POINCARÉ DUALITY PAIRS OF DIMENSION THREE

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ABSTRACT. We extend Hendriks' classification theorem and Turaev's realisation and splitting theorems for PD^3–complexes to the relative case of PD^3–pairs. The results for PD^3–complexes are recovered by restricting the results to the case of PD^3–pairs with empty boundary. Up to oriented homotopy equivalence, PD^3–pairs are classified by their fundamental triple consisting of the fundamental group system, the orientation character and the image of the fundamental class under the classifying map. Using the derived module category we provide necessary and sufficient conditions for a given triple to be realised by a PD^3–pair. The results on classification and realisation yield splitting or decomposition theorems for PD^3–pairs, that is, conditions under which a given PD^3–pair decomposes as interior or boundary connected sum of two PD^3–pairs.

1. INTRODUCTION

A Poincaré duality complex of dimension n, or PD^n–complex, is a CW–complex exhibiting n–dimensional equivariant Poincaré duality. We may thus regard PD^n–complexes as homotopy generalisations of manifolds. Similarly, Poincaré duality pairs of dimension n, or PD^n–pairs, are homotopy generalisations of manifolds with boundary.

A PD^n–pair is a pair of CW–complexes, $(X, \partial X)$, where X is connected and ∂X is a PD^{n-1}–complex, together with an orientation character $\omega \in \mathrm{H}^1(X; \mathbb{Z}/2\mathbb{Z})$ and a fundamental class $[X, \partial X] \in \mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega)$, such that

$$\cap[X, \partial X] : H^r(X; B) \longrightarrow H_{n-r}(X, \partial X; B^\omega)$$

is an isomorphism for every $r \in \mathbb{Z}$ and every left module B over the integral group ring of the fundamental group of X.

Note that we do not require X to be finitely dominated. Wall [19] showed that the cellular chain complex $C(X)$ of the universal cover of X is chain homotopy equivalent to a complex of finitely generated projective Λ–modules, vanishing except in dimensions r with $0 \leq r \leq n$. Furthermore, $\pi_1(X; *)$ is finitely generated and almost finitely presentable, and X is dominated by a finite complex if and only if $\pi_1(X; *)$ is finitely presentable.

In Section 2 we review material well-known to experts concerning the definition and properties of the relative twisted cap products but not readily available in the literature, as well as results on the algebraic sums of chain pairs satisfying Poincaré duality needed for the discussion of connected sums of Poincaré duality pairs. One of the main results is the generalisation of a theorem by Browder [4] to the non–simply connected case.

Section 3 is concerned with the homotopy classification of PD^3–pairs. An oriented homotopy equivalence of PD^3–pairs, $(X, \partial X)$ and $(Y, \partial Y)$, induces an isomorphism

$$(\varphi, \{\varphi_i\}_{i\in J}) : \{\kappa_i : \pi_1(\partial X_i, *) \to \pi_1(X, *)\}_{i\in J} \to \{\rho_i : \pi_1(\partial Y_i, *) \to \pi_1(Y, *)\}_{i\in J}$$

of their π_1–systems such that

$$\varphi^*(\omega_Y) = \omega_X \quad \text{and} \quad \varphi_*(c_{X*}([X, \partial X])) = c_{Y*}([Y, \partial Y]), \tag{1}$$

where c_X and c_Y are classifying maps and $\{\partial X_i\}$ and $\{\partial Y_i\}$ are the connected components of ∂X and ∂Y respectively. The converse holds for PD^3–pairs with aspherical boundary components.

Putting $\mu_X := c_{X*}([X, \partial X])$, we call $(\{\kappa_i\}_{i\in J}, \omega_X, \mu_X)$ the *fundamental triple* of the PD^3–pair $(X, \partial X)$. Two fundamental triples $(\{\kappa_i\}_{i\in J}, \omega, \mu)$ and $(\{\kappa'_i\}_{i\in J}, \omega', \mu')$ are *isomorphic* if there is an isomorphism $(\varphi, \{\varphi_i\}_{i\in J}) : \{\kappa_i\}_{i\in J} \to \{\kappa'_i\}_{i\in J}$ with $\varphi^*(\omega') = \omega$ and $\varphi_*(\mu) = \mu'$.

Theorem 1.1 ([CLASSIFICATION]). *Two PD^3–pairs $(X, \partial X)$ and $(Y, \partial Y)$ with apsherical boundary components are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic.*

The case $\partial X = \emptyset$ yields Hendriks' Classification Theorem for PD^3–complexes.

Section 4 investigates which triples $(\{\kappa_i\}_{i\in J}, \omega, \mu)$ consisting of a π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}$, $\omega \in \mathrm{H}^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in \mathrm{H}_3(G, \{G_i\}_{i\in J}; \mathbb{Z}^\omega)$ are realised by PD^3–pairs. We introduce the derived module category [10] which is needed for the formulation of the realisation condition. Given a finitely presentable group G and $\omega \in \mathrm{H}^1(G; \mathbb{Z}/2\mathbb{Z})$, Turaev defined a homomorphism

$$\nu : \mathrm{H}_3(G; \mathbb{Z}^\omega) \longrightarrow [F, I],$$

where F is some $\mathbb{Z}[G]$–module, I is the augmentation ideal and $[A, B]$ denotes the group of homotopy classes of $\mathbb{Z}[G]$–morphisms from A to B. If

$$(\{\kappa_i : G_i \to G\}_{i\in J}, \omega, \mu)$$

is the fundamental triple of a PD^3–pair, then $\nu(\mu)$ is a class of homotopy equivalences of $\mathbb{Z}[G]$–modules. The converse holds for π_1–injective triples $(\{\kappa_i : G_i \to G\}_{i\in J}, \omega, \mu)$ with G finitely presentable.

Theorem 1.2 ([REALISATION]). *There is a PD^3–pair realising the π_1–injective triple $(\{\kappa_i : G_i \to G\}_{i\in J}, \omega, \mu)$ if and only if $\nu(\mu)$ is a class of homotopy equivalences of $\mathbb{Z}[G]$–modules.*

The case $\partial X = \emptyset$ yields Turaev's Realisation Theorem for PD^3–complexes. Precise statements of the results on realisation are contained in Section 4. As the assumption of π_1–injectivity is indispensable for the method used, the question remains whether the realisation theorem holds without it.

Finally, in Section 5, we use the Classification and Realisation Theorems to show that a PD^3–pair decomposes as connected sum if and only if its π_1–system decomposes as free product. There are two distinct notions of connected sum for PD^3–pairs reflecting the situation of manifolds with boundary. We introduce the interior connected sum of two PD^3–pairs as well as the connected sum along boundary components.

If the PD^3–pair $(X, \partial X)$ is orientedly homotopy equivalent to the interior connected sum of two PD^3–pairs $(X_k, \partial X_k)$ with π_1–systems $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell \in J_k}$ for $k = 1, 2$, then the π_1–system of $(X, \partial X)$ is isomorphic to $\{\iota_k \circ \kappa_{k\ell} : G_{k\ell} \to G_1 * G_2\}_{\ell \in J_k, k=1,2}$, where $\iota_k : G_k \to G_1 * G_2$ denotes the inclusion of the factor in the free product of groups for $k = 1, 2$. We then say that the π_1–system of $(X, \partial X)$ *decomposes as free product*. The Classification Theorem and the Realisation Theorem allow us to show that the converse holds for finitely dominated PD^3–pairs with aspherical boundaries in the case of π_1–injectivity.

Theorem 1.3 ([DECOMPOSITION I]). *Let $(X, \partial X)$ be a finitely dominated* PD^3*–pair with aspherical boundary components. Then $(X, \partial X)$ decomposes as interior connected sum of two π_1–injective* PD^3*–pairs if and only if its π_1–system decomposes as free product of two injective π_1–systems.*

If the PD^3–pair $(X, \partial X)$ is orientedly homotopy equivalent to the boundary connected sum of two PD^3–pairs $(X_k, \partial X_k)$ with π_1–systems $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell \in I_k}$, $k = 1, 2$, along the boundary components $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$, the π_1–system of $(X, \partial X)$ is isomorphic to $\{\kappa : K \to G_1 * G_2,\ \iota_k \circ \kappa_{k\ell} : G_{k\ell} \to G_1 * G_2\}_{\ell \in I_k, \ell \neq \ell_k, k=1,2}$, where $K := \pi_1(\partial X_{1,\ell_1} \sharp \partial X_{2,\ell_2}; *)$ and $\kappa : K \to G_1 * G_2$ is induced by the inclusion of the connected sum of the boundary components $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ in the connected sum of the pair. We then say that the π_1–system of $(X, \partial X)$ *decomposes as free product along $G_{1\ell_1}$ and $G_{2\ell_2}$*. As for the interior connected sum, the converse holds for finitely dominated PD^3–pairs with non–empty aspherical boundaries in the case of π_1–injectivity.

Theorem 1.4 ([DECOMPOSITION II]). *Let $(X, \partial X)$ be a finitely dominated* PD^3*–pair with non–empty aspherical boundary components. Then $(X, \partial X)$ decomposes as boundary connected sum of two π_1–injective* PD^3*–pairs $(X_k, \partial X_k)$, $k = 1, 2$, along $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$, if and only if its π_1–system decomposes as free product of two injective π_1–systems along $\pi_1(\partial X_{1\ell_1}; *)$ and $\pi_1(\partial X_{2\ell_2}; *)$.*

A recent paper by Wall [19] contains a different formulation of the first decomposition theorem.

Given an orientable PD^3–pair $(X, \partial X)$, where ∂X is an aspherical 2–manifold, Crisp's algebraic loop theorem [5] allowed him to construct a π_1–injective PD^3–pair $(\hat{X}, \partial \hat{X})$ with the same fundamental group. Using the decomposition theorem for interior connected sums, Crisp showed that $(\hat{X}, \partial \hat{X})$ is homotopy equivalent to the connected sum of a finite number of aspherical PD^3–pairs and a PD^3–complex with virtually free fundamental group. The result holds for the non–orientable case with the addition that the connected summand with virtually free fundamental group may be a PD^3–pair whose boundary is a disjoint union of copies of $\mathbb{R}P^2$.

2. The Relative Twisted Cap Products and Algebraic Sums

Let $\Lambda := \mathbb{Z}[G]$ be the integral group ring of the group G and $\mathrm{aug} : \Lambda \to \mathbb{Z}$ the *augmentation homomorphism*, defined by $\mathrm{aug}(g) := 1$ for all $g \in G$. Its kernel is the *augmentation ideal*, I. A cohomology class, $\omega \in \mathrm{Hom}(G; \mathbb{Z}/2\mathbb{Z})$, determines a homomorphism from G to $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, which, in turn, gives rise to the anti–isomorphism $\overline{} : \Lambda \to \Lambda$ defined by $\overline{g} := (-1)^{\omega(g)} g^{-1}$ for every $g \in G$. This anti–isomorphism allows us to associate a left Λ–module with every right Λ–module and vice versa. Namely, given a right Λ–module A we define a left action on the set underlying A by $\lambda.a := a.\overline{\lambda}$ for every $a \in A$ and $\lambda \in \Lambda$. Proceeding analogously for a left Λ–module B, we obtain a left Λ–module ${}^{\omega}\!A$ and a right Λ–module B^{ω}.

