DOUBLE SHUFFLE RELATIONS AND RENORMALIZATION OF MULTIPLE ZETA VALUES

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Abstract. In this paper we present some of the recent progresses in multiple zeta values (MZVs). We review the double shuffle relations for convergent MZVs and summarize generalizations of the sum formula and the decomposition formula of Euler for MZVs. We then discuss how to apply methods borrowed from renormalization in quantum field theory and from pseudodifferential calculus to partially extend the double shuffle relations to divergent MZVs.

1. Introduction

The purpose of this paper is to give a survey of recent developments in multiple zeta values (MZVs). We emphasize on the double shuffle relations which underlie the algebraic relations among the convergent MZVs, and on renormalization methods that aim to extend the double shuffle relations to MZVs outside of the convergent range of the nested sums defining MZVs. We also provide background on double shuffle relations and renormalization, as well as the closely related Rota-Baxter algebras and some analytic tools in pseudodifferential calculus in view of renormalization.

1.1. **Double shuffle relations and Euler's formulas.** A **multiple zeta value (MZV)** is the special value of the complex valued function

$$\zeta(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

at positive integers $s_1, \cdots, s_k$ with $s_1 \geqslant 2$ to insure the convergence of the nested sum. MZVs are natural generalizations of the Riemann zeta values $\zeta(s)$ to multiple variables. The two variable case (double zeta values) was already studied by Euler.

MZVs in the general case were introduced 1990s with motivations from number theory [70], combinatorics [40] and quantum field theory [11]. Since then the subject has turned into an active area of research that involves many areas of mathematics and mathematical physics [13]. Its number theoretic significance can be seen from the fact that all MZVs are periods of mixed Tate motives over $\mathbb{Z}$ and the conjecture that all periods of mixed Tate motives are rational combinations of MZVs [25, 27, 69].

It has been discovered that the analytically defined MZVs satisfy many algebraic relations. Further it is conjectured that these algebraic relations all follow from the combination of two algebra structures: the shuffle relation and the stuffle (harmonic shuffle or quasi-shuffle) relation [46]. This remarkable conjecture not only links the analytic study of MZVs to the algebraic study of double shuffle relations, but also implies the more well-known conjecture on the algebraic independence of $\zeta(2), \zeta(2k+1), k \geqslant 1$, over $\mathbb{Q}$.

Date: March 14, 2011.

Many results on algebraic relations among MZVs can be regarded as generalizations of Euler's sum formula and decomposition formula on double zeta values which preceded the general developments of multiple zeta values by over two hundred years. We summarize these results in Section 3. With the non-experts in mind, we first give in Section 2 preliminary concepts and results on double shuffle relations for MZVs and the related Rota-Baxter algebras.

1.2. **Renormalization.** Values of the Riemann zeta function at negative integers are defined by analytic continuation and possess significant number theory properties (Bernoulli numbers, Kummer congruences, p-adic L-functions, $\cdots$). Thus it would be interesting to similarly study MZVs outside of the convergent domain of the corresponding nested sums. However, most of the MZVs remain undefined even after the analytic continuation. To bring new ideas into the study, we introduce the method of renormalization from quantum field theory.

Renormalization is a process motivated by physical insight to extract finite values from divergent Feynman integrals in quantum field theory, after adding in a so-called counter-term. Despite its great success in physics, this process was well-known for its lack of a solid mathematical foundation until the seminal work of Connes and Kreimer [14, 15, 16, 50]. They obtained a Hopf algebra structure on Feynman graphs and showed that the separation of Feynman integrals into the renormalized values and the counter-terms comes from their algebraic Birkhoff decomposition similar to the Birkhoff decomposition of a loop map.

The work of Connes and Kreimer establishes a bridge that allows an exchange of ideas between physics and mathematics. In one direction, their work provides the renormalization of quantum field theory with a mathematical foundation which was previously missing, opening the door to further mathematical understanding of renormalization. For example, the related Riemann-Hilbert correspondence and motivic Galois groups were studied by Connes and Marcolli [17], and motivic properties of Feynman graphs and integrals were studied by Bloch, Esnault and Kreimer [5]. See [2, 11, 56] for more recent studies on the motivic aspect of Feynman rules and renormalization.

In the other direction, the mathematical formulation of renormalization provided by the algebraic Birkhoff decomposition allows the method of renormalization dealing with divergent Feynman integrals in physics to be applied to divergent problems in mathematics that could not be dealt with in the past, such as the divergence in multiple zeta values [36, 37, 73, 55] and Chen symbol integrals [54, 55]. We survey these studies on renormalization in mathematics in Sections 5 and 6 after reviewing in Section 4 the general framework of algebraic Birkhoff decomposition in the context of Rota-Baxter algebras. We further present an alternative renormalization method using Speer's generalized evaluators [67] and show it leads to the same renormalized double zeta values as the algebraic Birkhoff decomposition method.

We hope our paper will expose this active area to a wide range of audience and promote its further study, to gain a more thorough understanding of the double shuffle relations for convergent MZVs and to establish a systematical renormalization theory for the divergent MZVs. One topic that we find of interest is to compare the various renormalization methods presented in this paper from an abstract point of view in terms of a renormalization group yet to be described in this context, again motivated by the study in quantum field theory. With implications back to physics in mind, we note that MZVs offer a relatively handy and tractable field of experiment for such issues when compared with the very complicated Feynman integral computations.

Acknowledgements: L. Guo acknowledges the support from NSF grant DMS-0505643 and thanks JAMI at Johns Hopkins University for its hospitality. S. Paycha is grateful to D. Manchon for his comments on a preliminary version of part of this paper. B. Zhang acknowledges the support from NSFC grant 10631050.

2. Double shuffle relations for convergent multiple zeta values

All rings and algebras in this paper are assumed to be unitary unless otherwise specified. Let **k** be a commutative ring whose identity is denoted by 1.

2.1. **Rota-Baxter algebras.** Let $\lambda \in \mathbf{k}$ be fixed. A unitary (resp. nonunitary) **Rota–Baxter k-algebra (RBA) of weight** λ is a pair (R, P) in which R is a unitary (resp. nonunitary) **k**-algebra and $P : R \to R$ is a **k**-linear map such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy),\ \forall x,\ y \in R. \tag{1}$$

In some references such as [55], the notation $\theta = -\lambda$ is used.

We will mainly consider the following Rota-Baxter operators in this paper. See [19, 30, 64] for other examples.

Example 2.1. (The integration operator) Define the integration operator

$$I(f)(x) = \int_0^x f(t)dt \tag{2}$$

on the algebra $C[0, \infty)$ of continuous functions $f(x)$ on $[0, \infty)$. Then it follows from the integration by parts formula that I is a Rota-Baxter operator of weight 0 [4].

Example 2.2. (The summation operator) Consider the summation operator [75]

$$P(f)(x) := \sum_{n\geqslant 1} f(x+n).$$

Under certain convergency conditions, such as $f(x) = O(x^{-2})$ and $g(x) = O(x^{-2})$, $P(f)(x)$ and $P(g)(x)$ define absolutely convergent series and we have

$$\begin{aligned} P(f)(x)P(g)(x) &= \sum_{m\geqslant 1} f(x+m) \sum_{n\geqslant 1} g(x+n) \\ &= \sum_{n>m\geqslant 1} f(x+m)g(x+n) + \sum_{m>n\geqslant 1} f(x+m)g(x+n) + \sum_{m\geqslant 1} f(x+m)g(x+m) \\ &= P(fP(g))(x) + P(gP(f))(x) + P(fg)(x). \end{aligned} \tag{3}$$

Thus P is a Rota-Baxter operator of weight 1.

Example 2.3. (The partial sum operator) The operator P defined on sequences $\sigma : \mathbb{N} \to \mathbb{C}$ by:

$$P(\sigma)(n) = \sum_{k=0}^{n} \sigma(k) \tag{4}$$

satisfies the Rota-Baxter relation with weight -1. Similarly, the operator $Q = P - Id$ which acts on sequences $\sigma : \mathbb{N} \to \mathbb{C}$ by:

$$Q(\sigma)(n) = \sum_{k=0}^{n-1} \sigma(k) \tag{5}$$

satisfies the Rota-Baxter relation with weight 1.

Example 2.4. (Laurent series) Let $A = \mathbf{k}[\varepsilon^{-1}, \varepsilon]]$ be the algebra of Laurent series. Define $\Pi : A \to A$ by

$$\Pi(\sum_n a_n \varepsilon^n) = \sum_{n<0} a_n \varepsilon^n.$$

Then Π is a Rota-Baxter operator of weight -1.

2.2. **Shuffles, quasi-shuffles and mixable shuffles.** We briefly recall the construction of shuffle, stuffle and quasi-shuffle products in the framework of mixable shuffle algebras [32, 33].

Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra *that is not necessarily unitary*. For a given $\lambda \in \mathbf{k}$, the **mixable shuffle algebra of weight** λ **generated by** A (with coefficients in $\mathbf{k}$) is $\mathrm{MS}(A) = \mathrm{MS}_{\mathbf{k},\lambda}(A)$ whose underlying $\mathbf{k}$-module is that of the tensor algebra

$$T(A) = \bigoplus_{k \geq 0} A^{\otimes k} = \mathbf{k} \oplus A \oplus A^{\otimes 2} \oplus \cdots \tag{6}$$

equipped with the **mixable shuffle product** $\diamond_\lambda$ **of weight** λ defined as follows.

For pure tensors $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n \in A^{\otimes n}$, a **shuffle** of $\mathfrak{a}$ and $\mathfrak{b}$ is a tensor list of a_i and b_j without change the natural orders of the a_is and the b_js. More precisely, for $\sigma \in \Sigma_{k,\ell} := \{\tau \in S_{k+\ell} \mid \tau^{-1}(1) < \cdots < \tau^{-1}(k), \tau^{-1}(k+1) < \cdots < \tau^{-1}(k+\ell)\}$, the shuffle of $\mathfrak{a}$ and $\mathfrak{b}$ by σ is

$$\mathfrak{a} ш_\sigma \mathfrak{b} := c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(k+\ell)}, \qquad \text{where } c_i = \begin{cases} a_i, & 1 \leqslant i \leqslant k, \\ b_{i-k}, & k+1 \leqslant i \leqslant k+\ell \end{cases}$$

The **shuffle product** of $\mathfrak{a}$ and $\mathfrak{b}$ is

$$\mathfrak{a} ш \mathfrak{b} := \sum_{\sigma \in \Sigma_{k,\ell}} \mathfrak{a} ш_\sigma \mathfrak{b}.$$

More generally, for a fixed $\lambda \in \mathbf{k}$, a **mixable shuffle** (of weight λ) of $\mathfrak{a}$ and $\mathfrak{b}$ is a shuffle of $\mathfrak{a}$ and $\mathfrak{b}$ in which some (or *none*) of the pairs $a_i \otimes b_j$ are merged into $\lambda\, a_i b_j$. Then the **mixable shuffle product of weight** λ is defined by

$$\mathfrak{a} \diamond_\lambda \mathfrak{b} = \sum \text{mixable shuffles of } \mathfrak{a} \text{ and } \mathfrak{b} \tag{7}$$

where the subscript λ is often suppressed when there is no danger of confusion. For example,

$$a_1 \diamond_\lambda (b_1 \otimes b_2) := \underbrace{a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1}_{\text{shuffles}} + \underbrace{\lambda(a_1 b_1) \otimes b_2 + \lambda b_1 \otimes (a_1 b_2)}_{\text{merged shuffles}}.$$

With $\mathbf{1} \in \mathbf{k}$ as the unit, this product makes $T(A)$ into a commutative $\mathbf{k}$-algebra that we denote by $\mathrm{MS}_{\mathbf{k},\lambda}(A)$. See [32] for further details on the mixable shuffle product. When $\lambda = 0$, we simply have the shuffle product which is also defined when A is only a $\mathbf{k}$-module, treated as an algebra with the zero multiplication.

We have the following relation between mixable shuffle product and free commutative Rota-Baxter algebras. A Rota-Baxter algebra homomorphism $f : (R, P) \to (R', P')$ between Rota-Baxter $\mathbf{k}$-algebras (R, P) and (R', P') is a $\mathbf{k}$-algebra homomorphism $f : R \to R'$ such that $f \circ P = P' \circ f$.

Theorem 2.5. *([32]) The tensor product algebra $Ш(A) := Ш_{\mathbf{k},\lambda}(A) = A \otimes \mathrm{MS}_{\mathbf{k},\lambda}(A)$, with the linear operator $P_A : Ш(A) \to Ш(A)$ sending $\mathfrak{a} \to 1 \otimes \mathfrak{a}$, is the free commutative Rota-Baxter algebra generated by A in the following sense. Let $j_A : A \to Ш(A)$ be the canonical inclusion map. Then for any Rota-Baxter **k**-algebra (R, P) and any **k**-algebra homomorphism $\varphi : A \to R$, there exists a unique Rota-Baxter **k**-algebra homomorphism $\tilde{\varphi} : (Ш(A), P_A) \to (R, P)$ such that $\varphi = \tilde{\varphi} \circ j_A$ as **k**-algebra homomorphisms.*

The product $\diamond_\lambda$ can also be defined by the following recursion [18, 37, 55] which provides the connection between mixable shuffle algebras and quasi-shuffle algebras of Hoffman [42]. First we define the multiplication by $A^{\otimes 0} = \mathbf{k}$ to be the scalar product. In particular, $\mathbf{1}$ is the identity. For any $m, n \geqslant 1$ and $\mathfrak{a} := a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$, $\mathfrak{b} := b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$, define $a \diamond_\lambda b$ by induction on the sum $m + n \geqslant 2$. When $m + n = 2$, we have $a = a_1$ and $b = b_1$. We define

$$a \diamond_\lambda b = a_1 \otimes b_1 + b_1 \otimes a_1 + \lambda a_1 b_1.$$

Assume that $\mathfrak{a} \diamond_\lambda \mathfrak{b}$ has been defined for $m + n \geqslant k \geqslant 2$ and consider $\mathfrak{a}$ and $\mathfrak{b}$ with $m + n = k + 1$. Then $m + n \geqslant 3$ and so at least one of m and n is greater than 1. We define

$$\mathfrak{a} \diamond_\lambda \mathfrak{b} = \begin{cases} a_1 \otimes b_1 \otimes \cdots \otimes b_n + b_1 \otimes (a_1 \diamond_\lambda (b_2 \otimes \cdots \otimes b_n)) \\ \qquad + \lambda (a_1 b_1) \otimes b_2 \otimes \cdots \otimes b_n, \ \text{when } m = 1, n \geqslant 2, \\ a_1 \otimes ((a_2 \otimes \cdots \otimes a_m) \diamond_\lambda b_1) + b_1 \otimes a_1 \otimes \cdots \otimes a_m \\ \qquad + \lambda (a_1 b_1) \otimes a_2 \otimes \cdots \otimes a_m, \ \text{when } m \geqslant 2, n = 1, \\ a_1 \otimes ((a_2 \otimes \cdots \otimes a_m) \diamond_\lambda (b_1 \otimes \cdots \otimes b_n)) + b_1 \otimes ((a_1 \otimes \cdots \otimes a_m) \diamond_\lambda (b_2 \otimes \cdots \otimes b_n)) \\ \qquad + \lambda (a_1 b_1)((a_2 \otimes \cdots \otimes a_m) \diamond_\lambda (b_2 \otimes \cdots \otimes b_n)), \ \text{when } m, n \geqslant 2. \end{cases}$$

Here the products by $\diamond_\lambda$ on the right hand side of the equation are well-defined by the induction hypothesis.

Let S be a semigroup and let $\mathbf{k}S = \sum_{s \in S} \mathbf{k}\, s$ be the semigroup nonunitary **k**-algebra. A canonical **k**-basis of $(\mathbf{k}S)^{\otimes k}$, $k \geqslant 0$, is the set $S^{\otimes k} := \{s_1 \otimes \cdots \otimes s_k \mid s_i \in S, 1 \leqslant i \leqslant k\}$. Let S be a graded semigroup $S = \coprod_{i \geqslant 0} S_i$, $S_i S_j \subseteq S_{i+j}$ such that $|S_i| < \infty$, $i \geqslant 0$. Then the mixable shuffle product $\diamond_1$ of weight 1 is identified with the **quasi-shuffle product** $*$ defined by Hoffman [42, 18, 37].

Notation 2.6. To simplify the notation and to be consistent with the usual notations in the literature on multiple zeta values, we will identify $s_1 \otimes \cdots \otimes s_k$ with the concatenation $s_1 \cdots s_k$ unless there is a danger of confusion. We also denote the weight 1 mixable shuffle product $\diamond_1$ by $*$ and denote the corresponding mixable algebra $\mathrm{MS}_{\mathbf{k},1}(A)$ by $\mathcal{H}^*_A$. Similarly, when A is taken to be a **k**-module, we denote the weight zero mixable shuffle algebra $\mathrm{MS}_{\mathbf{k},0}(A)$ by $\mathcal{H}^{Ш}_A$.

Yet another interpretation of the mixable shuffle or quasi-shuffle product can be given in terms of order preserving maps that are called **stuffle** in the study of MZVs but could be traced back to Cartier's work [12] on free commutative Rota-Baxter algebras.

