

# Bifunctor cohomology and Cohomological finite generation for reductive groups

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## Abstract

Let  $G$  be a reductive linear algebraic group over a field  $k$ . Let  $A$  be a finitely generated commutative  $k$ -algebra on which  $G$  acts rationally by  $k$ -algebra automorphisms. Invariant theory tells that the ring of invariants  $A^G = H^0(G, A)$  is finitely generated. We show that in fact the full cohomology ring  $H^*(G, A)$  is finitely generated. The proof is based on the strict polynomial bifunctor cohomology classes constructed in [22]. We also continue the study of bifunctor cohomology of  $\Gamma^*(\mathfrak{gl}^{(1)})$ .

## 1 Introduction

Consider a linear algebraic group  $G$ , or linear algebraic group scheme  $G$ , defined over a field  $k$ . So  $G$  is an affine group scheme whose coordinate algebra  $k[G]$  is finitely generated as a  $k$ -algebra. We say that  $G$  has the cohomological finite generation property (CFG) if the following holds. Let  $A$  be a finitely generated commutative  $k$ -algebra on which  $G$  acts rationally by  $k$ -algebra automorphisms. (So  $G$  acts from the right on  $\text{Spec}(A)$ .) Then the cohomology ring  $H^*(G, A)$  is finitely generated as a  $k$ -algebra.

Here, as in [12, I.4], we use the cohomology introduced by Hochschild, also known as ‘rational cohomology’.

Our main result confirms a conjecture of the senior author:

**Theorem 1.1** *Any reductive linear algebraic group over  $k$  has property (CFG).*

The proof will be based on the ‘lifted’ universal classes [22] constructed by the junior author for this purpose. Originally [22] was the end of the proof, but for the purpose of exposition we have changed the order.

If the field  $k$  has characteristic zero, then the theorem just reiterates a standard fact in invariant theory. Indeed the reductive group is then linearly reductive and rational cohomology vanishes in higher degrees for any linearly reductive group.

So we further assume  $k$  has positive characteristic  $p$ . In this introduction we will also take  $k$  algebraically closed. (One easily reduces to this case, cf. [24, Lemma 2.3], [12, I 4.13], [25].) We will say that  $G$  acts on the algebra  $A$  if  $G$  acts rationally by  $k$ -algebra automorphisms.

Let us say that  $G$  has the finite generation property (FG), or a positive solution to Hilbert’s 14-th problem, if the following holds. If  $G$  acts on a finitely generated commutative  $k$ -algebra  $A$ , then the ring of invariants  $A^G = H^0(G, A)$  is finitely generated as a  $k$ -algebra. Observe that, unlike Hilbert, we do *not* require that  $A$  is a domain.

It is obvious that (CFG) implies (FG). We will see that our main result can also be formulated as follows

**Theorem 1.2** *A linear algebraic group scheme  $G$  over  $k$  has property (CFG) if and only if it has property (FG).*

Let us give some examples. The first example is a finite group  $G$ , viewed as a discrete algebraic group over  $k$ . It is well known to have property (FG), [24, Lemma 2.4], and the proof goes back to Emmy Noether 1926 [17]. Thus we recover the finite generation theorem of Evens, at least over our field  $k$ :

**Theorem 1.3 (Evens 1961 [6])** *A finite group has property (CFG), over  $k$ .*

As our proof of Theorem 1.2 does not rely on theorem 1.3, we get a new proof of 1.3, albeit much longer than the original proof. Note that the setting of Evens is more general: Instead of a field he allows an arbitrary noetherian base. This suggests a direction for further work.

If  $G$  is a linear algebraic group over  $k$ , we write  $G_r$  for its  $r$ -th Frobenius kernel, the scheme theoretic kernel of the  $r$ -th iterate  $F^r : G \rightarrow G^{(r)}$  of the Frobenius homomorphism [12, I Ch. 9]. It is easy to see that  $G_r$  has property (FG). More generally it is easy to see [24, Lemma 2.4] that any finite group scheme over  $k$  has property (FG). (A group scheme is finite if its coordinate

ring is a finite dimensional vector space.) Indeed one has [24, Theorem 3.5]:

**Theorem 1.4 (Friedlander and Suslin 1997 [9])** *A finite group scheme has property (CFG).*

But we do not get a new proof of this theorem, as our proof of the main result relies heavily on the specific information in [9, section 1]. Recall that the theorem of Friedlander and Suslin was motivated by a desire to get a theory of support varieties for infinitesimal group schemes. Our problem has the same origin. Then it started to get a life of its own and became a conjecture.

