mathrm{ad}(u_Q)} & & \Big\downarrow{\scriptstyle \mathrm{ad}(\widetilde{g})} \\ A_{Q_2,\widetilde{G}} & \longrightarrow & \widetilde{G} \end{array} \tag{63}$$

for some $g \in G$ and a lifting $\widetilde{g}$ of g in $\widetilde{G}$.

The element $\widetilde{g} \in \widetilde{G}$ defines an isomorphism (62) as follows. We assume that $z_i \in V_i$ satisfy the condition $u_V(z_1) = z_2$. Let Π_1 (resp., Π_2) be the fundamental groupoid of $S-\cup C_i^1$ (resp., $S-\cup C_i^2$), where $C_i^1=\phi_1^{-1}(x_i)$ and similarly for C_i^2, and let $j_i:\Pi_i\to\Pi,\ i=1,2$, be the obvious embeddings.

The element $g\in G$ appearing in the left-hand side of (63) represents an element of $\mathrm{Iso}(j_1,j_2)$ (see 8.2.2); its lifting $\widetilde{g}$ gives an element of $\mathrm{Iso}(j_1,j_2;\Sigma)$ as in the diagram (55). This defines an isomorphism between $(\Pi_1;j_1^*\Sigma)$ and $(\Pi_2;j_2^*\Sigma)$ which is precisely the isomorphism $\mathcal{C}_{Y_1}\to u_Y^*(\mathcal{C}_{Y_2})$ we need.

Another choice of lifting $u_Q:Q_1\to Q_2$ of u_Y leads to different g and $\widetilde{g}$. The difference is, however, not very serious. If u'_Q is another lifting, one has $u'_Q=u_Q\circ\gamma$ where $\gamma\in\Gamma'_0=\langle D_1^{k_1},\dots,D_r^{k_r}\rangle$. Thus the lifting u'_Q gives rise to the pair $g'\in G,\ \widetilde{g}'\in\widetilde{G}$ where

$$g'=g\gamma,\ \widetilde{g}'=\widetilde{g}\widetilde{\gamma}$$

and $\widetilde{\gamma}$ is the canonical lifting of γ.

Since $\widetilde{\gamma}$ is a product of Dehn twists along the components C'_{ij} of $\rho^{-1}(C_i)$, the induced element of $\mathrm{Iso}((\Pi_1;j_1^*\Sigma),(\Pi_2;j_2^*\Sigma))$ is the same.

The continuous map

$$v_\rho:\overline{\mathfrak{T}}\longrightarrow \mathfrak{Adm}_{g,n,d}$$

is constructed. $\square$

8.5.3. The map v_ρ on the level of points. To make sure we constructed exactly what was announced at the beginning of Section 8, let us describe the image $v_\rho(X,\phi)$ for arbitrary $(X,\phi)\in\overline{\mathfrak{T}}$.

We can assume that (X,ϕ) belongs to the image $\alpha(V)$ of an orbifold chart (V,H,α) of $G\backslash\overline{\mathfrak{T}}$.

The admissible covering $\mathcal{C}_V$ of V was constructed by gluing admissible coverings $\mathcal{C}_{\mathcal{Y}}$ and $\mathcal{C}_i$ of $\mathcal{Y}$ and of $\mathcal{P}_i$ respectively, see 8.5.1. Let $X=X_v$ for $v\in V$. The intersection $\mathcal{Y}\cap X$ is $X-c_v^{-1}\{x_1,\dots,x_r\}$ where $c_v:X=X_v\to X_z$ is the restriction of the contraction to X_v. An admissible covering of X is uniquely determined by its restriction to $X\cap\mathcal{Y}$; Since the G-markings of X and of X_z are compatible, the restriction of the admissible covering on $X\cap\mathcal{Y}$ induced from $\mathcal{C}_V$ is the same as the one described in 8.1. Therefore, $v_\rho(X,\phi)$ is presented by the admissible covering of X described in 8.1.

8.5.4. Admissible H-coverings. If $\rho:\widetilde{S}\to S$ is an H-covering, the resulting admissible coverings $\mathcal{C}_V$ of $(V,\widetilde{H},\alpha)$ acquire an H-action. Since the balancedness condition is verified at each point by 8.5.3 and 8.1.2, the admissible covering of $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$ becomes an admissible H-covering. Thus, a map

$$v_{\rho,H}:\overline{\mathfrak{T}}\longrightarrow \mathfrak{Adm}_{g,n}(H) \tag{64}$$

is defined.