For positive integers k and ℓ, denote $[k] = \{1, \cdots, k\}$ and $[k+1, k+\ell] = \{k+1, \cdots, k+\ell\}$. Define

$$\mathfrak{I}_{k,\ell} = \left\{ (\varphi, \psi) \,\middle|\, \begin{array}{l} \varphi : [k] \to [k+\ell], \psi : [\ell] \to [k+\ell] \text{ are order preserving} \\ \text{injective maps and } \mathrm{im}(\varphi) \sqcup \mathrm{im}(\psi) = [k+\ell] \end{array} \right\} \tag{8}$$

Let $\mathfrak{a} \in A^{\otimes k}$, $\mathfrak{b} \in A^{\otimes \ell}$ and $(\varphi, \psi) \in \mathfrak{I}_{k,\ell}$. We define $\mathfrak{a} Ш_{(\varphi,\psi)} \mathfrak{b}$ to be the tensor whose i-th factor is

$$(\mathfrak{a} Ш_{(\varphi,\psi)} \mathfrak{b})_i = \begin{cases} a_j & \text{if } i = \varphi(j) \\ b_j & \text{if } i = \psi(j) \end{cases} = a_{\varphi^{-1}(i)} b_{\psi^{-1}(i)}, \quad 1 \leqslant i \leqslant k + \ell, \tag{9}$$

with the convention that $a_\emptyset = b_\emptyset = 1$. Then we have

$$\mathfrak{a}ш\mathfrak{b} = \sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell}} \mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}. \tag{10}$$

More generally, for $0 \leqslant r \leqslant \min(k,\ell)$, define

$$\mathfrak{J}_{k,\ell,r} = \left\{(\varphi,\psi) \;\middle|\; \begin{array}{l}\varphi : [k] \to [k+\ell-r], \psi : [\ell] \to [k+\ell-r] \text{ are order preserving} \\ \text{injective maps and } \operatorname{im}(\varphi) \cup \operatorname{im}(\psi) = [k+\ell-r]\end{array}\right\}$$

Clearly, $\mathfrak{J}_{k,\ell,0} = \mathfrak{J}_{k,\ell}$. Let $\mathfrak{a} \in A^{\otimes k}$, $\mathfrak{b} \in A^{\otimes \ell}$ and $(\varphi,\psi) \in \mathfrak{J}_{k,\ell,r}$. We define $\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}$ to be the tensor whose i-th factor is

$$(\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b})_i = \left\{\begin{array}{ll} a_j & \text{if } i = \varphi(j), i \notin \operatorname{im}\psi \\ b_j & \text{if } i = \psi(j), i \notin \operatorname{im}\varphi \\ a_j b_{j'} & \text{if } i = \varphi(j), i = \psi(j') \end{array}\right\} = a_{\varphi^{-1}(i)} b_{\psi^{-1}(i)}, \quad 1 \leqslant i \leqslant k+\ell-r,$$

with the convention that $a_\emptyset = b_\emptyset = 1$. Then we have [32, 36]

$$\mathfrak{a} \diamond_\lambda \mathfrak{b} = \sum_{r=0}^{\min(k,\ell)} \lambda^r \Big(\sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell,r}} \mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}\Big). \tag{11}$$

In particular,

$$\mathfrak{a} * \mathfrak{b} = \sum_{r=0}^{\min(k,\ell)} \Big(\sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell,r}} \mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}\Big) = \sum_{(\varphi,\psi)\in\overline{\mathfrak{J}}_{k,\ell}} \mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}$$

where $\overline{\mathfrak{J}}_{k,\ell} = \cup_{r=0}^{\min(k,\ell)} \mathfrak{J}_{k,\ell,r}$.

Equivalently, let $\operatorname{stfl}(k,\ell,r)$ denote the set of surjective maps from $[k+\ell]$ to $[k+\ell-r]$ that preserve the natural orders of $[k]$ and $\{k+1,\cdots,k+\ell\}$. Let

$$\operatorname{stfl}(k,\ell) = \bigcup_{r=0}^{\min(k,\ell)} \operatorname{stfl}(k,\ell,r).$$

Then

$$(a_1 \otimes \cdots \otimes a_k) * (a_{k+1} \otimes \cdots \otimes a_{k+\ell}) = \sum_{\pi\in\operatorname{stfl}(k,\ell)} c_1^\pi \otimes \cdots \otimes c_{k+\ell-r}^\pi, \quad c_i^\pi = \prod_{j\in\pi^{-1}(i)} a_j. \tag{12}$$

A **connected filtered Hopf algebra** is a Hopf algebra (H,Δ) with $\mathbf{k}$-submodules $H^{(n)}$, $n \geqslant 0$ of H such that

$$H^{(n)} \subseteq H^{(n+1)}, \quad \cup_{n\geqslant 0} H^{(n)} = H, \quad H^{(p)}H^{(q)} \subseteq H^{(p+q)},$$
$$\Delta(H^{(n)}) \subseteq \textstyle\sum_{p+q=n} H^{(p)} \otimes H^{(q)}, \quad H^{(0)} = \mathbf{k} \text{ (connectedness)}.$$

On the algebra $\mathrm{MS}_{\mathbf{k},\lambda}(A)$ further define

$$\begin{aligned}\Delta(a_1 \otimes \cdots \otimes a_n) &= 1 \bigotimes (a_1 \otimes \cdots \otimes a_n) + a_1 \bigotimes (a_2 \otimes \cdots \otimes a_n) \\ &\quad + \cdots + (a_1 \otimes \cdots a_{n-1}) \bigotimes a_n + (a_1 \otimes \cdots \otimes a_n) \otimes 1.\end{aligned} \tag{13}$$

Then Δ extends by linearity to a linear map $\mathrm{MS}_{\mathbf{k},\lambda}(A) \to \mathrm{MS}_{\mathbf{k},\lambda}(A) \bigotimes \mathrm{MS}_{\mathbf{k},\lambda}(A)$.

Theorem 2.7. *([37, 42, 55]) The triple* $(\mathrm{MS}_{\mathbf{k},\lambda}(A), \diamond_\lambda, \Delta)$, *together with the unit* $u : \mathbf{k} \hookrightarrow \mathrm{MS}_{\mathbf{k},\lambda}(A)$ *and the counit* $\varepsilon : \mathrm{MS}_{\mathbf{k},\lambda}(A) \to \mathbf{k}$ *projecting onto the direct summand* $\mathbf{k} \subseteq \mathrm{MS}_{\mathbf{k},\lambda}(A)$, *equips* $\mathrm{MS}_{\mathbf{k},\lambda}(A)$ *with the structure of a connected filtered Hopf algebra with the filtration* $\mathrm{MS}(A)^{(n)} := \sum_{i\leqslant n} A^{\otimes i}$.

We also have the following easy extension of Hoffman's isomorphism between the shuffle Hopf algebra and the quasi-shuffle Hopf algebra (see also [18]). Recall the notation $\mathcal{H}^*_A = \mathrm{MS}_{\mathbf{k},1}(A)$ and $\mathcal{H}^{Ш}_A = \mathrm{MS}_{\mathbf{k},0}(A)$.

Theorem 2.8. ([42, 55]) *Let* $\mathbf{k}$ *be a* $\mathbb{Q}$*-algebra. There is an isomorphism of Hopf algebras :*

$$\exp : \mathcal{H}^{Ш}_A \xrightarrow{\sim} \mathcal{H}^*_A. \tag{14}$$

Hoffman's isomorphism (14) is built explicitly as follows. Let $\mathcal{P}(n)$ be the set of compositions of the integer n, i.e. the set of sequences $I = (i_1, \ldots, i_k)$ of positive integers such that $i_1+\cdots+i_k = n$. For any $u = v_1 \otimes \cdots \otimes v_n \in T(A)$ and any composition $I = (i_1, \ldots, i_k)$ of n we set:

$$I[u] := (v_1 \cdot \cdots \cdot v_{i_1}) \otimes (v_{i_1+1} \cdot \cdots \cdot v_{i_1+i_2}) \otimes \cdots \otimes (v_{i_1+\cdots+i_{k-1}+1} \cdot \cdots \cdot v_n).$$

Then the isomorphism exp is defined by

$$\exp u = \sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)} \frac{1}{i_1!\cdots i_k!} I[u].$$

Moreover ([42], Lemma 2.4), the inverse log of exp is given by :

$$\log u = \sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)} \frac{(-1)^{n-k}}{i_1\cdots i_k} I[u].$$

2.3. **Double shuffle of MZVs and related conjectures.** A multiple zeta value (MZV) is defined to be

$$\zeta(s_1, \cdots, s_k) := \sum_{n_1>\cdots>n_k\geqslant 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} \tag{15}$$

where $s_i \geqslant 1$ and $s_1 > 1$ are integers. As is well-known, an MZV has an integral representation due to Kontsevich [51]

$$\zeta(s_1, \cdots, s_k) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{|\vec{s}|-1}} \frac{dt_1}{f_1(t_1)} \cdots \frac{dt_{|\vec{s}|}}{f_{|\vec{s}|}(t_{|\vec{s}|})} \tag{16}$$

Here $|\vec{s}| = s_1 + \cdots + s_k$ and

$$f_j(t) = \begin{cases} 1 - t_j, & j = s_1, s_1 + s_2, \cdots, s_1 + \cdots + s_k, \\ t_j, & \text{otherwise.} \end{cases}$$

The MZVs spanned the following $\mathbb{Q}$-subspace of $\mathbb{R}$

$$\mathbf{MZV} := \mathbb{Q}\{\zeta(s_1, \cdots, s_k) \mid s_i \geqslant 1, s_1 \geqslant 2\} \subseteq \mathbb{R}.$$

Since the summation operator in Eq. (15) and the integral operator in Eq. (2) are both Rota-Baxter operators (of weight 1 and 0 respectively) by Example 2.2 and Example 2.1, it can be expected that the multiplication of two MZVs follows the multiplication rule in a free Rota-Baxter algebra and thus in a mixable shuffle algebra. This viewpoint naturally leads to the following double shuffle relations of MZVs.

For the sum representation of MZVs in Eq. (15), consider the semigroup

$$Z := \{z_s \mid s \in \mathbb{Z}_{\geqslant 1}, \quad z_s \cdot z_t = z_{s+t}, s, t \geqslant 1.\}$$

With the convention in Notation 2.6, we denote the quasi-shuffle algebra $\mathcal{H}^* := \mathcal{H}^*_{\mathbb{Q}Z}$ which contains the subalgebra

$$\mathcal{H}^*_0 := \mathbb{Q} \oplus \Big(\bigoplus_{s_1>1} \mathbb{Q} z_{s_1} \cdots z_{s_k} \Big).$$

Then the multiplication rule of MZVs according to their summation representation follows from the fact that the linear map

$$\zeta^* : \mathcal{H}^*_0 \to \mathbf{MZV}, \quad z_{s_1,\cdots,s_k} \mapsto \zeta(s_1, \cdots, s_k) \tag{17}$$

is an algebra homomorphism [41, 46].

For the integral representation of MZVs in Eq. (16), consider the set $X = \{x_0, x_1\}$. With the convention in Notation 2.6, we denote the shuffle algebra $\mathcal{H}^{Ш} := \mathcal{H}^{Ш}_{\mathbb{Q}X}$ which contains subalgebras

$$\mathcal{H}^{Ш}_0 := \mathbb{Q} \oplus x_0 \mathcal{H}^{Ш} x_1 \subseteq \mathcal{H}^{Ш}_1 := \mathbb{Q} \oplus \mathcal{H}^{Ш} x_1 \subseteq \mathcal{H}^{Ш}.$$

Then the multiplication rule of MZVs according to their integral representations follows from the statement that the linear map

$$\zeta^{Ш} : \mathcal{H}^{Ш}_0 \to \mathbf{MZV}, \quad x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \mapsto \zeta(s_1, \cdots, s_k)$$

is an algebra homomorphism [41, 46].

There is a natural bijection of $\mathbb{Q}$-vector spaces (but *not* algebras)

$$\eta : \mathcal{H}^{Ш}_1 \to \mathcal{H}^*, \quad 1 \leftrightarrow 1, \ x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \leftrightarrow z_{s_1,\cdots,s_k}.$$

that restricts to a bijection of vector spaces $\eta : \mathcal{H}^{Ш}_0 \to \mathcal{H}^*_0$. Then the fact that MZVs can be multiplied in two ways is reflected by the commutative diagram

$$\begin{tikzcd}
\mathcal{H}^{Ш}_0 \arrow[rr, "\eta"] \arrow[dr, "\zeta^{Ш}"'] & & \mathcal{H}^*_0 \arrow[dl, "\zeta^*"] \\
& \mathbf{MZV} &
\end{tikzcd}$$

Through η, the shuffle product $Ш$ on $\mathcal{H}^{Ш}_1$ and $\mathcal{H}^{Ш}_0$ transports to a product $Ш_*$ on $\mathcal{H}^*$ and $\mathcal{H}^*_0$. That is, for $w_1, w_2 \in \mathcal{H}^*_0$, define

$$w_1 Ш_* w_2 := \eta(\eta^{-1}(w_1) Ш \eta^{-1}(w_2)). \tag{18}$$

Then the **double shuffle relation** is simply the set

$$\{w_1 Ш_* w_2 - w_1 * w_2 \mid w_1, w_2 \in \mathcal{H}^*_0\}$$

and the **extended double shuffle relation** [46, 63, 75] is the set

$$\{w_1 Ш_* w_2 - w_1 * w_2, \ z_1 Ш_* w_2 - z_1 * w_2 \mid w_1, w_2 \in \mathcal{H}^*_0\}. \tag{19}$$

Theorem 2.9. **(**[41, 46, 63]**)** *Let $I_{\mathbf{EDS}}$ be the ideal of $\mathcal{H}^*_0$ generated by the extended double shuffle relation in Eq. (19). Then $I_{\mathbf{EDS}}$ is in the kernel of ζ^*.*

It is conjectured that $I_{\mathbf{EDS}}$ is in fact the kernel of ζ^*. A consequence of this conjecture is the irrationality of $\zeta(2n+1), n \geqslant 1$.

3. Generalizations of Euler's formulas

We begin with stating Euler's sum and decomposition formulas in Section 3.1. Generalizations of Euler's sum formula are presented in Section 3.2 and generalizations of Euler's decomposition formula are presented in Section 3.3.

3.1. **Euler's sum and decomposition formulas.** Over two hundred years before the general study of multiple zeta values was started in the 1990s, Goldback and Euler had already considered the two variable case, the double zeta values [23, 66]

$$\zeta(s_1, s_2) := \sum_{n_1 > n_2 \geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2}}.$$

Among Euler's major discoveries on double zeta values are his **sum formula**

$$\sum_{i=2}^{n-1} \zeta(i, n-i) = \zeta(n)$$

expressing one Riemann zeta values as a sum of double zeta values and the **decomposition formula**

$$\zeta(r)\zeta(s) = \sum_{k=0}^{s-1} \binom{r+k-1}{k} \zeta(r+k, s-k) + \sum_{k=0}^{r-1} \binom{s+k-1}{k} \zeta(s+k, r-k), \quad r, s \geqslant 2, \tag{20}$$

expressing the product of two Riemann zeta values as a sum of double zeta values.

A major aspect of the study of MZVs is to find algebraic and linear relations among MZVs, such as Euler's formulas. Indeed a large part of this study can be viewed as generalizations of Euler's formulas.

3.2. **Generalizations of Euler's sum formula.** Soon after MZVs were introduced, Euler's sum formula was generalized to MZVs [40, 28, 71] as the well-known sum formula, followed by quite a few other generalizations that we will next summarize.

3.2.1. *Sum formula.* The first generalization of Euler's sum formula is the sum formula conjectured in [40]. Let

$$I(n,k) = \{(s_1, \cdots, s_k) \mid s_1 + \cdots + s_k = n, s_i \geqslant 1, s_1 \geqslant 2\}. \tag{21}$$

For $\vec{s} = (s_1, \cdots, s_k) \in I(n,k)$, define the **multiple zeta star value (or non-strict MZV)**

$$\zeta^\star(s_1, \cdots, s_k) = \sum_{n_1 \geqslant \cdots \geqslant n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}. \tag{22}$$

Note the subtle different between the notations ζ^* in Eq. (17) and $\zeta^\star$ in Eq. (22).

Theorem 3.1. (**Sum formula**) *For positive integers $k < n$ we have*

$$\sum_{\vec{s} \in I(n,k)} \zeta(\vec{s}) = \zeta(n), \qquad \sum_{\mathbf{k} \in I(n,k)} \zeta^\star(\vec{s}) = \binom{n-1}{k-1} \zeta(n). \tag{23}$$

The case of $k = 3$ was proved by M. Hoffman and C. Moen [43] and the general case was proved by Zagier [71] with another proof given by Granville [28]. Later S. Kanemitsu, Y. Tanigawa, M. Yoshimoto [48] gave a proof for the case of $k = 2$ using Mellin transformation.

J.-I. Okuda and K. Ueno [62] gave the following version of the sum formula

$$\sum_{k=r}^{n} \binom{k-1}{r-1} \Big(\sum_{\vec{s} \in I(n,k)} \zeta(\vec{s}) \Big) = \binom{n-1}{r} \zeta(n)$$

for $n > r \geqslant 1$ from which they deduced the sum formula Eq. (23).

3.2.2. *Ohno's generalized duality theorem.* Another formula conjectured in [40] is the **duality formula**. To state the duality formula, we need an involution τ on the set of finite sequences of positive integers whose first element is greater than 1. If

$$\vec{s} = (1 + b_1, \underbrace{1, \cdots, 1}_{a_1-1}, \cdots, 1 + b_k, \underbrace{1, \cdots, 1}_{a_k-1}),$$

then

$$\tau(\vec{s}) = (1 + a_k, \underbrace{1, \cdots, 1}_{b_k-1}, \cdots, 1 + a_1, \underbrace{1, \cdots, 1}_{b_1-1}).$$

Theorem 3.2. (Duality formula)

$$\zeta(\vec{s}) = \zeta(\tau(\vec{s})).$$

This formula is an immediate consequence of the integral representation in Eq. (16).

Y. Ohno [57] provided a generalization of both the sum formula and the duality formula.

Theorem 3.3. (**Generalized Duality Formula** [57]) *For any index set* $\vec{s} = (s_1, \cdots, s_k)$ *with* $s_1 \geqslant 2, s_2 \geqslant 1, \cdots, s_k \geqslant 1$, *and a nonnegative integer* ℓ, *set*

$$Z(s_1, \cdots, s_k; \ell) = \sum_{\substack{c_1 + \cdots + c_k = \ell \\ c_i \geqslant 0}} \zeta(s_1 + c_1, \cdots, s_k + c_k).$$

Then

$$Z(\vec{s}; \ell) = Z(\tau(\vec{s}); \ell).$$

When $\ell = 0$, this is just the duality formula. When $\vec{s} = (k + 1)$ and $\ell = n - k - 1$, this becomes the sum formula.

3.2.3. *Sum formulas with further conditions on the variables.* M. Hoffman and Y. Ohno [44] gave a cyclic generalization of the sum formula.

