DEGENERATE PRINCIPAL SERIES REPRESENTATIONS AND THEIR HOLOMORPHIC EXTENSIONS

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Abstract. Let $X = H/L$ be an irreducible real bounded symmetric domain realized as a real form in an Hermitian symmetric domain $D = G/K$. The intersection S of the Shilov boundary of D with X defines a distinguished subset of the topological boundary of X and is invariant under H and can also be realized as $S = H/P$ for certain parabolic subgroup P of H. We study the spherical representations $Ind_P^H(\lambda)$ of H induced from P. We find formulas for the spherical functions in terms of the Macdonald ${}_2F_1$ hypergeometric function. This generalizes the earlier result of Faraut-Koranyi for Hermitian symmetric spaces D. We consider a class of H-invariant integral intertwining operators from the representations $Ind_P^H(\lambda)$ on $L^2(S)$ to the holomorphic representations of G on D restricted to H. We construct a new class of complementary series for the groups $H = SO(n,m)$, $SU(n,m)$ (with $n-m > 2$) and $Sp(n,m)$ (with $n-m > 1$). We realize them as a discrete component in the branching rule of the analytic continuation of the holomorphic discrete series of $G = SU(n,m)$, $SU(n,m) \times SU(n,m)$ and $SU(2n,2m)$ respectively.

1. Introduction

Since the work of Kashiwara and Vergne [14] on the tensor product decomposition of metaplectic representations and of Howe [13] on the dual pair correspondence [12], there has been intensive study on the branching rule of minimal and singular representations under various subgroups; see e. g. [18, 20] and references therein. The purpose of the present paper is to find certain irreducible discrete parts of the branching of scalar holomorphic representations π_ν of an Hermitian Lie group G of higher rank under a symmetric subgroup H, with the Riemannian symmetric domain $X = H/L$ being a real form of the Hermitian symmetric domain $D = G/K$. For larger parameter ν it is equivalent [39] to the regular action of H on $L^2(X)$, whose decomposition is well-known [10] and is a continuous sum of the principal series representations induced on the minimal Iwasawa parabolic subgroup. However for smaller parameter ν the decomposition is rather complicated with the continuous parts being integration over various hyperplanes in the complex dual of the real Cartan subalgebra and with discrete parts, and the full decomposition is not known; see [26] for some examples. It is thus worthwhile to find the discrete components which in certain sense are the most interesting part. A pivot example of such cases is when X is itself a complex bounded

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symmetric domain G/K realized as a diagonal part in the domain $X \times \bar{X} = G \times G/K \times K$. In this case the branching rule above is the tensor product decomposition of $\pi_\nu \otimes \overline{\pi_\nu}$ of the holomorphic representation with its complex conjugate. However for smaller parameter ν there might have some discrete components. A full decomposition is done in [40] for ν being the last Wallach point, and in this case there are finitely many discrete components appearing for the non-tube type domain $SU(r, r+b)/S(U(r) \times U(r+b))$, $r > 1$, $b > 2$, and they are then some complementary series representations. We thus get a realization of them in the space $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$, namely the space of Hilbert-Schmidt operators on $\mathcal{H}_\nu$, which can be viewed as a quantization of the complementary series representation; this is another part of our motivation. In the rank one case the appearance of the discrete parts in the tensor product $\pi_\nu \otimes \overline{\pi_\nu}$ for the group $SU(n, 1)$ has been studied in [37] for $n = 1$ and for $n > 1$ in [6]; similar results hold for the branching of holomorphic representation of $SU(n, 1)$ under $SO(n, 1)$ and $Sp(\frac{n}{2}, 1)$ [36]. Certain examples of higher rank cases have also been studied earlier in [30] and [26]. In [27] Neretin and Olshanski discovered that certain complementary series representation for $SO(p, q)$ appears in the branching of the minimal representation of $SU(p, q)$; see also [31]. Here we give a systematic study of the appearance of complementary series in the branching of the holomorphic representation π_ν for all groups G and H.

We will consider representations of H that are induced from some maximal parabolic subgroup P. More precisely we consider the boundary $S = H/P$ defined as the intersection of the Shilov boundary of D with the topological closure of X, which we may call the real Shilov boundary of X. The corresponding spherical functions on $X = H/L$ can be realized as certain Poisson integral, and they have natural analytic continuation to the whole domain D. We find first an expansion formula for the spherical functions in terms of the L-invariant polynomials on D, which are certain hypergeometric functions studied earlier by Faraut-Koranyi and Macdonald. For that purpose we generalize our earlier results [41] about characterizing the L-invariant polynomials in terms of the Jack symmetric polynomials to non-tube type domains, and we find their Fock space norm using the result of Dunkl [4]; combining this with the result of Faraut-Koranyi [7] we find then their Bergman space norm. Even though the computations are rather technical the end results on the spherical functions are simple and appealing; see Theorem 5.1.

We study consequently the question of describing those spherical representations that appear discretely in the branching of the holomorphic representations of G under H. It turns out this happens only, roughly speaking, when ν is smaller and when $H = SO(n, m)$, $SU(n, m)$ and $Sp(n, m)$ with $n - m$ sufficiently larger than ν. The result of this type is heuristically plausible as the the spaces of singular holomorphic representations are certain Sobolev type space and may contain H-invariant subspaces holomorphic functions with boundary values (called also trace in classical analysis) on certain boundary components of $X = H/L$, and the subspaces may form certain complementary series representation; conversely complementary series representations of H might usually be realized on space of functions on boundary components of X with extra smoother property (see e.g. [15] for some precise statements and related conjectures) and they may have holomorphic continuation on D. However the precise statement is rather subtly. The parameter for the induced representations are outside the unitary range so that we may call them complementary series; in rank one case they are precisely the known complementary series. We prove that when ν in the continuous part of the Wallach set and is small compared with the root multiplicity b there appear discrete parts in the branching of the holomorphic representation of G under H.

Our results on realization of the complementary series representation as discrete components in the holomorphic representations can also be viewed as representation theoretic study of the holomorphic extension of spherical functions. Roughly we are mostly concerned with constructing unitary spherical representation of H so that its holomorphic extension is a discrete component in the unitary holomorphic representation of the larger group G. For a general Riemannian symmetric space H/L the holomorphic extension of spherical functions is of considerable interests; see e.g. [23]. Also Kobayashi [17] has recently introduced some geometric concept characterizing the multiplicity free branching rule.

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For the reader's convenience we list the main symbols used in the paper.

(1) $D = G/K$, a bounded symmetric type domain in a complex vector space $V_{\mathbb{C}} = V^{\mathbb{C}}$ of rank r' with root multiplicity $(1, a', 2b')$.
(2) $X = H/L$, an irreducible real bounded symmetric type domain in a real form $V \subset V^{\mathbb{C}}$ of rank r with root multiplicity $(\iota - 1, a, 2b)$.
(3) $S = Le = H/P$ a distinguished boundary component in the topological boundary of X.
(4) $\mathfrak{h} = \mathfrak{l} + \mathfrak{q}$, the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{a} \subset \mathfrak{h}$ a maximal abelian subspace of $\mathfrak{p}$, $\Sigma(\mathfrak{h}, \mathfrak{a})$ the root system.
(5) $h(z, \bar{w})$ an irreducible polynomial on $V_{\mathbb{C}} \times \overline{V_{\mathbb{C}}}$, the Bergman reproducing kernel is $h(z, \bar{w})^{-p}$ with $p = a'(r' - 1) + 2 + b'$ the genus of domain $\mathbb{D}$.
(6) For a tuple (partition) $\underline{\mathbf{m}} = (m_1, \cdots, m_r)$ with $m_j \in \mathbb{N}$ (non negative integers) and $m_1 \geq \cdots \geq m_r \geq 0$ the generalized Pochammer symbol is

$$(c)_{\underline{\mathbf{m}},\beta} = \prod_{j=1}^{r} (c - \beta(j-1))_{m_j} = \prod_{j=1}^{r} \prod_{k=1}^{m_j} (c - \beta(j-1) + k - 1).$$

2. Preliminaries

2.1. **Bounded symmetric domains D and holomorphic representations.** We recall very briefly in this and next subsections some preliminary results on bounded symmetric domains and fix notation; see e.g. [7, 24] and references therein.

Let $D = G/K$ be as in the previous section an irreducible bounded symmetric domain in a d-dimensional complex vector space $V_{\mathbb{C}} = \mathbb{C}^d$ of rank r'. (The symbol r will be reserved for the rank of the real bounded symmetric domain X in next subsection.) Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ be the Harish-Chandra decomposition of its complexification. Let $(1, a', 2b')$ be the root multiplicities (or Peirce invariants in terms of the Jordan triple) of the real root system of $\mathfrak{g}$, with 1 being that of the longest roots, $2b'$ of the shortest roots, and a the middle. The rank r' and the multiplicities form a quadruple characterizing D

$$\boxed{D : (r', 1, a', 2b')} \tag{2.1}$$

and will be compared with that for the subdomains X below. According to the classification [9] of bounded symmetric domains the possible value of $(r', a', 2b')$ is

$$(r', 2, 2b), \quad (r', 1, 0), \quad (r', 4, 0), \quad (r', 4, 2), \quad (2, a', 0), \quad (2, 6, 8), \quad (3, 8, 0). \tag{2.2}$$

The space $V_{\mathbb{C}} = \mathfrak{p}^+$ has a Jordan triple structure so that the subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$, when realized as a space of holomorphic vector fields on $\mathbb{D}$, consists of vector fields of the form

$$\xi_v = \xi_v(z) = v - Q(z)\bar{v}, \quad v \in V_{\mathbb{C}}, \tag{2.3}$$

where $Q(z) : \bar{V_{\mathbb{C}}} \mapsto V_{\mathbb{C}}$ is quadratic in z. We denote $\{x\bar{y}z\} = D(x, \bar{y})z$ the Jordan triple product

$$\{x\bar{y}z\} = D(x, \bar{y})z = (Q(x + z) - Q(x) - Q(z))\bar{y}.$$

We fix a K-invariant Hermitian inner product $(\cdot, \cdot)$ on $V_{\mathbb{C}}$ so that a minimal tripotent has norm 1. We let $dm(z)$ be the corresponding Lebesgue measure. The Bergman reproducing kernel is up to a positive constant of the form $h(z, \bar{w})^{-p}$ where p is the genus of $\mathbb{D}$, defined by $p = a(r' - 1) + 2 + b'$, and $h(z, \bar{w})$ is an irreducible polynomial holomorphic in z and anti-holomorphic in w. In particular the function $h(z, \bar{w})$ satisfies the following transformation property under the group G,

$$h(gz, \overline{gw}) = J_g(z)^{\frac{1}{p}} h(z, \bar{w}) \overline{J_g(w)^{\frac{1}{p}}}, \quad g \in G, \tag{2.4}$$

where J_g is the Jacobian of the holomorphic mapping g.