Take chain complexes C and D of left Λ–modules and a chain complex E of right Λ–modules. Define $E \otimes_\Lambda D$ and $\mathrm{Hom}_\Lambda(C, D)$ as usual with the conventions

$$\partial^{E\otimes_\Lambda D}|_{E_i\otimes_\Lambda D_j} = \partial^E \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \partial^D;$$
$$\partial\big((f_i)_{i\in\mathbb{Z}}\big)|_{\mathrm{Hom}_\Lambda(C,D)_n} := (\partial^D f_i)_{i\in\mathbb{Z}} - \big((-1)^n f_i \partial^C\big)_{i\in\mathbb{Z}}.$$

Treating right Λ–modules A and left Λ–modules B as chain complexes concentrated at level zero, we define

$$\mathrm{H}_n(C; A) := \mathrm{H}_n(A\otimes_\Lambda C); \quad \mathrm{H}^k(C; B) := \mathrm{H}_{-k}(\mathrm{Hom}_\Lambda(C, B)).$$

The group G acts diagonally on $C\otimes D$ via $g(c_i \otimes d_j) = gc_i \otimes gd_j$ for $g \in G$ and $c_i \otimes d_j \in C_i \otimes D_j$. On the level of chains, the twisted slant operation is the chain map

$$/ : \big(\mathrm{Hom}_\Lambda(C, B)\big)_{-k}\otimes\big(\mathbb{Z}^\omega\otimes_\Lambda(C\otimes D)\big)_n \longrightarrow B^\omega\otimes_\Lambda D_{n-k},$$
$$\varphi/(z\otimes \sum_{i+j=n} c_i\otimes d_j) := z\varphi(c_k) \otimes d_{n-k}.$$

Passing to homology and composing with the homomorphism

$$\alpha : \mathrm{H}C\otimes\mathrm{H}D \to \mathrm{H}(C\otimes D), [x]\otimes[y] \mapsto [x\otimes y],$$

yields the twisted slant operation

$$/ : \mathrm{H}^k(C; B)\otimes\mathrm{H}_n(C\otimes D; \mathbb{Z}^\omega) \to \mathrm{H}_{n-k}(D; B^\omega).$$

Let $\varepsilon : P \twoheadrightarrow \mathbb{Z}$ be an augmented chain complex of left Λ–modules with equivariant diagonal $\Delta : P \to P\otimes P$, that is, $(\varepsilon\otimes\mathrm{id})\Delta(c) = (\mathrm{id}\otimes\varepsilon)\Delta(c) = c$, and let Q be a subcomplex of P such that the inclusion $\iota : Q \rightarrowtail P$ is a map of augmented chain complexes with equivariant diagonal, that is, $\Delta\circ\iota = (\iota\otimes\iota)\circ\Delta$. Then (P, Q) is called a *geometric chain pair*, $Q \overset{\iota}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} D := P/Q$ a *short exact sequence of augmented chain complexes with compatible diagonals* and the chain map

$$\Delta_{\mathrm{rel}} : D \longrightarrow P\otimes D, d \mapsto (\mathrm{id}\otimes\pi)(\Delta(\mathrm{p})), \quad \text{where} \quad \mathrm{d} = \pi(\mathrm{p}),$$

is called the *relative equivariant diagonal.* Composing the relative diagonal and the twisted slant operation, we obtain the *relative twisted cap product*

$$\cap : \mathrm{Hom}_\Lambda(P, B)_{-k}\otimes(\mathbb{Z}^\omega\otimes_\Lambda D)_n \to (B^\omega\otimes_\Lambda D)_{n-k}, (\varphi, (z\otimes d)) \mapsto \varphi/(z\otimes\Delta_{\mathrm{rel}}(d))$$

for every left Λ–module B. Passing to homology and composing with α, we obtain the relative twisted cap product

$$\cap : \mathrm{H}^k(P; B)\otimes\mathrm{H}_n(D; \mathbb{Z}^\omega) \to \mathrm{H}_{n-k}(D; B^\omega).$$

Similarly, the relative diagonal $\Delta'_{\mathrm{rel}} : D \longrightarrow D\otimes P, d \mapsto (\pi\otimes\mathrm{id})(\Delta(\mathrm{p}))$, where $d = \pi(p)$, yields the relative twisted cap product

$$\cap : \mathrm{H}^k(D; B)\otimes\mathrm{H}_n(D; \mathbb{Z}^\omega) \to \mathrm{H}_{n-k}(P; B^\omega).$$

Note that the relative cap products reduce to the absolute cap product when Q is trivial.

If M is a Λ–bimodule, then Λ acts on the left on $M\otimes_\Lambda B$ via $\lambda.(m\otimes b) := (\lambda.m)\otimes b$ and on the right on $\mathrm{Hom}_\Lambda(B, M)$ via $(\varphi.\lambda)(b) := \varphi(b).\lambda$ for all left Λ–modules $B, \lambda \in \Lambda, b \in B, m \in M$ and $\varphi \in \mathrm{Hom}_\Lambda(B, M)$. In particular, $B^* :=^\omega \mathrm{Hom}_\Lambda(B, \Lambda)$ is a left Λ–module. Any left Λ–module A gives rise to the functors $A^\omega\otimes_\Lambda-$ and

$\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-,\Lambda), A))$ from the category ${}_\Lambda\mathcal{M}$ of left Λ–modules to the category $\mathcal{A}b$ of abelian groups and there is a natural transformation

$$\eta_B : A^\omega \otimes_\Lambda B \longrightarrow \mathrm{Hom}_\Lambda(B^*, A) \tag{2}$$

given by

$$\eta_B(a \otimes b) : \ {}^\omega\mathrm{Hom}_\Lambda(B, \Lambda) \longrightarrow A, \quad \varphi \longmapsto \overline{\varphi(b)}a$$

for every left Λ–module B. When we restrict the two functors to the category of finitely generated free left Λ–modules, the natural transformation η becomes a natural equivalence. For Λ–bimodules M we may view ${}^\omega\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-,\Lambda), M)$ and ${}^\omega M^\omega \otimes_\Lambda -$ as functors from the category of left Λ–modules to itself and in this case the natural transformation η respects the additional left Λ–module structure. Identifying the left Λ–module B with ${}^\omega\Lambda^\omega \otimes_\Lambda B$ for $M = \Lambda$, the natural equivalence η becomes the evaluation homomorphism from B to its double dual

$$(B^*)^* = {}^\omega\,\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-,\Lambda), \Lambda).$$

Let $Q \rightarrowtail P \twoheadrightarrow D$ be a short exact sequence of augmented chain complexes with compatible diagonals, so that (P, Q) is a geometric chain pair. Then the chain map given by taking the cap product with a cycle $1 \otimes x \in \mathbb{Z}^\omega \otimes_\Lambda D_n$ is almost chain homotopic to its dual.

Lemma 2.1. *Let $1 \otimes x \in \mathbb{Z}^\omega \otimes_\Lambda D_n$ be a cycle. Then the diagram*

$$\begin{array}{ccc} D_k^* & \xrightarrow{\ \theta\ } & ({}^\omega\Lambda^\omega \otimes_\Lambda D_k)^* \\ \downarrow{\scriptstyle \cap 1\otimes x} & & \downarrow{\scriptstyle (\cap 1 \otimes x)^*} \\ {}^\omega\Lambda^\omega \otimes_\Lambda P_{n-k} & \xrightarrow[\ \eta_{P_{n-k}}\ ]{} & (P_{n-k}^*)^* \end{array}$$

commutes up to chain homotopy, where the isomorphism θ is given by $\theta(\varphi)(\lambda \otimes d) := \overline{\lambda}\varphi(d)$ for every $\varphi \in D_k^*, \lambda \in \Lambda$ and $d \in D_k$.

Proof. Use that any two diagonals are chain homotopic [2]. □

The short exact sequence $Q \overset{\iota}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} D$ of augmented chain complexes of free Λ–modules with compatible diagonals splits and stays split short exact when we tensor or apply the Hom_Λ–functor. Given a right Λ–module M, denote the connecting homomorphisms of

$$\mathbb{Z}^\omega \otimes_\Lambda Q \rightarrowtail \mathbb{Z}^\omega \otimes_\Lambda P \twoheadrightarrow \mathbb{Z}^\omega \otimes_\Lambda D,$$
$$M \otimes_\Lambda Q \rightarrowtail M \otimes_\Lambda P \twoheadrightarrow M \otimes_\Lambda D \quad \text{and}$$
$${}^\omega\mathrm{Hom}_\Lambda(D, {}^\omega M) \rightarrowtail {}^\omega\mathrm{Hom}_\Lambda(P, {}^\omega M) \twoheadrightarrow {}^\omega\mathrm{Hom}_\Lambda(Q, {}^\omega M)$$

by δ_*, δ'_* and δ^* respectively. Standard arguments [2] show

Theorem 2.2 (Cap Product Ladder)**.** *For all $y \in H_n(D;\mathbb{Z}^\omega)$, the diagram*

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^r(D;{}^\omega M) & \xrightarrow{\pi^*} & H^r(P;{}^\omega M) & \xrightarrow{\iota^*} & H^r(Q;{}^\omega M) \\
 & \downarrow{\scriptstyle \cap y} & & \downarrow{\scriptstyle \cap y} & & \downarrow{\scriptstyle \cap \delta_* y} \\
\cdots \longrightarrow & H_{n-r}(P;M) & \longrightarrow & H_{n-r}(D;M) & \xrightarrow{\delta'_*} & H_{n-r-1}(Q;M)
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{\delta^*} & H^{r+1}(D;{}^\omega M) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \cap y} & \\
\longrightarrow & H_{n-r-1}(P;M) & \longrightarrow \cdots
\end{array}$$

commutes, up to sign.

The geometric chain pair (P,Q) is called a *Poincaré chain pair* if there is an element $\nu \in H_n(P;Q;\mathbb{Z}^\omega)$ of infinite order such that

$$\cap\nu : \mathrm{H}^k(P;{}^\omega M)\otimes\mathrm{H}_n(D;\mathbb{Z}^\omega) \to \mathrm{H}_{n-k}(D;M)$$

is an isomorphism of abelian groups for every right Λ–module M and every $k \in \mathbb{Z}$. We then call ν the *fundamental class* of (P,Q). For the definition of connected sums of PD^{n}–pairs we need a generalisation of Browder's result [4] concerning cap products and sums of Poincaré chain pairs to the case of non–trivial fundamental groups.

