

IRREDUCIBLE 4-MANIFOLDS WITH ABELIAN NON-CYCLIC FUNDAMENTAL GROUP OF SMALL RANK

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In this paper we construct several irreducible 4-manifolds, both small and arbitrarily large, with abelian non-cyclic fundamental group. The manufacturing procedure allows us to fill in numerous points in the geography plane of symplectic manifolds with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z}_q \oplus \mathbb{Z}_p$ ($\gcd(p, q) \neq 1$). We then study the botany of these points for $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$.

1. MANUFACTURED MANIFOLDS

The main results in this paper are:

Theorem 1. *Let G be either $\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$ or $\mathbb{Z}_q \oplus \mathbb{Z}_p$. Let $n \geq 1$ and $m \geq 1$. For each of the following pairs of integers*

- (1) $(c, \chi) = (7n, n)$,
- (2) $(c, \chi) = (5n, n)$,
- (3) $(c, \chi) = (4n, n)$,
- (4) $(c, \chi) = (2n, n)$,
- (5) $(c, \chi) = ((6 + 8g)n, (1 + g)n)$ (for $g \geq 0$),
- (6) $(c, \chi) = (7n + (6 + 8g)m, n + (1 + g)m)$,
- (7) $(c, \chi) = (7n + 5m, n + m)$,
- (8) $(c, \chi) = (7n + 4m, n + m)$,
- (9) $(c, \chi) = (7n + 2m, n + m)$,
- (10) $(c, \chi) = ((6 + 8g)n + 5m, (1 + g)n + m)$ (for $g \geq 0$),
- (11) $(c, \chi) = ((6 + 8g)n + 4m, (1 + g)n + m)$ (for $g \geq 0$),
- (12) $(c, \chi) = ((6 + 8g)n + 2m, (1 + g)n + m)$ (for $g \geq 0$),
- (13) $(c, \chi) = (5n + 4m, n + m)$,
- (14) $(c, \chi) = (5n + 2m, n + m)$,
- (15) $(c, \chi) = (4n + 2m, n + m)$ and

there exists a symplectic irreducible 4-manifold X with

$$\pi_1(X) = G \text{ and } (c_1^2(X), \chi_h(X)) = (c, \chi).$$

Proposition 2. *Fix $\pi_1(X) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is a prime number greater than two. Let (c, χ) be any pair of integers given in Theorem 1 such that $n + m \geq 2$. There exists an infinite family $\{X_n\}$ of homeomorphic, pairwise non-diffeomorphic irreducible smooth non-symplectic 4-manifolds realizing the coordinates (c, χ) .*

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The characteristic numbers are given in terms of $\chi_h = 1/4(e+\sigma)$ and $c_1^2 = 2e+3\sigma$, where e is the Euler characteristic of the manifold X and σ its signature.

The geography problem for abelian fundamental groups of small rank has already been previously studied with great success. In Gompf's gorgeous paper [14] where the symplectic sum operation was introduced, infinitely many minimal symplectic 4-manifolds with $b_2^+ > 1$ were constructed. Gompf also constructed a new family of symplectic spin 4-manifolds with any prescribed fundamental group. In [4], [5] and [6], more and smaller symplectic manifolds were constructed.

Other construction techniques have also been implemented. For the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_p$, examples with big Euler characteristic were constructed using genus 2 Lefschetz fibrations in [16] and [18]. Results studying the symplectic geography for prescribed fundamental groups appeared in [6] and [4]. Concerning the botany, J. Park in [17] constructed infinitely many smooth structures on big 4-manifolds with finitely generated fundamental group.

The addition of Luttinger surgery (cf. [15], [3]) into the manufacturing procedure has provided clean constructions to study rather effectively the geography of simply connected 4-manifolds (cf. [4], [1], [2]). On the botany part, the technique of doing of using a nullhomologous torus as a dial in order to change the smooth structure developed in [9] and [8] has proven succesful to study the botany. In this paper, we apply these efforts to manifolds with the three given fundamental groups.

Our results provide manifolds with both $12\chi - c$ small and arbitrarily large. Most of the points filled in by Theorem 1 were not yet considered elsewhere. For example, the point $(7, 1)$ corresponds to the smallest manifold built up to now. A blunt overlap occurs for the points $(6 + 8g, 1 + g)$, $(5, 1)$ and $(4, 1)$, which have been filled in already by constructions given in [4] and [5]; we are using their constructions to build larger manifolds, thus filling in considerably many more points. The existence of at least two smooth structures on complex surfaces with finite non-cyclic fundamental groups was first studied in [11]. Proposition 2 takes advantage of the recent techniques and offers a myriad of new exotic irreducible 4-manifolds with finite abelian, yet non-cyclic fundamental group hosting infinitely many smooth structures; it includes the smallest manifold with such π_1 known to posses this quality.