Theorem 3.4. (Cyclic sum formula) *For any positive integers* $s_1, \cdots, s_k$ *with some* $s_i \geqslant 2$,

$$\sum_{i=1}^{k} \zeta(s_i + 1, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}) = \sum_{\{i \mid s_i \geqslant 2\}} \sum_{j=0}^{s_i-2} \zeta(s_i - j, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}, j + 1).$$

Y. Ohno and N. Wakabayashi [59] gave a cyclic sum formula for non-strict MZVs and used it to prove the sum formula Eq. (23).

Theorem 3.5. (**Cyclic sum formula in the non-strict case**) *For positive integers $k < n$ and $(s_1, \cdots, s_k) \in I(n, k)$ we have*

$$\sum_{i=1}^{k} \sum_{j=0}^{s_i-2} \zeta^\star(s_i - j, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}, j+1) = n\zeta(n+1), \tag{24}$$

where the empty sums are zero.

M. Eie, W.-C. Liaw and Y. L. Ong [22] gave a generalization of the sum formula by allowing a more general form in the arguments in the MZVs.

Theorem 3.6. *For all positive integers n, k with $n > k$, and a nonnegative integer p,*

$$\sum_{\substack{s_1+\cdots+s_k=n\\ s_1\geqslant 2}} \zeta(s_1, \cdots, s_k, \{1\}^p) = \sum_{\substack{c_1+\cdots+c_{p+1}=n+p\\ c_1\geqslant n-k+1}} \zeta(c_1, \cdots, c_{p+1}).$$

When $p = 0$, this becomes the sum formula.

Y. Ohno and D. Zagier [60] studied another kind of sum with certain restrictive conditions. Let

$$I(n, k, r) = \{(s_1, \cdots, s_k) \mid s_i \in \mathbb{Z}_{\geqslant 1}, s_1 + \cdots + s_k = n, \#\{s_i \mid s_i \geqslant 2\} = r\}$$

and put

$$G(n, k, r) = \sum_{\vec{s}\in I(n,k,r)} \zeta(\vec{s}).$$

They studied the associated generating function

$$\Phi(x, y, z) = \sum_{r\geqslant 1, k\geqslant r, n\geqslant k+r} G(n, k, r) x^{n-k-r} y^{k-r} z^{r-1} \in \mathbb{R}[x, y, z]$$

and proved the following

Theorem 3.7. *We have*

$$\Phi(x, y, z) = \frac{1}{xy - z}\left(1 - \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} S_n(x, y, z)\right)\right),$$

where $S_n(x, y, z)$ are given by the identity

$$\log\left(1 - \frac{xy - z}{(1-x)(1-y)}\right) = \sum_{n=2}^{\infty} \frac{S_n(x, y, z)}{n}$$

and the requirement that $S_n(x, y, z^2)$ is a homogeneous polynomial of degree n. In particular, all of the coefficients $G(n, k, r)$ can be expressed as polynomials in $\zeta(2), \zeta(3), \ldots$ with rational coefficients.

3.2.4. *Sum formulas for q-MZVs.* The concept of q-multiple zeta values (q-MZVs, or multiple q-zeta values) was introduced as a "quantumization" of MZVs that recovers MZVs when $q \mapsto 1$ [9, 72].

For positive integers $s_1, \cdots, s_k$ with $s_1 \geqslant 2$, define the q-MZV

$$\zeta_q(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k \geqslant 1} \frac{q^{n_1(s_1-1)+\cdots+n_k(s_k-1)}}{[n_1]^{s_1} \cdots [n_k]^{s_k}}$$

and the non-strict q-MZV

$$\zeta_q^\star(s_1, \cdots, s_k) = \sum_{n_1 \geqslant \cdots \geqslant n_k \geqslant 1} \frac{q^{n_1(s_1-1)+\cdots+n_k(s_k-1)}}{[n_1]^{s_1} \cdots [n_k]^{s_k}},$$

where $[n] = \frac{1-q^n}{1-q}$.

D. M. Bradley [9] proved the q-analogue of the sum formula for ζ_q.

Theorem 3.8. (**q-analogue of the sum formula**) *For positive integers $0 < k < n$ we have*

$$\sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1+\cdots+s_k=n}} \zeta_q(s_1, \cdots, s_k) = \zeta_q(n). \tag{25}$$

Y. Ohno and J.-I. Okuda [58] gave the following q-analogue of the cyclic sum formula (24) and then used it to prove a q-analogue of the sum formula for $\zeta_q^\star$.

Theorem 3.9. (**q-analogue of the cyclic sum formula**) *For positive integers $0 < k < n$ and $(s_1, \cdots, s_k) \in I(n,k)$ we have*

$$\sum_{i=1}^{k} \sum_{j=0}^{s_i-2} \zeta_q^\star(s_i - j, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}, j+1) = \sum_{\ell=0}^{k} (n-\ell) \binom{k}{\ell} (1-q)^\ell \zeta_q(n-\ell+1),$$

where the empty sums are zero.

Theorem 3.10. (**q-analogue of the sum formula in the non-strict case**) *For positive integers $0 < k < n$ we have*

$$\sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1+\cdots+s_k=n}} \zeta_q^\star(s_1, \cdots, s_k) = \frac{1}{n-1} \binom{n-1}{k-1} \sum_{\ell=0}^{k-1} (n-1-\ell)(1-q)^\ell \zeta_q(n-\ell). \tag{26}$$

3.2.5. *Weighted sum formulas.* In the other direction to generalize Euler's sum formula, there is the weighted version of Euler's sum formula recently obtained by Ohno and Zudilin [61].

Theorem 3.11. (**Weighted Euler's sum formula** [61]) *For any integer $n \geqslant 2$, we have*

$$\sum_{i=2}^{n-1} 2^i \zeta(i, n-i) = (n+1)\zeta(n). \tag{27}$$

They applied it to study multiple zeta star values. By the sum formula, Eq. (27) is equivalent to the following equation

$$\sum_{i=2}^{n-1} (2^i - 1)\zeta(i, n-i) = n\zeta(n). \tag{28}$$

As a generalization of Eq. (28), two of the authors proved the following

Theorem 3.12. (**Weighted sum formula** [35]) *For positive integers $k \geqslant 2$ and $n \geqslant k+1$, we have*

$$\sum_{\substack{s_i\geqslant 1, s_1\geqslant 2\\ s_1+\cdots+s_k=n}} \Big[2^{s_1-1} + (2^{s_1-1}-1)\Big(\big(\sum_{i=2}^{k-1} 2^{S_i-s_1-(i-1)}\big) + 2^{S_{k-1}-s_1-(k-2)}\Big)\Big]\zeta(s_1,\cdots,s_k) = n\zeta(n),$$

where $S_i = s_1 + \cdots + s_i$ for $i = 1, \cdots, k-1$.

3.3. **Generalizations of Euler's decomposition formula.** Unlike the numerous generalizations of Euler's sum formula, no generalization of Euler's decomposition formula to MZVs, neither proved nor conjectured, had been given until [34] even though Euler's decomposition formula was recently revisited in connection with modular forms [24] and weighted sum formula [61] on weighted sum formula of double zeta values, and was generalized to the product of two q-zeta values [10, 72].

3.3.1. *Euler's decomposition formula and double shuffle.* The first step in generalizing Euler's decomposition formula is to place it as a special case in a suitable broader context. In [34], Euler's decomposition formula was shown to be a special case of the double shuffle relation. We give a proof of Euler's formula in this context before presenting its generalization in the next subsection.

We recall that the extended double shuffle relation is the set

$$\{w_1 ш_* w_2 - w_1 * w_2,\ z_1 ш_* w_2 - z_1 * w_2 \mid w_1, w_2 \in \mathcal{H}_0^*\}.$$

Thus the determination of the double shuffle relation amounts to computing the two products $*$ and $ш_*$.

It is straightforward to compute the product $*$, either from its recursive definition in Eq. (8) or its explicit interpretation as mixable shuffles in Eq. (7) and stuffles in Eq. (11) or (12). For example, to determine the double shuffle relation from multiplying two Riemann zeta values $\zeta(r)$ and $\zeta(s)$, $r, s \geqslant 2$, one uses their sum representations and easily gets the quasi-shuffle relation

$$\zeta(r)\zeta(s) = \zeta(r,s) + \zeta(s,r) + \zeta(r+s).$$

On the other hand, computing the product $ш_*$ is more involved as can already be seen from its definition in Eq. (18). One first needs to use their integral representations to express $\zeta(r)$ and $\zeta(s)$ as iterated integrals of dimensions r and s, respectively. One then uses the shuffle relation to express the product of these two iterated integrals as a sum of $\binom{r+s}{r}$ iterated integrals of dimension $r+s$. Finally, these last iterated integrals are translated back to MZVs and give the shuffle relation of $\zeta(r)\zeta(s)$. As an illustrating example, consider $\zeta(100)\zeta(200)$. The quasi-shuffle relation is simply $\zeta(100)\zeta(200) = \zeta(100,200) + \zeta(200,100) + \zeta(300)$, but the shuffle relation is a large sum of $\binom{300}{100}$ shuffles of length (dimension) 300. As we will show below, an explicit formula for this is precisely Euler's decomposition formula (20).

Theorem 3.13. *For $r, s \geqslant 2$, we have*

$$z_r ш_* z_s = \sum_{k=0}^{s-1} \binom{r+k-1}{k} z_{r+k} z_{s-k} + \sum_{k=0}^{r-1} \binom{s+k-1}{k} z_{s+k} z_{r-k}.$$

Via the algebra homomorphism ζ^* in Eq. (17) this theorem immediately gives Euler's decomposition formula. Applying to the above example, we have

$$\zeta(100)\zeta(200) = \sum_{k=0}^{199} \binom{100+k-1}{k} \zeta(100+k, 200-k) + \sum_{k=0}^{99} \binom{200+k-1}{k} \zeta(200+k, 100-k).$$

Proof. Following the definition of $\amalg_*$ in Eq. (18), we have

$$z_r \amalg_* z_s = \eta(x_0^{r-1} x_1 \amalg x_0^{s-1} x_1).$$

So we just need to prove

$$x_0^{r-1} x_1 \amalg x_0^{s-1} x_1 = \sum_{k=0}^{s-1} \binom{r+k-1}{k} x_0^{r+k-1} x_1 x_0^{s-k-1} x_1 + \sum_{k=0}^{r-1} \binom{s+k-1}{k} x_0^{s+k-1} x_1 x_0^{r-k-1} x_1$$

since $\amalg_* (x_0^{r+k-1} x_1 x_0^{s-k-1} x_1) = z_{r+k} z_{s-k}$ and $\amalg_* (x_0^{s+k-1} x_1 x_0^{r-k-1} x_1) = z_{s+k} z_{r-k}$. This has a direct shuffle proof [8]. But we use the description of order preserving maps of shuffles in order to motivate the general case.

By Eq. (10), we have

$$x_0^{r-1} x_1 \amalg x_0^{s-1} x_1 = \sum_{(\varphi,\psi)\in \mathfrak{I}(r,s)} x_0^{r-1} x_1 \amalg_{(\varphi,\psi)} x_0^{s-1} x_0.$$

Since φ and ψ are order preserving, we have the disjoint union $\mathfrak{I}(r, s) = \mathfrak{I}(r, s)' \sqcup \mathfrak{I}(r, s)''$ where

$$\mathfrak{I}(r, s)' = \{(\varphi, \psi) \in \mathfrak{I}(r, s) \mid \psi(s) = r + s\}$$

and

$$\mathfrak{I}(r, s)'' = \{(\varphi, \psi) \in \mathfrak{I}(r, s) \mid \varphi(r) = r + s\}.$$

Again by the order preserving property, for $(\varphi, \psi) \in \mathfrak{I}(r, s)'$, we must have $\varphi(r) = r + k$ where $k \geqslant 0$. Thus for such (φ, ψ), we have

$$x_0^{r-1} x_1 \amalg_{(\varphi,\psi)} x_0^{s-1} x_1 = x_0^{r-1+k} x_1 x_0^{s-1-k} x_1$$

since $\operatorname{im}\varphi \sqcup \operatorname{im}\psi = [r+s]$. For fixed $k \geqslant 0$, $\varphi(r) = r+k$ means that there are k elements $i_1, \cdots, i_k$ from $[s-1]$ such that $\psi(i_j) \in [r+k-1]$ since $\psi(s) = r+s$. Thus $k \geqslant s-1$ and, since ψ is order preserving, we have $\{i_1, \cdots, i_k\} = [k]$. Further there are $\binom{r+k-1}{k}$ such ψ's since $\psi([k])$ can take any k places in $[r+k-1]$ in increasing order and then $\phi([r])$ takes the rest places in increase order. Thus

$$\sum_{(\varphi,\psi)\in \mathfrak{I}(r,s)'} x_0^{r-1} x_1 \amalg_{(\varphi,\psi)} x_0^{s-1} x_1 = \sum_{k=0}^{s-1} \binom{r+k-1}{k} x_0^{r+k-1} x_1 x_0^{s-k-1} x_1.$$

By a similar argument, we have

$$\sum_{(\varphi,\psi)\in \mathfrak{I}(r,s)''} x_0^{r-1} x_1 \amalg_{(\varphi,\psi)} x_0^{s-1} x_1 = \sum_{k=0}^{r-1} \binom{s+k-1}{k} x_0^{s+k-1} x_1 x_0^{r-k-1} x_1.$$

This completes the proof. □

3.3.2. *Generalizations of Euler's decomposition formula.* In a recent work [34], two of the authors generalized Euler's decomposition formula in two directions, from the product of one variable functions to that of multiple variables and from multiple zeta values to multiple polylogarithms.

A **multiple polylogarithm value** [7, 25, 26] is defined by

$$\mathrm{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) := \sum_{n_1>\cdots>n_k\geqslant 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

where $|z_i|\leqslant 1$, $s_i\in\mathbb{Z}_{\geqslant 1}$, $1\leqslant i\leqslant k$, and $(s_1,z_1)\neq(1,1)$. When $z_i=1, 1\leqslant i\leqslant k$, we obtain the multiple zeta values $\zeta(s_1,\cdots,s_k)$. With the notation of [7], we have

$$\mathrm{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) = \lambda\left(\begin{array}{c} s_1,\cdots,s_k \\ b_1,\cdots,b_k\end{array}\right) := \sum_{n_1>n_2\cdots>n_k\geqslant 1} \frac{(\frac{1}{b_1})^{n_1}(\frac{b_1}{b_2})^{n_2}\cdots(\frac{b_{k-1}}{b_k})^{n_k}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}},$$

where $(b_1,\cdots,b_k)=(z_1^{-1},(z_1z_2)^{-1},\cdots,(z_1\cdots z_k)^{-1})$.

To state the result, let k and ℓ be positive integers and let $\mathfrak{I}_{k,\ell}$ be as defined in Eq. (8). Let $\vec{r}=(r_1,\cdots,r_k)\in\mathbb{Z}_{\geqslant 1}^k$, $\vec{s}=(s_1,\cdots,s_\ell)\in\mathbb{Z}_{\geqslant 1}^\ell$ and $\vec{t}=(t_1,\cdots,t_{k+\ell})\in\mathbb{Z}_{\geqslant 1}^{k+\ell}$ with $|\vec{r}|+|\vec{s}|=|\vec{t}|$. Here $|\vec{r}|=r_1+\cdots+r_k$ and similarly for $|\vec{s}|$ and $|\vec{t}|$. Denote $R_i=r_1+\cdots+r_i$ for $i\in[k]$, $S_i=s_1+\cdots+s_i$ for $i\in[\ell]$ and $T_i=t_1+\cdots+t_i$ for $i\in[k+\ell]$. For $(\varphi,\psi)\in\mathfrak{I}_{k,\ell}$ and $i\in[k+\ell]$, define

$$h_{(\varphi,\psi),i}=h_{(\varphi,\psi),(\vec{r},\vec{s}),i}=\begin{cases} r_j & \text{if } i=\varphi(j)\\ s_j & \text{if } i=\psi(j)\end{cases} = r_{\varphi^{-1}(i)}s_{\psi^{-1}(i)},$$

with the convention that $r_\emptyset=s_\emptyset=1$.

With these notations, we define

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)=\begin{cases} \binom{t_i-1}{h_{(\varphi,\psi),i}-1} & \begin{array}{l}\text{if } i=1, \text{if } i-1,i\in\mathrm{im}(\varphi)\\ \text{or if } i-1,i\in\mathrm{im}(\psi),\end{array}\\ \binom{t_i-1}{T_i-R_{|\varphi^{-1}([i])|}-S_{|\psi^{-1}([i])|}} = \binom{t_i-1}{\sum_{j=1}^{i}t_j-\sum_{j=1}^{i}h_{(\varphi,\psi),j}} & \text{otherwise.}\end{cases} \tag{29}$$

For $\vec{a}\in(S^1)^k$ and $\vec{b}\in(S^1)^\ell$, as in Eq. (9), define

$$\vec{a}\,ш_{(\varphi,\psi)}\vec{b}=(a_{\varphi^{-1}(1)}b_{\psi^{-1}(1)},\cdots,a_{\varphi^{-1}(k+\ell)}b_{\psi^{-1}(k+\ell)}). \tag{30}$$

Theorem 3.14. ([34]) *Let k,ℓ be positive integers. Let $\vec{r}\in\mathbb{Z}_{\geqslant 1}^k$ and $\vec{s}\in\mathbb{Z}_{\geqslant 1}^\ell$. Let $\vec{a}=(a_1,\cdots,a_k)\in(S^1)^k$ and $\vec{b}=(b_1,\cdots,b_\ell)\in(S^1)^\ell$ such that $\left[\begin{array}{c} r_1\\ a_1\end{array}\right]\neq\left[\begin{array}{c}1\\1\end{array}\right]$ and $\left[\begin{array}{c} s_1\\ b_1\end{array}\right]\neq\left[\begin{array}{c}1\\1\end{array}\right]$. Then*

$$\lambda\left(\begin{array}{c}\vec{r}\\ \vec{a}\end{array}\right)\lambda\left(\begin{array}{c}\vec{s}\\ \vec{b}\end{array}\right)=\sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|}\ \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}}\Big(\prod_{i=1}^{k+\ell}c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\Big)\lambda\left(\begin{array}{c}\vec{t}\\ \vec{a}\,ш_{(\varphi,\psi)}\vec{b}\end{array}\right).$$

where $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ is given in Eq. (29) and $\vec{a}\,ш_{(\varphi,\psi)}\vec{b}$ is given in Eq. (30).