8.6. Compatibilities

The augmented Teichmüller spaces $\overline{\mathfrak{T}}_{g,n}$ as well as the stacks of admissible coverings $\mathfrak{Adm}_{g,n,d}$ have various gluing operations giving rise to (a sort of) modular operads, see 7.3.

In this subsection we will describe the compatibility of these structures with the map v_ρ.

We also describe functoriality of $v_{\rho,H}$ with respect to the change of H.

The proofs of the properties 8.6.1–8.6.3 are given in 8.6.4–8.6.5. Basically, the properties follow directly from the construction of an admissible covering of $[\widetilde{G}\backslash\overline{\mathfrak{T}}_S]$ described in 8.5.2.

8.6.1. Functoriality for $v_{\rho,H}$. We will now describe functoriality for the maps $v_{\rho,H}$. Let $\rho:\widetilde{S}\to S$ be an H-covering and let $f:H\to H'$ be a finite group homomorphism. This defines an H'-covering $\rho':\widetilde{S}'\to S$ obtained by *induction* along $H\to H'$. If f is injective, $\widetilde{S}'$ consists of $[H':H]$ copies of $\widetilde{S}$. If f is surjective, $\widetilde{S}'$ is the quotient of $\widetilde{S}$ by the group $\mathrm{Ker}(f)$.

One has

Proposition. *A group homomorphism $f:H\to H'$ induces a map of the stacks*

$$f_*:\mathfrak{Adm}_{g,n}(H)\longrightarrow\mathfrak{Adm}_{g,n}(H').$$

Moreover, the following diagram

$$\begin{tikzcd}
\overline{\mathfrak{T}}(S) \arrow[r, "="] \arrow[d, "v_{\rho,H}"'] & \overline{\mathfrak{T}}(S) \arrow[d, "v_{\rho',H'}"'] \\
\mathfrak{Adm}_{g,n}(H) \arrow[r, "f_*"] & \mathfrak{Adm}_{g,n}(H')
\end{tikzcd}$$

is 2-commutative.

8.6.2. Factorization (gluing two bordered surfaces). Let $S_1\in\mathcal{S}_{g_1,n_1},\ S_2\in\mathcal{S}_{g_2,n_2}$ be two bordered surfaces. Choose a boundary component in each one of S_i and let $S=S_1\circ S_2\in\mathcal{S}_{g,n}$ where $g=g_1+g_2,\ n=n_1+n_2-2$.

Fix a finite covering $\rho:C\to S$ and let $\rho_i:C_i\to S_i$ be the induced covering of $S_i,\ i=1,2$.

Let Υ_d denote the (discrete) groupoid of finite multisets of weight d: its objects are pairs (X,w) where X is a finite set and $w:X\to\mathbb{Z}_{>0}$ satisfies $\sum w(x)=d$.

Let $\pi:C\to X$ be an admissible covering of degree d. Then any $x\in X$ defines an object of Σ_d: this is the set-theoretic preimage $\pi^{-1}(x)$ with the weight function defined by the multiplicities of the points of $\pi^{-1}(x)$. The covering C is non-ramified at x if and only if all points of $\pi^{-1}(x)$ have weight one. An admissible covering $\mathcal{C}\xrightarrow{\pi}\mathcal{X}\to V$ of degree d and a choice of a puncture $s:V\to\mathcal{X}$ defines a map $V\to\Upsilon_d$ which assigns to $v\in V$ the fiber of the map $\mathcal{C}_v\to\mathcal{X}_v$ at $\pi(v)$. This map is locally constant. Thus, the map

$$F_s:\mathfrak{Adm}_{g,n,d}\longrightarrow\Upsilon_d$$

of orbifolds is defined. In particular, a choice of boundary components of $S_i,\ i=1,2$, defines a pair of maps $\mathfrak{Adm}_{g_i,n_i,d}\to\Upsilon_d,\ i=1,2$.