We denote by $\mathcal{F}(V_{\mathbb{C}})$ the Fock space of entire functions on $V_{\mathbb{C}}$. Let $\nu > p - 1 = a(r' - 1) + 1 + b$ and consider the probability measure $d\mu_\nu(z) = c'_\nu h(z, \bar{z})^{\nu - p} dm(z)$ where c'_ν is the normalization constant, and the corresponding weighted Bergman space $\mathcal{H}_\nu = \mathcal{H}_\nu(\mathbb{D})$ of holomorphic functions f so that

$$\|f\|_\nu^2 = \int_{\mathbb{D}} |f(z)|^2 d\mu_\nu(z) < \infty.$$

It has reproducing kernel $h(z, \bar{w})^{-\nu}$. The group G acts unitarily on $\mathcal{H}_\nu$ via the following

$$\pi_\nu(g) f(z) = J_{g^{-1}}(z)^{\frac{\nu}{p}} f(g^{-1}z), \tag{2.5}$$

and it forms a unitary projective representation of G. We let $\mathcal{O}(D)$ be the space of all holomorphic functions on D. The formula (2.5) defines also a representation of G on the space $\mathcal{O}(D)$.

The representation has an analytic continuation in ν and the whole set of ν so that it still defines an irreducible unitary representation on a proper subspace of holomorphic functions is given by the so-called Wallach set

$$W = \{0, \frac{a'}{2}, \cdots, \frac{a'}{2}(r'-1)\} \cup (\frac{a}{2}(r'-1), \infty).$$

The corresponding Hilbert space for $\nu \in W$ will also be denoted by $\mathcal{H}_\nu$ and the norm by $\|\cdot\|_\nu$. The discrete points in the set will be also referred as singular Wallach point (to differ the discrete component in the branching rule).

We summarize some related results (see e.g. [7]) in the following

Theorem 2.1. Let $D = G/K$ be as above. The space $\mathcal{P}$ of holomorphic polynomials on $V_{\mathbb{C}}$ decomposes into irreducible subspaces under K, with multiplicity one as:

$$\mathcal{P} \cong \sum_{\underline{\mathbf{n}}\geq 0} \mathcal{P}_{\underline{\mathbf{n}}}. \tag{2.6}$$

Each $\mathcal{P}_{\underline{\mathbf{n}}}$ is of lowest weight $-\underline{\mathbf{n}} = -(n_1\gamma_1 + \cdots + n_{r'}\gamma_{r'})$ with $n_1 \geq \cdots \geq n_{r'} \geq 0$ and $\gamma_1 > \cdots > \gamma_r$ the Harish-Chandra strongly orthogonal roots. For each nonzero $f \in \mathcal{P}_{\underline{\mathbf{n}}}$ it holds

$$\|f\|^2_{\mathcal{F}} = (\nu)_{\underline{\mathbf{n}}, \frac{a'}{2}} \|f\|^2_\nu,$$

where $(\nu)_{\underline{\mathbf{n}}, \frac{a'}{2}}$ is the generalized Pochammer symbol in Section 1. The reproducing kernel $h(z, \bar{w})^{-\nu}$ has the following expansion

$$h(z, \bar{w})^{-\nu} = \sum_{\underline{\mathbf{n}}} (\nu)_{\underline{\mathbf{n}}, \frac{a'}{2}} K_{\underline{\mathbf{n}}}(z, \bar{w}), \tag{2.7}$$

where $K_{\underline{\mathbf{n}}}(z, \bar{w})$ is the reproducing kernel of $\mathcal{P}_{\underline{\mathbf{n}}}$ in the Fock space. In particular for $\nu = \frac{a'}{2}(j-1)$, $1 \leq j \leq r'$, we have

$$h(z, \bar{w})^{-\nu} = \sum_{\underline{\mathbf{n}}; n_j = 0} (\nu)_{\underline{\mathbf{n}}} K_{\underline{\mathbf{n}}}(z, \bar{w}), \quad \mathcal{H}_\nu = \sum^{\oplus}_{\underline{\mathbf{n}}: n_j = 0} \mathcal{P}_{\underline{\mathbf{n}}}. \tag{2.8}$$

2.2. **Real forms X of D.** Let $V \subset V_{\mathbb{C}}$ be a real form of $V_{\mathbb{C}}$, $V_{\mathbb{C}} = V + iV$ and let $X = V \cap D$ be the corresponding real form of D. X is called a real bounded symmetric domain if the real involution with respect to V preserves the domain D. In this case V is a real Jordan triple and X is a Riemannian symmetric space, $X = H/L$, with induced metric from that of D, realized as a bounded domain in V. Here we take H the connected component of the subgroup of G preserving X. The most well-studied case is when H/L is the symmetric cone in the Siegel tube domain G/K, namely Type A below. To have a some what unified treatment we exclude the rank one case and we will only consider those irreducible X.

Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{h} = \mathfrak{l} + \mathfrak{q}$ be the Cartan decomposition. We let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}$. We fix a frame of minimal tripotents $\{e_1, \cdots, e_r\}$ of V. The corresponding vector fields (see 2.3), $\xi_j = \xi_{e_j}$, $j = 1, \cdots, r$ form a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{q}$. We will also view $\mathfrak{a}$ as a subspace of V. We let

$$e := e_1 + \cdots + e_r \in V, \quad \xi := \xi_{e_1} + \cdots + \xi_{e_r} \in \mathfrak{a}, \tag{2.9}$$

e being a fixed maximal tripotent in V. The root system $\Sigma = \Sigma(\mathfrak{h}, \mathfrak{a})$ of $(\mathfrak{h}, \mathfrak{a})$ is of type A,

$$\Sigma = \{\frac{\beta_j - \beta_k}{2}\},$$

with common multiplicity a or types B, BC or D, which we write as

$$\Sigma = \{\pm\beta_j, \frac{\beta_j \pm \beta_k}{2}, \pm\frac{\beta_j}{2}\}$$

with respective multiplicities $\iota - 1, a, 2b$ with the interpretation that the corresponding multiplicities $2b = 0$ for type C, $\iota - 1 = 0$ for type B, and $2b = \iota - 1 = 0$ for type D. Here $\{\beta_j\}$ is a basis for $\mathfrak{a}^*$ normalized by

$$\beta_j(\xi_k) = 2\delta_{j,k}, \quad j, k = 1 \cdots, r.$$

(We write the multiplicity as $\iota - 1$ since $\iota = 1, 2, 4$ has the interpretation as the dimension of the real, complex, quaternionic fields.) We will view type B as a special case of type BC with the multiplicity $\iota - 1 = 0$. We order the roots so that $\beta_1 > \cdots > \beta_r$ (and $\beta_r > 0$ for types B and BC) and denote

$$\rho = \frac{1}{2} \sum_{\gamma \in \Sigma^+} m_\gamma \gamma \tag{2.10}$$

the half-sum of positive roots.

We get also a quadruple characterizing X in D of (2.1),

$$\boxed{X : (r, \iota - 1, a, 2b)}. \tag{2.11}$$

We list the corresponding quadruples $(r, \iota - 1, a, 2b)$ and classify them according to the root system; see e. g. [24], [11] and [19].

Type $BC \times BC$. The complex domain is $D \times \bar{D} = (G \times G)/(K \times K)$ where D is an irreducible bounded symmetric (tube or non-tube type) domain of rank r in $\mathbb{C}^d$, and and $X = H/L = G/K = D$ viewed as the diagonal part in $D \times \bar{D} = (G \times G)/(K \times K)$. (The complex domain $D \times \bar{D}$ is reducible so there is some abuse of definition in §2.2.) The quadruple (2.11) becomes

$$\boxed{\text{Type } BC \times BC : (r, \iota - 1, a, 2b) = (r', 1, a', 2b')} \tag{2.12}$$

where $(r', 1, a', 2b')$ is as in (2.1) in §2.1.

Type A. The list of (G, H) is in [8]. In this case we have $r' = r$, $a' = a$ and $b = 0$, namely D is of tube type,

$$\boxed{\text{Type A:} \quad (r, a) = (r, 2),\ (r, 1),\ (r, 4),\ (2, a),\ (3, 8).} \tag{2.13}$$

Type BC. $(\mathfrak{h}, \mathfrak{l}) = (\mathfrak{sp}(l, r), \mathfrak{sp}(l) \times \mathfrak{sp}(r))$ $(l > r)$ with $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2l, 2r), \mathfrak{su}(2l) \times \mathfrak{su}(2r))$ $(l > r)$. The rank and the root multiplicities are related by

$$\boxed{\text{Type BC:} \quad (r, \iota - 1, a, 2b) = (r, 3, 4, 4(l - r)),\ (r', a', 2b') = (2r, 2, 4(l - r))}, \tag{2.14}$$

where $2r = r'$, $2a' = a$.

Type B. $(H,L)=(SO_0(l,r),SO(l)\times SO(r))$ $(l>r)$, $(G,K)=(SU(l,r),S(U(l)\times U(r))$ or $(H,L)=(SO(2r+1,\mathbb{C}),SO(2r+1))$, $(G,K)=(SO^*(2(2r+1),U(2r+1))$, and

$$\boxed{\text{Type B-1}: \quad (r,\iota-1,a,2b)=(r',1,1,1-r), \quad (r',a',2b')=(r',2,2(l-r))}. \tag{2.15}$$

or

$$\boxed{\text{Type B-2:}\quad (r,\iota-1,a,2b)=(r',1,2,2), \quad (r',a',2b')=(r',4,4)}. \tag{2.16}$$

Here $r'=r$, $a'=2a$.