It is a theorem of Nagata [16], [19, Ch. 2] that geometrically reductive groups (or group schemes [3]) have property (FG). (Springer [19] deletes ‘geometrically’ in the terminology.) Conversely, it is elementary [24, Th. 2.2] that property (FG) implies geometric reductivity. (Here it is essential that in property (FG) one allows any finitely generated commutative  $k$ -algebra on which  $G$  acts.) So our main result states that property (CFG) is equivalent to geometric reductivity.

Now Haboush has shown [12, II.10.7] that reductive groups are geometrically reductive, and Popov [18] has shown that a linear algebraic group with property (FG) is reductive. (Popov allows only reduced algebras, so his result is even stronger.) Waterhouse has completed the classification by showing [25] that a linear algebraic group scheme  $G$  (he calls it an ‘algebraic affine group scheme’) is geometrically reductive exactly when the connected component  $G_{\text{red}}^o$  of its reduced subgroup  $G_{\text{red}}$  is reductive. So this is also a characterization of the  $G$  with property (CFG).

Let us now give a consequence of (CFG). We say that  $G$  acts on an  $A$ -module  $M$  when it acts rationally on  $M$  such that the structure map  $A \otimes M \rightarrow M$  is a  $G$ -module map.

**Theorem 1.5** *Let  $G$  have property (CFG). Let  $G$  act on the finitely generated commutative  $k$ -algebra  $A$  and on the noetherian  $A$ -module  $M$ . Then  $H^*(G, M)$  is a noetherian  $H^*(G, A)$ -module. In particular, if  $G$  is reductive and  $A$  has a good filtration, then  $H^*(G, M)$  is a noetherian  $A^G$ -module,  $H^i(G, M)$  vanishes for large  $i$ , and  $M$  has finite good filtration dimension.*

**Proof** See [24, Lemma 3.3, proof of 4.7]. One puts an algebra structure on  $A \oplus M$  and uses that  $A \otimes k[G/U]$  also has a good filtration.  $\square$

As special case we mention

**Theorem 1.6** *Let  $G = GL_n$ ,  $n \geq 1$ . Let  $G$  act on the finitely generated commutative  $k$ -algebra  $A$  and on the noetherian  $A$ -module  $M$ . If  $A$  has a good filtration, then  $H^*(G, M)$  is a noetherian  $A^G$ -module,  $H^i(G, M)$  vanishes for large  $i$ , and  $M$  has finite good filtration dimension.*

This theorem is proved directly in [21], with functorial resolution of the ideal of the diagonal in a product of Grassmannians. It will be used in our proof of the main theorems.

Now let us start discussing the proof of the main result. First of all one has the following variation on the ancient transfer principle [11, Chapter Two].

**Lemma 1.7** ([24, Lemma 3.7]) *Let  $G$  be a linear algebraic group over  $k$  with property (CFG). Then any geometrically reductive subgroup scheme  $H$  of  $G$  also has property (CFG).*

As every geometrically reductive linear algebraic group scheme is a subgroup scheme of  $GL_n$  for  $n$  sufficiently large, we only have to look at the  $GL_n$  to prove the main theorems, Theorem 1.1 and Theorem 1.2. Therefore we further assume  $G = GL_n$  with  $n > 1$ . (Or  $n \geq p$  if you wish.) (In [23] we used  $SL_n$  instead of  $G = GL_n$ , but also explained it hardly makes any difference.)

We have  $G$  act on  $A$  and wish to show  $H^*(G, A)$  is finitely generated. If  $A$  has a good filtration [12], then there is no higher cohomology and invariant theory (Haboush) does the job. A general  $A$  has been related to one with a good filtration by Grosshans. He defines a filtration  $A_{\leq 0} \subseteq A_{\leq 1} \cdots$  on  $A$  and embeds the associated graded  $\text{gr } A$  into an algebra with a good filtration  $\text{hull}_{\nabla} \text{gr } A$ . He shows that  $\text{gr } A$  and  $\text{hull}_{\nabla} \text{gr } A$  are also finitely generated and that there is a flat family parametrized by the affine line with special fiber  $\text{gr } A$  and general fiber  $A$ . We write  $\mathcal{A}$  for the coordinate ring of the family. It is a graded algebra and one has natural homomorphisms  $\mathcal{A} \rightarrow \text{gr } A$ ,  $\mathcal{A} \rightarrow A$ . Mathieu has shown [15], cf. [23, Lemma 2.3], that there is an  $r > 0$  so that  $x^{p^r} \in \text{gr } A$  for every  $x \in \text{hull}_{\nabla} \text{gr } A$ . We have no bound on  $r$ , which is the main reason that our results are only qualitative. One sees that  $\text{gr } A$  is a noetherian module over the  $r$ -th Frobenius twist  $(\text{hull}_{\nabla} \text{gr } A)^{(r)}$  of  $\text{hull}_{\nabla} \text{gr } A$ . So we do not quite have the situation of theorem 1.6, but it is close. We have to untwist. Untwisting involves  $G^{(r)} = G/G_r$  and we end up looking at the

