MULTIGRADED REGULARITY AND THE KOSZUL PROPERTY

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ABSTRACT. We give a criterion for the section ring of an ample line bundle to be Koszul in terms of multigraded regularity. We discuss an application to polytopal semigroup rings.

1. INTRODUCTION

Let A be an ample line bundle on a projective variety X over a field k, and let $R(A) = \bigoplus_{\ell \geq 0} H^0\left(X, A^\ell\right)$ be the section ring associated to A. Recall that a graded k algebra $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ is called *Koszul* (or *wonderful*) if k admits a linear free resolution over R. It is well known that a Koszul algebra is generated in degree 1, and that the ideal of relations between its generators is generated by quadrics.

The purpose of this note is to give criteria for the section ring of an ample line bundle to be Koszul in terms of the regularity of the line bundle. The following theorem illustrates the flavour of our main result, Theorem 3.

Theorem 1. *Let A be an ample line bundle on a projective variety X over an infinite field k. Assume that $H^i(X, A^{m-i}) = 0$ for $i > 0$. Then the section ring $R(A^m) = \bigoplus_{\ell \geq 0} H^0(X, A^{\ell m})$ is Koszul.*

It is well known that section rings of high enough powers of ample line bundles are Koszul (see [Bac86]). Moreover, Eisenbud and Reeves [ERT94] give criteria for Veronese subalgebras of graded k-algebras to be Koszul in terms of the algebraic Castelnuovo-Mumford regularity. We illustrate the relationship between our theorem and these criteria in the end of Section 2.

Sufficient criteria for powers of ample line bundles to have Koszul section ring are known for curves [VF93, But94, Pol95, PP97, CRV01], homogeneous spaces [IM94, Bez95, Rav95], elliptic ruled surfaces [GP96b], abelian varieties [Kem89], and toric varieties [BGT97]. More generally, there are criteria for certain adjoint line bundles on smooth projective varieties to have Koszul section ring, see [Par93].

The Koszul property has also been studied for points in projective spaces, see [CTV01, Kem92, Pol06], and toric varieties admitting additional combinatorial structure, see [Stu96, PRS98, HHR00, OH99]. The Koszul property appears naturally in many areas of mathematics; see for example [PP05] for an introduction to Koszul algebras from different perspectives.

We will prove a more general version of Theorem 1 in terms of multigraded regularity (compare [MS04] and [HSS06]), see Theorem 3. It generalizes a result for line bundles on surfaces by Gallego and Purnaprajna [GP96a, Theorem 5.4] that they use to give exact criteria for line bundles on elliptic ruled surfaces to have a Koszul section ring. The proof is based on a vanishing theorem due to Lazarsfeld and uses methods very similar to those in [GP96a] and [HSS06].

As an application, we show how the criteria for polytopal semigroup rings to be Koszul due to Bruns, Gubeladze and Trung [BGT97, Theorem 1.3.3.] can be improved if multiples of the polytope do not contain interior lattice points, see Section 3. Theorem 3 is part of my thesis [Her06].

2. Multigraded regularity and proof of theorem

Let X be a projective variety over a field k. We will assume for the remainder of the paper that k is infinite. Let $B_1, \ldots, B_r$ be globally generated line bundles on X. For $\mathbf{u} \in \mathbb{Z}^r$, we let $B^{\mathbf{u}} := B_1^{u_1} \otimes \cdots \otimes B_r^{u_r}$ and $|\mathbf{u}| = u_1 + \cdots + u_r$. Let $\mathcal{B} = \{B^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{N}^r\} \subset \mathrm{Pic}(X)$ be the submonoid of $\mathrm{Pic}(X)$ generated by $B_1, \ldots, B_r$.

Definition 2. Let L be a line bundle on X. A sheaf $\mathcal{F}$ is called L-regular with respect to $B_1, \ldots, B_r$ if

$$H^i\left(X, \mathcal{F} \otimes L \otimes B^{-\mathbf{u}}\right) = 0$$

for all $i > 0$ and for all $\mathbf{u} \in \mathbb{N}^r$ with $|\mathbf{u}| = i$.