Corollary 3.15. *Let $\vec{r} \in \mathbb{Z}_{\geqslant 1}^k$ and $\vec{s} \in \mathbb{Z}_{\geqslant 1}^\ell$ with $r_1, s_1 \geqslant 2$. Then*

$$\zeta(\vec{r})\,\zeta(\vec{s}) = \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|} \Big(\sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}} \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\Big)\zeta(\vec{t})$$

where $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ is given in Eq. (29).

4. The algebraic framework of Connes and Kreimer on renormalization

The Algebraic Birkhoff Decomposition of Connes and Kreimer is a fundamental result in their ground breaking work [15] on Hopf algebra approach to renormalization of perturbative quantum field theory (pQFT). This decomposition also links the physics theory of renormalization to Rota-Baxter algebra that has evolved in parallel to the development of QFT renormalization for several decades.

The introduction of Rota-Baxter algebra by G. Baxter [4] in 1960 was motivated by Spitzer's identity [68] that appeared in 1956 and was regarded as a remarkable formula in the fluctuation theory of probability. Soon Atkinson [3] proved a simple yet useful factorization theorem in Rota-Baxter algebras. The identity of Spitzer took its algebraic form through the work of Cartier, Rota and Smith [12, 65] (1972).

It was during the same period when the renormalization theory of pQFT was developed, through the the work of Bogoliubov and Parasiuk [6] (1957), Hepp [39](1966) and Zimmermann [74] (1969), later known as the BPHZ prescription.

Recently QFT renormalization and Rota-Baxter algebra are tied together through the algebraic formulation of Connes and Kreimer for the former and a generalization of classical results on Rota-Baxter algebras in the latter [20, 21]. More precisely, generalizations of Spitzer's identity and Atkinson factorization give the twisted antipode formula and the algebraic Birkhoff decomposition in the work of Connes and Kreimer.

We recall the algebraic Birkhoff decomposition in Section 4.1, prove the Atkinson factorization in Section 4.2 and derive the algebraic Birkhoff decomposition from the Atkinson factorization in Section 4.3.

4.1. Algebraic Birkhoff decomposition.

For a $\mathbf{k}$-algebra A and a $\mathbf{k}$-coalgebra C, we define the convolution of two linear maps f, g in $\mathrm{Hom}(C, A)$ to be the map $f \star g \in \mathrm{Hom}(C, A)$ given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes g} A \otimes A \xrightarrow{m} A.$$

Theorem 4.1. (Algebraic Birkhoff Decomposition) *Let H be a connected filtered Hopf algebra over $\mathbb{C}$. Let (A, Π) be a commutative Rota-Baxter algebra of weight -1 with $\Pi^2 = \Pi$.*

(a) *For $\phi \in \mathrm{char(H, A)}$, there are unique linear maps $\phi_- : H \to \mathbf{k} + \Pi(A)$ and $\phi_+ : H \to \mathbf{k} + (\mathrm{id} - \Pi)(A)$ such that*

$$\phi = \phi_-^{\star(-1)} \star \phi_+. \tag{31}$$

(b) *The elements* ϕ_- *and* ϕ_+ *take the following forms on* $\ker\varepsilon$.

$$\phi_-(x) = -\Pi(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')), \tag{32}$$

$$\phi_+(x) = \tilde{\Pi}(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')), \tag{33}$$

where we have used the notation $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum_{(x)} x' \otimes x''$ *with* $x', x'' \in \ker\varepsilon$.

(c) *The linear maps* ϕ_- *and* ϕ_+ *are also algebra homomorphisms.*

We call ϕ_+ the **renormalization** of ϕ and call ϕ_- the **counter-term**. Here is roughly how the renormalization method can be applied through the Algebraic Birkhoff Decomposition. See the tutorial article [31] for further details, examples and references.

Theorem 4.1 can applied to renormalization as follows. Suppose there is a set of divergent formal expressions, such as MZVs with not necessarily positive arguments, that carries a certain algebraic combinatorial structure and from which we would like to extract finite values. On one hand, we first apply a suitable regularization (deformation) to each of these formal expressions so that the formal expression can be viewed as a singular value of the deformation function. Expanding around the singular point gives a Laurent series in $\mathbf{k}[\varepsilon^{-1}, \varepsilon]]$. On the other hand, the algebraic combinatorial structure of the formal expressions, inherited by the deformation functions, can be abstracted to a free object in a suitable category. This free object parameterizes the deformation functions and often gives a Hopf algebra H. Thus the parametrization gives a morphism $\phi : H \to \mathbf{k}[\varepsilon^{-1}, \varepsilon]]$ in the suitable category. Upon applying the Algebraic Birkhoff Decomposition, we obtain $\phi_+ : H \to \mathbf{k}[[\varepsilon]]$ which, composed with $\varepsilon \mapsto 0$, gives us well-defined values in $\mathbf{k}$.

4.2. **Atkinson factorization.**

The following is the classical result of Atkinson.

Theorem 4.2. (Atkinson Factorization) *Let* (R, P) *be a Rota–Baxter algebra of weight* $\lambda \neq 0$. *Let* $a \in R$. *Assume that* b_ℓ *and* b_r *are solutions of the fixed point equations*

$$b_\ell = 1 + P(b_\ell a), \qquad b_r = 1 + (\mathrm{id}_R - P)(ab_r). \tag{34}$$

Then

$$b_\ell(1 + \lambda a)b_r = 1.$$

Thus

$$1 + \lambda a = b_\ell^{-1} b_r^{-1} \tag{35}$$

if b_ℓ *and* b_r *are invertible.*

We note that the factorization (35) depends on the existence of invertible solutions of Eq. (34) that we will address next.

Definition 4.3. *A* **filtered k-algebra** *is a* $\mathbf{k}$*-algebra* R *together with a decreasing filtration* R_n, $n \geqslant 0$, *of nonunitary subalgebras such that*

$$\bigcup_{n\geqslant 0} R_n = R, \quad R_n R_m \subseteq R_{n+m}.$$

It immediately follows that $R_0 = R$ and each R_n is an ideal of R. A filtered algebra is called **complete** if R is a complete metric space with respect to the metric defined by the subsets $\{R_n\}$. Equivalently, a filtered **k**-algebra R with $\{R_n\}$ is complete if $\cap_n R_n = 0$ and if the resulting embedding

$$R \to \bar{R} := \varprojlim R/R_n$$

of R into the inverse limit is an isomorphism.

A Rota-Baxter algebra (R, P) is called **complete** if there are submodules $R_n \subseteq R, n \geqslant 0$, such that (R, R_n) is a complete algebra and $P(R_n) \subseteq R_n$.

Theorem 4.4. (Existence and uniqueness of the Atkinson factorization) *Let (R, P, R_n) be a complete Rota-Baxter algebra. Let a be in R_1.*

(a) *The equations in (34) have unique solutions b_ℓ and b_r. Further b_ℓ and b_r are invertible. Hence Atkinson Factorization (35) exists.*

(b) *If λ has no non-zero divisors in R_1 and $P^2 = -\lambda P$ (in particular if $P^2 = -\lambda P$ on R), then there are unique $c_\ell \in 1 + P(R)$ and $c_r \in 1 + (\mathrm{id}_R - P)(R)$ such that*

$$1 + \lambda a = c_\ell c_r.$$

4.3. **From Atkinson factorization to algebraic Birkhoff decomposition.** We now derive the Algebraic Birkhoff Decomposition of Connes and Kreimer in Theorem 4.1 from Atkinson Factorization in Theorem 4.4. Adapting the notations in Theorem 4.1, let H be a connected filtered Hopf algebra and let (A, Q) be a commutative Rota-Baxter algebra of weight $\lambda = -1$ with $Q^2 = Q$, such as the pair (A, Q) in Theorem 4.1 (see also Example 2.4). The increasing filtration on H induces a decreasing filtration $R_n = \{f \in \mathrm{Hom}(H, A) \mid f(H^{n-1}) = 0\}, n \geqslant 0$ on $R := \mathrm{Hom}(H, A)$, making it a complete algebra. Further define

$$P : R \to R, \quad P(f)(x) = Q(f(x)), f \in \mathrm{Hom}(H, A), x \in H.$$

Then it is easily checked that P is a Rota-Baxter operator of weight -1 and $P^2 = P$. Thus (R, R_n, P) is a complete Rota-Baxter algebra.

Now let $\phi : H \to A$ be a character (that is, an algebra homomorphism). Consider $e-\phi : H \to A$. Then

$$(e - \phi)(1_H) = e(1_H) - \phi(1_H) = 1_H - 1_H = 0.$$

Thus $e - \phi$ is in R_1. Take $e - \phi$ to be our a in Theorem 4.4, we see that there are unique $c_\ell \in P(R_1)$ and $c_r \in P(R_1)$ such that

$$\phi = c_\ell c_r.$$

Further, by Theorem 4.2, for $b_\ell = c_\ell^{-1}$, $b_\ell = e + P(b_\ell \star (e - \phi))$. Thus for $x \in \ker \varepsilon = \ker e$, we have

$$\begin{aligned}
b_\ell(x) &= P(b_\ell \star (e - \phi))(x) \\
&= \sum_{(x)} Q(b_\ell(a_{(1)})(e - \phi)(a_{(2)})) \\
&= Q(b_\ell(1_H)(e - \phi)(x) + \sum_{(a)} b_\ell(x')(e - \phi)(x'') + b_\ell(x)(e - \phi)(1_H) \\
&= -Q(\phi(x) + \sum_{(x)} b_\ell(x')\phi(x'')).
\end{aligned}$$

In the last equation we have used $e(a) = 0, e(a'') = 0$ by definition. Since $b_\ell(1_H) = 1_H$, we see that $b_\ell = \phi_-$ in Eq. (32).

Further, we have

$$c_r = c_\ell^{-1}\phi = b_\ell\phi = -b_\ell(e-\phi) + b_\ell = -b_\ell(e-\phi) + e + P(b_\ell(e-\phi)) = e - (\mathrm{id}-P)(b_\ell(e-\phi)).$$

With the same computation as for b_ℓ above, we see that $c_r = \phi_+$ in Eq. (33).

5. Heat-kernel type regularization approach to the renormalization of MZVs

To extend the double shuffle relations to MZVs with non-positive arguments, we have to make sense of the divergent sums defining these MZVs. For this purpose, we adapt the renormalization method from quantum field theory in the algebraic framework of Connes-Kreimer recalled in the last section. We will give three approaches including the approach in this section using a heat-kernel type regularization, named after a similar process in physics. Since examples and motivations of this approach can already be found elsewhere [30, 36, 37], we will be quite sketchy in this section. More details will be given to the two other approaches in Section 6.

5.1. **Renormalization of MZVs.** Consider the abelian semigroup

$$\mathfrak{M} = \left\{\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] \,\middle|\, (s,r) \in \mathbb{Z}\times\mathbb{R}_{>0}\right\} \tag{36}$$

with the multiplication

$$\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} s' \\ r' \end{smallmatrix}\right] = \left[\begin{smallmatrix} s+s' \\ r+r' \end{smallmatrix}\right].$$

With the notation in Section 2.2, we define the Hopf algebra

$$\mathcal{H}_{\mathfrak{M}} := \mathrm{MS}_{\mathbb{C},1}(\mathbb{C}\mathfrak{M})$$

with the quasi-shuffle product $*$ and the deconcatenation coproduct Δ in Section 2.2. For $w_i = \left[\begin{smallmatrix} s_i \\ r_i \end{smallmatrix}\right] \in \mathfrak{M},\ i = 1,\cdots,k$, we use the notations

$$\vec{w} = (w_1,\ldots,w_k) = \left[\begin{smallmatrix} s_1,\ldots,s_n \\ r_1,\ldots,r_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right],\ \text{where } \vec{s} = (s_1,\ldots,s_k), \vec{r} = (r_1,\ldots,r_k).$$

For $\vec{w} = \left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right] \in \mathfrak{M}^k$ and $\varepsilon \in \mathbb{C}$ with $\mathrm{Re}(\varepsilon) < 0$, define the **directional regularized MZV**:

$$Z(\left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right];\varepsilon) = \sum_{n_1>\cdots>n_k>0} \frac{e^{n_1 r_1 \varepsilon}\cdots e^{n_k r_k \varepsilon}}{n_1^{s_1}\cdots n_k^{s_k}} \tag{37}$$

It converges for any $\left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right]$ and is regarded as the regularization of the **formal MZV**

$$\zeta(\vec{s}) = \sum_{n_1>\cdots>n_k>0} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} \tag{38}$$

which converges only when $s_i > 0$ and $s_1 > 1$. It is related to the multiple polylogarithm

$$\mathrm{Li}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{n_1>\cdots>n_k>0} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

by a change of variables $z_i = e^{r_i\varepsilon}, 1 \leqslant i \leqslant k$.

This regularization defines an algebra homomorphism [36]:

$$\tilde{Z} : \mathcal{H}_{\mathfrak{M}} \to \mathbb{C}[T][[\varepsilon,\varepsilon^{-1}], \tag{39}$$

In the same way, for

$$\mathfrak{M}^- = \{[{s \atop r}] \mid (s,r) \in \mathbb{Z}_{\leq 0} \times \mathbb{R}_{>0}\}, \tag{40}$$

$\tilde{Z}$ restricts to an algebra homomorphism

$$\tilde{Z} : \mathcal{H}_{\mathfrak{M}^-} \to R := \mathbb{C}[[\varepsilon, \varepsilon^{-1}]. \tag{41}$$

Since both $(\mathbb{C}[T][\varepsilon^{-1}, \varepsilon]], \Pi)$ and $(\mathbb{C}[\varepsilon^{-1}, \varepsilon]], \Pi)$, with Π defined in Example 2.4, are commutative Rota-Baxter algebras with $\Pi^2 = \Pi$, we have the decomposition

$$\tilde{Z} = \tilde{Z}_-^{-1} \star \tilde{Z}_+$$

by the algebraic Birkhoff decomposition in Theorem 4.1 and obtain

Theorem 5.1. *(*[36, 37]*) The map $\tilde{Z}_+ : \mathcal{H}_{\mathfrak{M}} \to \mathbb{C}[T][[\varepsilon]]$ is an algebra homomorphism which restricts to an algebra homomorphism $\tilde{Z}_+ : \mathcal{H}_{\mathfrak{M}^-} \to \mathbb{C}[[\varepsilon]]$.*

Because of Theorem 5.1, the following definition is valid.

Definition 5.2. *For $\vec{s} = (s_1, \ldots, s_k) \in \mathbb{Z}^k$ and $\vec{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k_{>0}$, define the* **renormalized directional MZV** *by*

$$\zeta([{\vec{s} \atop \vec{r}}]) = \lim_{\varepsilon \to 0} \tilde{Z}_+([{\vec{s} \atop \vec{r}}]; \varepsilon). \tag{42}$$

Here $\vec{r}$ is called the **direction vector**.

As a consequence of Theorem 5.1, we have

Corollary 5.3. *The renormalized directional MZVs satisfy the quasi-shuffle relation*

$$\zeta([{\vec{s} \atop \vec{r}}])\zeta([{\vec{s}' \atop \vec{r}'}]) = \zeta([{\vec{s} \atop \vec{r}}] * [{\vec{s}' \atop \vec{r}'}]). \tag{43}$$

Here the right hand side is defined in the same way as in Eq. (8).

Definition 5.4. *For $\vec{s} \in \mathbb{Z}^k_{>0} \cup \mathbb{Z}^k_{\leqslant 0}$, define*

$$\overline{\zeta}(\vec{s}) = \lim_{\delta \to 0^+} \zeta([{\vec{s} \atop |\vec{s}| + \delta}]), \tag{44}$$

where, for $\vec{s} = (s_1, \cdots, s_k)$ and $\delta \in \mathbb{R}_{>0}$, we denote $|\vec{s}| = (|s_1|, \cdots, |s_k|)$ and $|\vec{s}| + \delta = (|s_1| + \delta, \cdots, |s_k| + \delta)$. These $\overline{\zeta}(\vec{s})$ are called the **renormalized MZVs** *of the multiple zeta function $\zeta(u_1, \cdots, u_k)$ at $\vec{s}$.*

Theorem 5.5. [36]

(a) *The limit in Eq. (44) exists for any $\vec{s} = (s_1, \cdots, s_k) \in \mathbb{Z}^k_{>0} \cup \mathbb{Z}^k_{\leqslant 0}$.*

(b) *When s_i are all positive with $s_1 > 1$, we have $\zeta([{\vec{s} \atop \vec{r}}]) = \zeta(\vec{s})$ independent of $\vec{r} \in \mathbb{Z}^k_{>0}$. In particular, we have $\overline{\zeta}(\vec{s}) = \zeta(\vec{s})$.*

(c) *When s_i are all positive, we have $\overline{\zeta}(\vec{s}) = \zeta([{\vec{s} \atop \vec{s}}])$. Further, $\overline{\zeta}(\vec{s})$ agrees with the regularized MZV $Z^*_{\vec{s}}(T)$ defined by Ihara-Kaneko-Zagier* [46].

(d) *When s_i are all negative, we have $\overline{\zeta}(\vec{s}) = \zeta([{\vec{s} \atop -\vec{s}}]) = \lim_{\vec{r} \to -\vec{s}} \zeta([{\vec{s} \atop \vec{r}}])$. Further, these values are rational numbers.*

(e) *The value* $\overline{\zeta}(\vec{s})$ *agrees with* $\zeta(\vec{s})$ *whenever the latter is defined by analytic continuation. Furthermore,*

(f) *the set* $\{\overline{\zeta}(\vec{s})\big|\vec{s}\in\mathbb{Z}^k_{>0}\}$ *satisfies the quasi-shuffle relation;*

(g) *the set* $\{\overline{\zeta}(\vec{s})\big|\vec{s}\in\mathbb{Z}^k_{\leqslant 0}\}$ *satisfies the quasi-shuffle relation.*

The following table lists $\overline{\zeta}(-a,-b)$ for $0\leqslant a,b\leqslant 6$.