Hochschild–Serre spectral sequence

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \text{gr } A)) \Rightarrow H^{i+j}(G, \text{gr } A).$$

One may write  $H^i(G/G_r, H^j(G_r, \text{gr } A))$  also as  $H^i(G, H^j(G_r, \text{gr } A)^{(-r)})$ . By Friedlander and Suslin  $H^*(G_r, \text{gr } A)^{(-r)}$  is a noetherian module over the graded algebra  $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \text{hull}_{\nabla} \text{gr } A$ . Here the  $\#$  refers to taking a dual,  $S^*$  refers to a symmetric algebra over  $k$ , and the  $(2p^{i-1})$  indicates in what degree one puts a copy of the dual of the adjoint representation  $\mathfrak{gl}_n$ . By the fundamental work [1] of Akin, Buchsbaum, Weyman, which is also of essential importance in [21], one knows that  $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \text{hull}_{\nabla} \text{gr } A$  has a good filtration. So  $H^*(G_r, \text{gr } A)^{(-r)}$  has finite good filtration dimension and page 2 of our Hochschild–Serre spectral sequence is noetherian over its first column  $E_2^{0*}$ . By Friedlander and Suslin  $H^*(G_r, \text{gr } A)^{(-r)}$  is a finitely generated algebra and by invariant theory  $E_2^{0*}$  is thus finitely generated, so  $E_2^{**}$  is finitely generated. The spectral sequence is one of graded commutative differential graded algebras in characteristic  $p$ , so the  $p$ -th power of an even cochain in a page passes to the next page. It easily follows that all pages are finitely generated. As page 2 has only finitely many columns by 1.6, cf. [21, 2.3], this explains why the abutment  $H^*(G, \text{gr } A)$  is finitely generated. We are getting closer to  $H^*(G, A)$ .

The filtration  $A_{\leq 0} \subseteq A_{\leq 1} \cdots$  induces a filtration of the Hochschild complex [12, I.4.14] whence a spectral sequence

$$E(A) : E_1^{ij} = H^{i+j}(G, \text{gr}_{-i} A) \Rightarrow H^{i+j}(G, A).$$

It lives in the second quadrant, but as  $E_1^{**}$  is a finitely generated  $k$ -algebra this causes no difficulty with convergence: given  $m$  there will be only finitely many nonzero  $E_1^{m-i, i}$ . (Compare [23, 4.11]. Note that in [23] the  $E_1$  page is mistaken for an  $E_2$  page.) All pages are again finitely generated, so we would like the spectral sequence to stop, meaning that  $E_s^{**} = E_\infty^{**}$  for some finite  $s$ . There is a standard method to achieve this [6], [9]. One must find a ‘ring of operators’ acting on the spectral sequence and show that some page is a noetherian module for the chosen ring of operators. As the ring of operators we take  $H^*(G, \mathcal{A})$ . Indeed  $E(A)$  is acted on by the trivial spectral sequence  $E(\mathcal{A})$  whose pages equal  $H^*(G, \mathcal{A})$ , see [23, 4.11]. And  $H^*(G, \mathcal{A})$  also acts on our Hochschild–Serre spectral sequence through its abutment. If we can show that one of the pages of the Hochschild–Serre spectral sequence is a noetherian module over  $H^*(G, \mathcal{A})$ , then that will do the trick, as then the

abutment  $H^*(G, \text{gr } A)$  is noetherian by [9, Lemma 1.6]. And this abutment is the first page of  $E(A)$ .