Suppose $B = B_1 + B_2, B_0 = B_1 \cap B_2, A \subseteq B, A_i = B_i \cap A$ for $i = 1,2$, and $A_0 = A_1 \cap A_2$ are chain complexes of free Λ–modules such that all pairs arising are geometric chain pairs. Let ∂_0 be the connecting homomorphism of the short exact sequence

$$\mathbb{Z}^\omega\otimes_\Lambda(B_0/A_0) \rightarrowtail \mathbb{Z}^\omega\otimes_\Lambda(B_1/A_1 \oplus B_2/A_2) \twoheadrightarrow \mathbb{Z}^\omega\otimes_\Lambda(B/A),$$

and, for $i = 1,2$, let η_i denote the Λ–morphism rendering the diagram

$$\begin{array}{ccc}
H_q(B,A;\mathbb{Z}^\omega) & \xrightarrow{\eta_i} & H_q(B_1,B_0+A_1;\mathbb{Z}^\omega) \\
 & \searrow & \downarrow \\
 & & H_q(B,B_2+A;\mathbb{Z}^\omega)
\end{array}$$

commutative. Standard arguments [2] show

Theorem 2.3. *Any two of the following conditions imply the third:*

1. *(B,A) is a Poincaré chain pair with fundamental class $\nu \in H_n(B,A;\mathbb{Z}^\omega)$;*
2. *(B_0,A_0) is a Poincaré chain pair with fundamental class $\partial_0\nu$ an element of $H_{n-1}(B_0,A_0;\mathbb{Z}^\omega)$;*
3. *for $i = 1,2,$ (B_i,B_0+A_i) is a Poincaré chain pair with fundamental class $\eta_i\nu \in H_n(B_i,B_0+A_i;\mathbb{Z}^\omega)$.*

3. Homotopy Classification of PD^3–Pairs

Given a CW–complex X, fix a base point $*$ in X, put $G := \pi_1(X;*)$ and $\Lambda := \mathbb{Z}[G]$. Let $p : \tilde{X} \to X$ denote the universal covering of X and $C(X)$ the cellular chain complex of $\tilde{X}$, viewed as a complex of left Λ–modules. Then, for a right Λ–module A and a left Λ–module B,

$$\mathrm{H}_r(X;A) := \mathrm{H}_r(C(X);A) \quad \text{and} \quad \mathrm{H}^r(X;B) := \mathrm{H}^r(C(X);B).$$

A *connected* PD^n*–complex* is a triple $(X, \omega, [X])$, where X is a connected CW–complex, $\omega \in \mathrm{H}^1(X;\mathbb{Z}/2\mathbb{Z})$ and $[X] \in \mathrm{H}_n(X;\mathbb{Z}^\omega)$ such that

$$\cap\,[X] : \mathrm{H}^r(X;B) \to \mathrm{H}_{n-r}(X;B^\omega); \quad \alpha \mapsto \alpha \cap [X] \tag{3}$$

is an isomorphism of abelian groups for every $r \in \mathbb{Z}$ and every left Λ–module B. We often write X for the triple $(X, \omega, [X])$ and call ω the *orientation character* and $[X]$ the *fundamental class* of X. A PD^n–complex $(X, \omega, [X])$ is a finite disjoint union of connected PD^n–complexes $(X_i, \omega_i, [X]_i)$, $i \in J$, where $\Lambda = \bigoplus_{i\in J} \mathbb{Z}[G_i]$, $\omega = (\omega_i)_{i\in J}$ and $[X] = ([X_i])_{i\in J}$. Note that it is enough to demand that (3) is an isomorphism for $B = \Lambda$ [15].

Every n–dimensional manifold is homotopy equivalent to a CW–complex and thus determines a PD^n–complex. But not every PD^3–complex is homotopy equivalent to a 3–manifold. Wall [15] showed that the class of finite groups which are fundamental groups of PD^3–complexes coincides with the class of finite groups with periodic cohomology of period 4. By Milnor's results [12], some of these groups are not realisable as fundamental groups of 3–manifolds, the simplest such group being the permutation group S_3. Swan [13] explicitly constructed a PD^3–complex K with fundamental group S_3.

Given a pair, $(X, \partial X)$, of CW–complexes, let $C(\partial X)$ denote the subcomplex of $C(X)$ generated by the cells lying above ∂X and put $C(X, \partial X) := C(X)/C(\partial X)$. Then $C(X, \partial X)$ is called the *relative cellular complex* and

$$C(\partial X) \rightarrowtail C(X) \twoheadrightarrow C(X, \partial X)$$

the *short exact sequence of cellular chain complexes* of the pair $(X, \partial X)$. We define

$$\mathrm{H}_r(X, \partial X; A) := \mathrm{H}_r(C(X, \partial X); A) \quad \text{and} \quad \mathrm{H}^r(X, \partial X; B) := \mathrm{H}^r(C(X, \partial X); B)$$

for every right Λ–module A and every left Λ–module B, and denote the connecting homomorphism of $\mathbb{Z}^\omega \otimes_\Lambda C(\partial X) \rightarrowtail \mathbb{Z}^\omega \otimes_\Lambda C(X) \twoheadrightarrow \mathbb{Z}^\omega \otimes_\Lambda C(X, \partial X)$ by δ_*.

Definition 1. *The quadruple $(X, \partial X, \omega_X, [X, \partial X])$ is a* connected PD^n–pair *if $(X, \partial X)$ is a pair of CW–complexes with X connected, $(\partial X, \omega_{\partial X}, [\partial X])$ is a* PD^{n-1}*–complex, where ω_X induces $\omega_{\partial X_i}$ on the connected components ∂X_i of ∂X, and $[X, \partial X]$ is an element of $\mathrm{H}_n(X, \partial X; \mathbb{Z}^\omega)$ with $\delta_*[X, \partial X] = [\partial X]$, such that*

$$\cap\,[X, \partial X] : \mathrm{H}^r(X;B) \to \mathrm{H}_{n-r}(X, \partial X; B^\omega), \alpha \mapsto \alpha \cap [X, \partial X] \tag{4}$$

is an isomorphism for every left Λ–module B and every $r \in \mathbb{Z}$. We often write $(X, \partial X)$ for the PD^n*–pair and call ω_X the* orientation character *and $[X, \partial X]$ the* fundamental class *of $(X, \partial X)$.*

Again it is enough to demand that (4) is an isomorphism for $B = \Lambda$ [15]. As for PD^n–complexes, we may define a general PD^n–pair as a finite disjoint union of connected PD^n–pairs. Every manifold with boundary determines a PD^n–pair. We obtain examples of PD^n–pairs which are not homotopy equivalent to a manifold with boundary by taking a PD^n–complex which is not homotopy equivalent to a manifold and removing the interior of an n–cell.

Poincaré duality manifests itself on the level of chains, namely, for a PD^n–pair, $(X, \partial X)$, the cap product with a representative of the fundamental class defines a chain map of degree n which is a chain homotopy equivalence.

Definition 2. *The* PD^n*–pairs* $(X_1, \partial X_1)$ *and* $(X_2, \partial X_2)$ *are* orientedly homotopy equivalent *if there is a homotopy equivalence* $f : (X_1, \partial X_1) \to (X_2, \partial X_2)$ *of pairs with* $f^*(\omega_{X_2}) = \omega_{X_1}$ *and* $f_*([X_1, \partial X_1]) = [X_2, \partial X_2]$.

A family, $\{\kappa_i : G_i \to G\}_{i\in J}$, of group homomorphisms is also called a π_1*–system* and an *Eilenberg–Mac Lane pair* of type $K(\{\kappa_i : G_i \to G\}_{i\in J}; 1)$ is a pair $(X, \partial X)$ such that X is an Eilenberg–Mac Lane complex of type $K(G; 1)$, the connected components $\{\partial X_i\}_{i\in J}$ of ∂X are Eilenberg–Mac Lane complexes of type $K(G_i; 1)$ and there is an isomorphism

$$(\varphi, \{\varphi_i\}_{i\in J}) : \{\rho_i : G_i \to G\}_{i\in J} \to \{\kappa_i : \pi_1(\partial X_i, *) \to \pi_1(X, *)\}_{i\in J}$$

of π_1–systems, that is, for each $i \in J$ the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\varphi_i} & \pi(\partial X_i, *) \\ {\scriptstyle\rho_i}\downarrow & & \downarrow{\scriptstyle\kappa_i} \\ G & \xrightarrow{\varphi} & \pi_1(X, *) \end{array}$$

commutes up to conjugacy. Note that we do not require the homomorphisms κ_i to be injective as in the standard definition given by Bieri–Eckmann [1]. For any π_1–system, $\{\kappa_i : G_i \to G\}_{i\in J}$, there is an Eilenberg–Mac Lane pair of type $K(\{\kappa_i : G_i \to G\}_{i\in J}; 1)$, which is uniquely determined up to homotopy equivalence of pairs. Further, for any pair, $(X, \partial X)$, of CW–complexes there is map of pairs

$$c_X : (X, \partial X) \longrightarrow K(G, \{G_i\}_{i\in J}; 1),$$

called the *classifying map*, which is uniquely determined up to homotopy of pairs and induces an isomorphism of π_1–systems.

An oriented homotopy equivalence of PD^n–pairs, $(X, \partial X)$ and $(Y, \partial Y)$, induces an isomorphism

$$(\varphi, \{\varphi_i\}_{i\in J}) : \{\kappa_i : \pi_1(\partial X_i, *) \to \pi_1(X, *)\}_{i\in J} \to \{\rho_i : \pi_1(\partial Y_i, *) \to \pi_1(Y, *)\}_{i\in J}$$

of their π_1–systems such that

$$\varphi^*(\omega_Y) = \omega_X \quad \text{and} \quad \varphi_*(c_{X*}([X, \partial X])) = c_{Y*}([Y, \partial Y]), \tag{5}$$

where c_X and c_Y are classifying maps and $\{\partial X_i\}$ and $\{\partial Y_i\}$ are the connected components of ∂X and ∂Y respectively. The Classification Theorem states that there is a converse for $n = 3$.

Now put $\mu_X := c_{X*}([X, \partial X])$ and restrict attention to the case $n = 3$.

As the homology and cohomology sequences of any Eilenberg–Mac Lane pair $K(\{\kappa_i : G_i \to G\}_{i\in J}; 1)$ are isomorphic to those of the group pair $(G, \{G_i\}_{i\in J})$ [1], we may identify $c_{X*}([X, \partial X]$ and ω_X with their images in $\mathrm{H}^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{H}_3(G, \{G_i\}_{i\in J}; \mathbb{Z}^\omega)$, respectively.

Definition 3. *The triple* $(\{\kappa_i\}_{i\in J}, \omega_X, \mu_X)$ *is the* fundamental triple *of the* PD^3*–pair* $(X, \partial X)$.

Two triples $(\{\kappa_i\}_{i\in J},\omega,\mu)$ *and* $(\{\kappa'_i\}_{i\in J},\omega',\mu')$ *are* isomorphic *if there is an isomorphism* $(\varphi,\{\varphi_i\}_{i\in J}):\{\kappa_i\}_{i\in J}\to\{\kappa'_i\}_{i\in J}$ *of* π_1*–systems such that* $\varphi^*(\omega')=\omega$ *and* $\varphi_*(\mu)=\mu'$.

Note that the fundamental triple of a PD^3–pair is uniquely determined up to isomorphism of triples.

Proof of Classification. The proof generalises Turaev's alternative proof of Hendriks' result for the absolute case.

Given two PD^3–pairs, $(X,\partial X)$ and $(Y,\partial Y)$, with aspherical boundaries and isomorphic fundamental triples, we use homological algebra and obstruction theory to construct a map $f:(X,\partial X)\to(Y,\partial Y)$ of pairs such that

(i) f induces a homotopy equivalence $\hat{f}:\partial X\to\partial Y$;
(ii) $f_*:\pi_1(X,*)\to\pi_1(Y,*)$ is an isomorphism respecting the orientation character;
(iii) f has degree one, that is, $f_*([X,\partial X])=[Y,\partial Y]$.

Then f is a homotopy equivalence of PD^3–pairs by Poincaré duality, the Five Lemma and Whitehead's Theorem.