Proposition. (1) *Gluing of S_i defines a canonical operation*

$$\mathfrak{Adm}_{g_1,n_1,d} \times_{\Upsilon_d} \mathfrak{Adm}_{g_2,n_2,d} \xrightarrow{\iota} \mathfrak{Adm}_{g,n,d}. \tag{65}$$

(2) *The product of maps v_{ρ_1} and v_{ρ_2}*

$$v_{\rho_1} \times v_{\rho_2} : \overline{\mathfrak{T}}(S_1) \times \overline{\mathfrak{T}}(S_2) \longrightarrow \mathfrak{Adm}_{g_1,n_1,d} \times \mathfrak{Adm}_{g_2,n_2,d}$$

is canonically factored through the map

$$\mathfrak{Adm}_{g_1,,n_1,d} \times_{\Upsilon_d} \mathfrak{Adm}_{g_2,n_2,d} \longrightarrow \mathfrak{Adm}_{g_1,n_1,d} \times \mathfrak{Adm}_{g_2,n_2,d}.$$

(3) *The following diagram*

$$\begin{tikzcd} \overline{\mathfrak{T}}(S_1) \times \overline{\mathfrak{T}}(S_2) \arrow[r, "\iota"] \arrow[d, "v_{1,2}"'] & \overline{\mathfrak{T}}(S) \arrow[d, "v_\rho"] \\ \mathfrak{Adm}_{g_1,n_1,d} \times_{\Upsilon_d} \mathfrak{Adm}_{g_2,n_2,d} \arrow[r, "\iota"] & \mathfrak{Adm}_{g,n,d} \end{tikzcd} \tag{66}$$

is 2*-commutative. Here $v_{1,2}$ is defined by $v_{\rho_1} \times v_{\rho_2}$ via (2).*

8.6.3. Factorization (gluing two boundary components). Let $S \in \mathcal{S}_{g,n}$ be a bordered surface. Gluing a pair of boundary components in S we get a surface $\bar{S} \in \mathcal{S}_{g+1,n-2}$ together with a canonical map $S \to \bar{S}$. Fix a finite covering $\rho : C \to \bar{S}$ and let $\rho_S : C_S \to S$ be the induced covering of S.

The choice of two boundary components in S defines a map

$$\mathfrak{Adm}_{g,n,d} \to \Upsilon_d \times \Upsilon_d$$

as in 8.6.2.

Proposition. (1) *Gluing of two boundary components of S defines a canonical operation*

$$\mathfrak{Adm}_{g,d,n} \times_{\Upsilon_d \times \Upsilon_d} \Upsilon_d \longrightarrow \mathfrak{Adm}_{g+1,n-2,d}. \tag{67}$$

(2) *The map*

$$v_{\rho_S} : \overline{\mathfrak{T}}(S) \longrightarrow \mathfrak{Adm}_{g,n,d}$$

is canonically factored through the projection onto the first factor

$$\mathfrak{Adm}_{g,d,n} \times_{\Upsilon_d \times \Upsilon_d} \Upsilon_d \longrightarrow \mathfrak{Adm}_{g,d,n}.$$

(3) *The following diagram*

$$\begin{tikzcd} \overline{\mathfrak{T}}(S) \arrow[r, "\iota"] \arrow[d, "v'_{\rho_S}"'] & \overline{\mathfrak{T}}(\bar{S}) \arrow[d, "v_\rho"] \\ \mathfrak{Adm}_{g,n,d} \times_{\Upsilon_d \times \Upsilon_d} \Upsilon_d \arrow[r, "\iota"] & \mathfrak{Adm}_{g+1,n-2,d} \end{tikzcd} \tag{68}$$

is 2*-commutative. Here v'_{ρ_S} is defined by v_{ρ_S} via (2).*

8.6.4. Operations for $\mathfrak{Adm}$. The induction operation

$$f_* : \mathfrak{Adm}_{g,n}(H) \to \mathfrak{Adm}_{g,n}(H')$$

can be constructed separately for the case $f : H \to H'$ is injective or surjective.

If f is injective and if $\mathcal{C} \to \mathcal{X} \to V$ is an admissible H-covering, its image under f_* is given by $\mathcal{C}' \to \mathcal{X} \to V$ where $\mathcal{C}'$ is disjoint union of $[H' : H]$ copies of $\mathcal{C}$.

If f is surjective, $\mathcal{C}'$ is the quotient of $\mathcal{C}$ by the action of $\mathrm{Ker}(f)$.

To define the gluing operation (65), we have to construct, given two families $\mathcal{C}_i \to \mathcal{X}_i \to V$, $i = 1, 2$, of admissible coverings, together with a choice of punctures $s_i : V \to \mathcal{X}_i$ and an isomorphism $\theta : F_{s_1} \to F_{s_2}$ in Υ_d, a glued up family $\mathcal{C} \to \mathcal{X} \to V$. We have already described (see 7.3.3) how to get $\mathcal{X}$ as the colimit of the diagram $\mathcal{X}_1 \longleftarrow V \to \mathcal{X}_2$. Similarly, the choice of punctures $s_1,\ s_2$ and of θ define a one-to-one correspondence between the punctures of $\mathcal{C}_1$ over s_1 and the punctures of $\mathcal{C}_2$ over s_2. The coproduct of $\mathcal{C}_1$ and $\mathcal{C}_2$ under an appropriate number of copies of V gives the admissible covering $\mathcal{C}$.