(45)

$\overline{\zeta}(-a,-b)$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
$b=0$	$\frac{3}{8}$	$\frac{1}{12}$	$\frac{1}{120}$	$-\frac{1}{120}$	$-\frac{1}{252}$	$\frac{1}{252}$	$\frac{1}{240}$
$b=1$	$\frac{1}{24}$	$\frac{1}{288}$	$-\frac{1}{240}$	$\frac{83}{64512}$	$\frac{1}{504}$	$-\frac{3925}{2239488}$	$-\frac{1}{480}$
$b=2$	$-\frac{1}{120}$	$-\frac{1}{240}$	0	$\frac{1}{504}$	$-\frac{319}{437400}$	$-\frac{1}{480}$	$\frac{2494519}{1362493440}$
$b=3$	$-\frac{1}{240}$	$-\frac{71}{35840}$	$\frac{1}{504}$	$\frac{1}{28800}$	$-\frac{1}{480}$	$\frac{114139507}{139519328256}$	$\frac{1}{264}$
$b=4$	$\frac{1}{252}$	$\frac{1}{504}$	$\frac{319}{437400}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$-\frac{41796929201}{26873437500000}$
$b=5$	$\frac{1}{504}$	$\frac{32659}{15676416}$	$-\frac{1}{480}$	$-\frac{21991341}{25836912640}$	$\frac{1}{264}$	$\frac{1}{127008}$	$-\frac{691}{65520}$
$b=6$	$-\frac{1}{240}$	$-\frac{1}{480}$	$-\frac{2494519}{1362493440}$	$\frac{1}{264}$	$\frac{41796929201}{26873437500000}$	$-\frac{691}{65520}$	0

5.2. **The differential structure.** The shuffle relation for convergent MZVs from their integral representations does not directly generalize to renormalized MZVs due to the lack of a suitable integral representation. However a differential variation of the shuffle relation might exist for renormalized MZVs. One evidence is the following differential version of the algebraic Birkhoff decomposition [37] for renormalized MZVs and further progress will be discussed in a paper under preparation. We first recall some concepts.

(a) A **differential algebra** is a pair (A,d) where A is an algebra and d is a **differential operator**, that is, such that $d(xy)=d(x)y+xd(y)$ for all $x,y\in A$. A differential algebra homomorphism $f:(A_1,d_1)\to(A_2,d_2)$ between two differential algebras (A_1,d_1) and (A_2,d_2) is an algebra homomorphism $f:A_1\to A_2$ such that $f\circ d_1=d_2\circ f$.

(b) A **differential Hopf algebra** is a pair (H,d) where H is a Hopf algebra and $d:H\to H$ is a differential operator such that

$$\Delta(d(x))=\sum_{(x)}\left(d(x_{(1)})\bigotimes x_{(2)}+x_{(1)}\bigotimes d(x_{(2)})\right). \tag{46}$$

(c) A **differential Rota-Baxter algebra** is a triple (A, Π, d) where (A, Π) is a Rota-Baxter algebra and $d : R \to R$ is a differential operator such that $P \circ d = d \circ P$.

Theorem 5.6. (Differential Algebraic Birkhoff Decomposition) [37] *Under the same assumption as in Theorem 4.1, if in addition (H, d) is a differential Hopf algebra, (A, Π, ∂) is a commutative differential Rota-Baxter algebra, and $\phi : H \to A$ is a differential algebra homomorphism, then the maps ϕ_- and ϕ_+ in Theorem 4.1 are also differential algebra homomorphisms.*

Theorem 5.7. ([37])

(a) *For $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] \in \mathfrak{M}$, define $d(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]) = r\left[\begin{smallmatrix} s-1 \\ r \end{smallmatrix}\right]$. Extend d to $\mathcal{H}_{\mathfrak{M}} = \oplus_{k\geqslant 0}(\mathbf{k}\mathfrak{M})^{\otimes k}$ by defining, for $\mathfrak{a} := a_1 \otimes \cdots \otimes a_k \in (\mathbf{k}\mathfrak{M})^{\otimes k}$,*

$$d(\mathfrak{a}) = \sum_{i=1}^{k} a_{i,1} \otimes \cdots \otimes a_{i,k}, \quad a_{i,j} = \begin{cases} a_j, & j \neq i, \\ d(a_j), & j = i. \end{cases} \tag{47}$$

Then $(\mathcal{H}_A, d)$ is a differential Hopf algebra.

(b) *The triple $(\mathbb{C}[\varepsilon^{-1}, \varepsilon]], \Pi, \frac{d}{d\varepsilon})$ is a commutative differential Rota-Baxter algebra.*

(c) *The map $\tilde{Z} : \mathcal{H}_{\mathfrak{M}} \to \mathbb{C}[[\varepsilon, \varepsilon^{-1}]$ defined in Eq. (41) is a differential algebra homomorphism.*

(d) *The algebra homomorphism $\tilde{Z}_+ : \mathcal{H}_{\mathfrak{M}^-} \to \mathbb{C}[[\varepsilon]]$ in Theorem 5.1 is a differential algebra homomorphism.*

6. Renormalization of multiple zeta values seen as nested sums of symbols

We present two more approaches to renormalize multiple zeta functions at non-positive integers, both of which lead to MZVs which obey stuffle relations. Like the renormalization method described in the previous section, they both give rise to rational multiple zeta values at non-positive integers and we check that the two methods yield the same double multiple zeta values at non-positive integer arguments. This presentation is based on joint work of one of the authors with D. Manchon [55] in which multiple zeta functions are viewed as particular instances of nested sums of symbols and where the algebraic Birkhoff decomposition approach is used to renormalize multiple zeta functions at poles. Here, we furthermore present an alternative renormalization method based on generalized evaluators used in physics [67].

6.1. **A class of symbols.** For a complex number b, a smooth function $f : \mathbb{R} - \{0\} \to \mathbb{C}$ is called **positively homogeneous of degree** b if $f(t\xi) = t^b f(\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}$.

The symbols which were originally defined on $\mathbb{R}^n$ are now defined on $\mathbb{R}$ which is sufficient for our needs in this paper. We call a smooth function $\sigma : \mathbb{R} \to \mathbb{C}$ a **symbol** if there is a real number a such that for any non-negative integer γ, there is a positive constant C_γ with

$$|\partial^\gamma \sigma(\xi)| \leq C_\gamma (1 + |\xi|)^{a-\gamma}, \quad \forall \xi \in \mathbb{R}.$$

For a complex number a and a non-negative integer j, let $\sigma_{a-j} : \mathbb{R} - \{0\} \to \mathbb{C}$ be a smooth and positively homogeneous function of degree $a - j$. We write $\sigma \sim \sum_{j=0}^{\infty} \sigma_{a-j}$ if, for any non-negative integer N and non-negative integer γ, there is a positive constant $C_{\gamma,N}$ such that

$$\left| \partial^\gamma \left(\sigma(\xi) - \sum_{j=0}^{N} \sigma_{a-j}(\xi) \right) \right| \leq C_{\gamma,N} (1 + |\xi|)^{\mathrm{Re}\,(a) - N - 1 - \gamma}, \quad \forall \xi \in \mathbb{R} - \{0\},$$

where $\mathrm{Re}(a)$ stands for the real part of a.

For any complex number a and any non-negative integer k, a symbol $\sigma : \mathbb{R} \to \mathbb{C}$ is called a **log-polyhomogeneous of log-type k and order a** if

$$\sigma(\xi) = \sum_{l=0}^{k} \sigma_l(\xi) \log^l |\xi|, \quad \sigma_l(\xi) \sim \sum_{j=0}^{\infty} \sigma_{a-j,l}(\xi) \tag{48}$$

with $\sigma_{a-j,l}(\xi)$ positively homogeneous of degree $a - j$.

Let $\mathcal{S}^{a,k}$ denote the linear space over $\mathbb{C}$ of log-polyhomogeneous symbols on $\mathbb{R}$ of log-type k and order a. Then we have $\mathcal{S}^{a,k} \subseteq \mathcal{S}^{a,k+1}$. Let $\mathcal{S}^{*,k}$ denote the linear span over $\mathbb{C}$ of all $\mathcal{S}^{a,k}$ for $a \in \mathbb{C}$. Then $\mathcal{S}^{*,0}$ corresponds to the algebra of classical symbols on $\mathbb{R}$. We also define

$$\mathcal{S}^{*,*} := \bigcup_{k=0}^{\infty} \mathcal{S}^{*,k}$$

which is an algebra for the ordinary product of functions filtered by the log-type [52] since the product of two symbols of log-types k and k' respectively is of log-type $k + k'$. The union $\bigcup_{a\in\mathbb{Z}} \bigcup_{k=0}^{\infty} \mathcal{S}^{a,k}$ is a subalgebra of $\mathcal{S}^{*,*}$, and $\bigcup_{a\in\mathbb{Z}} \mathcal{S}^{a,0}$ is a subalgebra of $\mathcal{S}^{*,0}$.

Let $\mathcal{P}^{\alpha,k}$ be the algebra of **positively supported** symbols, i.e. symbols in $\mathcal{S}^{\alpha,k}$ with support in $(0, +\infty)$ so that they are non-zero only at positive arguments. We keep *mutatis mutandis* the above notations; in particular $\mathcal{P}^{*,0}$ is a subalgebra of the filtered algebra $\mathcal{P}^{*,*}$.

For $\sigma \in \mathcal{P}^{\alpha,k}$ we call $\operatorname{fp}_{\xi\to\infty} \sigma(\xi) := \sigma_{0,0}(\xi)$ the **finite part** at zero (so named since it it reminiscent of Hadamard's finite parts) of such a symbol σ which corresponds to the constant term in the expansion.

The following rather elementary statement is our main motivation here for introducing log-polyhomogeneous symbols.

Proposition 6.1. [54] *The operator I defined in (2) on the algebra $C[0, \infty)$ by*

$$f \mapsto \left(\xi \mapsto I(f)(\xi) = \int_0^{\xi} f(t)dt\right)$$

maps $\mathcal{P}^{*,k-1}$ to $\mathcal{P}^{*,k}$ for any positive integer k.

By Proposition 6.1, for any σ in $\mathcal{P}^{*,k}$, the primitive $I(\sigma)(\xi)$ has an asymptotic behavior as $\xi \to \infty$ of the type (48) with k replaced by $k+1$. The constant term defines the cut-off regularized integral (see e.g. [52]):

$$-\!\!\!\!\!\!\int_0^{\infty} \sigma(t)\, dt := \operatorname{fp}_{\xi\to\infty} \int_0^{\xi} \sigma(t)\, dt.$$

6.2. **Nested sums of symbols and their pole structures.**

6.2.1. *Nested sums.* Recall that the operator I on $\mathcal{P}^{*,*}$ defined by Eq. (2) satisfies the weight zero Rota-Baxter relation (1). On the other hand the operator P defined by Eq. (4) satisfies the Rota-Baxter relation with weight $\lambda = -1$ and the operator $Q = P - Id$ in Eq. (5) satisfies the Rota-Baxter relation with weight $\lambda = 1$.

The Rota-Baxter operators P and I relate by means of the Euler-MacLaurin formula which compares discrete sums with integrals. For $\sigma \in \mathcal{P}^{*,*}$ the Euler-MacLaurin formula (see e.g. [38])

reads:

$$
\begin{aligned}
P(\sigma)(N) - I(\sigma)(N) &= \frac{1}{2}\sigma(N) + \sum_{k=2}^{2K} \frac{B_k}{k!}\sigma^{(k-1)}(N) \\
&+ \frac{1}{(2K+1)!}\int_0^N \overline{B_{2K+1}}(x)\sigma^{(2K+1)}(x)\,dx. \qquad (49)
\end{aligned}
$$

with $\overline{B_k}(x) = B_k(x-[x])$. Here $B_k(x) = \sum_{i=0}^k \binom{k}{i} B_{k-i}\, x^k$ are the Bernoulli polynomials of degree k, the B_i being the Bernoulli numbers, defined by the generating series:

$$\frac{t}{e^t-1} = \sum_i \frac{B_i}{i!}t^i.$$

Since $B_k(1) = B_k$ for any $k \geqslant 2$, setting $x = 1$ we have

$$B_k = \sum_{i=0}^k \binom{k}{i} B_{k-i} = \sum_{i=0}^k \binom{k}{i} B_i, \quad \forall k \geqslant 2. \qquad (50)$$

The Euler-MacLaurin formula therefore provides an interpolation of $P(\sigma)$ by a symbol.

Proposition 6.2. [55] *For any $\sigma \in \mathcal{P}^{a,k}$, the discrete sum $P(\sigma)$ can be interpolated by a symbol $\overline{P}(\sigma)$ in $\mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$ (i.e. $\overline{P}(\sigma)(n) = P(\sigma)(n) = \sum_{k=0}^n \sigma(k), \quad \forall n \in \mathbb{N}$) such that*

$$\overline{P}(\sigma) - I(\sigma) \in \mathcal{P}^{a,k}.$$

The operator $\overline{Q} := \overline{P} - Id : \mathcal{P}^{a,k} \to \mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$ interpolates Q.

By Proposition 6.2, given a symbol σ in $\mathcal{P}^{a,k}$, the interpolating symbol $\overline{P}(\sigma)$ lies in $\mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$. It follows that the discrete sum $P(\sigma)(N) = \overline{P}(\sigma)(N)$ has an asymptotic behavior for large N given by finite linear combinations of expressions of the type (48) with k replaced by $k+1$ and a by $a+1$ or 0. Picking the finite part, for any $\sigma \in \mathcal{P}^{*,*}$ we define the following cut-off sum:

$$\sum\!\!\!\!\!\!-_{0}^{\infty} \sigma := \operatorname*{fp}_{N\to\infty} P(\sigma)(N) = \operatorname*{fp}_{N\to\infty} \sum_{k=0}^N \sigma(k), \qquad (51)$$

which extends the ordinary discrete sum $\sum_0^\infty$ on L^1-symbols. If σ has non-integer order, we have $\sum\!\!\!\!\!-_{0}^{\infty} \sigma = \operatorname*{fp}_{N\to\infty} \sum_{k=0}^{N+K} \sigma(k)$ for any integer K, so that in particular $\sum\!\!\!\!\!-_{0}^{\infty} \sigma = \operatorname*{fp}_{N\to\infty} Q(\sigma)(N)$ since the operators P and Q only differ by an integer in the upper bound of the sum.

With the help of the interpolation map described in Proposition 6.2, we can assign to a tensor product $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$ of (positively supported) classical symbols, two log-polyhomogeneous symbols defined inductively in the degree k of the tensor product, which interpolate the nested iterated sum

$$\sum_{0 \leqslant n_k \leqslant n_{k-1} \leqslant \cdots \leqslant n_2 \leqslant n_1} \sigma_1(n_1)\cdots\sigma_k(n_k) = \sigma_1\, P\Big(\cdots \sigma_{k-2}\, P(\sigma_{k-1}\, P(\sigma_k))\ldots\Big),$$

$$\sum_{0 \leqslant n_k < n_{k-1} < \cdots < n_2 < n_1} \sigma_1(n_1)\cdots\sigma_k(n_k) = \sigma_1\, Q\Big(\cdots \sigma_{k-2}\, Q(\sigma_{k-1}\, P(\sigma_k))\ldots\Big).$$

In the following we will only consider the second class of symbols, including their regularization, renormalization and application to multiple zeta values. A parallel approach applies to the first class of symbols with application to non-strict multiple zeta values in Eq. (22) [61, 75].

Theorem 6.3. [55] *Given $\sigma_i \in \mathcal{P}^{\alpha_i,0}$, $i = 1,\dots,k$, setting $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$, the function $\widetilde{\sigma}$ defined by:*

$$\widetilde{\sigma} := \sigma_1 \, \overline{Q}\Big(\cdots \sigma_{k-2} \, \overline{Q}(\sigma_{k-1} \, \overline{Q}(\sigma_k))...\Big) \tag{52}$$

which interpolates nested sums in the following way:

$$\widetilde{\sigma}(n_1) \quad = \sum_{0 \leqslant n_k < n_{k-1} < \cdots < n_2 < n_1} \sigma_1(n_1) \cdots \sigma_k(n_k), \quad \forall n_1 \in \mathbb{N},$$

lies in $\mathcal{P}^{*,k-1}$ as linear combinations of (positively supported) symbols in $\mathcal{P}^{\alpha_1+\cdots+\alpha_j+j-1,j-1}$, $j \in \{1,\dots,k\}$.

On the grounds of this result, we define the cut-off nested discrete sum of a tensor product of (positively supported) classical symbols.

Definition 6.4. For $\sigma_1,\dots,\sigma_k \in \mathcal{P}^{*,0}$ and $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$ we call

$$\sum_{<}^{\text{Chen}} \sigma := \sum_0^\infty \widetilde{\sigma} = \sum_{0<n_k<\cdots<n_1} \sigma_1(n_1)\cdots\sigma_k(n_k)$$

the **cut-off nested sum** of $f = \sigma_1 \otimes \cdots \otimes \sigma_k$.

6.2.2. *The pole structure of nested sums of symbols.* To build meromorphic extensions, we combine the cut-off sum $\sum_0^\infty$ introduced in (51) with holomorphic deformations of the symbol in the integrand.

A family $\{a(z)\}_{z\in\Omega}$ in a topological vector space $\mathcal{A}$ which is parameterized by a complex domain Ω, is holomorphic at $z_0 \in \Omega$ if the corresponding function $f : \Omega \to \mathcal{A}$ admits a Taylor expansion in a neighborhood N_{z_0} of z_0

$$a(z) = \sum_{k=0}^{\infty} a^{(k)}(z_0)\, \frac{(z-z_0)^k}{k!}$$

which is convergent, uniformly on compact subsets of N_{z_0} (i.e. locally uniformly), with respect to the topology on $\mathcal{A}$. The vector spaces of functions we consider here are $C(\mathbb{R},\mathbb{C})$ and $C^\infty(\mathbb{R},\mathbb{C})$ equipped with their usual topologies, namely uniform convergence on compact subsets, and uniform convergence of all derivatives on compact subsets respectively.