It is not difficult to see that X is homologically two–dimensional for any PD^3–pair, $(X,\partial X)$. By definition, the boundary ∂X is a PD^2–complex and Eckmann, Müller and Linnell ([7] and [6]) showed that every connected PD^2–complex is homotopy equivalent to a closed surface. If the boundary is not empty, we may apply the mapping cylinder construction and hence assume that the components $\{\partial X_i\}_{i\in J}$ of ∂X are closed surfaces which are collared in X, the collar being a 3–manifold with boundary. Splitting off a 3–cell from the collar of one of the boundary components, we obtain $X=X'\cup_g e^3$ where $g:S^2\to X'$.

By definition [16], the connected CW–complex X satisfies $\mathrm{D_n}$ if $\mathrm{H}_i(\tilde{X})=0$ for $i>n$ and $\mathrm{H}^{n+1}(X;B)=0$ for all left Λ–modules B. Poincaré duality implies that X satisfies D_2 for every PD^3–pair $(X,\partial X)$. We obtain [2]

Lemma 3.1. *Let* $(X,\partial X)$ *be a* PD^3*–pair. Then* $X=X'\cup_g e^3$ *where* $g:S^2\to X'$ *and* X' *satisfies* D_2 *and is thus homologically two–dimensional.*

Take a connected PD^3–pair, $(X,\partial X)$, with aspherical boundary components and $X=X'\cup_g e^3$ as above. Without loss of generality, we may assume that $\partial X\cap e^3=\emptyset$. We denote the subcomplex of $C(X)$ generated by the cells lying in $\tilde{X}':=p^{-1}(X')$ by $C(X')$. By Theorem [E] in [16], we may assume that X and hence X' are geometrically 3–dimensional. As X' is homologically 2–dimensional, $C(X')$ is a chain complex of free left Λ–modules with $\mathrm{H}^k(C(X');B)=0$ for $k\geq 3$ and for every (left) Λ–module B. Hence Lemma 3.6 in [2] implies that $C(X')$ decomposes into the direct sum of the two subcomplexes

$$D:\quad 0\;\to\;C_3(X')\;\twoheadrightarrow\;\mathrm{im}\partial'_2$$

$$E:\quad 0\;\to\;\Lambda[e]\;\twoheadrightarrow\;S\;\to\;C_1(X)\;\to\;C_0(X),$$

where $D\simeq 0$, S is the cokernel of the boundary operator $\partial'_2:C_3(X')\to C_2(X')$, the chain e corresponds to the 3–cell attached via g and $\Lambda[e]$ denotes the free Λ-module with generator e. As before, we may assume that the connected components of ∂X are closed surfaces, so that $C_i(\partial X)=0$ for $i>2$.

Lemma 3.2. *Let $p : \tilde{X} \to X$ be the universal covering. Then the components of $p^{-1}(\partial X)$ are open surfaces and hence the boundary operator $C_2(\partial X) \to C_1(\partial X)$ is injective.*

Proof. Suppose the PD^3–pair $(X, \partial X)$ has π_1–system $\{\kappa_i : G_i \to G\}_{i \in J}$ and connected boundary components $\{\partial X_j\}_{j \in J}$ which are closed aspherical surfaces. Suppose further that $\kappa_i(G_i)$ is finite in G for some $i \in J$. Passing to $(X^+, \partial X^+)$, where X^+ is the orientable covering space of X and ∂X^+ is the inverse image of ∂X under the covering map, the image of the fundamental group of each component of ∂X^+ over ∂X_i is still finite. Thus we may assume that $(X, \partial X)$ is orientable.

We replace X by $X' = X \cup_{j \neq i} H_j$, where $(H_j, \partial X_j)$ is a connected PD^3–pair for $j \in J, j \neq i$. By Theorem 2.3, $(X', \partial X_i)$ is a PD^3–pair. Thus we may assume without loss of generality that ∂X is connected. Poincaré duality and the Hurewicz homomorphism together with our assumptions yield $\mathrm{H}_1(\partial X; \mathbb{Q}) = 0$, so that $\partial X = S^2$. Hence, if all components of ∂X are aspherical, all components of $p^{-1}(\partial X)$ are open surfaces. □

Lemma 3.2 implies $\mathrm{im}\partial_2' \cap C_2(\partial X) = 0$ and hence the relative chain complex $C(X, \partial X)$ of the pair $(X, \partial X)$ decomposes into the direct sum of the two subcomplexes

$$D: \quad 0 \to C_3(X') \twoheadrightarrow \mathrm{im}\partial_2' \quad \text{and}$$

$$E: \quad 0 \to \Lambda[e] \to \tilde{S} \to C_1(X, \partial X) \to C_0(X, \partial X),$$

where $\tilde{S} = S/C_2(\partial X)$ and $D \simeq 0$.

Let $\{e_m^2\}_{m \in M}$ be a collection of two–cells of $\tilde{X}$ such that above every two–cell in $X \setminus \partial X$ there lies exactly one cell from this collection. Then the collection $\{e_m\}_{m \in M}$ of chains represented by these cells comprises a basis of the Λ–module $C_2(X, \partial X)$. Suppose $\partial_2(e) = \sum a_m e_m$, where $\partial_2 : C_3(X, \partial X) \to C_2(X, \partial X)$ denotes the boundary operator of the relative complex and $a_m \in \Lambda$ for $m \in M$.

Lemma 3.3. *The chain $1 \otimes e$ is a relative cycle representing the homology class $[X, \partial X]$. Further, $I = \mathrm{im}(\partial_2^E)^*$ and is generated by $\{\overline{a_m}\}_{m \in M}$.*

For a proof we refer the reader to [2].

Given connected PD^3–pairs, $(X, \partial X)$ and $(Y, \partial Y)$, with aspherical boundary components and isomorphic fundamental triples, we must construct a map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ satisfying (i) – (iii). Suppose the isomorphism of fundamental triples is given by

$$(\varphi, \{\varphi_i\}_{i \in J}) : \{\kappa_i : G_i \to G\}_{i \in J} \to \{\rho_i : H_i \to H\}_{i \in J}.$$

We may assume without loss of generality that Y is contained in $(K, \partial Y) := K(\{\rho_i : H_i \to H\}_{i \in J}; 1)$. Let $q : \tilde{K} \to K$ be the universal covering, put $\Lambda_X := \mathbb{Z}[G]$, $\Lambda_Y := \mathbb{Z}[H]$ and let I_X and I_Y denote the kernel of aug_X and aug_Y respectively. By Lemma 3.3, $X = X' \cup e^3$ and $Y = Y' \cup e'^3$, where X' and Y' are homologically two–dimensional, and if e and e' are chains representing the 3–cells e^3 and e'^3, then $1 \otimes e$ and $1 \otimes e'$ represent $[X, \partial X]$ and $[Y, \partial Y]$ respectively.

Further, we may assume that the connected components of ∂X and ∂Y are closed surfaces, and that there is a homeomorphism $\tilde{g} : \partial X \longrightarrow \partial Y$ which induces the isomorphisms $\varphi_i : G_i \to H_i$ on the fundamental groups of the connected components of ∂X and ∂Y respectively. It is not difficult to construct a cellular map of pairs $g' : (X', \partial X) \longrightarrow (Y', \partial Y)$ with $g'|_{\partial X} = \tilde{g}$. As $\pi_2(K;*) = 0$, the map g' gives rise to a cellular map of pairs $g : (X, \partial X) \longrightarrow (K, \partial Y)$ with $g|_{\partial X} = \tilde{g}$. Let $g_* : C(X, \partial X) \longrightarrow C(K, \partial Y)$ be the chain map induced by g. Then $\varphi_*(c_{X*}([X, \partial X])) = c_{Y*}([Y, \partial Y])$ implies $1 \otimes g_*(e) - 1 \otimes e' = 0$ in $\mathrm{H}_3(K, \partial Y; \mathbb{Z}^{\omega_Y})$, where we identify e' with its inclusion in K. Thus $g_*(e) - e'$ is contained in $\mathrm{im}(\partial_3 : C_4(K, \partial Y) \to C_3(K, \partial Y)) + \overline{I_Y} C_3(K, \partial Y)$, and hence

$$\partial_2(g_*(e) - e') \in \overline{I_Y} \partial_2(C_3(K, \partial Y)). \tag{6}$$

The group isomorphism $\varphi : G \to H$ induces a ring isomorphism $\varphi : \Lambda_X \to \Lambda_Y$ and $\varphi^*(\omega_Y) = \omega_X$ implies $\varphi(\overline{\lambda}) = \overline{\varphi(\lambda)}$ for every $\lambda \in \Lambda_X$. By Lemma 3.3, $\overline{I_X}$ is generated by $\{a_m\}_{m \in M}$, where $\partial_2(e) = \sum_{m \in M} a_m e_m$. Hence $\overline{I_Y} = \overline{(\varphi(I_X))} = \varphi(\overline{I_X})$ is generated as right Λ–module by $\{\varphi(a_m)\}_{m \in M}$ and we obtain

$$\partial_2(g_*(e)) - \partial_2(e') = g_*(\partial_2(e)) - \partial_2(e') = \sum_{m \in M} \sum_k \varphi(a_m) \lambda_{mk} d_k,$$

where $d_k = \partial_2 c_k$ for some $c_k \in C_3(K, \partial Y)$. In other words, d_k is represented by the gluing map $f_k : S^2 \to q^{-1}(Y' \backslash \partial Y)$ of a 3–cell in $q^{-1}(Y' \backslash \partial Y)$. We obtain $\partial_2(e') = \sum_m \varphi(a_m)(g_*(e_m) - \sum \lambda_{mk} d_k)$ and as $D^2 \# S^2 \simeq D^2$, we may modify $g'|_{X^{[2]}}$ on the interior of two–cells to obtain a map of pairs $h : (X', \partial X) \longrightarrow (Y', \partial Y)$ with

$$h_*(\partial_2(e)) = h_*(\sum_{m \in M} a_m e_m) = \sum_{m \in M} \varphi(a_m)(g_*(e_m) - \sum \lambda_{km} d_k) = \partial_2(e'). \tag{7}$$

Since all components of ∂Y are aspherical, the components of $q^{-1}(\partial Y)$ are open surfaces by Lemma 3.2. Hence the long exact homology sequence of the pair $(Y', \partial Y)$ yields an injective homomorphism $\mathrm{H}_2(q^{-1}(Y'); \Lambda_Y) \rightarrowtail \mathrm{H}_2(q^{-1}(Y'), q^{-1}(\partial Y); \Lambda_Y)$. As $\delta_*[Y, \partial Y] = [\partial Y]$, the class $[\partial_2 e']$ is contained in the image of this homomorphism. Therefore $\pi_2(Y') \cong \mathrm{H}_2(q^{-1}(Y'); \Lambda_Y)$ and (7) imply that the composition of the attaching map of the 3–cell e^3 (represented by e) and the map h is homotopy equivalent to the attaching map of the 3–cell ${e'}^3$ (represented by e').

We conclude that h extends to a map of pairs $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ satisfying (i) – (iii), showing that the PD^3–pairs $(X, \partial X)$ and $(Y, \partial Y)$ are homotopy equivalent. □

4. Realisation of Invariants by PD^3–Pairs

We need the following result from the *derived module category*, also called the *projective homotopy category of modules.*

Theorem 4.1. *Let Λ be a ring with unit. A homotopy equivalence $\varphi : A \to B$ of Λ–modules factors as*

$$A \overset{\iota}{\rightarrowtail} A \oplus P \overset{\tilde{\varphi}}{\twoheadrightarrow} B \oplus Q \overset{\pi}{\twoheadrightarrow} B,$$

where P and Q are projective and ι and π are the natural inclusion and projection respectively.