The assumption $gcd(p, q) \neq 1$ serves the sole purpose of emphasizing that the results in this paper are disjoint from the cyclic case studied in [20]. We feel the results presented here deserve their own space and they should not be buried in a long paper for several reasons. Amongst them is the employment of the homeomorphism criteria for finite groups of odd order (cf. [11]) given in Section 6.3.

The blueprint of the paper is as follows. The geography is addressed first; Section 2 starts by describing the ingredients we will use to build the manifolds of Theorem 1. The manufacturing procedure starts later on in this section. The results that allow us to conclude irreducibility are presented in Section 3. The fourth section takes care of the fundamental group calculations. The fifth section gathers up our

efforts into the proof of Theorem 1. The last part of the paper goes into the botany, where Section 6 takes on the existence of the exotic smooth structures claimed in Proposition 2.

2. RAW MATERIALS

The following definition was introduced in [1].

Definition 3. An ordered triple (X, T_1, T_2) consisting of a symplectic 4-manifold X and two disjointly embedded Lagrangian tori T_1 and T_2 is called a *telescoping triple* if

- (1) The tori T_1 and T_2 span a 2-dimensional subspace of $H_2(X; \mathbb{R})$.
- (2) $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism $\pi_1(X - (T_1 \cup T_2)) \rightarrow \pi_1(X)$. In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2))$.
- (3) The image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \rightarrow \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$.
- (4) The homomorphism induced by inclusion $\pi_1(T_2) \rightarrow \pi_1(X)$ is an isomorphism.

The telescoping triple is called *minimal* if X itself is minimal. Notice the importance of the order of the tori. The meridians μ_{T_1}, μ_{T_2} in $\pi_1(X - (T_1 \cup T_2))$ are trivial and the relevant fundamental groups are abelian. The push off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of T_i represents a well defined element of $\pi_1(X - (T_1 \cup T_2))$ which is independent of the choices of framing and base-point.

The first condition assures us that the Lagrangian tori T_1 and T_2 are linearly independent in $H_2(X; \mathbb{R})$. This allows for the symplectic form on X to be slightly perturbed so that one of the T_i remains Lagrangian while the other becomes symplectic. The symplectic form can also be perturbed in such way that both tori become symplectic. If we were to consider a symplectic surface F in X disjoint from T_1 and T_2 , the perturbed symplectic form can be chosen so that F remains symplectic.

Removing a surface from a 4-manifold usually introduces new generators into the fundamental group of the resulting manifold. The second condition indicates that the meridians are nullhomotopic in the complement and, thus, the fundamental group of the manifold and the fundamental group of the complement of the tori in the manifold coincide.

Out of two telescoping triples, one is able to produce another telescoping triple as follows. If both X and X' are symplectic manifolds, then the symplectic sum along the symplectic tori $X \#_{T_2, T_1'} X'$ has a symplectic structure ([14]). If both X and X' are minimal, then the resulting telescoping triple is minimal too (by Usher's theorem cf. [21]).

Proposition 4. (cf. [1]). *Let (X, T_1, T_2) and (X', T_1', T_2') be two telescoping triples. Then for an appropriate gluing map the triple*

$$(X \#_{T_2, T'_1} X', T_1, T'_2)$$

is again a telescoping triple.

The Euler characteristic and the signature of $X \#_{T_2, T'_1} X'$ are given by $e(X) + e(X')$ and $\sigma(X) + \sigma(X')$.

We refer the reader to theorems 20 and 13 and to proposition 12 in [4] for the proof and for more details. The building blocks we will use are gathered together in the following theorem.

Theorem 5. • *There exists a minimal telescoping triple (A, T_1, T_2) with $e(A) = 5$, $\sigma(A) = -1$.*

- *For each $g \geq 0$, there exists a minimal telescoping triple (B_g, T_1, T_2) satisfying $e(B_g) = 6 + 4g$, $\sigma(B_g) = -2$.*
- *There exists a minimal telescoping triple (C, T_1, T_2) with $e(C) = 7$, $\sigma(C) = -3$.*
- *There exists a minimal telescoping triple (D, T_1, T_2) with $e(D) = 8$, $\sigma(D) = -4$.*
- *There exists a minimal telescoping triple (F, T_1, T_2) with $e(F) = 10$, $\sigma(F) = -6$.*

The manifolds B_g , D and F were already built in [1]. They are taken out of the constructions given in [4] by the following mechanism. The main goal of [4] is to construct simply connected 4-manifolds by applying Luttinger surgery to symplectic sums. If one is careful about the fundamental fundamental group calculations, the procedure can be interrupted by NOT performing two surgeries, and thus obtain a symplectic manifold with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, the skipped surgeries have to be chosen carefully so that the unused Lagrangian tori comply with the requirements and the pieces can then be aligned into a telescoping triple.