Type D. We have $(\mathfrak{h},\mathfrak{l})=(\mathfrak{so}(r,r),\mathfrak{so}(r)\times\mathfrak{so}(r))$ with $(\mathfrak{g},\mathfrak{k})=(\mathfrak{su}(r,r),\mathfrak{su}(r)\times\mathfrak{u}(r))$, or $(\mathfrak{h},\mathfrak{l})=(\mathfrak{su}^*(8),\mathfrak{sp}(4))=(\mathfrak{sl}(4,\mathbb{H}),\mathfrak{sp}(4))$ with $(\mathfrak{g},\mathfrak{k})=(\mathfrak{e}_{7(-25)},\mathfrak{e}_6\oplus\mathfrak{so}(2))$.

$$\boxed{\text{Type D-1: } (r,\iota-1,a,2b)=(r,0,1,0),\ (r',a',2b')=(r,2,0)}. \tag{2.17}$$

or

$$\boxed{\text{Type D-2: } (r,\iota-1,a,2b)=(3,0,4,0),\ (r',a',2b')=(3,8,0)}. \tag{2.18}$$

We have here $r'=r$, $a'=2a$.

Remark 2.2. The above list can be deduced from Loos [24], where it is done according to the classification of Jordan triples. Note that the rank two domain in the real octonions $\mathbb{O}^2$ is not listed here, since it is isomorphic as symmetric space to the tube domain of 2×2-quaternionic matrices. The realization of $Sp(2,2)/Sp(2)\times Sp(2)$ as a real form in $E_{6(-14)}/Spin(10)\times SO(2)$ is not listed here either as it is realized inside $SU(4,4)/S(U(4)\times U(4))$. The realization of the exceptional rank one domain H/L with $(\mathfrak{h},\mathfrak{l})=(\mathfrak{f}_{4(-20)},\mathfrak{spin}(9))$ as real form in G/K with $(\mathfrak{g},\mathfrak{k})=(\mathfrak{e}_{6(-20)},\mathfrak{spin}(10)\times\mathfrak{so}(2))$ is also not listed as we have excluded the rank one case.

3. L-invariant polynomials and their Fock-Fischer norms.

3.1. **Jack polynomials.** Let $\Omega_{\underline{\mathbf{m}}}=\Omega_{\underline{\mathbf{m}}}^{(2/a)}$ be the Jack symmetric polynomials with multiplicity $\frac{a}{2}$ normalized by

$$\Omega_{\underline{\mathbf{m}}}(1^r)=1.$$

Here we use the abbreviation $1^r=(1,\cdots,1)$. In the standard notation [25] it is

$$\Omega_{\underline{\mathbf{m}}}(x_1,\cdots,x_r)=\frac{J_m^{(2/a)}(x_1,\cdots,x_r)}{J_m^{(\frac{2}{a})}(1^r)}.$$

Following [38] we introduce

$$q:=q_{a/2}:=1+\frac{a}{2}(r-1). \tag{3.1}$$

and

$$\pi_{\underline{\mathbf{m}}}:=\pi_{\underline{\mathbf{m}},\frac{a}{2}}:=\prod_{1\le i<j\le r}\frac{m_i-m_j+\frac{a}{2}(j-1)}{\frac{a}{2}(j-1)}\frac{(\frac{a}{2}(j-i+1))_{m_i-m_j}}{(\frac{a}{2}(j-i-1)+1)_{m_i-m_j}}. \tag{3.2}$$

(This is denoted by $d_{\underline{\mathbf{m}}}$ in [38, §4].)

3.2. **Type** $BC \times BC$**.** Consider the complex bounded symmetric domain $D = H/L = G/K \subset V_{\mathbb{C}} = \mathbb{C}^d$ realized as a real form in $D \times \bar{D} \subset V_{\mathbb{C}} \times \bar{V}_{\mathbb{C}}$. The parameter is now $(r, a, b) = (r', a', b')$. The space $\mathcal{P}$ is

$$\mathcal{P} = \mathcal{P}(V_{\mathbb{C}}) \otimes \overline{\mathcal{P}(V_{\mathbb{C}})}. \tag{3.3}$$

Under $K \times K$ it is decomposed as

$$\mathcal{P} = \sum_{\underline{\mathbf{n}} = \underline{\mathbf{m}} \times \underline{\mathbf{m}}'} \mathcal{P}_{\underline{\mathbf{m}}}(V_{\mathbb{C}}) \otimes \overline{\mathcal{P}_{\underline{\mathbf{m}}'}(V_{\mathbb{C}})}. \tag{3.4}$$

The following lemma follows immediately from Theorem 2.1. All the Pochammer product $(\sigma)_{\underline{\mathbf{m}}}$ in here are understood as $(\sigma)_{\underline{\mathbf{m}}, a/2}$.

Lemma 3.1. In the decomposition (3.4), $\mathcal{P}_{\underline{\mathbf{n}}}^L \neq 0$ if and only if $\underline{\mathbf{n}} = (\underline{\mathbf{m}}, \underline{\mathbf{m}})$, in which case the polynomial

$$p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}(x) = \frac{(d/r)_{\underline{\mathbf{m}}}}{d(\underline{\mathbf{m}})} K_{\underline{\mathbf{m}}}(x, x), \quad d(\underline{\mathbf{m}}) = \mathrm{dim} P_{\underline{\mathbf{m}}}, \quad \underline{\mathbf{n}} = \underline{\mathbf{m}} \times \underline{\mathbf{m}},$$

is the unique K-invariant polynomial in $\mathcal{P}_{\underline{\mathbf{n}}}$ normalized by $p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}(e) = 1$. The restriction of $p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}(x)$ on $\mathfrak{a} = \{x = \sum x_j e_j; x_j \in \mathbb{R}\} \subset V_{\mathbb{C}}$ is the Jack symmetric polynomial, $p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}(x) = \Omega_{\underline{\mathbf{m}}}(x_1^2, \cdots, x_r^2)$. Its norms in the Fock and Bergman spaces are given by

$$\|p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}\|^2_{\mathcal{F} \otimes \overline{\mathcal{F}}} = \frac{((d/r)_{\underline{\mathbf{m}}})^2}{d(\underline{\mathbf{m}})}$$

and

$$\|p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}\|^2_{\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}} = \frac{1}{(\nu)^2_{\underline{\mathbf{m}}}} \|p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}\|^2_{\mathcal{F} \otimes \overline{\mathcal{F}}} = \frac{(d/r)^2_{\underline{\mathbf{m}}}}{(\nu)^2_{\underline{\mathbf{m}}} d(\underline{\mathbf{m}})}.$$

To compare with Proposition 3.6 below we write $\|p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}\|^2_{\mathcal{F} \otimes \overline{\mathcal{F}}}$ in terms of $\pi_{\underline{\mathbf{m}}, a'/2}$ defined in (3.2). The dimension $d(\underline{\mathbf{m}})$ is computed in [35, Lemmas 2.5 and 2.6] and is given by

$$d(\underline{\mathbf{m}}) = \frac{(d/r)_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}} \pi_{\underline{\mathbf{m}}}. \tag{3.5}$$

Thus

$$\|p_{(\underline{\mathbf{m}}, \underline{\mathbf{m}})}\|^2_{\mathcal{F} \otimes \overline{\mathcal{F}}} = \frac{(d/r)_{\underline{\mathbf{m}}} (q)_{\underline{\mathbf{m}}}}{\pi_{\underline{\mathbf{m}}}}.$$

3.3. **Type A.** The following is proved in [7].

Lemma 3.2. In the decomposition (2.6), each component $\mathcal{P}_{\underline{\mathbf{m}}}$ has a unique L-invariant polynomial

$$p_{\underline{\mathbf{m}}}(x) = \frac{(d/r)_{\underline{\mathbf{m}}}}{d(\underline{\mathbf{m}})} K_{\underline{\mathbf{m}}}(x, e), \quad d(\underline{\mathbf{m}}) = \mathrm{dim} P_{\underline{\mathbf{m}}}, \quad \underline{\mathbf{n}} = \underline{\mathbf{m}},$$

normalized by $p_{\underline{\mathbf{m}}}(e) = 1$. The restriction of $p_{\underline{\mathbf{m}}}(x)$ on $\mathfrak{a}$ is the Jack symmetric polynomial, $p_{\underline{\mathbf{m}}}(x) = \Omega_{\underline{\mathbf{m}}}(x_1, \cdots, x_r)$. Its norms in the Fock and Bergman spaces are given by

$$\|p_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}} = \frac{(q)_{\underline{\mathbf{m}}}}{\pi_{\underline{\mathbf{m}}}}, \quad \|p_{\underline{\mathbf{m}}}\|^2_{\mathcal{H}_\nu} = \frac{(q)_{\underline{\mathbf{m}}}}{(\nu)_{\underline{\mathbf{m}}} \pi_{\underline{\mathbf{m}}}}.$$

3.4. **Types B, BC, C, D.** In this section we will generalize the result in [41] to the non-tube case; some of which are quite similar to that of tube domain while others can be proved by using the results there. We will be rather brief.

The following lemma can be proved by using the classification theory of spherical pairs [22]; for tube domains (namely types A, B, D) it is also a consequence of the Cartan - Helgason theorem [10, Chapter V, Theorem 4.1].

Lemma 3.3. In the decomposition (2.6), $\mathcal{P}^L_{\underline{\mathbf{n}}} \neq 0$ if and only if,

$$\text{Type BC} \quad \underline{\mathbf{n}} = (\underline{\mathbf{m}}, \underline{\mathbf{m}}) := (m_1, m_1, m_2, m_2, \ldots, m_r, m_r) = \sum_{j=1}^r m_j(\gamma_{2j-1} + \gamma_{2j}), \tag{3.6}$$

$$\text{Type B} : \underline{\mathbf{n}} = 2\underline{\mathbf{m}} = (2m_1, 2m_2, \ldots, 2m_r) = \sum_{j=1}^r 2m_j\gamma_j, \tag{3.7}$$

$$\begin{aligned} \text{Type D} : \underline{\mathbf{n}} = 2\underline{\mathbf{m}} + m &= (2m_1 + m, 2m_2 + m, \ldots, 2m_r + m) \\ &= \sum_{j=1}^r (2m_j + m)\gamma_j, \quad m = 0, 1, \end{aligned} \tag{3.8}$$

in which case $\mathcal{P}^L_{\underline{\mathbf{n}}}$ is one dimensional. Here $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$.

We will find the L-invariant polynomials in $\mathcal{P}_{\underline{\mathbf{n}}}$ in the previous Lemma in terms of the Weyl group invariant orthogonal polynomials studied by [4], and compute their Fock space norm. We will denote the subspace $\mathbb{R}e_1 + \cdots + \mathbb{R}e_r$ of V also by $\mathfrak{a}$.