Obeserve that for $r = 1$, this is the usual definition for Castelnuovo-Mumford regularity, compare [Laz04, 1.8.4.].

Now we are ready to state the main theorem.

Theorem 3. *Let $B_1, \ldots, B_r$ be a set of globally generated line bundles on X generating a semigroup $\mathcal{B}$. Let $A \in \mathcal{B}$ be an ample line bundle such that $A \otimes B_i^{-1} \in \mathcal{B}$ for all $i = 1, \ldots, r$. If A is $\mathcal{O}_X$-regular with*

respect to $B_1, \ldots, B_r$, *then the section ring* $R(A) = \bigoplus_{\ell \geq 0} H^0(X, A^\ell)$ *is Koszul.*

In the proof we will use the following generalization of Mumford's theorem to the multigraded case. For ease of notation, we fix $B_1, \ldots, B_r$, and say a sheaf $\mathcal{F}$ is L-regular if it is L-regular with respect to $B_1, \ldots, B_r$.

Theorem 4 ([HSS06], Theorem 2.1). *Let* $\mathcal{F}$ *be L-regular. Then for all* $\mathbf{u} \in \mathbb{N}^r$,

    (1) $\mathcal{F}$ *is* $(L \otimes B^{\mathbf{u}})$*-regular;*
    (2) *the natural map*

$$H^0\left(X, \mathcal{F} \otimes L \otimes B^{\mathbf{u}}\right) \otimes H^0\left(X, B^{\mathbf{v}}\right) \to H^0\left(X, \mathcal{F} \otimes L \otimes B^{\mathbf{u}+\mathbf{v}}\right)$$

        is surjective for all $\mathbf{v} \in \mathbb{N}^r$;
    (3) $\mathcal{F} \otimes L \otimes B^{\mathbf{u}}$ *is generated by its global sections, provided there exists* $\mathbf{w} \in \mathbb{N}^r$ *such that* $B^{\mathbf{w}}$ *is ample.*

Observe that the proof of Theorem 4 in [HSS06] only requires k to be infinite.

We will also make ample use of the following lemma.

Lemma 5 ([HSS06], Lemma 2.2). *Let* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be a short exact sequence of coherent* $\mathcal{O}_X$*-modules. If* $\mathcal{F}$ *is L-regular,* $\mathcal{F}''$ *is* $\left(L \otimes B_j^{-1}\right)$*-regular for all* $1 \leq j \leq r$ *and* $H^0\left(X, \mathcal{F} \otimes L \otimes B_j^{-1}\right) \to H^0\left(X, \mathcal{F}'' \otimes L \otimes B_j^{-1}\right)$ *is surjective for all* $1 \leq j \leq r$, *then* $\mathcal{F}'$ *is also L-regular.*

The main tool of the proof is a vanishing theorem for a family of vector bundles associated to an ample and globally generated line bundle A as follows. To a globally generated vector bundle E is associated a vector bundle M_E, the kernel of the evaluation map

$$(1) \qquad\qquad 0 \to M_E \to H^0\left(X, E\right) \otimes \mathcal{O}_X \to E \to 0.$$

For $h \in \mathbb{N}$, we define vector bundles $M^{(h)}$ inductively, by letting $M^{(0)} = A$ and $M^{(h)} = M_{M^{(h-1)}} \otimes A$, provided $M^{(h-1)}$ is globally generated.

Lemma 6 (Lazarsfeld, see [Par93, Lemma 1]). *Let* X *be a projective variety, let* A *be an ample line bundle on* X, *and let* $R(A) = \bigoplus_{\ell \geq 0} H^0\left(X, A^\ell\right)$ *be the section ring associated to* A. *Assume that the vector bundles* $M^{(h)}$ *are globally generated for all* $h \geq 0$. *If* $H^1\left(X, M^{(h)} \otimes A^\ell\right) = 0$ *for all* $\ell \geq 0$ *then* $R(A)$ *is Koszul. Moreover, if* $H^1\left(X, A^\ell\right) = 0$ *for all* $\ell \geq 1$, *the converse also holds.*

Observe that the proof of this lemma is valid for projective varieties over any field.