Definition 6.5. Let k be a non-negative integer, and let Ω be a domain in $\mathbb{C}$. A **simple holomorphic family of log-polyhomogeneous symbols** $\sigma(z) \in \mathcal{S}^{*,k}$ parameterized by Ω is a holomorphic family $\sigma(z)(\xi) := \sigma(z,\xi)$ of smooth functions on $\mathbb{R}$ such that:

(a) the order $\alpha : \Omega \to \mathbb{C}$ is holomorphic on Ω,
(b) $\sigma(z)(\xi) = \sum_{l=0}^k \sigma_l(\xi)\, \log^l |\xi|$ with

$$\sigma_l(z)(\xi) \sim \sum_{j \geqslant 0} \sigma(z)_{\alpha(z)-j,l}(\xi).$$

Here $\sigma(z)_{\alpha(z)-j,l}$ positively homogeneous of degree $\alpha(z) - j$,
(c) for any positive integer N there is some positive integer K_N such that the remainder term

$$\sigma_{(N)}(z)(\xi) := \sigma(z)(\xi) - \sum_{l=0}^{k} \sum_{j=0}^{K_N} \sigma(z)_{\alpha(z)-j,l}(\xi) \log^l |\xi| = o(|\xi|^{-N})$$

is holomorphic in $z \in \Omega$ as a function of ξ and verifies for any $\epsilon > 0$ the following estimates:

$$\partial_\xi^\beta \partial_z^k \sigma_{(N)}(z)(\xi) = o(|\xi|^{-N-|\beta|+\epsilon})$$

locally uniformly in $z \in \Omega$ for $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

A **holomorphic family of log-polyhomogeneous symbols** is a finite linear combination (over $\mathbb{C}$) of simple holomorphic families.

It follows from the Euler-MacLaurin formula (see e.g. [38, 55]) that for any holomorphic family $\sigma(z)$ of symbols in $\mathcal{P}^{*,*}$, we have

$$\sum_{n=0}^{\infty} \sigma(z)(n) = -\!\!\!\!\!\!\int_0^\infty \sigma(z)(\xi)\, d\xi + C(\sigma(z))$$

with $z \mapsto C(\sigma(z))$ a holomorphic function at zero. Hence, $z \mapsto \sum_{n=0}^{\infty} \sigma(z)(n)$ and $z \mapsto -\!\!\!\!\!\!\int_0^\infty \sigma(z)(\xi)\, d\xi$ have the same pole structure. Results by Kontsevich and Vishik [49] for classical symbols and their generalization by Lesch [52] to log-polyhomogeneous symbols, and relative to the pole structure of cut-off integrals of holomorphic families of symbols, therefore carry out to discrete cut-off sums of holomorphic (positively supported) log-polyhomogeneous symbols. Let us briefly recall the notion of holomorphic regularization inspired by [49].

Definition 6.6. A **holomorphic regularization procedure** on $\mathcal{S}^{*,*}$ is a map

$$\begin{aligned} \mathcal{R} : \mathcal{S}^{*,*} &\to \mathrm{Hol}_\Omega\,(\mathcal{S}^{*,*}) \\ f &\mapsto \{\sigma(z) = \sigma_f(z)\}_{z\in\Omega} \end{aligned}$$

where Ω is an open subset of $\mathbb{C}$ containing 0, and $\mathrm{Hol}_\Omega\,(\mathcal{S}^{*,*})$ is the algebra of holomorphic families in $\mathcal{S}^{*,*}$, such that for any $f \in \mathcal{S}^{*,*}$,

(a) $\sigma(0) = f$,
(b) the holomorphic family $\sigma(z)$ can be written as a linear combination of simple ones:

$$\sigma(z) = \sum_{j=1}^{k} \sigma_j(z),$$

the holomorphic order $\alpha_j(z)$ of which verifies $\mathrm{Re}(\alpha_j'(z)) < 0$ for any $z \in \Omega$ and any $j \in \{1, \dots, k\}$.

A holomorphic regularization $\mathcal{R}$ is **simple** if, for any log-polyhomogeneous symbol $\sigma \in \mathcal{S}^{\alpha,k}$, the holomorphic family $\mathcal{R}(\sigma)$ is simple. Since we only consider simple holomorphic regularizations, we drop the explicit mention of simplicity.

A similar definition holds with suitable subalgebras of $\mathcal{S}^{*,*}$, e.g. classical symbols $\mathcal{S}^{*,0}$ instead of log-polyhomogeneous. Holomorphic regularization procedures naturally arise in physics:

Example 6.7. Let $z \mapsto \tau(z) \in \mathcal{S}^{*,0}$ be a holomorphic family of classical symbols such that $\tau(0) = 1$ and $\tau(z)$ has holomorphic order $\alpha(z)$ with $\mathrm{Re}(\alpha'(z)) < 0$. Then

$$\mathcal{R} : \sigma \mapsto \sigma(z) := \sigma\,\tau(z)$$

yields a holomorphic regularization on $\mathcal{S}^{*,*}$ as well as on $\mathcal{S}^{*,0}$. Choosing $\tau(z)(\xi) := \chi(\xi) + (1 - \chi(\xi))(H(z)\,|\xi|^{-z})$ where H is a scalar valued holomorphic map such that $H(0) = 1$, and where χ

is a smooth cut-off function which is identically one outside the unit interval and zero in a small neighborhood of zero, we get

$$\mathcal{R}(\sigma)(z)(\xi) = \chi(\xi)\sigma(\xi) + (1-\chi(\xi))(H(z)\,\sigma(\xi)\,|\xi|^{-z}).$$

Dimensional regularization commonly used in physics is of this type, where H is expressed in terms of Gamma functions which account for a "complexified" volume of the unit sphere. When $H \equiv 1$, such a regularization $\mathcal{R}$ is called Riesz regularization.

Proposition 6.8. *Given a holomorphic regularization $\mathcal{R} : \sigma \mapsto \sigma(z)$ on $\mathcal{P}^{*,k}$, for any $\sigma \in \mathcal{P}^{*,k}$, the map $z \mapsto \sum_0^\infty \sigma(z)$ is meromorphic with poles of order $\leqslant k+1$ in the discrete set $\alpha^{-1}(\{-1,0,1,2,\cdots\})$ whenever $\sigma(z)$ is a holomorphic family with order $\alpha(z)$ such that $\mathrm{Re}(\alpha'(z)) \neq 0$ for any z in Ω.*

Let $\Omega \subset \mathbb{C}$ be an open neighborhood of 0. Given symbols $\sigma_1, \cdots, \sigma_k \in \mathcal{P}^{*,0}$, and a holomorphic regularization $\mathcal{R}$ which sends σ_i to $\sigma_i(z)$ with order $\alpha_i(z)$, $z \in \Omega$, we build holomorphic perturbations in the complex multivariable $\underline{z} := (z_1, \cdots, z_k) \in \Omega^k$ of the symbols $\widetilde{\sigma}$ introduced in (52):

$$\widetilde{\sigma}(\underline{z}) \quad := \quad \sigma_1(z_1)\,\overline{Q}\Big(\cdots\sigma_{k-2}(z_{k-2})\,\overline{Q}(\sigma_{k-1}(z_{k-1})\,\overline{Q}(\sigma_k(z_k)))...\Big).$$

By Theorem 6.3, these are linear combinations of log-polyhomogeneous symbols of log-type $j-1$ and order $\alpha_1(z_1) + \cdots + \alpha_j(z_j) + j - 1$, $j \in \{1, \ldots, k\}$. Applying Proposition 6.8 to each of these symbols provides information on the pole structure of nested sums of (positively supported) classical symbols reminiscent of the pole structure of multiple zeta functions [1, 25, 73].

Theorem 6.9. *Fix symbols $\sigma_1, \cdots, \sigma_k \in \mathcal{P}^{*,0}$ and a holomorphic regularization $\mathcal{R}$ which sends σ_i to $\sigma_i(z)$ with order $\alpha_i(z)$.*

(a) *The map*

$$(z_1, \cdots, z_k) \quad \mapsto \quad \sum_{<}^{Chen} \sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)$$

is meromorphic with poles on a countable number of hypersurfaces

$$\sum_{i=1}^{j} \alpha_i(z_i) \in -j + \mathbb{N}_0,$$

of multiplicity j varying in $\{1, \cdots, k\}$. Here $\mathbb{N}_0$ stands for the set of non-negative integers.

(b) *Let $\sigma(z) := \sigma_1(z) \otimes \cdots \otimes \sigma_k(z)$ with $z \in \Omega$. Assume that the orders $\alpha_i(z)$ of the σ_i's are nonconstant affine with $\alpha'_j(0) = -q$ for any j in $\{1, \cdots, k\}$ and some positive real number q. The map $z \mapsto \sum_<^{Chen} \sigma(z)$ is meromorphic on Ω with poles $z \in (\sum_{i=1}^j \alpha_i(0) + j - \mathbb{N}_0)/(q\,j)$ of order $\leqslant j$.*

(c) *If $\mathrm{Re}(\alpha_1(z_1) + \cdots + \alpha_j(z_j)) < -j$ for any $j \in \{1, \ldots, k\}$, the nested sums converge and boil down to ordinary nested sums (independently of the perturbation). Setting $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ we have:*

$$\sum_{<}^{Chen,\mathcal{R}} \sigma \quad = \quad \lim_{z\to 0} \sum_{<}^{Chen} \sigma(z) = \sum_{<}^{Chen} \sigma.$$

6.3. **A twisted holomorphic regularization.** We now take $\mathcal{A}$ to be a subalgebra of $\mathcal{P}^{*,0}$ equipped with the ordinary product on functions. Any holomorphic regularization $\mathcal{R}$ on $\mathcal{A}$ with parameter space $\Omega \subset \mathbb{C}$ induces one on the tensor algebra $T(\mathcal{A})$:

$$\widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) := \mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k).$$

It is compatible with the shuffle product

$$\widetilde{\mathcal{R}}\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \ \text{ш}\ (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) = \widetilde{\mathcal{R}}\left(\sigma_1 \otimes \cdots \otimes \sigma_k\right) \ \text{ш}\, \widetilde{\mathcal{R}}\left(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}\right)$$

for any $\sigma_i \in \mathcal{A},\ i \in \{1, \cdots, k+l\}$.

Remark 6.10. *Note that* $\widetilde{\mathcal{R}}(\sigma_1 \,\text{ш}\, \sigma_2)(z_1, z_2) \neq \mathcal{R}(\sigma_1)(z_1) \,\text{ш}\, \mathcal{R}(\sigma_2)(z_2)$ *whereas* $\widetilde{\mathcal{R}}(\sigma_1 \,\text{ш}\, \sigma_2)(z_1, z_2) = (\mathcal{R}(\sigma_1) \,\text{ш}\, \mathcal{R}(\sigma_2))(z_1, z_2)$.

Let

$$\delta_k : \mathbb{C} \to \mathbb{C}^{\otimes k}, \quad z \mapsto z \cdot 1^{\otimes k},$$

be the diagonal map $\delta : \mathbb{C} \mapsto T(\mathbb{C})$ and δ^* the induced map on tensor products of holomorphic symbols

$$\delta_k^\star : T\left(\mathrm{Hol}_\Omega\left(\mathcal{A}\right)\right) \to \mathrm{Hol}_\Omega\left(T(\mathcal{A})\right), \quad \sigma \mapsto \sigma \circ \delta_k.$$

The regularization $\widetilde{\mathcal{R}}$ induces a one parameter holomorphic regularization:

$$\left(\delta^* \circ \widetilde{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) = \mathcal{R}(\sigma_1)(z) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z)$$

compatible with the shuffle product:

$$\begin{aligned} &\left(\delta^* \circ \widetilde{\mathcal{R}}\right)\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \ \text{ш}\ (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) \\ = \ &\left(\delta^* \circ \widetilde{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k) \ \text{ш}\ \left(\delta^* \circ \widetilde{\mathcal{R}}\right)\left(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}\right), \end{aligned} \tag{53}$$

for any $\sigma_i \in \mathcal{A},\ i \in \{1, \cdots, k+l\}$.

Twisting it by Hoffman's isomorphism in Theorem 2.8 yields a holomorphic regularization $\left(\delta^* \circ \tilde{\mathcal{R}}\right)*$ (denoted by $\mathcal{R}^*$ in [55]) on $T(\mathcal{A})$:

$$\left(\delta^* \circ \tilde{\mathcal{R}}\right)* := \exp \circ \left(\delta^\star \circ \tilde{\mathcal{R}}\right) \circ \log,$$

which is compatible with the stuffle product:

$$\left(\delta^* \circ \tilde{\mathcal{R}}\right)^* (\sigma * \tau) = \left(\delta^* \circ \tilde{\mathcal{R}}\right)^* (\sigma) * \left(\delta^* \circ \tilde{\mathcal{R}}\right)^* (\tau), \quad \forall \sigma, \tau \in T\left(\mathcal{A}\right). \tag{54}$$

Consequently, the following regularization

$$\widetilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) = \exp \circ \tilde{\mathcal{R}} \circ \log(\sigma_1 \otimes \cdots \otimes \sigma_k)\,(z_1, \cdots, z_k) \tag{55}$$

is compatible with stuffle relations after symmetrization in the complex variables z_i

$$\left(\widetilde{\mathcal{R}}^*(\sigma * \tau)\right)_{\mathrm{sym}} = \left(\tilde{\mathcal{R}}^*(\sigma) * \tilde{\mathcal{R}}^*(\tau)\right)_{\mathrm{sym}}, \quad \forall \sigma, \tau \in \mathcal{T}\left(\mathcal{A}\right), \tag{56}$$

where the subscript sym stands for symmetrization

$$f_{\mathrm{sym}}(z_1, \cdots, z_k) := \frac{1}{k!} \sum_{\tau \in \Sigma_k} f(z_{\tau(1)}, \cdots, z_{\tau(k)}),$$

over all the complex variables $z_1, \cdots, z_{k+l}$ if σ is a tensor of degree k and τ a tensor of degree l. Setting $z_1 = \cdots = z_{k+l} = z$ in (56) yields back (54) so that (56) can be seen as a polarization of

(54). Given symbols $\sigma_1, \cdots, \sigma_k$ in $\mathcal{P}^{*,0}$, and a holomorphic regularization $\mathcal{R} : \sigma \mapsto \sigma(z)$, sending σ_i to $\sigma_i(z)$ with order $\alpha_i(z)$, we are now ready to build a map

$$(z_1, \cdots, z_k) \mapsto \sum_{<}^{\text{Chen}} \widetilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k),$$

which, by Theorem 6.9, is meromorphic with poles on a countable number of hypersurfaces

$$\sum_{i=1}^{j} \alpha_i(z_i) \in -j + \mathbb{N}_0,$$

with multiplicity j varying in $\{1, \cdots, k\}$. In particular, if the holomorphic regularization $\mathcal{R}$ sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q, the hypersurfaces of poles are given by

$$\sum_{i=1}^{j} z_i \in \frac{\sum_{i=1}^{j} \alpha_i(0) + j - n}{q}, \quad n \in \mathbb{N}_0,$$

so that hyperplanes of poles containing the origin correspond to $\sum_{i=1}^{j} z_i = 0$ each of which with multiplicity j varying in $\{1, \cdots, k\}$.

6.4. **Meromorphic nested sums of symbols.** Let $\mathrm{Mer}_0(\mathbb{C})$ denote the germ of meromorphic functions in a neighborhood of zero in the complex plane and let $\mathrm{Hol}_0(\mathbb{C})$ be the germ of holomorphic functions at zero. We consider the (Grothendieck closure of the) tensor algebra

$$T\,(\mathrm{Mer}_0(\mathbb{C})) = \oplus_{k=0}^{\infty} T^k(\mathrm{Mer}_0(\mathbb{C}))$$

over $\mathrm{Mer}_0(\mathbb{C})$ and its subalgebra $T(\mathrm{Hol}_0(\mathbb{C})) := \oplus_{k=0}^{\infty} T^k(\mathrm{Hol}_0(\mathbb{C}))$ where we have set $T^k(\mathrm{Mer}_0(\mathbb{C})) := \hat{\otimes}^k \mathrm{Mer}_0(\mathbb{C})$, $T^k(\mathrm{Hol}_0(\mathbb{C})) := \hat{\otimes}^k \mathrm{Hol}_0(\mathbb{C})$, and where $\hat{\otimes}$ stands for the Grothendieck closure. They come equipped with the product:

$$(f_1 \otimes \cdots \otimes f_k) \bigotimes (f_{k+1} \otimes \cdots \otimes f_{k+l}) = f_1 \otimes \cdots \otimes f_k \otimes f_{k+1} \otimes \cdots \otimes f_{k+l}.$$

We consider the following linear extension of $T^k(\mathrm{Mer}_0(\mathbb{C}))$ which corresponds to germs at zero of meromorphic maps in severable variables with linear poles. Let $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty) := \oplus_{k=1}^{\infty} \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ where

$$\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k) := \left\{ \prod_{i=1}^{m} f_i \circ L_i \;\middle|\; f_i \in \mathrm{Mer}_0(\mathbb{C}), \quad L_i \in \left(\mathbb{C}^k\right)^* \right\}$$

or equivalently,

$$\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k) := \left\{ (z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L \in \left(\mathbb{C}^k\right)^*} (L(z_1, \cdots, z_k))^{m_L}} \;\middle|\; h \in \mathrm{Hol}_0\left(\mathbb{C}^k\right), \quad m_L \in \mathbb{N} \right\}.$$

Setting $m = k$ and $L_i(z_1, \cdots, z_k) = z_i$ yields a canonical injection

$$\begin{aligned} i : T^k(\mathrm{Mer}_0(\mathbb{C})) &\to \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k) \\ f_1 \otimes \cdots \otimes f_k &\mapsto \left((z_1, \cdots, z_k) \mapsto \prod_{i=1}^{k} f_i \circ L_i(z_1, \cdots, z_k) \right), \end{aligned}$$

and the tensor product on $T(\mathrm{Mer}_0(\mathbb{C}))$ extends to $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$, by

$$\left((z_1,\cdots,z_k)\mapsto\prod_{i=1}^m f_i\circ L_i(z_1,\cdots,z_k)\right)\bullet\left((z_1,\cdots,z_l)\mapsto\prod_{j=1}^n f_{m+j}\circ L_{i+j}(z_1,\cdots,z_l)\right)$$
$$=\left((z_1,\cdots,z_k,\cdots,z_{k+l})\mapsto\prod_{i=1}^m f_i\circ L_i(z_1,\cdots,z_k)\prod_{j=1}^n f_{m+j}\circ L_{m+j}(z_{k+1},\cdots,z_{k+l})\right) \tag{57}$$

which makes it a graded algebra.