Now we are in a situation similar to the one encountered by Friedlander and Suslin. Their problem was ‘surprisingly elusive’. To make their breakthrough they had to invent strict polynomial functors. Studying the homological algebra of strict polynomial functors they found universal cohomology classes  $e_r \in H^{2p^{r-1}}(G, \mathfrak{gl}_n^{(r)})$  with nontrivial restriction to  $G_1$ . That was enough to get through. We faced a similar bottleneck. We know from invariant theory and from [9] that page 2 of our Hochschild–Serre spectral sequence is noetherian over  $H^0(G, (\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathcal{A})^{(r)})$ . But we want it to be noetherian over  $H^*(G, \mathcal{A})$ . So if we could factor the homomorphism  $H^0(G, (\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathcal{A})^{(r)}) \rightarrow E_2^{0*}$  through  $H^*(G, \mathcal{A})$ , then that would do it. The universal classes  $e_j$  provide such a factorization on some summands, but they do not seem to help on the rest. One would like to have universal classes in more degrees so that one can map every summand of the form  $H^0(G, (\bigotimes_{i=1}^r S^{m_i}((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathcal{A})^{(r)})$  into the appropriate  $H^{2m}(G, \mathcal{A})$ , or even into  $H^{2m}(G, \mathcal{A}^{G_r})$ . The dual of  $S^{m_i}((\mathfrak{gl}_n)^\#)^{(r)}$  is  $\Gamma^{m_i}(\mathfrak{gl}_n^{(r)})$ . Thus one seeks nontrivial classes in  $H^{2mp^{i-1}}(G, \Gamma^m(\mathfrak{gl}_n^{(r)}))$ , to take cup product with. It turns out that  $r = i = 1$  is the crucial case and we seek nontrivial classes  $c[m] \in H^{2m}(G, \Gamma^m(\mathfrak{gl}_n^{(1)}))$ . The construction of such classes  $c[m]$  has been a sticking point at least since 2001. In [23] they were constructed for  $GL_2$ , but one needs them for  $GL_n$  with  $n$  large. The strict polynomial functors of Friedlander and Suslin do not provide a natural home for this problem, but the strict polynomial bifunctors [8] of Franjou and Friedlander do.

When the junior author found [22] a construction of nontrivial ‘lifted’ classes  $c[m]$ , this finished a proof of the conjecture. We present two proofs. The first proof continues the investigation of bifunctor cohomology in [22] and establishes properties of the  $c[m]$  analogous to those employed in [23]. Then the result follows as in the proof in [23] for  $GL_2$ . As a byproduct one also obtains extra bifunctor cohomology classes and relations between them. The second proof needs no more properties of the classes  $c[m]$  than those established in [22]. Indeed [22] stops exactly where the two arguments start to diverge. The second proof does not quite factor the homomorphism  $H^0(G, (\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathcal{A})^{(r)}) \rightarrow E_2^{0*}$  through  $H^*(G, \mathcal{A})$ , but argues by induction on  $r$ , returning to [9, section 1] with the new classes in hand. It is not hard to guess which author contributes which proof. The junior author goes first.

## 2 Acknowledgements

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### Part I

## The first proof

### 3 Main theorem and cohomological finite generation

We work over a field  $k$  of positive characteristic  $p$ . We keep the notations of [22]. In particular,  $\mathcal{P}_k(1, 1)$  denotes the category of strict polynomial bifunctors of [8]. The main result of part I is theorem 3.1, which states the existence of classes in the cohomology of the bifunctor  $\Gamma^*(\mathfrak{gl}^{(1)})$ . By [22, Thm 1.3], the cohomology of a bifunctor  $B$  is related to the cohomology of  $GL_{n,k}$  with coefficients in the rational representation  $B(k^n, k^n)$  by a map  $\phi_{B,n} : H_{\mathcal{P}}^*(B) \rightarrow H^*(GL_{n,k}, B(k^n, k^n))$  (natural in  $B$  and compatible with cup products). So our main result yields classes in the cohomology of  $GL_{n,k}$ , actually more classes (and more relations between them) than originally needed [23, Section 4.3] for the proof of the cohomological finite generation conjecture.

**Theorem 3.1** *Let  $k$  be a field of characteristic  $p > 0$ . There are maps  $\psi_\ell : \Gamma^\ell H_{\mathcal{P}}^*(\mathfrak{gl}^{(1)}) \rightarrow H_{\mathcal{P}}^*(\Gamma^\ell(\mathfrak{gl}^{(1)}))$ ,  $\ell \geq 1$  such that*

1.  $\psi_1$  is the identity map.
2. For all  $\ell \geq 1$  and for all  $n \geq p$ , the composite

$$\Gamma^\ell H_{\mathcal{P}}^*(\mathfrak{gl}^{(1)}) \xrightarrow{\psi_\ell} H_{\mathcal{P}}^*(\Gamma^\ell(\mathfrak{gl}^{(1)})) \xrightarrow{\phi_{\Gamma^\ell(\mathfrak{gl}^{(1)}),n}} H^*(GL_{n,k}, \Gamma^\ell(\mathfrak{gl}_n^{(1)}))$$

is injective. In particular, for all  $\ell \geq 1$ ,  $\psi_\ell$  is injective.