Proof. The proof is dual to the proof of Theorem 13.7 in [9]. □

Observation 1. *There is a projective Λ–module $\tilde{P}$ such that $P \oplus \tilde{P}$ is isomorphic to a free Λ–module F, so that φ factors as*

$$A \rightarrowtail A \oplus F \twoheadrightarrow B \oplus \tilde{Q} \twoheadrightarrow B$$

where $\tilde{Q} = Q \oplus \tilde{P}$ is projective. If the Λ–modules A and B are finitely generated, the projective Λ–modules P and Q are also finitely generated. Then $F \cong \Lambda^n$ for some $n \in \mathbb{N}$ and $\tilde{Q}$ is finitely generated projective.

Let Λ be the integral group ring of the group H and take $\omega \in \mathrm{H}^1(H; \mathbb{Z}/2\mathbb{Z})$. Given a chain complex $\ldots \to C_{r+1} \xrightarrow{\partial_r} C_r \to \ldots$ of left Λ–modules, put

$$\mathrm{G}_r(C) := \mathrm{coker}\partial_r = C_r/\mathrm{im}\partial_r,$$

and given a chain map $f : C \to D$, let $\mathrm{G}_r(f) : \mathrm{G}_r(C) \to \mathrm{G}_r(D)$ be the induced Λ–morphism of cokernels. Then $G = G_*$ is a functor from the category of chain complexes of left Λ–modules to itself. Writing C^* for ${}^{\omega}\mathrm{Hom}_\Lambda(C, \Lambda)$, we compose the functors G and ${}^{\omega}\mathrm{Hom}_\Lambda(-, \Lambda)$ to obtain the functor F (see [14] p.265) given by

$$F^r(C) = G_{-r}(C^*) = C^r/\mathrm{im}\partial^*_{r-1}. \tag{8}$$

Lemma 4.2. *Let $f, g : C \to D$ be chain homotopic maps of chain complexes over Λ. If D_n is projective, then $G_n(f) \simeq G_n(g)$ as Λ–morphisms.*

For a proof we refer the reader to [2].

Thus we may view G_n as a functor from the category of chain complexes of projective left Λ–modules and homotopy classes of chain maps to the derived module category.

Corollary 4.3. *Let $f : C \to D$ be a homotopy equivalence of chain complexes over Λ. If C_n and D_n are projective, then $G_n(f)$ is a homotopy equivalence of Λ–modules.*

Lemma 4.4. *Let (X, Y) be a pair of CW–complexes with X connected, $\omega \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ and $H_n(X, Y; \mathbb{Z}^\omega) \cong \mathbb{Z}$ generated by $[1 \otimes x]$. Then there is a chain $w_1 \in C_1(X)$, such that the Λ–morphism $\cap 1 \otimes x : C^*(X, Y) \to {}^{\omega}\Lambda^\omega \otimes_\Lambda C(X) \cong C(X)$ is given by*

$$\varphi \cap 1 \otimes x = \overline{\varphi(x)}.(1 + \partial_0 w_1)$$

for every cocycle $\varphi \in C^*(X, Y)$, where we identify $\lambda \otimes c \in {}^{\omega}\Lambda^\omega \otimes_\Lambda C(X)$ with $\overline{\lambda}.c \in C(X)$.

Proof. Let $\pi : C(X) \twoheadrightarrow C(X, \partial X)$ be the natural projection, take $y \in C_n(X)$ with $\pi(y) = x$ and assume $\Delta y = \sum y_i \otimes z_{n-i}$ with $y_i, z_i \in C_i(X)$. Then $y = (\mathrm{id} \otimes \varepsilon)\Delta(y)$ implies $y_n.\varepsilon(z_0) = y$. As $[1 \otimes x]$ is a generator, x and thus y are indivisible, so that $y = y_n$ and $\varepsilon(z_0) = 1$ up to sign. As X is connected, we may assume $C_0(X) = \Lambda$ and identify $\mathrm{im}\partial_1$ with $I = \ker \varepsilon$. Then $\varepsilon(z_0) = 1$ implies $z_0 = 1 + w_0$ with $w_0 \in I$. Hence $z_0 = 1 + \partial_0 w_1$ for some $w_1 \in C_1(X)$, so that $\varphi \cap 1 \otimes x = \overline{\varphi(x)}.(1 + \partial_0 w_1)$. □

Given a PD^3–pair, $(X, \partial X)$, take a cycle $1 \otimes x \in \mathbb{Z}^\omega \otimes_\Lambda C_3(X, \partial X)$ representing $[X, \partial X]$. Then $\cap 1 \otimes x : C^*(X, \partial X) \to^\omega \Lambda^\omega \otimes_\Lambda C(X) \cong C(X)$ is a chain homotopy equivalence. As both $C^*_2(X, \partial X)$ and $C_1(X)$ are free and hence projective, Corollary 4.3 implies that

$$G_{-2}(\cap 1 \otimes x) : F^2(C(X, \partial X)) = G_{-2}(C^*(X, \partial X)) \to G_1(C(X))$$

is a homotopy equivalence of Λ–modules. Composing with the isomorphism $\vartheta : G_1(C(X)) \to I$, given by $\vartheta([c]) := \partial_0(c)$, we obtain another homotopy equivalence of Λ–modules. The fact that $\cap 1 \otimes x$ is a chain map together with Lemma 4.4 yields $(\vartheta \circ G_{-2}(\cap 1 \otimes x))([\varphi]) = \overline{(\partial_2^* \varphi)(x)} + \overline{(\partial_2^* \varphi)(x)} \partial_0 w_1$ for every $\varphi \in C_2^*(X, \partial X)$ and some $w_1 \in C_1(X)$. As the Λ–morphism $F^2(C(X, \partial X)) \longrightarrow I,\ [\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)} \partial_0 w_1$ is null–homotopic,

$$F^2(C(X, \partial X)) \longrightarrow I,\ [\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)}$$

is a homotopy equivalence of Λ–modules.

Attaching cells of dimension three and larger to $(X, \partial X)$, we obtain an Eilenberg–Mac Lane pair $(K, \partial X)$ of type $K(\{\kappa_i : \pi_1(\partial X_i, *) \to \pi_1(X, *)\}_{i \in J}; 1)$ with the inclusion $\iota : (X, \partial X) \to (K, \partial X)$ a cellular classifying map. Identifying the cellular chain complexes of the pair $(X, \partial X)$ with their image under the chain map induced by ι, we obtain $C_i(K) = C_i(X), C_i(K, \partial X) = C_i(X, \partial X)$ for $i = 0, 1, 2$ and $[1 \otimes x] = [X, \partial X] = \iota_*([X, \partial X])$.

Lemma 4.5. *The Λ–morphism*

$$F^2(C(K, \partial X)) \longrightarrow I,\ [\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)} \tag{9}$$

is a homotopy equivalence of Λ–modules.

Given a chain complex C of free left Λ–modules, Turaev constructed a group homomorphism

$$\nu_{C,r} : \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C) \longrightarrow [F^r, I]$$

such that $\nu_{C(X,\partial X),2}([1 \otimes x]) = \nu_{C(K,\partial X),2}(\iota_*([X, \partial X]))$ is the homotopy class of the homotopy equivalence (9). Let δ denote the connecting homomorphism of the short exact sequence $\overline{I}C \rightarrowtail C \twoheadrightarrow \mathbb{Z}^\omega \otimes_\Lambda C$ and identify $c \in C_r$ with $1 \otimes c \in \Lambda^\omega \otimes_\Lambda C$. Then the natural transformation η of (2) yields the Λ–morphism $\eta : C_r \longrightarrow (C_r^*)^*,\ c \longmapsto \eta(c)$ given by $\eta(c)(\varphi) = \overline{\varphi(c)}$. It is not difficult to show that $\eta(c)$ factors through the cokernel $F^r(C)$ of ∂_{r-1}^* and that its image is contained in I. We obtain the Λ–morphism

$$\hat{\eta(c)} : F^r(C) \longrightarrow I,\ [\varphi] \longmapsto \overline{\varphi(c)},$$

whose homotopy class depends on the homology class of the cycle $c \in \overline{I}C$ only. The composition of the homomorphism $\mathrm{H}(\overline{I}C) \longrightarrow [F^r(C), I],\ [c] \longmapsto [\hat{\eta(c)}]$ with δ yields the homomorphism

$$\nu_{C,r} : \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C) \longrightarrow [F^r(C), I], \quad [1 \otimes c] \longmapsto [\hat{\eta(c)}] \tag{10}$$

represented by the Λ–morphism

$$F^r(C) \longrightarrow I,\ [\varphi] \longmapsto \overline{\varphi(\partial_r(x))}.$$

Lemma 4.6. *Suppose C is a chain complex of free left Λ–modules such that C_r is finitely generated and $H_r(C) = H_{r+1}(C) = 0$. Then $\nu_{C,r}$ is an isomorphism.*

For a proof we refer the reader to Lemma 2.5 in [14].

Suppose $(K, \partial X)$ is an Eilenberg–Mac Lane pair of type $K(\{\kappa_i : G_i \to G\}_{i \in J}; 1)$, $\omega \in \mathrm{H}^1(K, \mathbb{Z}/2\mathbb{Z})$ and $\mu \in \mathrm{H}_3(K, \partial X; \mathbb{Z}^\omega)$.

Theorem 4.7 (REALSIATION I)**.** *If $(\{\kappa_i\}_{i \in J}, \omega, \mu)$ is the fundamental triple of a* PD^3*–pair then $\nu_{C(K,\partial X),2}(\mu)$ is a homotopy equivalence of Λ–modules.*

Proof. Assume $(\{\kappa_i\}_{i\in J},\omega,\mu)$ is the fundamental triple of the PD^3–pair $(X,\partial X)$ and the Eilenberg–Mac Lane pair $(K,\partial X)$ of type $K(\{\kappa_i\}_{i\in J};1)$ was obtained by attaching cells of dimension three and larger to X. Take

$$1\otimes x\in\mathbb{Z}^\omega\otimes_\Lambda C_3(X,\partial X)\subseteq\mathbb{Z}^\omega\otimes_\Lambda C_3(K,\partial X)$$

with $[1\otimes x]=\mu$. Then $F^2(C(K,\partial X))\longrightarrow I$, $[\varphi]\longmapsto\overline{\varphi(\partial_2(x))}$ represents the class $\nu_{C(K,\partial X),2}(\mu)$ and is a homotopy equivalence of Λ–modules by Lemma 4.5.

Given another Eilenberg–Mac Lane pair $(L,\partial L)$ of type $K(\{\kappa_i\}_{i\in J};1)$, there is a homotopy equivalence $f:(K,\partial X)\to(L,\partial L)$ of pairs of CW–complexes inducing a chain homotopy equivalence $g^*:C^*(K,\partial X)\longrightarrow C^*(L,\partial L)$. Thus Corollary 4.3 implies that $F^2(g)=G_{-2}(g^*)$ is a homotopy equivalence of Λ–modules. The diagram

$$\begin{array}{ccc} F^2(C(L,\partial L)) & \xrightarrow{\nu_{C(L,\partial L),2}(f_*\mu)} & I \\ {\scriptstyle F^2(g)}\downarrow & & \| \\ F^2(C(K,\partial K)) & \xrightarrow{\nu_{C(K,\partial K),2}(\mu)} & I \end{array}$$

commutes and hence $\nu_{C(L,\partial L),2}(f_*\mu)$ is a homotopy equivalence of Λ–modules if and only if $\nu_{C(K,\partial K),2}(\mu)$ is one. □

For $\{\kappa_i:G_i\to G\}_{i\in J}$ to be the π_1–system of a PD^3–pair $(X,\partial X)$, the groups G_i must be surface groups for all $i\in J$ as the components of ∂X are PD^2–complexes by definition and thus homotopy equivalent to closed surfaces.