The gluing operation (67) is defined similarly.

8.6.5. Proof of 8.6.1–8.6.3. Proposition 8.6.1 results from the following obvious observation. Let $\mathcal{C}_V \to \mathcal{X}_V \to V$ (resp., $\mathcal{C}'_V \to \mathcal{X}_V \to V$) be an admissible covering constructed as in 8.5.1 for $\rho : \widetilde{S} \to S$ (resp., for $\rho' : \widetilde{S}' \to S$). Then $\mathcal{C}' = f_*(\mathcal{C})$.

To prove 8.6.2, let G be the finite index subgroup of $\Gamma(S)$ chosen as in 8.3 for the covering $\rho : C \to S = S_1 \circ S_2$; the groups $\Gamma(S_i)$ embed into $\Gamma(S)$; define $G_i = \Gamma_i \times_\Gamma G$.

The gluing operation

$$[G_1\backslash\overline{\mathfrak{T}}_1] \times [G_2\backslash\overline{\mathfrak{T}}_2] \longrightarrow [G\backslash\overline{\mathfrak{T}}]$$

defined in 7.3.4 extends trivially to its gerbe-version

$$[\widetilde{G}_1\backslash\overline{\mathfrak{T}}_1] \times [\widetilde{G}_2\backslash\overline{\mathfrak{T}}_2] \longrightarrow [\widetilde{G}\backslash\overline{\mathfrak{T}}],$$

where the groups $\widetilde{G}_i,\ \widetilde{G}$ are defined as in 8.3.

The property (3) of 8.6.2 results from 2-commutativity of the following diagram of complex orbifolds.

$$\begin{tikzcd}
[\widetilde{G}_1\backslash\overline{\mathfrak{T}}_1] \times [\widetilde{G}_2\backslash\overline{\mathfrak{T}}_2] \arrow[r, "\iota"] \arrow[d, "v_{1,2}"'] & {[\widetilde{G}\backslash\overline{\mathfrak{T}}]} \arrow[d, "v_\rho"'] \\
\mathfrak{Adm}_{g_1,n_1,d} \times_{\Upsilon_d} \mathfrak{Adm}_{g_2,n_2,d} \arrow[r, "\iota"] & \mathfrak{Adm}_{g,n,d}
\end{tikzcd} \qquad (69)$$

The latter results from the following observation. Let $(V_i, \widetilde{H}_i, \alpha_i),\ i = 1, 2$, and $(V, \widetilde{H}, \alpha)$ be the charts for the quotients $[\widetilde{G}_1\backslash\overline{\mathfrak{T}}_1]$, $[\widetilde{G}_2\backslash\overline{\mathfrak{T}}_2]$ and $[\widetilde{G}\backslash\overline{\mathfrak{T}}]$

respectively, so that a map

$$f_V : V_1 \times V_2 \longrightarrow V$$

realizes the gluing operation (69) as in 7.3.4. The spaces V_1, V_2 and V are bases of families of admissible coverings $\mathcal{C}_i \to \mathcal{X}_i \to V_i$ and $\mathcal{C} \to \mathcal{X} \to V$ constructed as in 8.5.1. Then the inverse image $f_V^*(\mathcal{C})$ identifies with the admissible covering based on $V_1 \times V_2$ obtained by gluing of

$$\mathcal{C}_1 \times V_2 \longrightarrow \mathcal{X}_1 \times V_2 \longrightarrow V_1 \times V_2$$

and

$$V_1 \times \mathcal{C}_2 \longrightarrow V_1 \times \mathcal{X}_2 \longrightarrow V_1 \times V_2$$

as described in 8.6.4.

The observation follows from the fact that an admissible covering of the family $(\mathcal{X}_1 \times V_2) \vee (V_1 \times \mathcal{X}_2)$ is uniquely defined by its restriction to the smooth locus of the exceptional curve $X_1 \vee X_2$ — see 8.5.1.

The proof of 8.6.3 goes along the same lines.

8.7. Associativity of the stringy orbifold cup-product

As an application of the results proved earlier in this section, we will show how they can be used in the study of orbifold cohomology.

Based on work of string theorists, Chen and Ruan in [15] (see also [19]) introduced a new invariant of almost complex orbifolds called the stringy orbifold cohomology ring. Multiplication in this ring is defined in a very non-trivial way and the proof of its associativity given in [15] and [19] involves various moduli spaces of stable Riemann surfaces with punctures.