To finish the proof of Theorem 5, we construct (A, T_1, T_2) and (C, T_1, T_2) by applying this mechanism to the constructions in [2]. This is done in the following two lemmas, where we follow the notation of [2].

Lemma 6. *There exists a telescoping triple (A, T_1, T_2) with $e(C) = 5$ and $\sigma(C) = -1$.*

Proof. This telescoping triple is obtained out of the construction of an exotic irreducible symplectic $\mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}$ given in [2]. The two surgeries to be skipped are $(a'_2 \times c', c', +1/p)$ and $(b'_1 \times c'', b'_1, -1)$ (the notation is explained in [9]). Rename the corresponding tori T_1 and T_2 . This procedure manufactures a minimal symplectic manifold A . Notice that the tori are linearly independent in $H_2(A; \mathbb{R})$. We need to check that such manifold has indeed $\pi_1 = \mathbb{Z}^2$ and that it contains the required tori.

Let us begin with the fundamental group calculations. By combining the relations coming from the surgeries $(a'_1 \times c', a'_1, -1)$ and $(a''_2 \times d', d', +1)$ that were performed on the $\Sigma_2 \times T^2$ block (see [2] for details) we have $\alpha_1 = a_1 = [b_1^{-1}, d^{-1}] = [b_1^{-1}, [b_2, c^{-1}]^{-1}] = [b_1^{-1}, [c^{-1}, b_2]] = 1$. The last commutator is trivial since b_1

commutes with both c_1 and b_2 . Substituting this in the relations coming from the surgeries applied to the building block $T^4\#\overline{\mathbb{C}\mathbb{P}^2}$, we obtain $\alpha_3 = a_2 = 1$ and $\alpha_4 = b_2 = 1$. By looking at the relations from the other building block we see $d = 1$. Note that the meridians of the surfaces along which the gluing is performed are trivial. Thus only two commuting generators survive in the group presentation.

We check that the meridian of the first torus is $\mu_{T_1} = [d^{-1}, b_2^{-1}] = 1$ and its Lagrangian push-offs are $m_{T_1} = c$ and $l_{T_1} = a_2 = 1$. For the torus T_2 one sees $\mu_{T_2} = [a_1^{-1}, d] = 1$ and its Lagrangian push-offs are $m_{T_2} = c$ and $l_{T_2} = b_1$. So, $\pi_1(A - (T_1 \cup T_2))$ is generated by the commuting elements b_1 and c . By a Mayer-Vietoris sequence we see $H_1(A - (T_1 \cup T_2)) = \mathbb{Z}^2$. Thus $\pi_1(A - (T_1 \cup T_2)) = \mathbb{Z}b_1 \oplus \mathbb{Z}c$. We conclude (A, T_1, T_2) is a telescoping triple. \square

Lemma 7. *There exists a telescoping triple (C, T_1, T_2) with $e(C) = 7$ and $\sigma(C) = -3$.*

Proof. We follow the construction of an exotic irreducible symplectic $\mathbb{C}\mathbb{P}^2\#4\overline{\mathbb{C}\mathbb{P}^2}$ given in [2]. The surgeries $(\alpha'_2 \times \alpha'_3, \alpha'_2, -1)$ in the $T^4\#2\overline{\mathbb{C}\mathbb{P}^2}$ block and $(\alpha''_2 \times \alpha'_4, \alpha'_4, -1)$ in the $T^4\#\overline{\mathbb{C}\mathbb{P}^2}$ block will NOT be performed. Call these tori T_2 and T_1 respectively and the resulting manifold C . Notice that they are linearly independent in $H_2(C; \mathbb{R})$.

We apply $(\alpha'_1 \times \alpha'_3, \alpha'_1, -1)$ on the $T^4\#2\overline{\mathbb{C}\mathbb{P}^2}$. This introduces the relation $\alpha_1 = [\alpha_2^{-1}, \alpha_4^{-1}]$. Using the commutator $[\alpha_2, \alpha_4] = 1$, one sees $\alpha_1 = 1$. The relation $\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]$ obtained by applying a Luttinger surgery on the $T^4\#\overline{\mathbb{C}\mathbb{P}^2}$ building block implies $\alpha_3 = 1$. The surfaces of genus 2 along which the symplectic sum is performed have trivial meridians.