Associated to the root system $\Sigma(\mathfrak{h}, \mathfrak{a})$ there are the Dunkl difference-differential operators [3],

$$D_j = \partial_j + \frac{1}{2}\sum_{\alpha \in \Sigma^+} m_\alpha \frac{\alpha(\xi_j)}{\alpha(x)}(1 - r_\alpha)$$

acting on polynomials $f(x)$ on $\mathfrak{a}$. (It is a realization on the space of polynomials of the Hecke-algebra of the tensor product of the symmetric algebra on $\mathfrak{a}$ and the Weyl group algebra, with ξ_j acting as D_j, and $w \in W$ acting by change of variables; see [29].)

We recall [41, Proposition 6.2] an isometric version of the Chevalley restriction theorem (see e. g. [10, 33] and references therein). We define a norm [4] on $\mathcal{P}(\mathfrak{a})^W$ by

$$\|p\|^2_B = p(D)p^*|_{x=0}$$

for Type B and

$$\|p\|^2_B = p(\frac{1}{2}D)p^*|_{x=0}$$

for Type BC, where for any polynomial $p(x)$, $x = x_1e_1 + \cdots + x_re_r \in \mathfrak{a}$, $p(D)$ is obtained by replacing the linear polynomial e^*_j by D_j and p^* obtained by taking the complex conjugate of the coefficients of the monomials in e^*_j. (The norm in [4] is defined by $p(D)p^*|_{x=0}$ for root systems of type B or type BC. The discrepancy here for type BC is due to the fact that the vectors e_j or the minimal tripotents in V have norm squares being twice of that of minimal tripotents in $V_{\mathbb{C}}$.)

Let

$$\mathrm{Res} = \mathrm{Res}_{\mathfrak{a}} : \mathcal{P}(V_{\mathbb{C}})^L \to \mathcal{P}(\mathfrak{a})^W$$

be the restriction map.

Lemma 3.4. The map Res is an isometric isomorphism between $\mathcal{P}(V_{\mathbb{C}})^L$ and the space $\mathcal{P}(\mathfrak{a})^W$ of Weyl group invariant polynomials on $\mathfrak{a}$.

It is proved in [4] that the polynomials $\Omega_{\underline{\mathbf{m}}}(x_1^2, \cdots, x_r^2)$ are then eigenfunctions of the operators $p(D_1, \cdots, D_r)$, where p are Weyl group invariant polynomials on $\mathfrak{a}$.

Proposition 3.5. For each $\underline{\mathbf{m}} = (m_1, \dots, m_r)$ there exists a unique polynomial $p_{\underline{\mathbf{n}}}$ in the space $\mathcal{P}_{\underline{\mathbf{n}}}^L$ with $\underline{\mathbf{n}}$ given by $\underline{\mathbf{m}}$ as in Lemma 3.3 such that

$$\mathrm{Res}\, p_{\underline{\mathbf{n}}}(x_1e_1 + \cdots + x_re_r) = \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2). \tag{3.9}$$

for types B and BC, and

$$\mathrm{Res}\, p_{\underline{\mathbf{n}}}(x_1e_1 + \cdots + x_re_r) = (x_1 \cdots x_r)^m\, \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2), \quad m = 0, 1, \tag{3.10}$$

for type D.

Proof. By Lemma 3.4 we have for each $\underline{\mathbf{m}}$ there exists a unique p in $\mathcal{P}(V_{\mathbb{C}})^L$ such that $\mathrm{Res}\, p = \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)$. We only need to prove that p is the the space $\mathcal{P}_{\underline{\mathbf{n}}}$. There is linear subspace $V_{\mathbb{C}}^0$ of $V_{\mathbb{C}}$ and symmetric tube domain $D_0 = G^0/K^0 \subset V_{\mathbb{C}}^0$ in $D = G/K$ of same rank r', and correspondingly there is a real tube domain $X_0 = H_0/L_0$ of rank r in the domain $X = H/L$; this can be proved abstractly or by checking the list in our classification. We view $\mathfrak{a}$ also as a Cartan subspace for the symmetric space X_0. The root system of X_0 is then of type D or type C. By [41, Propositions 7.6, 8.3] we see that there is a unique L_0 invariant polynomials q in $\mathcal{P}_{\underline{\mathbf{n}}}(V_{\mathbb{C}}^0)$ such that $\mathrm{Res}\, q = \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2) = \mathrm{Res}\, p$. That p belongs $\mathcal{P}_{\underline{\mathbf{n}}}(V_{\mathbb{C}})^L$ follows immediately from the fact the the isomorphism Res is compatible with the realization of α in V_0 or V. □

Next we compute the Fock-Fischer norm using Proposition 3.5 and the result of Dunkl [4].

Proposition 3.6. With the notation as in Proposition 3.5 we have the following formulas for the norm squares of the polynomial $p_{\underline{\mathbf{n}}}$ in the Fock space and Bergman spaces.

Type B:

$$\|p_{\underline{\mathbf{n}}}\|_{\mathcal{F}}^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} 2^{2|\underline{\mathbf{m}}|} (q)_{\underline{\mathbf{m}}, \frac{a}{2}} ((r-1)\frac{a}{2} + b + \frac{1}{2})_{\underline{\mathbf{m}}, \frac{a}{2}}$$

and

$$\|p_{\underline{\mathbf{n}}}\|_{\nu}^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} \frac{(q)_{\underline{\mathbf{m}}, \frac{a}{2}} ((r-1)\frac{a}{2} + b + \frac{1}{2})_{\underline{\mathbf{m}}, \frac{a}{2}}}{(\frac{\nu}{2})_{\underline{\mathbf{m}}, \frac{a}{2}} (\frac{\nu+1}{2})_{\underline{\mathbf{m}}, \frac{a}{2}}};$$

Type BC :

$$\|p_{\underline{\mathbf{n}}}\|_{\mathcal{F}}^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} (q)_{\underline{\mathbf{m}}, \frac{a}{2}} ((r-1)\frac{a}{2} + \frac{\iota + 2b}{2})_{\underline{\mathbf{m}}, \frac{a}{2}},$$

and

$$\|p_{\underline{\mathbf{n}}}\|_\nu^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} \frac{(q)_{\underline{\mathbf{m}},\frac{a}{2}}((r-1)\frac{a}{2}+\frac{\iota+2b}{2})_{\underline{\mathbf{m}},\frac{a}{2}}}{(\nu)_{\underline{\mathbf{m}},\frac{a}{2}}(\nu-\frac{a'}{2})_{\underline{\mathbf{m}},\frac{a}{2}}};$$

Type D :

$$\|p_{2\underline{\mathbf{m}}}\|_{\mathcal{F}}^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} 2^{2|\underline{\mathbf{m}}|}(q)_{\underline{\mathbf{m}},\frac{a}{2}}((r-1)\frac{a}{2}+b+\frac{1}{2})_{\underline{\mathbf{m}},\frac{a}{2}},$$

$$\|p_{2\underline{\mathbf{m}}+1}\|_{\mathcal{F}}^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} 2^r \prod_{j=1}^r (\frac{a}{2}(r-1)+\frac{1}{2}-\frac{a}{2}(j-1)) 2^{2|\underline{\mathbf{m}}|}(q)_{\underline{\mathbf{m}},\frac{a}{2}}(\frac{a}{2}(r-1)+\frac{3}{2})_{\underline{\mathbf{m}}},$$

$$\|p_{2\underline{\mathbf{m}}}\|_\nu^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} \frac{(q)_{\underline{\mathbf{m}},\frac{a}{2}}((r-1)\frac{a}{2}+\frac{1}{2})_{\underline{\mathbf{m}},\frac{a}{2}}}{(\frac{\nu}{2})_{\underline{\mathbf{m}},\frac{a}{2}}(\frac{\nu+1}{2})_{\underline{\mathbf{m}},\frac{a}{2}}};$$

$$\|p_{2\underline{\mathbf{m}}+1}\|_\nu^2 = \frac{1}{\pi_{\underline{\mathbf{m}}}} \prod_{j=1}^r \frac{\frac{a}{2}(r-1)+\frac{1}{2}-\frac{a}{2}(j-1)}{\frac{\nu}{2}-\frac{a}{2}(j-1)} \frac{(q)_{\underline{\mathbf{m}},\frac{a}{2}}((r-1)\frac{a}{2}+\frac{3}{2})_{\underline{\mathbf{m}},\frac{a}{2}}}{(\frac{\nu}{2}+\frac{1}{2})_{\underline{\mathbf{m}},\frac{a}{2}}(\frac{\nu}{2}+1)_{\underline{\mathbf{m}},\frac{a}{2}}};$$

Proof. By Lemma 3.4 and Proposition 3.5 we have

$$\|p_{\underline{\mathbf{n}}}\|_{\mathcal{F}}^2 = \|\mathrm{Res}\, p_{\underline{\mathbf{n}}}\|_B^2 = \|\Omega_{\underline{\mathbf{m}}}\|_B^2$$

for type B, and

$$\|p_{\underline{\mathbf{n}}}\|_{\mathcal{F}}^2 = 2^{-|\underline{\mathbf{n}}|}\|\Omega_{\underline{\mathbf{m}}}\|_B^2 = 2^{-2|\underline{\mathbf{m}}|}\|\Omega_{\underline{\mathbf{m}}}\|_B^2$$

for type BC, since $p_{\underline{\mathbf{n}}}$ is a polynomial of degree $|\underline{\mathbf{n}}| = 2|\underline{\mathbf{m}}|$. The right hand side is computed by Dunkl [4, Section 5], where the norm for type BC is defined as for type B. However to express the resulting formula as stated requires some rather technical computations, so we will adapt the notations there, his N, k, k_1, λ are our $r, \frac{a}{2}, \frac{2b+\iota-1}{2}, \underline{\mathbf{m}}$ etc., and the h below is the shifted hook length product defined in Definition 3.17 there. (Observe that for root system of type BC the constant k_1 in [4, Section 5, (5.1)] is a sum of two multiplicities.) The Jack polynomial $J_{\underline{\mathbf{m}}}$ is expressed in terms of the polynomial $j_{\underline{\mathbf{m}}}$ by [4, Section 3, p.465]

$$J_{\underline{\mathbf{m}}} = \frac{(r\frac{a}{2}+1)_{\underline{\mathbf{m}}} h(\underline{\mathbf{m}},1)}{(r\frac{a}{2}+1)_{\underline{\mathbf{m}}}(\frac{a}{2})^{|\underline{\mathbf{m}}|}(\#S_r\underline{\mathbf{m}})} j_{\underline{\mathbf{m}}}$$

with

$$J_{\underline{\mathbf{m}}}(1^r) = (r\frac{a}{2})_{\underline{\mathbf{m}}}(\frac{a}{2})^{-|\underline{\mathbf{m}}|}.$$