Definition 4. *The π_1–system $\{\kappa_i\}_{i\in J}$ is* injective *if κ_i is injective for every $i\in J$. The triple $(\{\kappa_i\}_{i\in J},\omega,\mu)$ is then called* π_1–injective.

Let $\{\kappa_i:G_i\to G\}_{i\in J}$ be an injective π_1–system with G finitely presentable and G_i a surface group for all $i\in J$. Furthermore, let $(K,\partial X)$ be an Eilenberg–Mac Lane pair of type $K(\{\kappa_i\}_{i\in J};1)$ such that the components ∂X_i of ∂X are all surfaces. Take $\omega\in\mathrm{H}^1(K;\mathbb{Z}/2\mathbb{Z})$ and $\mu\in\mathrm{H}_3(K,\partial X;\mathbb{Z}^\omega)$ such that $\delta_*\mu=[\partial X]$, where $[\partial X]$ is the fundamental class of the PD^2–complex ∂X and δ_* is the connecting homomorphism of $C(\partial X)\rightarrowtail C(X)\twoheadrightarrow C(X,\partial X)$. Following Turaev's construction in the absolute case, we construct a PD^3–pair realising $(\{\kappa_i\}_{i\in J},\omega,\mu)$.

Since G is assumed finitely presentable, we may also assume that K has finite 2–skeleton $K^{[2]}$, so that the Λ–modules $C_2(K,\partial X)$ and thus $F^2(C(K,\partial X))$ are finitely generated. Let $h:F^2(C(K,\partial X))\to I$ be a Λ–morphism representing $\nu_{C(K,\partial X),2}(\mu)$. Then h is a homotopy equivalence of Λ– modules which factors as

$$F^2(C(K,\partial X))\rightarrowtail F^2(C(K,\partial X))\oplus\Lambda^m\overset{j}{\rightarrowtail\!\!\!\twoheadrightarrow} I\oplus P\twoheadrightarrow I\ ,$$

by Theorem 4.1, where P is finitely generated and projective. Let $B=(e^0\vee e^2)\cup e^3$ be a 3–dimensional ball. If we replace K by $K\vee(\bigvee_{i=1}^m B)$, then $K^{[2]}$ is replaced by $K^{[2]}\vee(\bigvee_{i=1}^m e^2)$ and $F^2(C(K,\partial X))$ is replaced by $F^2(C(K,\partial X))\oplus\Lambda^m$. Thus we may assume without loss of generality that h factors as

$$F^2(C(K,\partial X))\overset{j}{\rightarrowtail\!\!\!\twoheadrightarrow} I\oplus P\twoheadrightarrow I, \tag{11}$$

where P is finitely generated and projective.

First we consider the case where P is free, that is, $P \cong \Lambda^n$, for some $n \in \mathbb{N}$. Denote the natural projection $C^2(K, \partial X) \twoheadrightarrow F^2(C(K, \partial X))$ by π, put $\tilde{\iota} := (\iota, \mathrm{id}_P) : I \oplus P \to \Lambda \oplus P$ and consider the Λ–morphism

$$(12) \qquad \varphi : C^2(K, \partial X) \xrightarrow{\pi} F^2(C(K, \partial X)) \xrightarrow{j} I \oplus P \xrightarrow{\tilde{\iota}} \Lambda \oplus P.$$

By definition, $\varphi \circ \partial_1^* = 0$ and hence $\mathrm{im}\varphi^* \subseteq \ker \partial_1$.

Let $p : \tilde{K} \to K$ be the universal covering. Since κ_i is injective for every $i \in J$, the components of $p^{-1}(\partial X)$ are universal covering spaces of Eilenberg–Mac Lane complexes, so that $\mathrm{H}_2(p^{-1}(\partial X)) = \mathrm{H}_1(p^{-1}(\partial X)) = 0$. The long exact homology sequence of the pair $(p^{-1}(K^{[2]}), p^{-1}(\partial X))$ and the Hurewicz Isomorphism Theorem imply

$$(13) \quad \mathrm{im}\varphi^* \subseteq \ker \partial_1 = \mathrm{H}_2(p^{-1}(K^{[2]}), p^{-1}(\partial X)) \cong \mathrm{H}_2(p^{-1}(K^{[2]})) \cong \pi_2(p^{-1}(K^{[2]})).$$

We may thus attach 3–cells to $K^{[2]}$ to obtain a pair $(X, \partial X)$ of CW–complexes whose relative cellular chain complex is given by

$$D : 0 \longrightarrow (\Lambda \oplus P)^* \xrightarrow{\varphi^*} C_2(K, \partial X) \longrightarrow C_1(K, \partial X) \longrightarrow C_0(K, \partial X).$$

As $\pi_2(K) = 0$, the inclusion

$$(K^{[2]}, \partial X) \to (K, \partial X)$$

extends to a map $f : (X, \partial X) \longrightarrow (K, \partial X)$. Since f induces an isomorphism of π_1–systems, we may view ω as an element of $\mathrm{H}^1(X; \mathbb{Z}/2\mathbb{Z})$.

Then $(X, \partial X)$ is a PD^3–pair realising $(\{\kappa_i\}_{i \in J}, \omega, \mu)$ since

Proposition 4.8.

(i) *$H_3(X, \partial X; \mathbb{Z}^\omega) \cong \mathbb{Z}$;*
(ii) *$f_*([X, \partial X]) = \mu$, where $[X, \partial X]$ generates $H_3(X, \partial X; \mathbb{Z}^\omega)$;*
(iii) *$\delta_*[X, \partial X] = [\partial X]$, where $[\partial X]$ is the fundamental class of the* PD^2*–complex ∂X and δ_* is the connecting homomorphism of the short exact sequence*

$$C(\partial X) \rightarrowtail C(X) \twoheadrightarrow C(X, \partial X);$$

(iv) *$\cap[X, \partial X] : H^r(X; {}^\omega\Lambda^\omega) \to H_{r-3}(X, \partial X; \Lambda)$ is an isomorphism for every $r \in \mathbb{Z}$.*

Proof. We only prove (iv), details for (i) – (iii) are contained in [2].

First observe that the definition of $(X, \partial X)$ implies $\mathrm{H}^2(X, \partial X; {}^\omega\Lambda^\omega) = 0$. Since $\mathrm{H}_1(X; \Lambda) = \mathrm{H}_1(C(X)) = 0$, the homomorphism

$$\cap[X, \partial X] : \mathrm{H}^2(X, \partial X; {}^\omega\Lambda^\omega) \to \mathrm{H}_1(X; \Lambda)$$

is an isomorphism.

As $\Lambda \otimes P$ is free, we may use the natural transformation η to identify the module ${}^\omega\mathrm{Hom}_\Lambda((\Lambda \oplus P)^*, {}^\omega\Lambda^\omega)$ with $\Lambda \oplus P$ and $(\varphi^*)^*$ with φ. Then $\mathrm{H}^3(X, \partial X; {}^\omega\Lambda^\omega) \cong (\Lambda \oplus P)/\mathrm{im}\varphi \cong \Lambda/I \cong \mathbb{Z}$. Clearly, $\mathrm{H}^3(X, \partial X; {}^\omega\Lambda^\omega)$ is generated by $\psi = (1, 0) \in (\Lambda^*)^* \oplus (P^*)^* = C_3^*(X, \partial X) = C_3^*(X)$. By Lemma 4.4, $[\psi] \cap [X, \partial X] = [\psi] \cap [1 \otimes x] = \overline{\psi(x)} = 1$, that is, $\cap[X, \partial X]$ maps ψ to a generator of $\mathrm{H}_0(X; \Lambda)$. Hence $\cap[X, \partial X] : \mathrm{H}^3(X, \partial X; {}^\omega\Lambda^\omega) \to \mathrm{H}_0(X; \Lambda)$ is an isomorphism.

Since ∂X is a PD^2–complex, $\cap[\partial X] : \mathrm{H}^r(\partial X; {}^\omega\Lambda^\omega) \longrightarrow \mathrm{H}_{2-r}(\partial X; \Lambda)$ is an isomorphism for every $r \in \mathbb{Z}$. Thus the Cap Product Ladder of $(X, \partial X)$ (Theorem 2.2) with $y = [X, \partial X]$, and the Five Lemma imply that $\cap[X, \partial X] : \mathrm{H}^r(X; {}^\omega\Lambda^\omega) \to \mathrm{H}_{r-3}(X, \partial X; \Lambda)$ is an isomorphism for $r = 2$ and $r = 3$. Identifying ${}^\omega\Lambda^\omega$ with Λ,

$\Lambda \otimes_\Lambda A$ with A and denoting ${}^\omega\mathrm{Hom}_\Lambda(A, \Lambda)$ by A^* for left Λ–modules A, we obtain the chain homotopy equivalence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{im}\partial_1^* & \rightarrowtail & C_2^*(X) & \longrightarrow & C_3^*(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \cap 1\otimes x} & & \downarrow{\scriptstyle \cap 1\otimes x} & & \downarrow \\
0 & \longrightarrow & \mathrm{im}\partial_1 & \rightarrowtail & C_1(X,\partial X) & \longrightarrow & C_0(X,\partial X) & \longrightarrow & 0.
\end{array}$$

Applying $^* = {}^\omega\mathrm{Hom}_\Lambda(-, \Lambda)$ yields the chain homotopy equivalence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(X,\partial X)^* & \longrightarrow & C_1(X,\partial X)^* & \longrightarrow & (\mathrm{im}\partial_1)^* & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle (\cap 1\otimes x)^*} & & \downarrow{\scriptstyle (\cap 1\otimes x)^*} & & \downarrow & & \downarrow \\
0 & \longrightarrow & (C_3(X)^*)^* & \longrightarrow & (C_2(X)^*)^* & \longrightarrow & (\mathrm{im}\partial_1^*)^* & \longrightarrow & 0,
\end{array}$$

which shows that $(\cap[1\otimes x])^*$ induces homology isomorphisms. By Lemma 2.1, the Λ–morphism $\cap(1\otimes x)$ induces isomorphisms in homology if and only if $(\cap 1\otimes x)^*$ does. Thus the homomorphism $\cap[X, \partial X] : \mathrm{H}^k(X, \partial X; {}^\omega\Lambda^\omega) \longrightarrow \mathrm{H}_{3-k}(X; \Lambda)$ is an isomorphism for $k = 0$ and $k = 1$. The Cap Product Ladder of $(X, \partial X)$ with $y = [X, \partial X]$ and the Five Lemma imply that $\cap[X, \partial X] : \mathrm{H}^r(X; {}^\omega\Lambda^\omega) \longrightarrow \mathrm{H}_{3-k}(X, \partial X; \Lambda)$ is an isomorphism for $r = 0$ and $r = 1$ and hence for every $r \in \mathbb{Z}$. □