The meridian of T_1 is $\mu_{T_1} = [a_1^{-1}, \alpha_4] = 1$ and its Lagrangian push-offs are $m_{T_1} = \alpha_2$ and $l_{T_1} = \alpha_3 = 1$. The meridian of T_2 is given by $\mu_{T_2} = [\alpha_1, \alpha_3^{-1}] = 1$ and its Lagrangian push-offs are $m_{T_2} = \alpha_4$ and $l_{T_2} = \alpha_2$. We have that $\pi_1(C - (T_1 \cup T_2))$ is generated by the commuting elements α_2 and α_4 . The Mayer-Vietoris sequence computes $H_1(C - (T_1 \cup T_2)) = \mathbb{Z}^2$, thus $\pi_1(C - (T_1 \cup T_2)) = \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_4$. Thus, (C, T_1, T_2) is a telescoping triple. \square

Remark 1. *One is able to realize the point $(c_1^2, \chi_h) = (3, 1)$ for the fundamental groups $\pi_1 = \mathbb{Z}^2$ and $\pi_1 = \mathbb{Z}$ during the manufacturing process of an exotic irreducible symplectic $\mathbb{C}\mathbb{P}^2\#6\overline{\mathbb{C}\mathbb{P}^2}$. Consider the symplectic sum of $T^4\#\overline{\mathbb{C}\mathbb{P}^2}$ and $T^2 \times S^2\#4\overline{\mathbb{C}\mathbb{P}^2}$ along a genus 2 surface given in [2]. The resulting minimal symplectic 4-manifold has a fundamental group with the following presentation*

$$\langle \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha_2] = 1, [\alpha_2, \alpha_3] = 1, \alpha_1^{-1} = \alpha_3^2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}.$$

If we apply the surgery $(\alpha''_2 \times \alpha'_4, \alpha'_4, -1)$, the relation $\alpha_4 = [\alpha_1, \alpha_3^{-1}]$ is introduced to the fundamental group presentation and we obtain a manifold with fundamental group

$$\pi_1 = \langle \alpha_1, \alpha_3 \mid \alpha_1^{-1} = \alpha_3^2 \rangle \cong \mathbb{Z}.$$

If we apply the surgery $(\alpha'_2 \times \alpha'_3, \alpha'_3, -1)$, the relation $\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]$ is introduced to the fundamental group presentation and we obtain a manifold with fundamental group $\pi_1 = \langle \alpha_2 \rangle = \mathbb{Z}$.

One can go on and build more telescoping triples out of these five by using Proposition 4. We proceed to do so now. Let us start by setting some useful notation. Let (X, T_1, T_2) be a telescoping triple. We will denote by $X_n := \#n(X)$ the manifold obtained by building the symplectic sum (cf. [14]) of n copies of X along the proper tori.

Proposition 8. *For each $n \geq 1$ and $m \geq 1$, the following minimal telescoping triples with the given Characteristic numbers exist:*

- (1) (A_n, T_1, T_2) satisfying $e(A_n) = 5n$ and $\sigma(A_n) = -n$.
- (2) (C_n, T_1, T_2) satisfying $e(C_n) = 7n$ and $\sigma(C_n) = -3n$.
- (3) (D_n, T_1, T_2) satisfying $e(D_n) = 8n$ and $\sigma(D_n) = -4n$.
- (4) (F_n, T_1, T_2) satisfying $e(F_n) = 10n$ and $\sigma(F_n) = -6n$.
- (5) $(\#n(B_g), T_1, T_2)$ satisfying $e(\#n(B_g)) = (6 + 4g)n$ and $\sigma(\#n(B_g)) = -2n$.
- (6) $(A_n \# m(B_g), T_1, T_2)$ satisfying $e(A_n \# m(B_g)) = 5n + (6 + 4g)m$ and $\sigma(A_n \# m(B_g)) = -n - 2m$.
- (7) $(A_n \# C_m, T_1, T_2)$ satisfying $e(A_n \# C_m) = 5n + 7m$ and $\sigma(A_n \# C_m) = -n - 3m$.
- (8) $(A_n \# D_m, T_1, T_2)$ satisfying $e(A_n \# D_m) = 5n + 8m$ and $\sigma(A_n \# D_m) = -n - 4m$.
- (9) $(A_n \# F_m, T_1, T_2)$ satisfying $e(A_n \# F_m) = 5n + 10m$ and $\sigma(A_n \# F_m) = -n - 6m$.
- (10) $(\#n(B_g) \# C_m, T_1, T_2)$ satisfying $e(\#n(B_g) \# C_m) = (6 + 4g)n + 7m$ and $\sigma(\#n(B_g) \# C_m) = -2n - 3m$.
- (11) $(\#n(B_g) \# D_m, T_1, T_2)$ satisfying $e(\#n(B_g) \# D_m) = (6 + 4g)n + 8m$ and $\sigma(\#n(B_g) \# D_m) = -2n - 4m$.
- (12) $(\#n(B_g) \# F_m, T_1, T_2)$ satisfying $e(\#n(B_g) \# F_m) = (6 + 4g)n + 10m$ and $\sigma(\#n(B_g) \# F_m) = -2n - 6m$.
- (13) $(C_n \# D_m, T_1, T_2)$ satisfying $e(C_n \# D_m) = 7n + 8m$ and $\sigma(C_n \# D_m) = -3n - 4m$.
- (14) $(C_n \# F_m, T_1, T_2)$ satisfying $e(C_n \# F_m) = 7n + 10m$ and $\sigma(C_n \# F_m) = -3n - 6m$.
- (15) $(D_n \# F_m, T_1, T_2)$ satisfying $e(D_n \# F_m) = 8n + 10m$ and $\sigma(D_n \# F_m) = -4n - 6m$.