The norm $j_{\underline{\mathbf{m}}}$ is (see p. 480-495, loc. cit.)

$$\|j_{\underline{\mathbf{m}}}\|_B^2 = 2^{2|\underline{\mathbf{m}}|}(r\frac{a}{2}+1)_{\underline{\mathbf{m}}}((r-1)\frac{a}{2}+\frac{\iota-1+2b}{2}+\frac{1}{2})_{\underline{\mathbf{m}}}(\#S_r\underline{\mathbf{m}})\mathcal{E}(\underline{\mathbf{m}}^R)\frac{h(\underline{\mathbf{m}},\frac{a}{2}+1)}{h(\underline{\mathbf{m}},1)}.$$

We find then

$$\|\Omega\|_B^2 = 2^{2|\underline{\mathbf{m}}|}\frac{((r-1)\frac{a}{2}+\frac{\iota-1+2b}{2}+\frac{1}{2})_{\underline{\mathbf{m}}}}{(r\frac{a}{2})_{\underline{\mathbf{m}}}} h(\underline{\mathbf{m}},1)h(\underline{\mathbf{m}},\frac{a}{2})$$

The shifted hook length product h is related to the upper and lower hook length products $h^*(\underline{\mathbf{m}})$ and $h_*(\underline{\mathbf{m}})$ (Stanley [32], Macdonald [25]) by

$$h(\underline{\mathbf{m}},1)h(\underline{\mathbf{m}},\frac{a}{2}) = (\frac{a}{2})^{2|\underline{\mathbf{m}}|}(h_*(\underline{\mathbf{m}})h^*(\underline{\mathbf{m}}))$$

and which, by [38, Proposition 4.1], can be further written in terms of the quantity $\pi_{\underline{\mathbf{m}}}$ defined in (3.2),

$$h_*(\underline{\mathbf{m}})h^*(\underline{\mathbf{m}}) = (\frac{2}{a})^{2|\underline{\mathbf{m}}|}\frac{(q)_{\underline{\mathbf{m}}}(r\frac{a}{2})_{\underline{\mathbf{m}}}}{(\pi)_{\underline{\mathbf{m}}}}.$$

Namely

$$\|\Omega\|_B^2 = 2^{2|\underline{\mathbf{m}}|}\frac{(q)_{\underline{\mathbf{m}}}}{\pi_{\underline{\mathbf{m}}}}((r-1)\frac{a}{2}+\frac{\iota-1+2b}{2}+\frac{1}{2})_{\underline{\mathbf{m}}},$$

proving our claim for the Fock space norm.

To find the Bergman space norm of $p_{\underline{\mathbf{n}}}$ we use Theorem 2.1. For Type B we have, $a'=2a$, $\iota=1$, $\underline{\mathbf{n}}=2\underline{\mathbf{m}}$,

$$\begin{aligned}\|p_{\underline{\mathbf{n}}}\|_\nu^2 &= \frac{1}{(\nu)_{\underline{\mathbf{n}},a'/2}}\|p_{\underline{\mathbf{n}}}\|_{\mathcal{F}}^2\\ &= \frac{1}{(\frac{\nu}{2})_{\underline{\mathbf{m}},\frac{a}{2}}(\frac{\nu+1}{2})_{\underline{\mathbf{m}},\frac{a}{2}}}\frac{1}{(q)_{\underline{\mathbf{m}}}}\pi_{\underline{\mathbf{m}}}((r-1)\frac{a}{2}+b+\frac{1}{2})_{\underline{\mathbf{m}}}.\end{aligned}$$

since

$$(\nu)_{\underline{\mathbf{n}},a'/2} = 2^{2|\underline{\mathbf{m}}|}(\frac{\nu}{2})_{\underline{\mathbf{m}},a/2}(\frac{\nu+1}{2})_{\underline{\mathbf{m}},a/2}, \tag{3.11}$$

for any ν.

For Type BC, it holds $a=2a'$, and

$$(\nu)_{\underline{\mathbf{n}},a'/2} = (\nu)_{\underline{\mathbf{m}},a/2}(\nu-a'/2)_{\underline{\mathbf{m}},a/2}, \tag{3.12}$$

and we get the last equality. □

4. Principal series representations on maximal boundaries and intertwining operators into the space $\mathcal{H}_\nu$. Type $BC\times BC$

The domain $X=H/L=G/K$ is the complex bounded symmetric of rank r with root multiplicity $(1,a,2b)$. We shall study spherical representations defined on the Shilov boundary of D in terms of Macdonald ${}_2F_1$ hypergeometric functions and their realization in the holomorphic representation $\mathcal{H}_\nu\otimes\overline{\mathcal{H}_\nu}$. First we recall some general factors about the hypergeometric series which we will need through the rest of the paper.

4.1. **Hypergeometric series with general parameter $\frac{2}{a}$.** In the following sections we will need the hypergeometric functions defined in terms of the Jack symmetric polynomials. Let $\alpha=(\alpha_l,\cdots,\alpha_k)$, $\beta=(\beta_l,\cdots,\beta_l)$ be two tuples of real positive numbers, such that that $\beta_1,\cdots,\beta_l>\frac{a}{2}(j-1)$ and $t=(t_1,\cdots,t_r)\in[0,1)^r$. We define

$${}_kF_l^{(2/a)}(\alpha;\beta;t) = \sum_{\underline{\mathbf{m}}}\frac{(\alpha_1)_{\underline{\mathbf{m}}}\cdots(\alpha_k)_{\underline{\mathbf{m}}}}{(\beta_1)_{\underline{\mathbf{m}}}\cdots(\beta_l)_{\underline{\mathbf{m}}}}\frac{\pi_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}}\Omega_{\underline{\mathbf{m}}}(t_1^2,\cdots,t_r^2).$$

Note that in case one of α's is $\frac{a}{2}(j-1)$ for some $1\le j\le r$ we have $(\alpha_1)_{\underline{\mathbf{m}}}\cdots(\alpha_k)_{\underline{\mathbf{m}}}=0$ and the sum is only over those with $m_j=0$, namely $\underline{\mathbf{m}}=(m_1,\cdots,m_{j-1},0,\cdots,0)$. We will suppress the upper index $2/a$ when no confusion would arise. The series ${}_2F_1$

has been well-studied as it is related the spherical functions on symmetric domains (see below).

The convergence property is similar to that of ${}_2F_1$, namely we have

Lemma 4.1. Suppose $\alpha_1, \cdots, \alpha_{l+1} > 0$ and $\beta_1, \cdots, \beta_l > \frac{a}{2}(j-1)$. The hypergeometric series ${}_{l+1}F_l(\alpha; \beta; t)$ is bounded on the set $[0,1)^r$ if and only if

$$\sum_{p=1}^{l+1} \alpha_p - \sum_{p=1}^{l} \beta_p < -\frac{a}{2}(r-1)$$

in which case the series $F(\alpha; \beta; 1^r)$ is convergent.

Proof. The sufficiency for $l = 1$ was proved in [7] for special values of (r, l) (corresponding to a complex bounded symmetric domain) and was generalized by Yan [38] to general (k, r), in the case $\alpha = (\alpha_1, \alpha_2)$ and $\beta = \beta_1$; the general case is exact the same (see e.g. [5]). The necessary part is essentially also proved in [38] with approximate behavior of the function near the certain boundary part of $[0,1)^r$, and we provide here the argument. Put $\varepsilon := \sum_{p=1}^{l+1} \alpha_p - \sum_{p=1}^{l} \beta_p + \frac{a}{2}(r-1)$. Consider t along the diagonal $(t_1, \cdots, t_r) = (t, \cdots, t)$, we have, by the Stirling formula (see [5], (2.9))

$${}_{l+1}F_l(\alpha; \beta; t) \asymp \sum_m \prod_{j=1}^{r} (1 + m_j)^{\varepsilon - q} \prod_{1 \le i < j \le r} (1 + m_i - m_j)^a t^{|\mathbf{m}|}.$$

Using the evaluation formula for j_m (see e.g. formula (2.13) in [5]) we see by elementary computations [38] that the above series behaves the same as $\log \frac{1}{1-t}$ for $\varepsilon = 0$ or $(1-t)^{\varepsilon + \frac{a}{2}(r-1)}$ for $\varepsilon > 0$, which is thus unbounded. $\square$

For the ${}_2F_1$-series the sum $F(\alpha; \beta; 1^r)$ has also been explicitly evaluated; see [28], [38] and [2].

4.2. **Induced representation of H on $L^2(S)$ and spherical functions.** Fix the maximal tripotent $e \in V$ and let $S = K \cdot e$. It is well understood that S is the Shilov boundary of D and $S = K/K_e = G/P$ where K_e and P is the isotropic subgroup of K and respectively G of e. Consider the root space decomposition of $\mathfrak{g}$ under the element $\xi \in \mathfrak{a}$ in (2.9),

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{n}_0 + \mathfrak{n}_+. \tag{4.1}$$

Then $P = MAN$ is a parabolic subgroup with Lie algebra $\mathfrak{n}_0 + \mathfrak{n}_+$ and A is the Lie group with Lie algebra $\mathbb{R}\xi$ and MA is the Levi component with Lie algebra $\mathfrak{n}_0$.

For $\lambda \in \mathbb{C}$ identified with the linear function $\lambda\xi^*$ we let $Ind_P^G(\lambda) = L^2(S)$ be the induced representation of G on $L^2(S) = L^2(S, dv)$, where dv is the normalized K-invariant measure on S; see [16, Chapter VII, §§1-2]. The group action is given by

$$U(\lambda, g)f(v) = |J_{g^{-1}}(v)|^{\frac{2n - i\lambda}{pr}} f(g^{-1}v) \tag{4.2}$$

where J is as in (2.5) the complex Jacobian at $v \in S$. In particular $Ind_P^G(\lambda)$ is unitary when λ is real.