Proof of Theorem 3. We will use induction on h to show that $M^{(h)}$ is $\mathcal{O}_X$-regular. In particular, by Theorem 4, (3), $M^{(h)}$ is globally generated, and M^{h+1} is defined.

Tensoring (1) for $E = M^{(h-1)}$ with A, we obtain the following short exact sequence

$$0 \to M^{(h)} \to H^0\left(X, M^{(h-1)}\right) \otimes A \to M^{(h-1)} \otimes A \to 0.$$

Then $H^0\left(X, M^{(h-1)}\right) \otimes A$ is $\mathcal{O}_X$ regular. Since $A \otimes B^{-e_j} \in \mathcal{B}$, it follows that $A \otimes B^{-e_j} \cong B^{u'}$ for $u' \in \mathbb{N}^r$. By the induction hypothesis and Theorem 4, (1) $M^{(h-1)}$ is $A \otimes B^{-e_j}$-regular for all j, and so $M^{(h-1)} \otimes A$ is B^{-e_j}-regular for all j. Similarly, by Theorem 4, (2), the natural map $H^0\left(X, M^{(h-1)}\right) \otimes H^0\left(X, A \otimes B^{-e_j}\right) \to H^0\left(X, M^{(h-1)} \otimes A \otimes B^{-e_j}\right)$ is surjective for all $1 \le j \le r$. Applying Lemma 5, we see that $M^{(h)}$ is $\mathcal{O}_X$-regular.

Theorem 4, (1) implies M^h is also B^u-regular for all $u \in \mathbb{N}^r$. Hence $H^1\left(X, M^{(h)} \otimes A^\ell\right) = 0$ for all $h \ge 0$ and $\ell \ge 0$, and Lemma 6 implies that $R(A)$ is Koszul. $\qquad\square$

Theorem 1 is the special case when $r = 1$.

Example 7. The fact that the section ring of high enough powers of ample line bundles is Koszul follows easily from this result: By Serre vanishing, L^d is $\mathcal{O}_X$-regular with respect to L for d large enough, hence the associated section ring is Koszul.

Remark 8. Let $R \cong k[x_0, \ldots, x_N]/I$, where $I \subset k[x_0, \ldots, x_N]$ is a homogeneous ideal. If I admits a quadratic Gröbner basis with respect to some monomial ordering, then R is Koszul. However, a Koszul algebra need not admit a presentation whose ideal admits a quadratic Gröbner basis, see [ERT94].

Remark 9. It is well known that if the section ring of a line bundle L is Koszul, then L satisfies Green's property N_1 (see [Laz04, 1.8.C] for an introduction to property N_p). On the other hand, Sturmfels [Stu00, Theorem 3.1] exhibited an example of a smooth projectively normal curve whose coordinate ring is presented by quadrics but is not Koszul. However, in many cases, criteria for line bundles to satisfy N_p imply that their section ring is Koszul and even that its ideal admits a quadratic Gröbner basis when $p \ge 1$. For example, the conditions for Theorem 3 agree with those of [HSS06] for a line bundle to satisfy N_1.

Eisenbud, Reeves and Totaro [ERT94] give criteria for Veronese subrings of finitely generated graded k-algebras to be Koszul in terms of algebraic regularity. Translating their result into the language of ample line bundles, we obtain a better bound than Theorem 1 for normally generated line bundles.

Definition 10. An ample line bundle is called *normally generated*, if the natural map

$$\underbrace{H^0(X, L) \otimes \cdots \otimes H^0(X, L)}_{m} \to H^0(X, L^m)$$

is surjective for all m.