The claim about minimality is proven in the next section.

3. MINIMALITY AND IRREDUCIBILITY

The following result allows us to conclude the irreducibility of the manufactured minimal 4-manifolds.

Theorem 9. *(Hamilton and Kotschick, [12]). Minimal symplectic 4-manifolds with residually finite fundamental groups are irreducible.*

Finite groups and free groups are well-known examples of residually finite groups. Since the direct products of residually finite groups are residually finite groups themselves, the previous result implies that all we need to worry about is producing minimal manifolds in order to conclude on their irreducibility. This endeavor follows from Usher's theorem.

Theorem 10. (Usher, [21]). *Let $X = Y \#_{\Sigma \equiv \Sigma} Y'$ be the symplectic sum where the surfaces have genus greater than zero.*

- (1) *If either $Y - \Sigma$ or $Y' - \Sigma'$ contains an embedded symplectic sphere of square -1 , then X is not minimal.*
- (2) *If one of the summands, say Y for definiteness, admits the structure of an S^2 -bundle over a surface of genus g such that Σ is a section of this S^2 -bundle, then X is minimal if and only if Y' is minimal.*
- (3) *In all other cases, X is minimal.*

This theorem implies that the manifolds of Proposition 8 are minimal.

4. LUTTINGER SURGERY AND ITS EFFECTS ON π_1

Let T be a Lagrangian torus inside a symplectic 4-manifold M . *Luttinger surgery* (cf. [15], [3]) is the surgical procedure of taking out a tubular neighborhood of the torus $\text{nbh}(T)$ in M and gluing it back in, in such way that the resulting manifold admits a symplectic structure. The symplectic form is unchanged away from a neighborhood of T . We proceed to give an overview of the process before we get into the fundamental group calculations.

The Darboux-Weinstein theorem (cf. [7]) implies the existence of a parametrization of a tubular neighborhood $T \times D^2 \rightarrow \text{nbh}(T) \subset M$ such that the image of $T \times \{d\}$ is Lagrangian for all $d \in D^2$. Let $d \in D - \{0\}$. The parametrization of the tubular neighborhood provides us with a particular type of push-off $F_d : T \times \{d\} \subset M - T$ called the *Lagrangian push-off* or *Lagrangian framing*. Let $\gamma \subset T$ be an embedded curve. Its image $F_d(\gamma)$ under the Lagrangian push-off is called the *Lagrangian push-off* of γ . These curves are used to parametrize the Luttinger surgery.

A *meridian* of T is a curve isotopic to $\{t\} \times \partial D^2 \subset \partial(\text{nbh}(T))$ and it is denoted by μ_t . Consider two embedded curves in T which intersect transversally in one point and consider their Lagrangian push-offs m_T and l_T . The group $H_1(\partial(\text{nbh}(T))) = H_1(T^3)$ is generated by μ_T, m_t and l_T . We take advantage of the commutativity of $\pi_1(T^3)$ and choose a basepoint t on $\partial(\text{nbh}(T))$, so that we can refer unambiguously to $\mu_T, m_T, l_T \in \pi_1(\partial(\text{nbh}(T)), t)$.

Under this notation, a general torus surgery is the process of removing a tubular neighborhood of T in M and glue it back in such a way that the curve representing $\mu_T^k m_T^p l_T^q$ bounds a disk for some triple of integers k, p , and q . In order to obtain a symplectic manifold after the surgery, we need to set $k = \pm 1$ (cf. [4]).

When the base point x of M is chosen off the boundary of the tubular neighborhood of T , the based loops μ_T, m_t and l_T are to be joined by the same path in $M - T$. By doing so, these curves define elements of $\pi_1(M - T, x)$. The 4-manifold Y resulting from Luttinger surgery on M has fundamental group

$$\pi_1(M - T)/N(\mu_T m_T^p l_T^q)$$

where $N(\mu_T m_T^p l_T^q)$ denotes the normal subgroup generated by $\mu_T m_T^p l_T^q$.