It is known that Harish-Chandra e-function with respect to the decomposition $G = KMAN$ is given by

$$e^{(i\lambda+\rho)(A(kg))} = \frac{h(z,\bar z)^{\frac{\sigma}{2}}}{h(z,\bar v)^{\sigma}}, \quad v = k^{-1}e \in S,\ k \in K$$

where σ is determined by λ and vice versa via

$$\sigma = \frac{1}{2r}(i\lambda+\rho)(\xi) = \frac{i\lambda}{2r} + \frac{1}{2}(1+b+\frac{a}{2}(r-1)), \quad i\lambda = 2r\sigma - \rho(\xi), \tag{4.3}$$

and $e^{A(kg)}$ stands for the $A = \exp(\mathbb{R}\xi)$-component in the decomposition; see e.g. [21]. (Similar formulas holds, [34], in the Siegel domain realization, for general linear functional λ on $\mathfrak{a}$.) The Poisson transform from $Ind_P^G(\lambda)$ into the space of eigenfunctions of G-invariant differential operators is given by

$$P_\lambda f(z) = \int_S \left(\frac{h(z,\bar z)}{|h(z,\bar v)|^2}\right)^{\sigma} f(v)dv, \tag{4.4}$$

which intertwines the induced representation $Ind_P^G(\lambda)$ and the regular action on X.

The corresponding spherical function can be expressed in terms of the hypergeometric function. We recall the following know result; see e. g. [7, 38, 5].

Lemma 4.2. The spherical function

$$\phi_\lambda(z) = P_\lambda 1(z) = \int_S \frac{h(z,\bar z)^\sigma}{h(z,\bar v)^\sigma h(v,\bar z)^\sigma} d\nu(v),$$

when restricted to the radial directions $z = t_1e_1 + \cdots + t_re_r$, is given by

$$\phi_\lambda(z) = (\prod_{j=1}^r (1-t_j^2))^\sigma\, {}_2F_1(\sigma,\sigma;1+b+\frac{a}{2}(r-1);t), \quad \sigma = i\frac{\lambda}{2r} + \frac{1}{2}(1+b+\frac{a}{2}(r-1))$$

4.3. **Discrete components of $(\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}, G\times G)$ under G for $\nu > \frac{a}{2}(r-1)$ for type one domains D.** In this section we will prove that the spherical representation in the previous section can be realized as a discrete component in the tensor product decomposition of $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$ for ν in the continuous part of the Wallach set only if D is Type I domains, and in that case we will find the exact such parameters λ and the explicit intertwining operator.

We consider the sesqui-holomorphic extension of the Poisson transform. More precisely, for $\nu, k \in \mathbb{C}$ and $f \in L^2(S)$ we define $T_{\nu,k}f(z,\bar w)$ to be the holomorphic function

$$T_{\nu,k}f(z,\bar w) = h(z,\bar w)^k \int_S \left(\frac{1}{h(z,\bar v)h(v,\bar w)}\right)^{\nu+k} f(v)dv, \quad (z,w) \in d\times \bar D. \tag{4.5}$$

Its restriction to the diagonal is up to a factor the Poission transform, viz

$$T_{\nu,k}f(z,\bar z) = h(z,\bar z)^{-\nu} P_\lambda f(z)$$

with $i\lambda$ determined by $\sigma = \nu + k$ as in (4.3):

$$i\lambda = 2r(\nu+k) - \rho(\xi). \tag{4.6}$$

The intertwining property of P_λ and the transformation formula of $h(z,\bar w)$ under G imply immediately

Lemma 4.3. Let $\nu, k \in \mathbb{C}$ and λ be as in (4.6). The operator $T_{\nu,k} : Ind_P^G(\lambda) = L^2(S) \to \mathcal{O}(D\times\bar{D})$, is a formal G-intertwining operator from the induced representation $Ind_P^G(\lambda)$ to the space $\mathcal{O}(D \times \bar{D})$ with the action $\pi_\nu \otimes \overline{\pi_\nu}$.

We determine when the image of $T_{\nu,k}$ is in the Hilbert space $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$.

Lemma 4.4. Let $\nu > \frac{a}{2}(r-1)$, $k \geq 0$ be an integer and λ be given in (4.6). The image of the constant function 1 in $Ind_P^G(\lambda)$ is mapped under $T_{\nu,k}$ into the Hilbert space $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$ if and only if

$$2\nu + 4k < 1 + b.$$

Furthermore the inequality has a possible solution if and only if when D is a type I non-tube type domain $SU(l,r)/S(U(l) \times U(r))$ with $a = 2$, $b = l - r > 2$ and

$$r - 1 < \nu + 2k < \frac{1}{2}(1 + l - r).$$

Proof. We write

$$T_{\nu,k}f(z, \bar{w}) = h(z, \bar{w})^k F(z, \bar{w}), \quad F(z, \bar{w}) := \int_S \left(\frac{1}{h(z, \bar{v})h(v, \bar{w})}\right)^{\nu+k} f(v)dv,$$

and we shall prove that $F(z, \bar{w})$ is in the space, and our results then follows since the function $h(z, \bar{w})^k$ for non negative integer k is a polynomial in z and $\bar{w}$, the multiplication operator by coordinate functions is a bounded operator on $\mathcal{H}_\nu$ (see [1]). The function F can be computed by using the expansion (2.7) (see also [7])

$$F(z, \bar{w}) = \sum_{\underline{\mathbf{m}}} \frac{(\nu + k)^2_{\underline{\mathbf{m}}}}{(d/r)_{\underline{\mathbf{m}}}} K_{\underline{\mathbf{m}}}(z, \bar{w}).$$

Its norm square in $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$, again by Theorem 2.1, is

$$\|F\|^2_{\mathcal{H}_\nu\otimes\overline{\mathcal{H}_\nu}} = \sum_{\underline{\mathbf{m}}} \left(\frac{(\nu + k)^2_{\underline{\mathbf{m}}}}{(d/r)_{\underline{\mathbf{m}}}}\right)^2 \frac{1}{(\nu)^2_{\underline{\mathbf{m}}}} d(\underline{\mathbf{m}})$$

which by (3.5) and the definition of hypergeometric function, is

$${}_4F_3(\nu + k, \nu + k, \nu + k, \nu + k; \nu, \nu, d/r; 1^r)$$

By Lemma 4.1, the series is convergent if and only if

$$4(\nu + k) - 2\nu - \frac{d}{r} < -\frac{a}{2}(r - 1).$$

Simplifying this is $2\nu + 4k < 1 + b$, proving the first part. Checking over the list (2.2) of bounded symmetric domains we see that this has a solution only if D is type one non-tube domain. The condition for ν and k is then $r-1 < \nu \leq \nu+2k < \frac{1}{2}(1+l-r)$. $\square$

Theorem 4.5. Let D be the type one domain $SU(l,r)/S(U(l) \times U(r))$ with $l - r > 2$ and

$$r - 1 < \nu < \frac{1}{2}(1 + l - r + 2(r - 1).$$

Let k be a nonnegative positive integer such that

$$0 \leq k < \frac{1}{4}(1 + l - r - 2\nu).$$

Then the spherical function ϕ_λ, for

$$i\lambda = 2r(\nu + k) - \rho(\xi),$$

is positive definite and the corresponding unitary spherical representation of G appears as a discrete component in the irreducible representation of $(\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}, G \times G)$.

Proof. Consider the linear span of the constant function 1 in $L^2(S)$ under the principal series representation,

$$\mathcal{S}_\lambda = \mathrm{span}\{U(g,\lambda)1; g \in G\}$$

It is a pre-Hilbert space with the inner product

$$(f, g) := (T_{\nu,k}f, T_{\nu,k}f)_{\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}}$$

It follows from the previous two Lemmas that this is well-defined and G-invariant. Its completion is then a spherical unitary representation, which in turn is irreducible since it is defined by a spherical function, and is realized as a discrete component in the tensor product via $T_{\nu,k}$. $\square$

4.4. **Discrete component of $(\mathcal{H}_\nu \otimes \overline{H_\nu}, G \times G)$ under G for $\nu = \frac{a}{2}(j-1)$ being a singular Wallach point.** We consider the tensor product $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$ with $\nu = \frac{a}{2}(j-1)$ being a singular Wallach point.

The operator $T_{\nu,k}$ intertwines the induced representation with the action $\pi_\nu \otimes \overline{\pi_\nu}$ on $\mathcal{O}(D \times \bar{D})$. However the space $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$ has K-types restriction (2.8). Thus only the operator $T_\nu = T_{\nu,0}$,

$$T_\nu f(z, w) := \int_S \frac{1}{h(z, \bar{v})^\nu h(v, \bar{w})^\nu} dv, \tag{4.7}$$

will be possibly an operator into $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$. Furthermore by some similar computation as in Lemma 4.3 (see also the proof below) this will happen possibly only for type I domains.

Theorem 4.6. Let D be the type one domain $SU(l, r)/S(U(l) \times U(r))$, $l - r > 2$ and let $\nu = j - 1$, $2 \leq j \leq r$, be a singular Wallach point. Suppose $l - r > 2j - 3$. The spherical function ϕ_λ, for λ given by $i\lambda = 2r(j-1) - r(1 + l - r + (r-1))$, is positive definite and the corresponding unitary spherical representation appears as a discrete component in the irreducible representation of $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$.

Proof. We prove that the image of the function 1 under T_ν is in the Hilbert space $\mathcal{H}_\nu \otimes \overline{\mathcal{H}_\nu}$, and the rest is proved by similar arguments as that of the previous Theorem. We have,

$$F(z, w) := (T_\nu 1)(z, w) = \sum_{\underline{\mathbf{m}}; m_j = 0} \frac{(\nu + k)^2_{\underline{\mathbf{m}}}}{(d/r')_{\underline{\mathbf{m}}}} K_{\underline{\mathbf{m}}}(z, w).$$

Its norm in

$$\sum_{\underline{\mathbf{m}}; m_j = 0} \frac{(\nu)^2_{\underline{\mathbf{m}}}}{(d/r)^2_{\underline{\mathbf{m}}}} d(\underline{\mathbf{m}}) = \sum_{\underline{\mathbf{m}}; m_j = 0} \frac{(\nu)^2_{\underline{\mathbf{m}}}}{(d/r)_{\underline{\mathbf{m}}}} \frac{\pi_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}}$$

which by Lemma 4.1 is convergent if

$$2\nu - d/r = 2v - (1 + b + (r-1)) < -(r-1)$$

namely if $l - r = b > 2\nu - 1 = 2(j-1) - 1 = 2j - 3$. □

5. Principal series representations on maximal boundaries and spherical functions. Intertwining operators into the space $\mathcal{H}_\nu$. Types A, B, BC, D

5.1. **Induced representations and spherical functions.** For the real domain X in D we let $S = L \cdot e$ be the orbit of e under L. Then $L \subset \partial_e D \cap V$, where $\partial_e D$ is the Shilov boundary of D. (In certain cases S is a true subset of $\partial_e D \cap V$.) Then S can be realized as $S = G/P$ where P is a parabolic subgroup of H with Lie algebra given by $\mathfrak{n}_0 + \mathfrak{n}_+$ in the decomposition

$$\mathfrak{h} = \mathfrak{n}_- + \mathfrak{n}_0 + \mathfrak{n}_+, \tag{5.1}$$

under the adjoint action $\xi = \xi_1 + \cdots + \xi_r$.