Theorem 4.9 (REALISATION II). *There is a* PD^3*–pair which realises the* π_1*–injective triple* $(\{\kappa_i\}_{i\in J}, \omega, \mu)$ *if and only if* $\nu_{C(K,\partial X),2}(\mu)$ *is a class of homotopy eqivalences.*

Proof. It only remains to investigate the general case, where the module P in the factorisation (11) of the homotopy equivalence h is finitely generated projective, but not necessarily free. Then there is a finitely generated projective Λ–module Q such that $P^* \oplus Q = \Lambda^n$. Attaching infinitely many 3–cells to $K^{[2]} \vee \left(\vee_{i=1}^\infty e^2\right)$ we obtain a pair $(X, \partial X)$ of CW–complexes whose relative cellular chain complex is chain homotopy equivalent to the complex

$$E: \quad \ldots \longrightarrow \Lambda^n \xrightarrow{\ \mathrm{pr}\ } \Lambda^n \xrightarrow{\ \mathrm{pr}'\ } \Lambda^n \xrightarrow{\ q\ } (\Lambda \oplus P)^* \oplus Q$$
$$\xrightarrow{\begin{bmatrix} \varphi^* & 0 \\ 0 & 0 \end{bmatrix}} C_2(K, \partial X) \longrightarrow C_1(K, \partial X) \longrightarrow C_0(K, \partial X),$$

where $\mathrm{pr} : \Lambda^n = P^* \oplus Q \to Q$ and $\mathrm{pr}' : \Lambda^n = P^* \oplus Q \to P^*$ are the canonical projections and $q(x) = (0, 0, \mathrm{pr}(x)) \in (\Lambda \oplus P)^* \oplus Q$ for $x \in \Lambda^n$. As E is a complex of finitely generated free Λ–modules, the proof that $(X, \partial X)$ realises $(\{\kappa_i\}_{i\in J}, \omega, \mu)$ is analogous to the proof in the case where the module P is free. □

Observation 2. *If the* PD^3*–pair* $(X, \partial X)$ *realises* $(\{\kappa_i\}_{i\in J}, \omega, \mu)$ *with* G *finitely presentable, Wall's results imply that* X *is in fact dominated by a finite cell–complex.*

Observation 3. *Note that* π_1*–injectivity guarantees the first isomorphism in (13) which allows us to attach* 3*–cells to* $K^{[2]}$ *such that* φ^* *is the boundary operator of the resulting relative cellular chain complex. Thus* π_1*–injectivity is an indispensable assumption for our method. The question remains whether the realisation theorem holds without this assumption.*

5. Connected Sums and Decomposition of PD^3–Pairs

5.1. **The Interior Connected Sum.** Given a PD^3–pair $(X, \partial X)$, we may assume that the components of ∂X are collared in X, the collar being a 3–manifold with boundary. Lemma 3.3 implies that there is a homologically 2–dimensional CW–complex K and a map $f : S^2 \to K$ such that

$$X \sim K \cup_f e^3.$$

We may assume without loss of generality that $f(S^2) \cap \partial X = \emptyset$. If $e \in C_3(X, \partial X)$ is the chain corresponding to the 3–cell e^3, then $1 \otimes e$ is a relative cycle representing the homology class $[X, \partial X]$. Wall [15] showed that the pair (K, f) is unique up to homotopy and orientation for $\partial X = \emptyset$. The argument extends to the case of PD^3–pairs with ∂X not necessarily empty, allowing for the definition of the interior connected sum of PD^3–pairs.

Proposition 5.1. *The pair (K, f) is unique up to homotopy, that is, if $X \sim K_1 \cup_{f_1} e^3 \sim K_2 \cup_{f_2} e^3$, there is a homotopy equivalence $g : K_1 \to K_2$ such that $g_*([f_1]) = [f_2]$, where g_* denotes the isomorphism of homotopy groups induced by h.*

For a proof we refer the reader to [2].

To define the interior connected sum of two PD^3–pairs $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$, write $X_\ell = K_\ell \cup_{f_\ell} e^3_\ell$ for $\ell = 1, 2$, and let $\iota_\ell : K_\ell \to K_1 \vee K_2$ be the inclusion of the first and second factor respectively. Then $\hat{f}_\ell := \iota_\ell \circ f_\ell : S^2 \longrightarrow K_1 \vee K_2$ determines an element of $\pi_2(K_1 \vee K_2; *)$, and we put $f_1 + f_2 := \hat{f}_1 + \hat{f}_2$.

Lemma and Definition 5.2. *Up to oriented homotopy equivalence, the pair*

$$(X, \partial X) := (X_1, \partial X_1)\sharp(X_2, \partial X_2) := \big((K_1 \vee K_2) \cup_{f_1+f_2} e^3, \partial X_1 \cup \partial X_2)\big)$$

is a PD^3*–pair uniquely determined by $(X_i, \partial X_i), i = 1, 2$. It is called the* interior connected sum *of $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$.*

Proof. Proposition 5.1 implies uniqueness up to oriented homotopy equivalence. Theorem 2.3 guarantees that $(X, \partial X)$ is indeed a PD^3–pair. To see this, put $G := \pi_1(X; *)$ and $G_k := \pi_1(X_k; *)$, $k = 1, 2$, so that $G = G_1 * G_2$. For $k = 1, 2$, let $\iota_k : G_k \to G$ be the canonical inclusion. Regarding $\mathbb{Z}[G]$ as a right $\mathbb{Z}[G_k]$–module via ι_k, we define the functor L_k from the category of $\mathbb{Z}[G_k]$–modules to the category of $\mathbb{Z}[G]$–modules by

$$L_k M := \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_k]} M \quad \text{and} \quad L_k\alpha := \mathrm{id} \otimes \alpha \tag{14}$$

for $\mathbb{Z}[G_k]$–modules M and $\mathbb{Z}[G_k]$–morphisms $\alpha : M \to N$. Let B denote the subcomplex of $C(X)$ containing the 3–cell attached via $f_1 + f_2$ together with its boundary and denote the boundary by ∂B. Then $(L_k(C(K_k)), L_k(C(\partial X_k)) + \partial B)$ is a Poincaré chain pair for $k = 1, 2$ and $(\partial B, \emptyset)$ is also a Poincaré chain pair. Applying Theorem 2.3 repeatedly, we see that $(C(X), C(\partial X)) = (L_1(C(K_1)) + L_2(C(K_2)) + B, L_1(C(\partial X_1)) + L_2(C(\partial X_2)))$ is a Poincaré chain pair, showing that $(X, \partial X)$ is a PD^3–pair. □

The notion of interior connected sums of PD^3–pairs is consistent with that of interior connected sums of manifolds with boundary. In the case of empty boundaries, the notion of interior connected sums of PD^3–pairs reduces to that of connected sums of PD^3–complexes [15].

Definition 5. *The free product of two π_1–systems $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell \in J_k}$, $k = 1, 2$, is the π_1–system*

$$\{\iota_k \circ \kappa_{k\ell} : G_{k\ell} \to G_1 * G_2\}_{\ell \in J_k, k=1,2},$$

where $\iota_k : G_k \to G_1 * G_2$ denotes the inclusion of the factor in the free product of groups.

A π_1–system isomorphic to such a free product is said to decompose as free product.

A decomposition of the PD^3–pair $(X, \partial X)$ as interior connected sum yields a decomposition of its π_1–system as free product. The Classification Theorem 1.1 and the Realisation Theorem 1.2 allow us to show that the converse holds for finitely dominated pairs with non–empty aspherical boundaries in the case of π_1–injectivity, thus proving Theorem 1.3.

Proof of Decomposition I. Suppose the π_1–system of the finitely dominated PD^3–pair $(X, \partial X)$ with aspherical boundary components decomposes as free product of the injective π_1–systems $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell \in J_k}, k = 1, 2$. Then $G := \pi_1(X; *) \cong G_1 * G_2$ and thus G_1 and G_2 are finitely presentable. For $i = 1, 2$, let $(K_i, \partial K_i)$ be an Eilenberg–Mac Lane pair of type $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell \in J_k}$ with finite 2–skeleton. Then the pair $(K, \partial K) = (K_1 \vee K_2, \partial K_1 \cup \partial K_2)$ is an Eilenberg–Mac Lane pair of type $\{\iota_k \circ \kappa_{k\ell} : G_{k\ell} \to G_1 * G_2\}_{\ell \in J_k, k=1,2}$ and $\mathrm{H}_3(K, \partial K; \mathbb{Z}^\omega) \cong \mathrm{H}_3(K_1, \partial K_1; \mathbb{Z}^{\omega_1}) \oplus \mathrm{H}_3(K_2, \partial K_2; \mathbb{Z}^{\omega_2})$, where $\omega_k \in \mathrm{H}^1(K_i; \mathbb{Z}^\omega)$ is the restriction of the orientation character $\omega_X \in \mathrm{H}^1(K; \mathbb{Z}^\omega)$ for $k = 1, 2$. Hence $\mu_X = \mu_1 + \mu_2$ with $\mu_k \in \mathrm{H}_3(K_k, \partial K_k; \mathbb{Z}^{\omega_k})$. It is now sufficient to show that the triple $(\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell \in J_k}, \mu_k, \omega_k)$ is realised by a PD^3–pair $(X_k, \partial X_k)$ for $k = 1, 2$. Then the interior connected sum of $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ realises the fundamental triple of $(X, \partial X)$ and the Classification Theorem 1.1 implies that $(X, \partial X)$ is orientedly homotopy equivalent to $(X_1, \partial X_1) \sharp (X_2, \partial X_2)$.

For $k = 1, 2$, let L_k be the functor defined by (14). Then

$$C_i(K, \partial K) = L_1(C_i(K_1, \partial K_1)) \oplus L_2(C_i(K_2, \partial K_2))$$

for $i \geq 1$, and the boundary morphism $\partial_i^{K, \partial K} : C_{i+1}(K, \partial K) \to C_i(K, \partial K)$ is the direct sum of $L_1(\partial_i^{K_1, \partial K_1})$ and $L_2(\partial_i^{K_2, \partial K_2})$. Thus

$$F^2(C(K, \partial K)) = L_1(F^2(C(K_1, \partial K_1))) \oplus L_2(F^2(C(K_2, \partial K_2)))$$

and $I(G) = L_1(I(G_1)) \oplus L_2(I(G_2))$, where the canonical inclusion $L_k(I(G_k)) \rightarrowtail I(G)$ is given by $\mu \otimes \lambda \mapsto \mu\lambda$ for $\mu \in \mathbb{Z}[G]$, and $\lambda \in I(G_k)$ is viewed as an element of $I(G)$.

For $k = 1, 2$, let $\varphi_k : F^2(C(K_k, \partial K_k)) \to I(G_k)$ be a $\mathbb{Z}[G_k]$–morphism representing the class $\nu_{C(K_k, \partial K_k),2}(\mu_k)$. Then the class $\nu_{C(K, \partial K),2}(\mu)$ of homotopy equivalences is represented by

$$\begin{array}{ccc} L_1(F^2(C(K_1, \partial K_1))) \oplus L_2(F^2(C(K_2, \partial K_2))) & = & F^2(C(K, \partial K)) \\ \Big\downarrow{\scriptstyle L_1(\varphi_1) \oplus L_2(\varphi_2)} & & \\ L_1(I(G_1)) \oplus L_2(I(G_2)) & = & I(G). \end{array}$$

It follows from the proof of the analogous proposition for the absolute case [14] that the $\mathbb{Z}[G_k]$–morphism φ_k is a homotopy equivalence of modules for $k = 1, 2$.