We proceed now with the fundamental group calculations needed to prove Theorem 1. To do so, we plug into the previous general picture the information we have for the telescoping triples. Let (X, T_1, T_2) be a telescoping triple. The fundamental group of X has the presentation $\langle t_1, t_2 | [t_1, t_2] = 1 \rangle$. Let us apply $+1/p$ Luttinger surgery on T_1 along l_{T_1} and call Y_1 the resulting manifold. Since the meridian μ_{T_1} is trivial we have

$$\pi_1(Y_1) = \pi_1(X - T)/N(\mu_T m_{T_1}^0 l_{T_1}^p) = \mathbb{Z} \oplus \mathbb{Z}/N(1 \cdot 1 \cdot l_{T_1}^p).$$

Thus, $\pi_1(Y_1) = \langle t_1, t_2 | [t_1, t_2] = 1, t_2^p = 1 \rangle$.

Let us apply now $+1/q$ Luttinger surgery on T_2 along m_{T_2} and call the resulting manifold Y_2 the resulting manifold. Since the meridian μ_{T_2} is trivial we have

$$\pi_1(Y_2) = \mathbb{Z} \oplus \mathbb{Z}_p/N(1 \cdot m_{T_2}^q \cdot 1).$$

Thus, $\pi_1(Y_1) = \langle t_1, t_2 | [t_1, t_2] = 1, t_1^q = 1 = t_2^p \rangle$.

The reader might have already noticed the symmetry of these calculations.

Proposition 11. *Let (X, T_1, T_2) be a minimal telescoping triple. Let l_{T_1} be a Lagrangian push off of a curve on T_1 and m_{T_2} the Lagrangian push off of a curve on T_2 so that l_{T_1} and m_{T_2} generate $\pi_1(X)$.*

- *The minimal symplectic 4-manifold obtained by performing either $+1/p$ Luttinger surgery on T_1 along l_{T_1} or $+1/p$ surgery on T_2 along m_{T_2} has fundamental group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_p$.*
- *The minimal symplectic 4-manifold obtained by performing $+1/p$ Luttinger surgery on T_1 along l_{T_1} and $+1/q$ surgery on T_2 along m_{T_2} has fundamental group isomorphic to $\mathbb{Z}_q \oplus \mathbb{Z}_p$.*

The proof is omitted. It is based on a repeated use of Lemma 2 in [4] and Usher's theorem (cf. [21]). The reader is suggested to look at the proofs of theorems 8, 10 and 13 of [4] for a blueprint to the proof.

5. PROOF OF THEOREM 1

Proof. The possible choices for characteristic numbers in Theorem 1 are in a one-to-one correspondence with the telescoping triples of Proposition 8. The enumeration indicates that, in order to produce the manifold in Theorem 1 with one of the choices for characteristic numbers claimed in item # (k), we start with the telescoping triple of item # (k) in Proposition 8 ($k \in \{1, 2, 3, 4, 5, \dots, 14, 15\}$). Let

$S := (X, T_1, T_2)$ be the chosen minimal telescoping triple. The manifolds of Theorem 1 are produced by applying Luttinger surgery to S according to the choice of characteristic numbers. By Proposition 11 we know that out of S one produces two symplectic manifolds: Y_1 with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_p$ and Y_2 with $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}_p$. Since Luttinger surgery does not change the Euler characteristic nor the signature, the resulting manifolds Y_1 and Y_2 share the same characteristic numbers as X .

Proposition 11 states that Y_1 and Y_2 are minimal. By Hamilton-Kotschick result, both of them are irreducible. The calculation of the characteristic numbers of Y_1 and Y_2 is straight-forward. Since our chosen S was arbitrary, this finishes the proof. \square

6. EXOTIC SMOOTH STRUCTURES ON 4-MANIFOLDS WITH ABELIAN FINITE NON-CYCLIC π_1

The purpose of this section is to put on display the exotic smooth structures for the manufactured manifolds having $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$, i.e., to prove Proposition 2.

6.1. Smooth Topological Prototype. We proceed to construct the underlying smooth manifold on which infinitely many exotic smooth structures will be displayed. Start with the product of a Lens space and a circle: $L(p, 1) \times S^1$. Its Euler characteristic is zero as well as its signature. Consider the map

$$\begin{aligned} L(p, 1) \times S^1 &\rightarrow L(p, 1) \times S^1 \\ \{pt\} \times \alpha &\mapsto \{pt\} \times \alpha^p \end{aligned}$$

We perform surgery on $L(p, 1) \times S^1$: cut out the loop α^p and glue in a disc in order to kill the corresponding generator

$$L(p, 1) \widetilde{\times} S^1 := L(p, 1) \times S^1 - (S^1 \times D^3) \cup S^2 \times D^2.$$

The resulting manifold has zero signature and Euler characteristic two. By the Seifert-Van Kampen theorem, one concludes $\pi_1(L(p, 1) \widetilde{\times} S^1) = \mathbb{Z}_p \oplus \mathbb{Z}_p$.