3. For all positive integers  $\ell, m$ , there are commutative diagrams

$$\begin{array}{ccc} H_{\mathcal{P}}^*(\Gamma^{\ell+m}(gl^{(1)})) & \xrightarrow{\Delta_{\ell,m}^*} & H_{\mathcal{P}}^*(\Gamma^{\ell}(gl^{(1)}) \otimes \Gamma^m(gl^{(1)})) \\ \psi_{\ell+m} \uparrow & & \psi_{\ell} \cup \psi_m \uparrow \\ \Gamma^{\ell+m} H_{\mathcal{P}}^*(gl^{(1)}) & \xrightarrow{\Delta_{\ell,m}} & \Gamma^{\ell} H_{\mathcal{P}}^*(gl^{(1)}) \otimes \Gamma^m H_{\mathcal{P}}^*(gl^{(1)}) , \end{array}$$

and

$$\begin{array}{ccc} H_{\mathcal{P}}^*(\Gamma^{\ell}(gl^{(1)}) \otimes \Gamma^m(gl^{(1)})) & \xrightarrow{m_{\ell,m}^*} & H_{\mathcal{P}}^*(\Gamma^{\ell+m}(gl^{(1)})) \\ \psi_{\ell} \cup \psi_m \uparrow & & \psi_{\ell+m} \uparrow \\ \Gamma^{\ell} H_{\mathcal{P}}^*(gl^{(1)}) \otimes \Gamma^m H_{\mathcal{P}}^*(gl^{(1)}) & \xrightarrow{m_{\ell,m}} & \Gamma^{\ell+m} H_{\mathcal{P}}^*(gl^{(1)}) , \end{array}$$

where  $m_{\ell,m}$  and  $\Delta_{\ell,m}$  denote the maps induced by the multiplication  $\Gamma^{\ell} \otimes \Gamma^m \rightarrow \Gamma^{\ell+m}$  and the diagonal  $\Gamma^{\ell+m} \rightarrow \Gamma^{\ell} \otimes \Gamma^m$ .

As a consequence, we obtain that [23, Th 4.4] is valid for any value of  $n$ :

**Corollary 3.2** *Let  $k$  be a field of positive characteristic. For all  $n > 1$ , there are classes  $c[m] \in H^{2m}(GL_{n,k}, \Gamma^m(\mathfrak{gl}_n^{(1)}))$  such that*

1.  $c[1]$  is the Witt vector class  $e_1$ ,
2.  $\Delta_{i,j^*}(c[i+j]) = c[i] \cup c[j]$  for  $i, j \geq 1$ .

**Proof** Arguing as in [22, Lemma 1.5] we notice that it suffices to prove the statement when  $n \geq p$ . By [22, Th 1.3], we have morphisms

$$\phi_{\Gamma^m(gl^{(1)}),n} : H_{\mathcal{P}}^*(\Gamma^m(gl^{(1)})) \rightarrow H^*(GL_{n,k}, \Gamma^m(\mathfrak{gl}_n^{(1)}))$$

compatible with the cup products, and for  $m = 1$  the map  $\phi_{\Gamma^1(gl^{(1)}),n}$  is an isomorphism. Let  $b[1]$  be the pre-image of the Witt vector class by  $\phi_{\Gamma^1(gl^{(1)}),n}$ . We define  $c[m] := (\phi_{\Gamma^m(gl^{(1)}),n} \circ \psi_m)(b[1]^{\otimes m})$ . Then  $c[1]$  is the Witt vector class since  $\psi_1$  is the identity, and by theorem 3.1(3) the classes  $c[i]$  satisfy condition 2.  $\square$

**Corollary 3.3** *The cohomological finite generation conjecture (Theorem 1.1) holds.*



**Proof** Let  $G$  be a reductive linear algebraic group acting on a finitely generated commutative  $k$ -algebra  $A$ . We want to prove that  $H^*(G, A)$  is finitely generated. To do this, it suffices to follow [23] and this is exactly what we do below. We keep the notations of the introduction.