Hence, by the Realisation Theorem 1.2, $(\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in I_k}, \mu_k, \omega_k)$ is realised by a PD^3–pair $(X_k, \partial X_k)$ for $k = 1, 2$. □

5.2. **The Boundary Connected Sum.** Let $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ be two PD^3–pairs with connected boundary components $\{\partial X_{1i}\}_{i\in I_1}$ and $\{\partial X_{2j}\}_{j\in I_2}$ respectively. Choosing $\ell_k \in I_k, k = 1, 2$, we may assume that $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ are collared in X_1 and X_2 respectively, and that there are discs $e_1^2 \subseteq \partial X_{1\ell_1}$ and $e_2^2 \subseteq \partial X_{2\ell_2}$. We denote the chains corresponding to e_1^2 and e_2^2 by e_1 and e_2 respectively, and the quotient of $X_1 \coprod X_2$ obtained by identifying e_1^2 and e_2^2 via an orientation reversing map by $X_1 \coprod X_2/\sim$. For subsets $A_i \subseteq X_i, i = 1, 2$, we denote the image of $A_1 \coprod A_2$ under the canonical projection $\pi : X_1 \coprod X_2 \to X_1 \coprod X_2/\sim$ by $A_1 \coprod A_2/\sim$. If we assume that $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ are closed surfaces, then $\partial X_{1\ell_1} \sharp \partial X_{2\ell_2} := (\partial X_{1\ell_1} \setminus e_1^2) \coprod (\partial X_{2\ell_2} \setminus e_2^2)/\sim$ is homotopy equivalent to the connected sum of $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ as 2–manifolds.

Lemma and Definition 5.3. *Up to oriented homotopy equivalence, the pair*

$$(X_1, \partial X_1)\natural_{\ell_1,\ell_2}(X_2, \partial X_2) := \big(X_1 \coprod X_2/\sim, (\partial X_1 \setminus e_1^2) \coprod (\partial X_2 \setminus e_2^2)/\sim\big)$$

is a PD^3*–pair determined by* $(X_i, \partial X_i), i = 1, 2,$ *and* $(\ell_1, \ell_2) \in I_1 \times I_2$*. It is called the* (ℓ_1, ℓ_2)–boundary connected sum *of* $(X_1, \partial X_1)$ *and* $(X_2, \partial X_2)$*.*

Proof. Note that $X_1 \coprod X_2/\sim$ is homotopy equivalent to the wedge $X_1 \vee X_2$. For $k = 1, 2$, we denote the fundamental triple of $(X_k, \partial X_k)$ by $(\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in I_k}, \mu_k, \omega_k)$. Putting $G := \pi_1(X; *)$, we obtain $G = G_1 * G_2$. For $k = 1, 2$, let L_k be the functor defined by (14).

Put $(X, \partial X) := (X_1, \partial X_1)\natural_{\ell_1,\ell_2}(X_2, \partial X_2)$. Furthermore, let D denote the subcomplex of $C(X)$ generated by $\pi(e_1) = \pi(e_2)$ and its boundary $\pi(\partial e_1) = \pi(\partial e_2)$, and let ∂D denote the subcomplex generated by $\pi(\partial e_1) = \pi(\partial e_2)$. Then the geometric pairs $(L_1(C(X_1)), L_1(C(\partial X_1)))$, $(L_2(C(X_2)), L_2(C(\partial X_2)))$ as well as $(D, \partial D)$ are Poincaré chain pairs. By Theorem 2.3,

$$(L_1(C(X_1)) + L_2(C(X_2)), L_1(C(\partial X_1) \setminus \Lambda_1[e_1]) + L_2(C(\partial X_2) \setminus \Lambda_2[e_2]))$$

is a Poincaré chain pair, where $\Lambda_i = \mathbb{Z}[G_i]$. Hence $(X, \partial X)$ is a PD^3–pair with fundamental triple

$$(\{\kappa : K \to G_1 * G_2,\ \iota_k \circ \kappa_{k\ell} : G_{k\ell} \to G_1 * G_2\}_{\ell\in I_k, \ell\neq\ell_k, k=1,2}, \mu_1 + \mu_2, \omega_1 + \omega_2),$$

where $K := \pi_1(\partial X_{1\ell_1} \sharp \partial X_{2\ell_2}; *)$ and $\kappa : K \to G_1 * G_2$ is induced by the inclusion $\partial X_{1\ell_1} \sharp \partial X_{2\ell_2} \to X_1 \coprod X_2/\sim$. As the fundamental triple does not depend on the choice of discs $e_i^2, i = 1, 2$, the Classification Theorem 1.1 implies that, up to oriented homotopy equivalence, the PD^3–pair $(X, \partial X)$ is uniquely determined by $(X_i, \partial X_i), i = 1, 2$, and $(\ell_1, \ell_2) \in I_1 \times I_2$. □

The notion of boundary connected sums of PD^3–pairs is consistent with that of boundary connected sums of manifolds with boundary.

Let $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in J_k}$ be two π_1–systems such that $G_{k\ell}$ is a surface group for every $\ell \in J_k$ and let $(K_k, \partial K_k)$ be an Eilenberg–Mac Lane pair of type $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in I_k}$ for $k = 1, 2$. We may assume that the components of ∂K_k are closed surfaces which are collared in K_k and that there are discs $e_1^2 \subseteq \partial K_{1,\ell_1}$ and $e_2^2 \subseteq \partial K_{2,\ell_2}$. Let e_1 and e_2 denote the chains corresponding to e_1^2 and e_2^2 respectively and let $K_1 \coprod K_2/\sim$ be the quotient of $K_1 \coprod K_2$ obtained by identifying e_1^2 and e_2^2 via an orientation reversing map. Put $\partial K_{1\ell_1} \sharp \partial K_{2\ell_2} := (\partial K_{1\ell_1} \setminus e_1^2) \coprod (\partial K_{2\ell_2} \setminus e_2^2)/\sim$

and $H := \pi_1(\partial K_{1\ell_1}\sharp\partial K_{2\ell_2})$. Finally, let $\kappa : H \to G_1 * G_2$ be the homomorphism induced by the inclusion $\partial K_{1\ell_1}\sharp\partial K_{2\ell_2} \to K_1 \coprod K_2/\sim$.

Definition 6. *The free product of the π_1–systems $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in J_k}$, $k = 1, 2$, along $G_{1\ell_1}$ and $G_{2\ell_2}$ is the π_1–system*

$$\{\kappa : H \to G_1 * G_2,\ \iota_k \circ \kappa_{k\ell} : G_{k\ell} \to G_1 * G_2\}_{\ell\in I_k,\ell\neq\ell_k,k=1,2},$$

where $\iota_k : G_k \to G_1 * G_2, k = 1, 2$, denote the inclusions of the factors in the free product of groups.

A π_1–system isomorphic to such a free product is said to decompose as free product along $G_{1\ell_1}$ and $G_{2\ell_2}$.

The proof of Lemma 5.3 shows that the π_1–system of the boundary connected sum of two PD^3–pairs $(X_k, \partial X_k)$ along $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ decomposes as free product along $G_{1\ell_1}$ and $G_{2\ell_2}$. The Classification Theorem 1.1 and the Realisation Theorem 1.2 allow us to show that the converse holds for finitely dominated pairs with non–empty aspherical boundaries in the case of π_1–injectivity, thus proving Theorem 1.4.

Proof of Decomposition II. Suppose the π_1–system of the finitely dominated PD^3–pair $(X, \partial X)$ with non–empty aspherical boundary components decomposes as free product of the injective π_1–systems $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in I_k}, k = 1, 2$, along $G_{1\ell_1}$ and $G_{2\ell_2}$. Then $G := \pi_1(X; *) \cong G_1 * G_2$ and thus G_1 and G_2 are finitely presentable. For $k = 1, 2$, let $(K_k, \partial K_k)$ be an Eilenberg–Mac Lane pair of type $\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in I_k}$ with finite 2–skeleton, let $K_1 \coprod K_2/\sim$ be defined as above and let L_k be the functor defined by (14), so that

$$C_3(K, \partial K) = L_1(C_3(K_1, \partial K_1)) \oplus L_2(C_3(K_2, \partial K_2)).$$

Take a cycle $1 \otimes x$ in $\mathbb{Z}^\omega \otimes_\Lambda C_3(K, \partial K)$ representing μ_X. Then there are cycles $1 \otimes x_k \in \mathbb{Z}^{\omega_k} \otimes_\Lambda C_3(K_k, \partial K_k)$ such that $1\otimes x = 1\otimes x_1 + 1\otimes x_2$, where ω_k denotes the restriction of the orientation character $\omega_X \in \mathrm{H}^1(K; \mathbb{Z}/2\mathbb{Z})$ to $\mathrm{H}^1(K_k; \mathbb{Z}/2\mathbb{Z})$ for $k = 1, 2$. Put $\mu_k := [1\otimes x_k]$ and note that

$$\begin{aligned} C_1(K, \partial K) &= L_1(C_1(K_1, \partial K_1)) \oplus L_2(C_1(K_2, \partial K_2)) \\ C_2(K, \partial K) &= L_1(C_2(K_1, \partial K_1)) \oplus L_2(C_2(K_2, \partial K_2)) \oplus \Lambda[e_1] \end{aligned}$$

and that the boundary morphism $\partial_1^{K,\partial K} : C_2(K, \partial K) \to C_1(K, \partial K)$ is the direct sum of $L_1(\partial_1^{K_1,\partial K_1})$ and $L_2(\partial_1^{K_2,\partial K_2})$. Thus

$$F^2(C(K, \partial K)) = L_1(F^2(C(K_1, \partial K_1))) \oplus L_2(F^2(C(K_2, \partial K_2))) \oplus \Lambda[e_1].$$

For $k = 1, 2$, let $\varphi_k : F^2(C(K_k, \partial K_k)) \to I(G_k)$ be a $\mathbb{Z}[G_k]$–morphism representing the class $\nu_{C(K_k,\partial K_k),2}(\mu_k)$. As $\partial_1^{K,\partial K}|_{\Lambda[e_1]} = 0$, the class $\nu_{C(K,\partial K),2}(\mu)$ of homotopy equivalences yields a homotopy equivalence

$$\begin{array}{ccc} L_1(F^2(C(K_1, \partial K_1))) \oplus L_2(F^2(C(K_2, \partial K_2))) & = & F^2(C(K, \partial K)) \\ \downarrow{\scriptstyle L_1(\varphi_1)\oplus L_2(\varphi_2)} & & \\ L_1(I(G_1)) \oplus L_2(I(G_2)) & = & I(G). \end{array}$$

By [14], the $\mathbb{Z}[G_k]$–morphism φ_k is a homotopy equivalence of modules for $k = 1, 2$ and the Realisation Theorem 1.2 implies that $(\{\kappa_{k\ell} : G_{k\ell} \to G_k\}_{\ell\in I_k}, \mu_k, \omega_k)$ is realised by a PD^3–pair $(X_k, \partial X_k)$ for $k = 1, 2$. The Classification Theorem 1.1

implies that $(X, \partial X)$ is orientedly homotopy equivalent to the (ℓ_1, ℓ_2)–boundary connected sum $(X_1, \partial X_1)\natural_{\ell_1,\ell_2}(X_2, \partial X_2)$. □

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