Since we are aiming at non-spin manifolds, our topological prototypes will have the shape

$$b_2^+ \mathbb{C}P^2 \# b_2^- \overline{\mathbb{C}P^2} \# L(p, 1) \widetilde{\times} S^1$$

but spin 4-manifolds with $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$ are also built in such a straight-forward manner.

6.2. An infinite family $\{X_n\}$. We apply now the procedure described in [9] and [8] to produce infinitely many distinct smooth structures on any of our topological prototypes. Let X_0 be the manifold obtained by applying $+1/p$ Luttinger surgery on T_2 along l_{T_2} to any of the manifolds from the telescoping triples previously constructed. Since X_0 is a minimal symplectic manifold with $b_2^+ = 2$, its Seiberg-Witten invariant is non-trivial by [19].

The infinite family $\{X_n\}$ is obtained by applying a $+n/p$ torus surgery to X_0 on T_1 along m_{T_1} . Notice that now $k = n$ according to our notation of Section 4; only the case $k = 1 = n$ produces a symplectic manifold. We take a closer look at the process to see that we comply with the hypothesis of Corollary 2 in [8].

The boundary of the tubular neighborhood of T_1 in X_0 is a 3-torus whose fundamental group is generated by the loops μ_{T_1} , m_{T_1} and l_{T_1} . Notice that in $\pi_1(X_0 - T_1)$, the meridian is trivial $\mu_{T_1} = 1$, $m_{T_1} = x$ and $l_{T_1} = 1$, where x is a generator in $\pi_1(X_0) = \mathbb{Z}_p \oplus \mathbb{Z}x$. The manifolds in the family $\{X_n\}$ can be described as the result of applying to X_0 a n/p surgery on T_1 along m_{T_1} , and so $\mu_{T_1}^n m_{T_1} = x^p$ is killed.

Let X be the manifold obtained from $X_0 - T_1$ by gluing a thick torus $T^2 \times D^2$ in a manner that $\gamma = S^1 \times \{1\} \times \{1\}$ is sent to l_{T_1} , $\lambda = \{1\} \times S^1 \times \{1\}$ is sent to μ_{T_1} , and $\mu_X = \{(1, 1)\} \times \partial D^2$ is sent to $m_{T_1}^{-p}$. If $n \neq 1$, the manifold X will not be symplectic, but in any case $\pi_1(X) = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Denote by $\Lambda \subset X$ the core torus of the surgery.

Notice that given the identifications on the loops during the surgery, $\lambda = \mu_{T_2} = 1$, thus it is nullhomotopic in $X_0 - T_1 = X - \Lambda$; in particular, λ is nullhomologous. The torus surgery kills one generator of H_1 and two generators of H_2 ; Λ is a nullhomologous torus. One obtains a manifold X_n by applying $1/n$ surgery on Λ along λ with $\pi_1(X_n) = \mathbb{Z}_p \oplus \mathbb{Z}_p$. The manifold X_0 can be recovered from X by applying a $0/1$ surgery on Λ along λ .

By Corollary 2 in [8], we produce an infinite family $\{X_n\}$ of pairwise non-diffeomorphic 4-manifolds. These manifolds will have the same cohomology ring as the corresponding topological prototype. Thus we have the following lemma.

Lemma 12. *There exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds with $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$ sharing the same Euler characteristic, signature and type as a given topological prototype constructed in the previous subsection.*

6.3. Homeomorphism Criteria. Now we need to see that the manifolds produced share indeed the same underlying topological prototype. Ian Hambleton and Matthias Kreck proved the needed homeomorphism criteria in [11] (theorem B). They showed that topological 4-manifolds with odd order fundamental group and large Euler characteristic are classified up to homeomorphism by explicit invariants.

The precise statement of their result includes a lower bound for the Euler characteristic in terms of an integer number $d(\pi)$, which depends on the fundamental group of the manifold. We proceed to explain the notation employed.

Let $\pi_1 = \pi$ be a finite group and let $d(\pi)$ be the minimal \mathbb{Z} -rank for the abelian group $\Omega^3 \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}$. One minimizes over all representatives of $\Omega^3 \mathbb{Z}$, the kernel of a projective resolution of length three (cf. [10]) of \mathbb{Z} over the group ring $\mathbb{Z}[\pi]$. In particular, $\Omega^3 \mathbb{Z}$ is a submodule of $\pi_2(X)$. The minimal representative is given by $\pi_2(K)$, where K is a two-complex with the given π_1 .