By lemma 1.7, the case  $G = GL_{n,k}$  suffices. As recalled in the introduction, there exists a positive integer  $r$  such that the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \text{gr } A)) \Rightarrow H^{i+j}(G, \text{gr } A)$$

stops for a finite good filtration dimension reason. Moreover it is a sequence of finitely generated algebras, and its second page is noetherian over its subalgebra  $E_2^{0*}$  (all this was first proved in [23, Prop 3.8], under some restrictions on the characteristic which were removed in [21]).

The composite  $\mathcal{A}^{G_r} \hookrightarrow \mathcal{A} \rightarrow \text{gr } A$  makes  $\text{gr } A$  into a noetherian module over  $\mathcal{A}^{G_r}$ . Hence, by [9, Thm 1.5] (with ‘ $C$ ’ =  $\mathcal{A}^{G_r}$ ) and by invariant theory [11, Thm 16.9],  $E_2^{0*} = H^0(G/G_r, H^*(G_r, \text{gr } A))$  (hence  $E_2^{**}$ ) is noetherian over  $H^0(G/G_r, \bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes \mathcal{A}^{G_r})$ .

Now we use the classes of corollary 3.2 as in section 4.5 and in the proof of corollary 4.8 of [23]. In this way, we factor the morphism  $H^0(G/G_r, \bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes \mathcal{A}^{G_r}) \rightarrow E_2^{0*}$  through the map  $H^{\text{even}}(G, \mathcal{A}) \rightarrow H^0(G/G_r, H^{\text{even}}(G_r, \mathcal{A})) = E_2^{0 \text{ even}}$  (the latter map is induced by restricting the cohomology from  $G$  to  $G_r$ ). So  $E_2^{**}$  is noetherian over  $H^{\text{even}}(G, \mathcal{A})$ . By [9, Lemma 1.6] (with ‘ $A$ ’ =  $H^{\text{even}}(G, \mathcal{A})$  and ‘ $B$ ’ =  $k$ ), we conclude that the map  $H^{\text{even}}(G, \mathcal{A}) \rightarrow H^*(G, \text{gr } A)$  (induced by  $\mathcal{A} \rightarrow \text{gr } A$ ) makes  $H^*(G, \text{gr } A)$  into a noetherian module over  $H^{\text{even}}(G, \mathcal{A})$ .

The proof finishes as described in the introduction (or in section 4.11 of [23]): the second spectral sequence

$$E(A) : E_1^{ij} = H^{i+j}(G, \text{gr}_{-i} A) \Rightarrow H^{i+j}(G, A)$$

is a sequence of finitely generated algebras. It is acted on by the trivial spectral sequence  $E(\mathcal{A})$  whose pages equal  $H^*(G, \mathcal{A})$ . But we have proved that  $E_1^{**}$  is noetherian over  $H^*(G, \mathcal{A})$ , so by the usual trick ([6], [9] or [24, Lemma 3.9]) the spectral sequence  $E(A)$  stops, which proves that  $H^*(G, A)$  is finitely generated.  $\square$

## 4 Proof of theorem 3.1

By [22, Prop 3.21], the divided powers  $\Gamma^\ell$  admit a twist compatible coresolution  $J_\ell$ . So by [22, Prop 3.18], we have a bicomplex  $A(J_\ell)$  whose totalization yield an  $H_{\mathcal{P}}^*$ -acyclic coresolution of  $\Gamma^\ell(gl^{(1)})$ . In particular the homology of the totalization of  $H_{\mathcal{P}}^0(A(J_\ell))$  computes  $H_{\mathcal{P}}^*(\Gamma^\ell(gl^{(1)}))$ .

The plan of the proof of theorem 3.1 is the following. First, we build the maps  $\psi_\ell$ . To be more specific, we build maps  $\vartheta_\ell$  which send each element of degree  $d$  of  $\Gamma^\ell(H_{\mathcal{P}}^*(gl^{(1)}))$  to a homogeneous cocycle of bidegree  $(0, d)$  in the bicomplex  $H_{\mathcal{P}}^0(A(J_\ell))$ . Our maps  $\psi_\ell$  will then be induced by the  $\vartheta_\ell$ .

Second, we show the relations between the classes on the cochain level. In this step, we encounter the following problem: the cup product of two classes is represented by a cocycle in the bicomplex  $H_{\mathcal{P}}^0(A(J_\ell) \otimes A(J_m))$  while we want to have it represented by a cocycle in  $H_{\mathcal{P}}^0(A(J_\ell \otimes J_m))$ . So we have investigate further the compatibility of the functor  $A$  with cup products.

Finally, we prove theorem 3.1(2) by reducing to one parameter subgroups.