Specializing to linear forms $\mathcal{L}_k := \{L\in\left(\mathbb{C}^k\right)^* \mid \exists\, J\subset\{1,\cdots,k\},\quad L(z_1,\cdots,z_k)=\sum_{j\in J}z_j\}$ gives rise to a subalgebra $\mathcal{L}\mathcal{M}_0(\mathbb{C}^\infty):=\oplus_{k=1}^\infty\mathcal{L}\mathcal{M}_0(\mathbb{C}^k)\subset\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ defined by:

$$\mathcal{L}\mathcal{M}_0(\mathbb{C}^k):=\left\{(z_1,\cdots,z_k)\mapsto\frac{h(z_1,\cdots,z_k)}{\prod_{L\in\mathcal{L}_k}(L(z_1,\cdots,z_k))^{m_L}}\quad\middle|\quad h\in\mathrm{Hol}_0\left(\mathbb{C}^k\right),\quad m_L\in\mathbb{N}\right\}.$$

For future use, we consider the map $\delta^*:\mathcal{L}\mathcal{M}_0(\mathbb{C}^k)\to\mathrm{Mer}_0(\mathbb{C})$ defined by

$$\delta_k^\star:\mathcal{L}\mathcal{M}_0(\mathbb{C}^k)\to\mathrm{Mer}_0(\mathbb{C}),\qquad f\mapsto f\circ\delta_k,$$

induced by the diagonal map $\delta:\mathbb{C}\mapsto T(\mathbb{C})$ previously defined.

By definition of the twisted regularization $\tilde{\mathcal{R}}^*$, the expressions $\sum_<^{\mathrm{Chen}}\tilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)$ are linear combinations of expressions of the type $\sum_<^{\mathrm{Chen}}\tau_1(u_1)\otimes\cdots\otimes\tau_l(u_k)$ with symbols $\tau_j(u_j)$ built from products of the $\sigma_i(z_i)$'s. It therefore follows from Theorem 6.9, that the functions $(z_1,\cdots,z_k)\mapsto\sum_<^{\mathrm{Chen}}\tilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)$ lie in $\mathcal{L}\mathcal{M}_0(\mathbb{C}^k)$. Since the stuffle relations are satisfied for convergent nested sums, given two tensor products $\sigma=\sigma_1\otimes\cdots\otimes\sigma_k$ and $\tau=\tau_1\otimes\cdots\otimes\tau_l$ of symbols in $\mathcal{P}$, setting $\sigma_i(z_i):=\mathcal{R}(\sigma_i)(z_i)$, for $\mathrm{Re}(z_i)$ sufficiently large we have:

$$\sum_<^{\mathrm{Chen}}\left(\tilde{\mathcal{R}}^*\left(\sigma_1\otimes\cdots\otimes\sigma_k\right)(z_1,\cdots,z_k)\right)*\left(\tilde{\mathcal{R}}^*\left(\tau_1\otimes\cdots\otimes\tau_l\right)(z_{k+1},\cdots,z_{k+l})\right)$$
$$=\left(\sum_<^{\mathrm{Chen}}\tilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right)\left(\sum_<^{\mathrm{Chen}}\tilde{\mathcal{R}}^*(\tau_1\otimes\cdots\otimes\tau_l)(z_{k+1},\cdots,z_{k+l})\right).$$

By analytic continuation (see for example [29], in particular the Identity Theorem in Chapter 1, Section A, or [45]), this holds as an identity of meromorphic functions. Since

$$\left(\tilde{\mathcal{R}}^*\left((\sigma_1\otimes\cdots\otimes\sigma_k)*(\tau_1\otimes\cdots\otimes\tau_l)\right)(z_1,\cdots,z_{k+l})\right)_{\mathrm{sym}}$$
$$=\left(\left(\tilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)*\tilde{\mathcal{R}}^*(\tau_1\otimes\cdots\otimes\tau_)\right)(z_{k+1},\cdots,z_{k+l})\right)_{\mathrm{sym}},$$

symmetrization in the variables z_i yields

$$\left(\sum_<^{\mathrm{Chen}}\left(\tilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)*\tilde{\mathcal{R}}^*(\tau_1\otimes\cdots\otimes\tau_l)\right)(z_1,\cdots,z_{k+l})\right)_{\mathrm{sym}}$$
$$=\left(\left(\sum_<^{\mathrm{Chen}}\tilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right)\left(\sum_<^{\mathrm{Chen}}\tilde{\mathcal{R}}^*(\tau_1\otimes\cdots\otimes\tau_l)(z_{k+1},\cdots,z_{k+l})\right)\right)_{\mathrm{sym}}.$$

This can be reformulated as follows.

Theorem 6.11. [55] *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}^{*,0}$ and let $\mathcal{R}$ be a holomorphic regularization which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q.*

(a) *The map*

$$\begin{aligned} \Psi^{\mathcal{R}} : (T(\mathcal{A}), *) &\to (\mathcal{L}\mathcal{M}_0(\mathbb{C}^\infty), \bullet) \\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \left((z_1, \cdots, z_k) \mapsto \sum_<^{Chen} \tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) \right), \end{aligned}$$

satisfies the following relation:

$$\left(\Psi^{\mathcal{R}}(\sigma * \tau)\right)_{\rm sym} = \left(\Psi^{\mathcal{R}}(\sigma) \bullet \Psi^{\mathcal{R}}(\tau)\right)_{\rm sym},$$

which holds as an equality of meromorphic functions in several variables. Here, as before the subscript sym stands for the symmetrization in the complex variables z_i.

(b) *After composition with δ^* this in turn gives rise to a map*

$$\begin{aligned} \psi^{\mathcal{R}} : (T(\mathcal{A}), *) &\to \mathrm{Mer}_0(\mathbb{C}) \\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \left(z \mapsto \delta^* \circ \sum_<^{Chen} \tilde{\mathcal{R}}^\star(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) \right), \end{aligned} \tag{58}$$

which is an algebra morphism. In other words, $\psi^{\mathcal{R}}$ satisfies the relation:

$$\psi^{\mathcal{R}}(\sigma * \tau) = \psi^{\mathcal{R}}(\sigma) \cdot \psi^{\mathcal{R}}(\tau),$$

which holds as an equality of meromorphic functions in one variable.

6.5. **Renormalized nested sums of symbols.** We want to extract finite parts from the meromorphic functions in Theorem 6.11 while preserving the stuffle relations using a renormalization procedure. Renormalized evaluators inspired from generalized evaluators used in physics provide a first renormalization procedure.

6.5.1. *Renormalized nested sums via renormalized evaluators.* We call regularized evaluator at zero on the germ $\mathrm{Mer}_0(\mathbb{C})$ of meromorphic functions around zero, any linear form on $\mathrm{Mer}_0(\mathbb{C})$ which extends the evaluation at zero $\mathrm{ev}_0 : h \mapsto h(0)$ on holomorphic germs at zero. The map $\mathrm{ev}_0^{\rm reg}$ defined by

$$\mathrm{ev}_0^{\rm reg} := \mathrm{ev}_0 \circ (I - \Pi),$$

where $\Pi : \mathrm{Mer}_0(\mathbb{C}) \to \mathrm{Mer}_0(\mathbb{C})$ as defined in Example 2.4 corresponds to the projection onto the pole part of the Laurent expansion at zero, is such a regularized evaluator at zero. When we need to specify the complex variable z we also write $\mathrm{ev}_{z=0}^{\rm reg}$. Following Speer [67] we introduce renormalized evaluators which correspond to his generalized evaluators.

Definition 6.12. *A renormalized evaluator Λ on a graded subalgebra $\mathcal{B} = \oplus_{k=0}^\infty \mathcal{B}_k$ of $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty) = \oplus_{k=0}^\infty \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ equipped with the product $\bullet$ introduced in (57), is a character on $\mathcal{B}$ which is compatible with the filtration induced by the grading and extends the ordinary evaluation at zero on holomorphic maps. Equivalently,*

(a) *Compatibility with the filtration: Let $\mathcal{B}^K := \oplus_{k=0}^K \mathcal{B}_k$ and $\Lambda_K := \Lambda_{|_{\mathcal{B}^K}}$. Then $\Lambda_{K+1|_{\mathcal{B}^K}} = \Lambda_K$.*

(b) *It coincides with the evaluation map at zero on holomorphic maps:*

$$\Lambda_{|_{T(\mathrm{Hol}_0(\mathbb{C}))}} = \mathrm{ev}_0.$$

(c) *It fulfills a multiplicativity property:*

$$\Lambda(f \bullet g) = \Lambda(f)\,\Lambda(g), \quad \forall f, g \in \mathcal{B}.$$

We call the evaluator symmetric if moreover for any f in $\mathcal{B}_k$ and τ in Σ_k, we have

$$\Lambda(f_\tau) = \Lambda(f), \quad \forall \tau \in \Sigma_k,$$

where we have set $f_\tau(z_1, \cdots, z_k) := f(z_{\tau(1)}, \cdots, z_{\tau(k)})$.

Example 6.13. *Any regularized evaluator at zero λ on $\mathrm{Mer}_0(\mathbb{C})$ uniquely extends to a renormalized evaluator $\tilde{\lambda}$ on the tensor algebra $(T\,(\mathrm{Mer}_0(\mathbb{C}))\,, \otimes)$ defined by*

$$\tilde{\lambda}(f_1 \otimes \cdots \otimes f_k) = \prod_{i=1}^{k} \lambda(f_i).$$

Example 6.14. *Any regularized evaluator λ on $\mathrm{Mer}_0(\mathbb{C})$ extends to renormalized evaluators Λ and Λ' on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ defined on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ by*

$$\Lambda := \lambda_{z_1} \circ \cdots \circ \lambda_{z_k}, \quad \Lambda' := \lambda_{z_k} \circ \cdots \circ \lambda_{z_1}$$

and to a symmetrized evaluator defined on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ by

$$\Lambda^{\mathrm{sym}} := \frac{1}{k!} \sum_{\tau \in \Sigma_k} \lambda_{z_{\tau(1)}} \circ \cdots \circ \lambda_{z_{\tau(k)}},$$

where λ_{z_i} stands for the evaluator λ implemented in the sole variable z_i, the others being kept fixed. Their restrictions to $T\,(\mathrm{Mer}_0(\mathbb{C}))$ all coincide with $\tilde{\lambda}$.

Example 6.15. *Take $\lambda := \mathrm{ev}_0^{\mathrm{reg}}$, and set with the above notations*

$$\mathrm{ev}_0^{\mathrm{ren}} := \Lambda; \quad \mathrm{ev}_0^{\mathrm{ren}'} := \Lambda', \quad \mathrm{ev}_0^{\mathrm{ren,sym}} := \Lambda^{\mathrm{sym}},$$

then given a holomorphic function $h(z_1, z_2)$ in a neighborhood of 0 and setting $f(z_1, z_2) := \frac{h(z_1, z_2)}{z_1 + z_2}$, we have

$$\mathrm{ev}_0^{\mathrm{ren}}(f) = \partial_1 h(0,0);\ \mathrm{ev}_0^{\mathrm{ren}'}(f) = \partial_2 h(0,0);\ \mathrm{ev}_0^{\mathrm{ren,sym}}(f) = \frac{\partial_1 h(0,0) + \partial_2 h(0,0)}{2} = \mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*(f),$$

though in general,

$$\mathrm{ev}_0^{\mathrm{ren,sym}} \neq \mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*.$$

Proposition 6.16. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}^{*,0}$ and let $\mathcal{R}$ be a holomorphic regularization which sends a symbol f to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q. Let $\mathcal{E}$ be a* symmetrized *renormalized evaluator on $\mathcal{L}\mathcal{M}_0$. The map*

$$\begin{array}{rcl} \Psi^{\mathcal{R},\mathcal{E}} : (T\,(\mathcal{A})\,, *) & \to & \mathbb{C} \\ \sigma_1 \otimes \cdots \otimes \sigma_k & \mapsto & \mathcal{E} \circ \Psi^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k) \end{array}$$

defines a character. In other words, $\Psi^{\mathcal{R},\mathcal{E}}$ satisfies the stuffle relation:

$$\Psi^{\mathcal{R},\mathcal{E}}(\sigma * \tau) \;=\; \Psi^{\mathcal{R},\mathcal{E}}(\sigma) \cdot \Psi^{\mathcal{R},\mathcal{E}}(\tau).$$

Remark 6.17. Here, we use the fact that for a symmetrized evaluator Λ we have $\Lambda(f) = \Lambda(f_{\mathrm{sym}})$ where as before the subscript “sym” stands for the symmetrization in the complex variables z_i.

This proposition gives rise to renormalized nested sums of symbols

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} \sigma_1\otimes\cdots\otimes\sigma_k := \Psi^{\mathcal{R},\mathcal{E}}(\sigma_1\otimes\cdots\otimes\sigma_k)$$

which obey stuffle relations:

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} (\sigma * \tau) = \left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} \sigma\right)\left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} \tau\right).$$

6.5.2. *Renormalized nested sums via algebraic Birkhoff decomposition.* On the other hand, the tensor algebra $T(\mathcal{A})$ can be equipped with the deconcatenation coproduct:

$$\Delta\left(\sigma_1\otimes\cdots\otimes\sigma_k\right) := \sum_{j=0}^{k}\left(\sigma_1\otimes\cdots\otimes\sigma_j\right)\bigotimes\left(\sigma_{j+1}\otimes\cdots\otimes\sigma_k\right)$$

which then inherits a structure of connected graded commutative Hopf algebra [42]. Using the convolution product $\star$ associated with the product and coproduct on $T(\mathcal{A})$ and since $\mathrm{Mer}_0(\mathbb{C})$ embeds into the Rota-Baxter algebra $\mathbb{C}[\varepsilon^{-1},\varepsilon]]$ we can implement an algebraic Birkhoff decomposition as in (31) to the map $\psi^{\mathcal{R}}$ in Eq. (58):

$$\psi^{\mathcal{R}} = \left(\psi_-^{\mathcal{R}}\right)^{\star(-1)}\star\psi_+^{\mathcal{R}}$$

associated with the minimal substraction scheme to build characters

$$\psi_+^{\mathcal{R}}(0) : (T(\mathcal{A}), *)\to\mathbb{C}.$$

Proposition 6.18. [55] *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}^{*,0}$ and let $\mathcal{R}$ be a holomorphic regularization which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - qz$ for some positive real number q. The map*

$$\begin{array}{rcl}\psi^{\mathcal{R},\mathrm{Birk}} : (T(\mathcal{A}), *) & \to & \mathbb{C}\\ \sigma_1\otimes\cdots\otimes\sigma_k & \mapsto & \psi_+^{\mathcal{R}}(0)(\sigma_1\otimes\cdots\otimes\sigma_k)\end{array}$$

defines a character

$$\psi^{\mathcal{R},\mathrm{Birk}}(\sigma*\tau) \;=\; \psi^{\mathcal{R},\mathrm{Birk}}(\sigma)\cdot\psi^{\mathcal{R},\mathrm{Birk}}(\tau).$$

The map yields an alternative set of renormalized nested sums of symbols

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} \sigma_1\otimes\cdots\otimes\sigma_k := \psi^{\mathcal{R},\mathrm{Birk}}(\sigma_1\otimes\cdots\otimes\sigma_k)$$

which obey stuffle relations:

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} (\sigma * \tau) = \left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} \sigma\right)\left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} \tau\right).$$

6.6. **Renormalized (Hurwitz) multiple zeta values at non-positive integers.**

6.6.1. *An algebra of symbols.* Since we consider both zeta and Hurwitz zeta functions, let us first observe that for any non-negative number v and any σ in $\mathcal{P}^{*,k}$, the map $\xi \mapsto t_v^*\sigma(\xi) := \sigma(\xi + v)$ defines a symbol in $\mathcal{P}^{*,k}$.

Let $\widetilde{\mathcal{A}}$ be the subalgebra of $\mathcal{P}^{*,0}$ generated by the continuous functions with support inside the interval $(0,1)$ and the set

$$\{\sigma \in \mathcal{P}^{*,0} \,\big|\, \exists v \in [0,+\infty), \exists s \in \mathbb{C},\ \sigma(\xi) = (\xi + v)^{-s} \text{ when } \xi \geq 1\}.$$

Consider the ideal $\mathcal{N}$ of $\widetilde{\mathcal{A}}$ of continuous functions with support inside the interval $(0,1)$. The quotient algebra $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{N}$ is then generated by elements $\sigma_{s,v} \in \mathcal{P}^{*,0}$ with $\sigma_{s,v}(\xi) = (\xi + v)^{-s}$ for $|\xi| \geq 1$. For any $v \in \mathbb{R}_+$ the subspace $\mathcal{A}_v$ of $\mathcal{A}$ generated by $\{\sigma_{s,v} \mid s \in \mathbb{C}\}$ is a subalgebra of $\mathcal{A}$. We equip $\mathcal{A}_v$ with the following holomorphic regularization on an open neighborhood Ω of 0 in $\mathbb{C}$:

$$\begin{aligned} \mathcal{R} : \mathcal{A}_v &\to \mathrm{Hol}_\Omega(\mathcal{A}_v) \\ \sigma_{s,v} &\mapsto (z \mapsto (1-\chi)\,\sigma_{s,v} + \chi\,\sigma_{s+z,v}) \end{aligned}$$

where χ is any smooth cut-off function which is identically one outside the unit ball and vanishes in a small neighborhood of 0.