Let $\lambda = \lambda\xi^*$ on $\mathbb{C}\xi$. We consider the induced representation $Ind_P^H(\lambda)$ with of H realized on $L^2(S)$.

Theorem 5.1. The spherical function $\phi_\lambda(z)$ is given by the integral

$$\phi_\lambda(z) = \int_S \frac{h(z,\bar{z})^{\frac{\sigma}{2}}}{h(z,\bar{v})^\sigma} dv.$$

Its restriction on $z = t_1 e_1 + \cdots + t_r e_r$, $|t_1|, \cdots, |t_j| < 1$, is further given (and uniquely determined) by

Type A:

$$\phi_\lambda(z) = \prod_{j=1}^r (1-t_j)^{\frac{\sigma}{2}} {}_1F_1(\sigma; 1 + \frac{a}{2}(r-1); t),\ \sigma = \frac{i\lambda}{r} + \frac{a}{2}(r-1);$$

Type B:

$$\phi_\lambda(z) = (\prod_{j=1}^r (1-t_j^2)^{\frac{\sigma}{2}} {}_2F_1(\frac{\sigma}{2}, \frac{\sigma+1}{2}; \frac{a}{2}(r-1) + b + \frac{1}{2}; t^2),\ \sigma = \frac{i\lambda}{r} + b + \frac{a}{2}(r-1);$$

Type BC:

$$\phi_\lambda(z) = (\prod_{j=1}^r (1-t_j^2)^{\sigma} {}_2F_1(\sigma, \sigma-1; \frac{a}{2}(r-1) + \frac{\iota+2b}{2}; t^2),\ \sigma = \frac{i\lambda}{2r} + \frac{1}{2}(\iota - 1 + b + 2a(r-1));$$

Type D:

$$\phi_\lambda(z) = (\prod_{j=1}^r (1-t_j^2)^{\frac{\sigma}{2}} {}_2F_1(\frac{\sigma}{2}, \frac{\sigma+1}{2}; \frac{a}{2}(r-1) + \frac{1}{2}; t^2)$$
$$+ \prod_{j=1}^r \frac{\frac{\sigma}{2} - \frac{a}{2}(j-1)}{\frac{a}{2}(r-1) - \frac{a}{2}(j-1) + \frac{1}{2}} (\prod_{j=1}^r t_j (1-t_j^2)^{\sigma/2})\, {}_2F_1(\frac{\sigma}{2}+1, \frac{\sigma+1}{2}; \frac{a}{2}(r-1) + \frac{3}{2}; t^2),$$

$$\sigma = \frac{i\lambda}{r} + \frac{a}{2}(r-1).$$

Proof. We claim first that the Harish-Chandra e-function is given by

$$e^{(i\lambda+\rho)(A(kg))} = \frac{h(z,\bar{z})^{\frac{\sigma}{2}}}{h(z,\bar{v})^{\sigma}}. \tag{5.2}$$

This formula, in the Siegel domain realization of X, is given in [39] generalizing that of Upmeier-Unterberger [34] for the complex case. Here it can be simply proved by using the transformation rule of $h(z,\bar{w})$ under the group H. We get thus the integral representation of ϕ_λ. We compute the integration using the Faraut-Koranyi expansion (2.7). We have (as $\bar{z}=z, \bar{v}=v$ for $z\in X$, $v\in S$ we will drop the bar)

$$\phi_\lambda(z) = h(z,z)^{\frac{\sigma}{2}} \int_S \sum_{\underline{\mathbf{n}}} (\sigma)_{\underline{\mathbf{n}},\frac{a'}{2}} K_{\underline{\mathbf{n}}}(z,v)dv = h(z,z)^{\frac{\sigma}{2}} \sum_{\underline{\mathbf{n}}} (\sigma)_{\underline{\mathbf{n}},\frac{a'}{2}} \int_S K_{\underline{\mathbf{n}}}(z,v)dv,$$

where the interchanging of the integration and the summation is justified by the uniform convergence of the expansion (2.7) on S for fixed $z\in D$. By the K-invariance of $K_{\underline{\mathbf{n}}}$, $K_{\underline{\mathbf{n}}}(kz,kv) = K_{\underline{\mathbf{n}}}(z,v)$, and the L-invariance of the measure dv we have $\int_S K_{\underline{\mathbf{n}}}(z,v)dv$ is a L-invariant polynomial in $\mathcal{P}_{\underline{\mathbf{n}}}$. Thus

$$\int_S K_{\underline{\mathbf{n}}}(z,v)dv = C_{\underline{\mathbf{n}}} p_{\underline{\mathbf{n}}}(z), \quad z\in D$$

for those $\underline{\mathbf{n}}$ given in Lemma 3.3 in terms of $\underline{\mathbf{m}}$ and for some constant $C_{\underline{\mathbf{n}}}$; otherwise it is zero. We claim that

$$C_{\underline{\mathbf{n}}} = \frac{1}{\langle p_{\underline{\mathbf{n}}}, p_{\underline{\mathbf{n}}}\rangle_{\mathcal{F}}}.$$

Indeed, the left hand side is an L-invariant element in $\mathcal{P}_{\underline{\mathbf{n}}}$ thus is a multiple of $p_{\underline{\mathbf{n}}}$ determined in Proposition 3.5. To find the constant we compute the norm square of the left hand in the Fock space. By definition of $K_{\underline{\mathbf{n}}}$ we have it is

$$\int_S \int_S K_{\underline{\mathbf{n}}}(w,v)dv\, dw,$$

which is, by the invariant of K_n under $L\subset K$ and that $S = L\cdot e$,

$$\int_S K_{\underline{\mathbf{n}}}(w,e)dw = C_{\underline{\mathbf{n}}} p_{\underline{\mathbf{n}}}(e) = C_{\underline{\mathbf{n}}}.$$

On the other hand, the squared norm of the right hand side is

$$C_n^2 \langle p_{\underline{\mathbf{n}}}, p_{\underline{\mathbf{n}}}\rangle_{\mathcal{F}},$$

proving our claim. Thus

$$\phi_\lambda(z) = h(z,z)^{\frac{\sigma}{2}} \sum_{\underline{\mathbf{n}}} (\sigma)_{\underline{\mathbf{n}},\frac{a'}{2}} \frac{1}{\langle p_{\underline{\mathbf{n}}}, p_{\underline{\mathbf{n}}}\rangle_{\mathcal{F}}} p_n(z).$$

For Type A we have $\underline{\mathbf{n}} = \underline{\mathbf{m}}$, $a = a'$ and

$$\frac{1}{\langle p_{\underline{\mathbf{n}}}, p_{\underline{\mathbf{n}}}\rangle_{\mathcal{F}}} = \frac{\pi_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}}$$

by Lemma 3.2. The function $\phi_\lambda(z)$ is, for $z = t_1e_1 + \cdots + t_re_r$,

$$\phi_\lambda(z) = \prod_{j=1}^{r}(1-t_j^2)^{\frac{\sigma}{2}} \sum_{\underline{\mathbf{m}}} (\sigma)_{\underline{\mathbf{n}},\frac{a'}{2}} \frac{\pi_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}} \Omega_{\underline{\mathbf{m}}}(t) = \prod_{j=1}^{r}(1-t_j^2)^{\frac{\sigma}{2}} {}_1F_1(\sigma; q; t)$$

For Type B, we have $\underline{\mathbf{n}} = 2\underline{\mathbf{m}}$, $a' = 2a$, the polynomial $p_{\underline{\mathbf{n}}}$ is the Jack polynomial $\Omega_{\underline{\mathbf{m}}}$ by Proposition 3.5, and its norm square $\langle p_{\underline{\mathbf{n}}}, p_{\underline{\mathbf{n}}}\rangle_{\mathcal{F}}$ is computed in Proposition 3.6. This gives, for $z = t_1e_1 + \cdots + t_re_r$,

$$\begin{aligned}
&\phi_\lambda(z)\\
&= h(z,z)^{\frac{\sigma}{2}} \sum_{\underline{\mathbf{n}}=2\underline{\mathbf{m}}} (\sigma)_{\underline{\mathbf{n}},\frac{a'}{2}} \left(\frac{1}{\pi_{\underline{\mathbf{m}}}} 2^{2|\underline{\mathbf{m}}|} (q)_{\underline{\mathbf{m}},\frac{a}{2}} ((r-1)\frac{a}{2} + b + \frac{1}{2})_{\underline{\mathbf{m}},\frac{a}{2}}\right)^{-1} \Omega_{\underline{\mathbf{m}}}(t^2)\\
&= h(z,z)^{\frac{\sigma}{2}} \sum_{\underline{\mathbf{m}}} 2^{2|\underline{\mathbf{m}}|} (\frac{\sigma}{2})_{\underline{\mathbf{m}},\frac{a}{2}} (\frac{\sigma+1}{2})_{\underline{\mathbf{m}},\frac{a}{2}} \left(\frac{1}{\pi_{\underline{\mathbf{m}}}} 2^{2|\underline{\mathbf{m}}|} (q)_{\underline{\mathbf{m}},\frac{a}{2}} ((r-1)\frac{a}{2} + b + \frac{1}{2})_{\underline{\mathbf{m}},\frac{a}{2}}\right)^{-1} \Omega_{\underline{\mathbf{m}}}(t^2)\\
&= h(z,z)^{\frac{\sigma}{2}} \sum_{\underline{\mathbf{m}}} \frac{(\frac{\sigma}{2})_{\underline{\mathbf{m}}} (\frac{\sigma+1}{2})_{\underline{\mathbf{m}}}}{((r-1)\frac{a}{2} + b + \frac{1}{2})_{\underline{\mathbf{m}}}} \frac{\pi_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}} \Omega_{\underline{\mathbf{m}}}(t^2)\\
&= h(z,z)^{\frac{\sigma}{2}} {}_2F_1(\frac{\sigma}{2}, \frac{\sigma+1}{2}; (r-1)\frac{a}{2} + b + \frac{1}{2}; t^2)
\end{aligned}$$

where in the second step we have written $(\sigma)_{\underline{\mathbf{n}},\frac{a'}{2}}$ in terms of $(c)_{\underline{\mathbf{n}},\frac{a}{2}}$ as in (3.11).