**Notations and sign conventions 4.1** If  $\mathfrak{A}$  is an additive category, we denote by  $Ch^{\geq 0}(\mathfrak{A})$  (resp.  $p\text{-}Ch^{\geq 0}(\mathfrak{A})$ , resp.  $\text{bi-}Ch^{\geq 0}(\mathfrak{A})$ ) the category of non-negative cochain complexes (resp.  $p$ -complexes, resp. bicomplexes) in  $\mathfrak{A}$ .

If  $\mathfrak{A}$  is equipped with a tensor product, then  $Ch^{\geq 0}(\mathfrak{A})$  inherits a tensor product. The differential of the tensor product  $C \otimes D$  involves a Koszul sign: the restriction of  $d_{C \otimes D}$  to  $C^i \otimes D^j$  equals  $d_C \otimes \text{Id} + (-1)^i \text{Id} \otimes d_D$ . The category  $p\text{-}Ch^{\geq 0}(\mathfrak{A})$  also inherits a tensor product, but the  $p$ -differential of  $C \otimes D$  does not involve any sign:  $d_{C \otimes D} = d_C \otimes \text{Id} + \text{Id} \otimes d_D$ .

Now we turn to bicomplexes. First, we may view a complex  $C^\bullet$  whose terms  $C^j$  are chain complexes as a bicomplex  $C^{\bullet, \bullet}$  whose object  $C^{i, j}$  is the  $i$ -th object of the complex  $C^j$  (*i.e.* the complexes  $C^j$  are the rows of  $C^{\bullet, \bullet}$ ). Thus we obtain an identification:

$$Ch^{\geq 0}(Ch^{\geq 0}(\mathfrak{A})) = \text{bi-}Ch^{\geq 0}(\mathfrak{A}).$$

Being a category of cochain complexes, the term on the left hand side has a tensor product. If  $C$  is a bicomplex, let us denote by  $d_C^{i, j} : C^{i, j} \rightarrow C^{i+1, j}$  its first differential, and by  $\partial_C^{i, j} : C^{i, j} \rightarrow C^{i, j+1}$  its second one. Then one checks that the tensor product on bicomplexes induced by the identification is such that the restriction of  $d_{C \otimes D}$  (resp.  $\partial_{C \otimes D}$ ) to  $C^{i_1, j_1} \otimes D^{i_2, j_2}$  equals  $d_C \otimes \text{Id} + (-1)^{i_1} \text{Id} \otimes d_D$  (resp.  $\partial_C \otimes \text{Id} + (-1)^{j_1} \text{Id} \otimes \partial_D$ ).

We define the totalization  $\text{Tot}(C)$  of a bicomplex  $C$  with the Koszul sign convention: the restriction of  $d_{\text{Tot}(C)}$  to  $C^{i,j}$  equals  $d_C + (-1)^i \partial_C$ . If  $C, D$  are two bicomplexes, there is a canonical isomorphism of complexes:  $\text{Tot}(C) \otimes \text{Tot}(D) \simeq \text{Tot}(C \otimes D)$  which sends an element  $x \otimes y \in C^{i_1, j_1} \otimes D^{i_2, j_2}$  to  $(-1)^{j_1 i_2} x \otimes y$ .

#### 4.1 Construction of the $\psi_\ell$ , $\ell \geq 1$

Let  $\ell$  be a positive integer. By [22, Prop 3.18, Prop 3.21] we have a bicomplex  $H_{\mathcal{P}}^0(A(J_\ell))$  whose homology computes the cohomology of the bifunctor  $\Gamma^\ell(gl^{(1)})$ . We now recall the description of the first two columns of this bicomplex. As in [22, Section 4], we denote by  $A_1$  the  $p$ -coresolution of  $gl^{(1)}$  obtained by precomposing the  $p$ -complex  $T(S^1)$  by the bifunctor  $gl$ . The symmetric group  $\mathfrak{S}_\ell$  acts on the  $p$ -complex  $A_1^{\otimes \ell}$  by permuting the factors of the tensor product (unlike the case of ordinary complexes, the action of  $\mathfrak{S}_\ell$  does not involve a Koszul sign since the tensor product of  $p$ -complexes does not involve any sign). Contracting the  $p$ -complex  $A_1^{\otimes \ell}$  and applying  $H_{\mathcal{P}}^0$ , we obtain an action of  $\mathfrak{S}_\ell$  on the ordinary complex  $H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]})$ . By [22, Lemma 4.2], the first two columns  $H_{\mathcal{P}}^*(A(J_\ell)^{0,\bullet}) \rightarrow H_{\mathcal{P}}^*(A(J_\ell)^{1,\bullet})$  of  $H_{\mathcal{P}}^0(A(J_\ell))$  equal