The result we will use in order to conclude on the homeomorphism type of our manifolds is the following

Theorem 13. (*Hambleton-Kreck, cf [11]*). *Let M be a closed oriented manifold of dimension four, and let $\pi_1(X) = \pi$ be a finite group of odd order. When $\omega_2(\tilde{X}) = 0$ (resp. $\omega_2(\tilde{X}) \neq 0$), assume that*

$$b_2(X) - |\sigma(X)| > 2d(\pi),$$

(resp. $> 2d(\pi) + 2$). *Then M is classified up to homeomorphism by the signature, Euler characteristic, type, Kirby-Siebenmann invariant, and fundamental class in $H_4(\pi, \mathbb{Z})/Out(\pi)$.*

Notice that since $p \geq 3$ is assumed to be a prime number, π_1 has odd order and no 2-torsion. Therefore, the type of the manifold is indicated by the parity of its intersection form over \mathbb{Z} . All of our manufactured manifolds are non-spin; since they are smooth, the Kirby-Siebenmann invariant vanishes.

For the finite groups $\pi = \mathbb{Z}_p \oplus \mathbb{Z}_p$, we claim

$$d(\pi) = 1.$$

We are indebted to Matthias Kreck for explaining us the argument [13]. Assume $\pi = \pi_1$ is a finite group and let K be a 2-complex with fundamental group π_1 . The minimal Euler characteristic of a K is given by $d(\pi)+1$. We claim $d(\pi) = 1$.

Consider the map from K to the Eilenberg-MacLane space $K(\pi, 1)$ which induces an isomorphism on π_1 . Then the induced map on $H_2(K; \mathbb{Z}_p)$ is surjective. Thus, the Euler characteristic of K is greater or equal than $3 - 2 + 1$. This implies $d(\pi)$ is greater or equal than 1.

To conclude now $d(\pi) = 1$, consider the standard presentation of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ given by

$$\langle x, y | x^p = 1, y^p = 1, [x, y] = 1 \rangle.$$

The 2-complex realising this presentation has Euler characteristic $2 = d(\pi) + 1$. Therefore, $d(\pi) = 1$ as claimed.

In order to conclude on the homeomorphism type of our manufactured manifolds, we only need to know the numerical invariants b_2^+ and b_2^- which need to satisfy

$$b_2(X) - |\sigma(X)| > 4.$$

6.4. Proof of Proposition 2. The proof of Proposition 2 is now clear if one rewrites it in the following form. Using $, n \geq 2$ if $m = 0$ or does not appear; $m \geq 2$ if $n = 0$

Proposition 14. *Assume $n + m \geq 2$. The manifolds*

$$b_2^+ \mathbb{C}P^2 \# b_2^- \overline{\mathbb{C}P^2} \# \widetilde{L(p, 1)} \times S^1$$

with the following coordinates admit infinitely many exotic irreducible smooth structures, only one of which is symplectic.

- (1) $(b_2^+, b_2^-) = (2n - 1, 3n - 1)$,
- (2) $(b_2^+, b_2^-) = (2n - 1, 5n - 1)$,
- (3) $(b_2^+, b_2^-) = (2n - 1, 6n - 1)$,
- (4) $(b_2^+, b_2^-) = (2n - 1, 8n - 1)$,
- (5) $(b_2^+, b_2^-) = ((2 + 2g)n - 1, (4 + 2g)n - 1)$,
- (6) $(b_2^+, b_2^-) = (2n + (2 + 2g)m - 1, 3n + (4 + 2g)m - 1)$,
- (7) $(b_2^+, b_2^-) = (2n + 2m - 1, 3n + 5m - 1)$,
- (8) $(b_2^+, b_2^-) = (2n + 2m - 1, 3n + 6m - 1)$,
- (9) $(b_2^+, b_2^-) = (2n + 2m - 1, 3n + 8m - 1)$,
- (10) $(b_2^+, b_2^-) = ((2 + 2g)n + 2m - 1, (4 + 2g)n + 5m - 1)$,
- (11) $(b_2^+, b_2^-) = ((2 + 2g)n + 2m - 1, (4 + 2g)n + 6m - 1)$,
- (12) $(b_2^+, b_2^-) = ((2 + 2g)n + 2m - 1, (4 + 2g)n + 8m - 1)$,
- (13) $(b_2^+, b_2^-) = (2n + 2m - 1, 5n + 6m - 1)$,
- (14) $(b_2^+, b_2^-) = (2n + 2m - 1, 5n + 8m - 1)$,
- (15) $(b_2^+, b_2^-) = (2n + 2m - 1, 6n + 8m - 1)$.

Proof. The infinite families are provided by Lemma 12. Choosing the topological prototype accordingly to the coordinates, by Theorem 12 and the discussion that follows we conclude on the homeomorphism type. Notice that the enumeration of the coordinates presented in Proposition 14 correspond exactly to the ones in Theorem 1. \square

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