The remaining types BC or D are done by the same method. Note that for type BC we have $r' = 2r$ and $h(z,z) = \prod_{j=1}^{r}(1-t_j^2)^2$. □

5.2. **Discrete components of $(\mathcal{H}_\nu, G)$ under H for $\nu > \frac{a'}{2}(r'-1)$.** In this section we will find and realize certain discrete components in the branching of the holomorphic representations $\mathcal{H}_\nu$ of G under H using the Poisson transform studied above. Similar computations as in Lemma 4.4 show that the operator $T_{\nu,k}$ defined in (4.7) below maps the spherical representation into the holomorphic representation $\mathcal{H}_\nu$ only if X is of Type B with $H = SO(r,l)$, $l > r$, or Type BC with $H = Sp(r,l)$, $l > r$. We will thus only consider those cases. The corresponding group G is then $SU(l,r)$ $(r' = r)$ or $SU(2l, 2r)$ $(r' = 2r)$.

Theorem 5.2. Let H be the group $SO_0(l,r)$ or $Sp(l,r)$. Let l, r satisfy $l - r > 2(r-1)$ for $H = SO_0(l,r)$ and $l - r \geq 2(r-1)$ for $H = Sp_0(l,r)$. Suppose $\nu > r' - 1$ is a be a point in the continuous part of the Wallach set of G, $\nu < \frac{l-r}{2}$ for $H = SO_0(l,r)$, and $\nu < l - r + \frac{3}{2}$ for $H = Sp(l,r)$. If $k \in \mathbb{N}$ such that

$$0 \leq k < \frac{1}{4}(\frac{l-r}{2} - \nu) \tag{5.3}$$

for $H = SO_0(r, r+b)$ and

$$0 \leq k < \frac{1}{8}(3 + 2(l-r) - 2\nu) \tag{5.4}$$

for $H = Sp(r, r+b)$, then the spherical function ϕ_λ with

$$i\lambda = \begin{cases} r(\nu+2k) - r(l-r+\frac{1}{2}(r-1)), & H = SO_0(l,r) \\ 2r(\nu+2k) - r(3+l-r+8(r-1)), & H = Sp(l,r), \end{cases} \tag{5.5}$$

is positive definite and appears as a discrete component in the irreducible representation of $(\mathcal{H}_\nu, G)$ under $H \subset G$.

Proof. The formula (5.2) for the Harish-Chandra e-function implies that the Poisson transform on S is given by

$$P_\lambda f(z) = \int_S \frac{h(z,\bar{z})^{\frac{\sigma}{2}}}{h(z,\bar{v})^\sigma} f(v) dv, \quad z \in X,$$

which intertwines the induced representation $Ind_P^H(\lambda)$ with the regular action X. The functions $h(z,z), h(z,v)$ are polynomials in $z \in X$ and thus have holomorphic extension to $z \in D$. Furthermore it is easy to see that they have no zeros on D, by, e.g., the explicit formula of h in terms of the determinant functions [24]. Thus $Pf(z)$ has a holomorphic extension on D, still denoted by $P_\lambda f(z)$. For non negative integer k satisfying (5.3) and (5.4) we put $\sigma := \nu + 2k$ with the corresponding λ as in Theorem 5.1; this is explicitly computed in (5.5). We define, for $f \in L^2(S)$,

$$\begin{aligned} T_{\nu,k} f(z) :&= h(z,z)^{-\frac{\nu}{2}} P_\lambda f(z) \\ &= h(z,z)^k \int_S \frac{1}{h(z,v)^\sigma} f(v) dv, \quad z \in D. \end{aligned} \tag{5.6}$$

The transformation formula (2.4) of h when restricted to X is

$$h(gz, \overline{gz}) = J_g(z)^{\frac{1}{p}} h(z,\bar{z}) \overline{J_g(z)^{\frac{1}{p}}} = J_g(z)^{\frac{2}{p}} h(z,z), \quad g \in H, \quad z \in X,$$

for the Jacobian J_g is real-valued for $x \in X$. It's holomorphic extension $h(z,z)$ satisfies then

$$h(gz, gz) = J_g(z)^{\frac{2}{p}} h(z,z).$$

Thus $T_{\nu,k}$ intertwines the induced representation $Ind_P^H(\lambda)$ with $(\mathcal{O}(D), H, \pi_\nu)$. We prove that $T_{\nu,k}$ maps the function 1 into $\mathcal{H}_\nu$ when k satisfies the stated condition. The rest is proved as in Theorem 4.5. We rewrite

$$T_{\nu,k} 1(z) = h(z,z)^k F(z), \quad F(z) := \int_S \frac{1}{h(z,v)^\sigma} f(v) dv, \quad z \in D.$$

and we shall prove that the function $F(z)$ is in $\mathcal{H}_\nu$, so is $T_\nu 1(z)$ since $h(z,z)^k$ is a polynomial of the coordinate functions and each of them defines a bounded multiplication operator on $\mathcal{H}_\nu$ [1].

The function $F(z)$ apart from the factor of $h(z,z)^{\frac{\sigma}{2}}$ is the holomorphic extension of the spherical function computed in Theorem 5.1. If $H = SO_0(l,r)$ we have,

$$F(z) = \sum_{\underline{\mathbf{m}}} \frac{(\frac{\nu+2k}{2})_{\underline{\mathbf{m}}} (\frac{\nu+2k+1}{2})_{\underline{\mathbf{m}}}}{(\frac{a}{2}(r-1)+b+\frac{1}{2})_{\underline{\mathbf{m}}}} \frac{\pi_{\underline{\mathbf{m}}}}{(q)_{\underline{\mathbf{m}}}} p_{\underline{\mathbf{n}}}(z).$$

Its norm square in $\mathcal{H}_\nu$ is, by Proposition 3.5, the hypergeometric function

$$\|F\|^2_{\mathcal{H}_\nu} = {}_4F_3\left(\frac{\sigma}{2}, \frac{\sigma}{2}, \frac{\sigma+1}{2}, \frac{\sigma+1}{2}; \frac{\nu}{2}, \frac{\nu+1}{2}, \frac{a}{2}(r-1)+b+\frac{1}{2}; 1^r\right). \tag{5.7}$$

The series is convergent if

$$2\sigma + 1 - (\nu + 1 + b + \frac{a}{2}(r-1)) < -\frac{a}{2}(r-1)$$

equivalently (recalling $\sigma = \nu + 2k$)

$$\nu + 4k < \frac{l-r}{2}$$

according to Lemma 4.1, which is guaranteed by our assumption on l, r and ν. Thus F is in $\mathcal{H}_\nu$, so is $T_\nu 1$.

If $H = Sp(l,r)$, which is of Type BC with $\underline{\mathbf{n}} = (\underline{\mathbf{m}}, \underline{\mathbf{m}})$, $a = 4 = 2a'$, the function F is

$$F(z) = \sum_{\underline{\mathbf{n}}=(\underline{\mathbf{m}},\underline{\mathbf{m}})} \frac{(\sigma)_{\underline{\mathbf{m}}}(\sigma-1)_{\underline{\mathbf{m}}}}{(\frac{a}{2}(r-1)+\frac{\iota+2b}{2})} p_{\underline{\mathbf{n}}}(z),$$

and

$$\|F\|^2_{\mathcal{H}_\nu} = {}_4F_3(\sigma, \sigma, \sigma-1, \sigma-1; \frac{a}{2}(r-1)+b+\frac{\iota}{2}, \nu, \nu-1; 1^r), \tag{5.8}$$

whose convergence is again determined by Lemma 4.1. The condition on the convergence is $l-r > \nu+4k-\frac{3}{2} > \frac{a'}{2}(r'-1)-\frac{3}{2} = (2r-1)-\frac{3}{2}$, namely $l-r \geq 2r-2 = 2(r-1)$. The condition on $k \geq 0$ is then obtained accordingly. $\square$

5.3. **Discrete components of** $(\mathcal{H}_\nu, G)$ **under** H **for** $\nu = \frac{a'}{2}(j-1)$ **being a singular Wallach point.** Let $H = SO_0(l,r)$ or $H = Sp(l,r)$ be as in the previous subsection. We fix $\nu = \frac{a'}{2}(j-1) = j-1$, $2 \leq j \leq r'$ be a singular Wallach point, where $r' = r$ and respectively $r' = 2r$. We consider the operator $f \mapsto T_\nu f := T_{\nu,0} f$ defined in (5.6), and the image $F = T_{\nu,0}1$ of the constant function 1. The norm square $\|F\|^2_{\mathcal{H}_\nu}$ is again a series as in (5.7) and (5.8). The condition for the convergence for the group $SO_0(l,r)$ is $l-r > j-1$ while as for the group $Sp(l,r)$ is $l-r > j-1-\frac{3}{2}$, namely $l-r \geq j-2$. Thus we have the following

Theorem 5.3. Let H be group $SO_0(l,r)$ or $Sp(l,r)$ and let $\nu = j-1$, $2 \leq j \leq r'$, be a singular Wallach point of G. Suppose $l-r > (j-1)$ for the group $H = SO_0(r, r+b)$, and $l-r \geq j-2$ for $H = Sp(r, r+b)$. The spherical function ϕ_λ, with λ determined by $(\nu, k) := (j-1, 0)$ as in (5.5), is positive definite and appears as a discrete component in the irreducible representation of $(\mathcal{H}_\nu, G)$ under $H \subset G$.

Note that in Theorems 4.5 and 5.3 we have only taken the operator $T_{\nu,k}$ with $k = 0$ when ν is a singular Wallach point. It would be interesting to refine the definition of $T_{\nu,k}$ so that we get also finitely many discrete components; indeed there are finitely many discrete components in the branching of $\pi_\nu \otimes \overline{\pi_\nu}$ when $\nu = 1$ is the last Wallach point [39].

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