Let $\mathcal{W}$ be the $\mathbb{C}$-vector space freely spanned symbols indexed by sequences $(u_1, \ldots, u_k)$ of real numbers. In other words, $\mathcal{W}$ is $T(W)$ where $W = \oplus_{u\in\mathbb{R}}\mathbb{R}x_u$ where we identify x_u with u for simplicity and set $x_u \cdot x_v = x_{u+v}, u, v \in \mathbb{R}$. We then define the stuffle product on $\mathcal{W}$ as usual in Eq. (8) or Eq. (11) with $\lambda = 1$. The map

$$\sigma : \mathcal{W} \to T(\mathcal{A}_v), \quad u = (u_1, \cdots, u_k) \mapsto \sigma_{u;v} := \sigma_{(u_1,\ldots,u_k;\,v)} := \sigma_{u_1;v} \otimes \cdots \otimes \sigma_{u_k;v}$$

induces a stuffle product on $\mathcal{T}(\mathcal{A}_v)$:

$$\sigma_{u;v} * \sigma_{u';v} := \sigma_{u*u';v}.$$

As before, we twist the regularization $\widetilde{\mathcal{R}}$ induced by $\mathcal{R}$ on $T(\mathcal{A}_v)$ by the Hoffman isomorphism (14) to build a twisted holomorphic regularization $\widetilde{\mathcal{R}}^*$ in several variables which satisfies

$$\left(\widetilde{\mathcal{R}}^*(\sigma_{u;v}) * \widetilde{\mathcal{R}}^*(\sigma_{u';v})\right)_{\mathrm{sym}} = \left(\widetilde{\mathcal{R}}^*(\sigma_{u*u';v})\right)_{\mathrm{sym}}$$

and a twisted holomorphic regularization $\delta^* \circ \widetilde{\mathcal{R}}^*$ in one variable compatible with the stuffle product:

$$\left(\delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u;v})\right) * \left(\delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u';v})\right) = \delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u*u';v}).$$

6.6.2. *Multiple zeta values renormalized via renormalized evaluators.* Let Ω be an open neighborhood of 0 in $\mathbb{C}$ and let $\mathcal{R} : \sigma \mapsto \{\sigma(z)\}_{z\in\Omega}$ be the holomorphic regularization procedure on $\widetilde{\mathcal{A}}$ previously introduced. The multiple Hurwitz zeta functions defined by:

$$\zeta(s_1, \ldots, s_k;\ v_1, \ldots, v_k) \quad := \quad \Psi^{\mathcal{R}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k})$$

are meromorphic in all variables with poles[1] on a countable family of hyperplanes $s_1 + \cdots + s_j \in\,]-\infty, j] \cap \mathbb{Z}$, j varying from 1 to k. When $v_1 = \cdots = v_k = v$, we set

$$\zeta(s_1, \ldots, s_k;\ v) := \zeta(s_1, \ldots, s_k;\ v_1, \ldots, v_k)$$

[1]When $k = 2$ and $v_1 = \cdots = v_l = v$ a more refined analysis actually shows that for some any negative real number v, poles actually only arise for $s_1 = -1$ or $s_1 + s_2 \in \{-2, -1, 0, 2, 4, 6, \cdots\}$.

in which case they satisfy the following relations:

$$\left(\zeta^{\mathcal{E}}(u * u'; v)\right)_{\mathrm{sym}} = \left(\zeta^{\mathcal{E}}(u; v)\, \zeta^{\mathcal{E}}(u'; v)\right)_{\mathrm{sym}}.$$

The renormalized multiple Hurwitz zeta values derived from a *symmetrized* renormalized evaluator $\mathcal{E}$ on $\mathcal{LM}_0(\mathbb{C}^\infty)$:

$$\zeta^{\mathcal{E}}(s_1, \ldots, s_k;\, v_1, \ldots, v_k) \quad := \quad \Psi^{\mathcal{R},\mathcal{E}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k})$$

denoted by $\zeta^{\mathcal{R},\mathcal{E}}(s_1, \ldots, s_k;\, v)$ when $v_1 = \cdots = v_k = v$, satisfy stuffle relations in that case:

$$\zeta^{\mathcal{E}}(u * u'; v) = \zeta^{\mathcal{E}}(u; v)\, \zeta^{\mathcal{E}}(u'; v).$$

Let us compute renormalized values in the case $k = 2$ using a renormalized evaluator. For any $a \in \mathbb{R}$ and $m \in \mathbb{N} - \{0\}$ we introduce the notation:

$$[a]_j := a(a-1)\cdots(a-j+1).$$

We extend this to $j = 0$ and $j = -1$ by setting: $[a]_0 := 1, [a]_{-1} := \frac{1}{a+1}$. Combining Definition (55)

$$\widetilde{\mathcal{R}}^*(\sigma_1 \otimes \sigma_2)(z_1, z_2) = \sigma_1(z_1) \otimes \sigma_2(z_2) - \frac{1}{2}(\sigma_1 \bullet \sigma_2)(z_1) + \frac{1}{2}\sigma_1(z_1) \bullet \sigma_2(z_2)$$

applied to the regularization

$$\mathcal{R}(\sigma_i)(z)(x) = (x+v)^{-s_i - z} \quad \text{of} \quad \text{order} \quad \alpha_i(z) = -s_i - z_i,$$

with the Euler-MacLaurin formula (49), and following [55] (see the proof of Theorem 9), we compute

$$\begin{aligned}
\zeta(s_1, s_2; v)(z_1, z_2) &= \Psi^{\mathcal{R}}(\sigma_{s_1,v} \otimes \sigma_{s_2,v})(z_1, z_2) \\
&= \sum_{<}^{\mathrm{Chen}} \sigma_1(z_1) \otimes \sigma_2(z_2) + \frac{1}{2}\sigma_1(z_1)\, \sigma_2(z_2) - \frac{1}{2}\,(\sigma_1\, \sigma_2)(z_1) \\
&= \sum_{j=0}^{2J_2} B_j\, \frac{[-s_2 - z_2]_{j-1}}{j!}\, (\zeta(s_1 + s_2 + z_1 + z_2 + j - 1;\, v) - \zeta(s_1 + z_1;\, v)) \\
&\quad + \frac{1}{2}\zeta(s_1 + s_2 + z_1 + z_2;\, v) - \frac{1}{2}\,\zeta(s_1 + s_2 + z_1;\, v) \\
&\quad + \frac{[-s_2 - z_2]_{2J_2+1}}{(2J_2+1)!} \sum_0^\infty \left((n+v)^{-s_1 - z_1} \int_1^n \overline{B_{2J_l+1}}(y)\, (y+v)^{-s_2 - z_2 - 2J_2 - 1}\, dy\right).
\end{aligned}$$

Hence, for non-positive integers $s_1 = -a_1, s_2 = -a_2$ and $2J_2 = a_1 + a_2 + 2$ we have:

$$\begin{aligned}
\zeta(-a_1, -a_2; v)(z_1, z_2) &= \sum_{j=0}^{a_1+a_2+2} B_j\, \frac{[a_2 - z_2]_{j-1}}{j!}\, (\zeta(-a_1 - a_2 + z_1 + z_2 + j - 1;\, v) - \zeta(-a_1 + z_1;\, v)) \\
&\quad + \frac{1}{2}\zeta(-a_1 - a_2 + z_1 + z_2;\, v) - \frac{1}{2}\,\zeta(-a_1 - a_2 + z_1;\, v) \qquad (59) \\
&\quad + \frac{[a_2 - z_2]_{a_2+2}}{(a_2+2)!} \sum_0^\infty \left((n+v)^{a_1 - z_1} \int_1^n \overline{B_{a_1+a_2+3}}(y)\, (y+v)^{-2}\, dy\right).
\end{aligned}$$

The last line on the r.h.s. is a holomorphic expression at zero on which all renormalized evaluators at zero vanish. The second line on the r.h.s is a linear combination of ordinary zeta functions at negative integers which are holomorphic at zero. Any evaluator Λ at zero vanishes on these terms; indeed, we have $\Lambda\left(\zeta(-a_1-a_2+z_1+z_2;\, v)-\zeta(-a_1-a_2+z_1;\, v)\right)=\zeta(-a_1-a_2;\, v)-\zeta(-a_1-a_2;\, v)=0$. Only when evaluated on the expression on the first line of the r.h.s can various evaluators differ.

We want to implement the symmetrized evaluator at zero

$$\mathrm{ev}_0^{\mathrm{ren,sym}}:=\frac{1}{2}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\circ\mathrm{ev}_{z_1=0}^{\mathrm{reg}}+\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\circ\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\right)$$

introduced in Example 6.15. We first compute

$$\begin{aligned}
&\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\zeta(-a_1,-a_2;v)(z_1,z_2)\right)\right)\\
=\;&\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\sum_{j=0}^{a_1+a_2+2}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+z_1+z_2+j-1;\,v)-\zeta(-a_1+z_1;\,v)\right)\right)\right)\\
=\;&\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\sum_{j=0}^{a_2+1}B_j\,\frac{[a_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+z_1+j-1;\,v)-\zeta(-a_1+z_1;\,v)\right)\right)\\
=\;&\frac{1}{a_2+1}\sum_{j=0}^{a_2+1}B_j\binom{a_2+1}{j}\left(\zeta(-a_1-a_2+j-1;\,v)-\zeta(-a_1;\,v)\right).
\end{aligned}$$

When $v=0$ this yields:

$$\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\zeta(-a_1,-a_2)(z_1,z_2)\right)\right):=\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\zeta(-a_1,-a_2)(z_1,z_2;0)\right)\right)$$

$$=\frac{1}{a_2+1}\sum_{j=0}^{a_2+1}B_j\binom{a_2+1}{j}\left(-\frac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}+\frac{B_{a_1+1}}{a_1+1}\right). \tag{60}$$

We next compute

$$\begin{aligned}
&\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\zeta(-a_1,-a_2;v)(z_1,z_2)\right)\right)\\
=\;&\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\frac{B_0}{a_2-z_2+1}\left(\zeta(-a_1-a_2+z_1+z_2-1;\,v)-\zeta(-a_1+z_1;\,v)\right)\right)\right)\\
&+\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\sum_{j=1}^{a_1+1}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+z_1+z_2+j-1;\,v)-\zeta(-a_1+z_1;\,v)\right)\right)\right)\\
&+\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\sum_{j=a_1+2}^{a_1+a_2+2}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+z_1+z_2+j-1;\,v)-\zeta(-a_1+z_1;\,v)\right)\right)\right)\\
=\;&\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\frac{B_0}{a_2+1}\left(\zeta(-a_1-a_2+z_2-1;\,v)-\zeta(-a_1;\,v)\right)\right)\\
&+\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\sum_{j=1}^{a_1+1}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+z_2+j-1;\,v)-\zeta(-a_1;\,v)\right)\right)\\
&+\sum_{j=1}^{a_2+1}B_{j+a_1+1}\,\partial_{z_2}\left(\frac{[a_2-z_2]_{j+a_1}}{(j+a_1+1)!}\right)_{|_{z_2=0}}\mathrm{Res}_{z_2=0}\left(\zeta(-a_2+z_2+j;\,v)\right)
\end{aligned}$$

$$= \frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j \binom{a_2+1}{j} \left(\zeta(-a_1-a_2+j-1;\ v) - \zeta(-a_1;\ v)\right)$$
$$+(-1)^{a_1+1} a_1! a_2! \, \frac{B_{a_1+a_2+2}}{(a_1+a_2+2)!},$$

since the only contribution to the residue comes from the term $j = a_1 + a_2 + 2$. Since $\zeta(-a) := \zeta(-a; 0) = -\frac{B_{a+1}}{a+1}$, this combined with (50) applied to $k = a_2 + 1$ yields

$$\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\zeta(-a_1,-a_2)(z_1,z_2)\right)\right) := \mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\zeta(-a_1,-a_2;0)(z_1,z_2)\right)\right)$$
$$(61) \qquad = -\frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j \binom{a_2+1}{j} \frac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2} + \frac{B_{a_1+1}}{a_1+1}\,\frac{B_{a_2+1}}{a_2+1}$$
$$+(-1)^{a_1+1} a_1! a_2! \, \frac{B_{a_1+a_2+2}}{(a_1+a_2+2)!}.$$

Combining (60) and (61) yields

$$\zeta^{\mathrm{ev}}(-a_1,-a_2) := \mathrm{ev}_0^{\mathrm{ren,sym}}\left(\zeta(-a_1,-a_2)(z_1,z_2)\right)$$
$$(62) \qquad = -\frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j \binom{a_2+1}{j} \frac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}$$
$$+\frac{B_{a_1+1}}{a_1+1}\,\frac{B_{a_2+1}}{a_2+1} + (-1)^{a_1+1} a_1! a_2! \, \frac{B_{a_1+a_2+2}}{2\,(a_1+a_2+2)!}.$$

Renormalized multiple zeta values of depth 2 at non-positive arguments obtained this way, are rational linear combinations of Bernoulli numbers, and hence rational numbers. More generally, an inductive procedure on k carried out in the same spirit as the proof of Theorem 10 in [55] shows that the renormalized multiple zeta values $\zeta^{\mathcal{E}}(s_1, \cdots, s_k; v)$ are rational values at non-positive integer arguments $s_1, \cdots, s_k$ whenever v is rational.

6.6.3. *Multiple zeta values renormalized via Birkhoff decomposition.* The renormalized multiple Hurwitz zeta values derived from a Birkhoff decomposition:

$$\zeta^{\mathrm{Birk}}(s_1,\ldots,s_k;\ v_1,\ldots,v_k) \quad := \quad \Psi^{\mathcal{R},\mathrm{Birk}}(\sigma_{s_1,v_1}\otimes\cdots\otimes\sigma_{s_k,v_k})$$

denoted by $\zeta^{\mathrm{Birk}}(s_1, \ldots, s_k;\ v)$ when $v_1 = \cdots = v_k = v$, satisfy stuffle relations

$$\zeta^{\mathrm{Birk}}(u * u'; v) = \zeta^{\mathrm{Birk}}(u; v)\,\zeta^{\mathrm{Birk}}(u'; v).$$

A striking holomorphy property arises at non-positive integer arguments [55] after implementing the diagonal map δ.

Proposition 6.19. *At non-positive integer arguments s_i and for a rational parameter v, the map*

$$z \mapsto \psi^{\mathcal{R}}\left(\sigma_{s_1,v}\otimes\cdots\otimes\sigma_{s_k,v}\right)(z)$$

defined in (58) is holomorphic at zero.

Consequently,

$$\zeta^{\mathrm{Birk}}(s_1,\ldots,s_k;\ v) = \lim_{z\to 0}\psi^{\mathcal{R}}(\sigma_{s_1,v}\otimes\cdots\otimes\sigma_{s_k,v}).$$

Let us compute double zeta values at non-positive integer arguments using Birkhoff decomposition. Setting $z_1 = z_2 = z$ in (59) leads to

$$\begin{aligned}\zeta(-a_1,-a_2;v)(z) &= \sum_{j=0}^{a_2+1} B_j \frac{[a_2-z]_{j-1}}{j!}\left(\zeta(-a_1-a_2+2z+j-1;\,v)-\zeta(-a_1+z;\,v)\right)\\ &\quad+\frac{[a_2-z]_{a_2+2}}{(a_2+2)!}\sum_0^\infty\left((n+v)^{a_1-z}\int_1^n \overline{B_{a_1+a_2+3}}(y)\,(y+v)^{-2}\,dy\right).\end{aligned}$$

Evaluating this expression at $z=0$ in a similar manner to the previous computation, yields:

$$\begin{aligned}\zeta^{\mathrm{Birk}}(-a_1,-a_2;v) &= \lim_{z\to 0}\left(\sum_{j=0}^{a_2+1} B_j \frac{[a_2-z]_{j-1}}{j!}\left(\zeta(-a_1-a_2+2z+j-1;\,v)-\zeta(-a_1+z;\,v)\right)\right)\\ &= \sum_{j=0}^{a_2+1} B_j \frac{[a_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+j-1;\,v)-\zeta(-a_1;\,v)\right)\\ &\quad+(-1)^{a_1+1}a_1!a_2!\,\frac{B_{a_1+a_2+2}}{2\,(a_1+a_2+2)!}.\end{aligned}$$

When $v=0$ this yields [55]:

$$\begin{aligned}\zeta^{\mathrm{Birk}}(-a_1,-a_2) &:= \zeta^{\mathrm{Birk}}(-a_1,-a_2;0)\\ &= -\frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j\binom{a_2+1}{j}\frac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}\\ &\quad+\frac{B_{a_1+1}}{a_1+1}\,\frac{B_{a_2+1}}{a_2+1}+(-1)^{a_1+1}a_1!a_2!\,\frac{B_{a_1+a_2+2}}{2\,(a_1+a_2+2)!}\end{aligned}$$

which coincides with (62).

Thus, renormalized double zeta values at non-positive integers obtained by two different methods – using the symmetrized renormalized evaluator $\mathrm{ev}_0^{\mathrm{ren,sym}}$ or a Birkhoff decomposition– coincide.

Formula (62) yields the following table of values $\zeta(-a_1,-a_2)$ for $a_1, a_2 \in \{0,\ldots,6\}$ derived in [55]:

(63)

$\zeta(-a,-b)$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
$b=0$	$\frac{3}{8}$	$\frac{1}{12}$	$\frac{7}{720}$	$-\frac{1}{120}$	$-\frac{11}{2\,520}$	$\frac{1}{252}$	$\frac{1}{224}$
$b=1$	$\frac{1}{24}$	$\frac{1}{288}$	$-\frac{1}{240}$	$-\frac{19}{10\,080}$	$\frac{1}{504}$	$\frac{41}{20\,160}$	$-\frac{1}{480}$
$b=2$	$-\frac{7}{720}$	$-\frac{1}{240}$	0	$\frac{1}{504}$	$\frac{113}{151\,200}$	$-\frac{1}{480}$	$-\frac{307}{166\,320}$
$b=3$	$-\frac{1}{240}$	$\frac{1}{840}$	$\frac{1}{504}$	$\frac{1}{28\,800}$	$-\frac{1}{480}$	$-\frac{281}{332\,640}$	$\frac{1}{264}$
$b=4$	$\frac{11}{2\,520}$	$\frac{1}{504}$	$-\frac{113}{151\,200}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$\frac{117\,977}{75\,675\,600}$
$b=5$	$\frac{1}{504}$	$-\frac{103}{60\,480}$	$-\frac{1}{480}$	$\frac{1}{1232}$	$\frac{1}{264}$	$\frac{1}{127\,008}$	$-\frac{691}{65\,520}$
$b=6$	$-\frac{1}{224}$	$-\frac{1}{480}$	$\frac{307}{166\,320}$	$\frac{1}{264}$	$-\frac{117\,977}{75\,675\,600}$	$-\frac{691}{65\,520}$	0

This table of values differs from the one derived in [36] (see Table (45)) with which it however matches for arguments (a, b) with $a + b$ odd and $b \neq 0$ and for diagonal arguments $(-a, -a)$.

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