Corollary 11. *Let A be a normally generated line bundle. Suppose A^m is $\mathcal{O}_X$-regular with respect to itself. Then if $d \geq \frac{m}{2}$, the ideal of the section ring associated to A^d admits a quadratic Gröbner basis; in particular, the section ring is Koszul.*

To see how this theorem follows from the criteria in [ERT94], we first review the notion of algebraic regularity.

Definition 12. Let $S = k[x_0, \ldots, x_N]$ be a polynomial ring over k. A finitely generated graded S-module M is m-regular if $\operatorname{Tor}_i^S(M, k)_j = 0$ for $j > i + m$ and $i \geq 0$.

Let $I \subset S$ be a homogeneous ideal, and let $R = S/I$. We denote with $R^{(d)} = \bigoplus_{m \in \mathbb{N}} R_{md}$ the d'th Veronese subalgebra of R. Keeping in mind that S/I is m-regular if and only if I is $(m + 1)$-regular, the following theorem is an easy consequence of the results proved in Eisenbud, Reeves and Totaro [ERT94].

Theorem 13 ([ERT94])**.** *If R is $(m - 1)$-regular and $d \geq \frac{m}{2}$, then the ideal of $R^{(d)}$ admits a quadratic Gröbner basis.*

Proof of Corollary 11. Since A is normally generated, it is very ample. Moreover, the section ring R associated to A is generated in degree 1, and it agrees with the homogeneous coordinate ring of the embedding $\iota \colon X \hookrightarrow \mathbb{P} := \mathbb{P}(H^0(X, A))$ induced by A. In particular, R is of the form S/I for $S = \operatorname{Sym}^\bullet H^0(X, A)$ and I a homogeneous ideal in S. Now A^m is $\mathcal{O}_X$-regular with respect to itself if and only if $\iota_* A$ is $\mathcal{O}_{\mathbb{P}}(m - 1)$-regular with respect to $\mathcal{O}_{\mathbb{P}}(1)$ if and only if $R(\iota_* A) = R(A)$ is $(m - 1)$-regular as a S-module, (see for example [Eis95, Exercise 20.20.] or [Laz04, 1.8.26.]). Since the section ring associated to A^d agrees with $R^{(d)}$, the corollary follows from Theorem 13. $\square$

3. Polytopal semigroup rings

The question which powers of ample line bundles on toric varieties have Koszul section ring was studied by Bruns, Gubeladze and Trung in [BGT97]. They prove that for an ample line bundle A on a toric variety X of dimension n, the section ring $R(A^n)$ is Koszul. Observe that this also follows easily from Theorem 1. In fact, since the higher cohomology of an ample line bundle on a toric variety vanishes, A^n is $\mathcal{O}_X$-regular.

A more careful study of the regularity of a line bundle on a toric variety shows that if r is the number of integer roots of the Hilbert polynomial of A, then A^{n-r} is $\mathcal{O}_X$-regular (see [HSS06, Lemma 4.1]), and we obtain the following Corollary.

Corollary 14. *Let A be an ample line bunlde on a toric variety X, and let r be the number of integer roots of the Hilbert polynomial of A. Then $R(A^{n-r})$ is Koszul.*

In terms of lattice polytopes and polytopal semigroup rings this can be rephrased as follows. Let $M \cong \mathbb{Z}^n$ be a lattice, and $P \subset M \otimes \mathbb{R} := M_{\mathbb{R}}$ be a lattice polytope. P determines a semigroup $S_P \subset M \times \mathbb{Z}$, the semigroup generated by $\{(p, 1) \in M \times \mathbb{Z} \mid p \in P \cap M\}$. Let $k[S_P]$ be the semigroup algebra associated to S_P.

Corollary 15. *Let P be a lattice polytope of dimension n, and let r be the largest positive integer such that rP does not contain any interior lattice points. Then the polytopal semigroup ring $k[S_{(n-r)P}]$ is Koszul.*

This follows from the fact that r is the number of integer roots of the Hilbert polynomial of the ample line bundle associated to P (see for example [HSS06, Section 4]).

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