In the forthcoming work [27] we will show that augmented Teichmüller spaces and their properties established in this paper provide a very natural tool for dealing with various orbifold cohomology theories.

Here we will only illustrate this by showing how to fix some problems in the proofs of associativity of the stringy orbifold cup-product given in [15] and in [19] (we elaborate on this in Remark 8.7.2 below).

Let $X = [Y/G]$ be an almost complex global quotient orbifold, i.e. Y is an almost complex manifold and G a finite group which acts on Y by diffeomorphisms preserving the almost complex structure. Proof of associativity of stringy orbifold cohomology cup-product reduces to the following statement.

Let g_1, g_2, g_3, g_4 be a quadruple of elements in G with $g_1 g_2 g_3 g_4 = 1$. Let

$$H = \langle g_1, g_2, g_3, g_4 \rangle \subset G$$

be the subgroup in G generated by these elements. Define two representations V_L and V_R of the group H as follows. Let $(C, p_1, p_2, p_3, p_4) \in \mathcal{M}_{0,4}$ be a nodal Riemann surface obtained by gluing two Riemann spheres C_1 and C_2 at a point p with punctures p_1, p_2 on the component C_1 and p_3, p_4 on C_2. Let

$$\pi : \widetilde{C} \to C$$

be the Galois covering of C with the Galois group H unramified outside of the punctures $\{p_1, p_2, p_3, p_4\} \subset C$, such that the monodromy around p_i is given by the action of $g_i \in H$. Let

$$V_L = H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$$

be the representation of H given by the action of H on $\widetilde{C}$.

Note that this covering depends on a choice of a marking of C, i.e. of an identification of the fundamental group $\pi_1(C - \{p_1, p_2, p_3, p_4\})$ with the free group

$$F_3 = \langle x_1, x_2, x_3, x_4 | \prod x_i = 1 \rangle.$$

Another representation of H denoted V_R is constructed by relabeling the marked points. Now we put the points p_1 and p_3 on C_1 and the points p_2 and p_4 on C_2.

The proof of associativity of stringy orbifold cup-product in [19] reduces to the following statement.

8.7.1. Lemma. *The representations V_L and V_R of the group H are isomorphic.*

Proof. Let S be a surface obtained by removing four open disks (holes) from S^2. The fundamental group of S can be identified with

$$F_3 = \langle x_1, x_2, x_3, x_4 | \prod x_i = 1 \rangle,$$

where x_i corresponds to the path going around the boundary of the ith hole. This gives an epimorphism $F_3 \to H$ and with it a canonical H-covering $\rho : \widetilde{S} \to S$.

Due to the result of Section 8.5.4 there exists a map (64)

$$v_{\rho,H} : \overline{\mathfrak{T}}(S) \longrightarrow \mathfrak{AOm}_{g,n}(H)$$

for certain g and n.

The tautological family of curves

$$\widetilde{C} \xrightarrow{\pi} C \xrightarrow{\sigma} \mathfrak{AOm}_{g,n}(H)$$

gives an H-equivariant vector bundle $\mathcal{V}$ on $\mathfrak{AOm}_{g,n}(H)$ defined by

$$\mathcal{V} = R^1(\sigma\pi)_*(\mathcal{O}_{\widetilde{C}})$$

which induces via $v_{\rho,H}$ an H-equivariant vector bundle

$$\mathcal{W} = v^*_{\rho,H}(\mathcal{V})$$

on $\overline{\mathfrak{T}}(S)$.

Representations V_L and V_R of H constructed above appear as fibers of $\mathcal{W}$ at two different boundary points of $\overline{\mathfrak{T}}(S)$ (they correspond to curves C and $\bar{C}$ with specific choices of marking). The desired isomorphism between V_L and V_R now follows from connectedness of $\overline{\mathfrak{T}}(S)$. $\square$

8.7.2. Remark. The proof of this lemma given in [19] uses the moduli stack $\mathfrak{M}_{g,n}$ instead of the augmented Teichmüller space $\overline{\mathfrak{T}}$.

This does not allow to take into account the dependence of the construction of relevant coverings on the choice of markings (which is equivalent to an identification of the fundamental group of the punctured surface with the free group). An attempt to resolve the issue by replacing $\mathfrak{M}_{g,n}$ with the stack $\mathfrak{Adm}$ hits the problem of high non-connectivity of $\mathfrak{Adm}$.

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