$$\underbrace{H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]})}_{\text{column of index 0}} \xrightarrow{\Pi^{(1-\tau_i)}} \underbrace{\bigoplus_{i=0}^{\ell-2} H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]})}_{\text{column of index 1}},$$

where  $\tau_i \in \mathfrak{S}_\ell$  is the transposition which exchanges  $i+1$  and  $i+2$  (and with the convention that the second column is null if  $\ell = 1$ ). Thus we have:

**Lemma 4.2** *Let  $\mathcal{Z}_\ell^{\text{even}}$  be the set of homogeneous cocycles of bidegree  $(0, d)$ ,  $d$  even, in the bicomplex  $H_{\mathcal{P}}^0(A(J_\ell))$ . Then  $\mathcal{Z}_\ell^{\text{even}}$  identifies as the set of even degree cocycles of the complex  $H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]})$ , which are invariant under the action of  $\mathfrak{S}_\ell$ .*

Now we turn to building a map  $\vartheta_\ell : \Gamma^\ell(H_{\mathcal{P}}^*(gl^{(1)})) \rightarrow \mathcal{Z}_\ell^{\text{even}}$ . In view of lemma 4.2, it suffices to build a  $\mathfrak{S}_\ell$ -equivariant map  $\vartheta_\ell : H_{\mathcal{P}}^*(gl^{(1)})^{\otimes \ell} \rightarrow H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]})$ .

Let us first recall what we know about  $H_{\mathcal{P}}^*(gl^{(1)})$ . By [8, Thm 1.5] and [9, Th. 4.5], the graded vector space  $H_{\mathcal{P}}^*(gl^{(1)})$  is concentrated in degrees  $2i$ ,  $0 \leq i < p$  and one dimensional in these degrees. Following [20], we denote

by  $e_1(i)$  a generator of degree  $2i$  of this graded vector space. The homology of the complex  $H_{\mathcal{P}}^0(A_{1[1]})$  computes the cohomology of the bifunctor  $gl^{(1)}$ . Thus we may choose for each integer  $i$ ,  $0 \leq i < p$ , a cycle  $z_i$  representing the cohomology class  $e_1(i)$  in this complex. The cycles  $z_i$  determine a graded map  $H_{\mathcal{P}}^*(gl^{(1)}) \rightarrow H_{\mathcal{P}}^0(A_{1[1]})$ . By [22, Prop 3.3], we may take cup products on the cochain level to obtain for each  $\ell \geq 1$  a map

$$H_{\mathcal{P}}^*(gl^{(1)})^{\otimes \ell} \rightarrow H_{\mathcal{P}}^0(A_{1[1]})^{\otimes \ell} \xrightarrow{\cup} H_{\mathcal{P}}^0((A_{1[1]})^{\otimes \ell}).$$

Moreover we define chain maps  $h_{\ell} : (A_{1[1]})^{\otimes \ell} \rightarrow (A_1^{\otimes \ell})_{[1]}$  by iterated use of [22, Prop 2.7]. More specifically,  $h_1$  is the identity and  $h_{\ell} = h_{A_1^{\otimes \ell-1}, A_1} \circ (h_{\ell-1} \otimes h_1)$ .

**Lemma 4.3** *Let  $\ell$  be a positive integer and let  $\vartheta_{\ell}$  be the composite*

$$\vartheta_{\ell} := H_{\mathcal{P}}^*(gl^{(1)})^{\otimes \ell} \rightarrow H_{\mathcal{P}}^0((A_{1[1]})^{\otimes \ell}) \xrightarrow{H_{\mathcal{P}}^0(h_{\ell})} H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]}).$$

*Then  $\vartheta_{\ell}$  satisfies the following two properties:*

- (1) *The image of  $\vartheta_{\ell}$  is contained in the set of even degree cocycles of  $H_{\mathcal{P}}^0((A_1^{\otimes \ell})_{[1]})$ .*
- (2)  *$\vartheta_{\ell}$  is  $\mathfrak{S}_{\ell}$ -equivariant.*

**Proof** The first property is straightforward from the definition of  $\vartheta_{\ell}$ . We prove the second one. The map  $H_{\mathcal{P}}^*(gl^{(1)})^{\otimes \ell} \rightarrow H_{\mathcal{P}}^0((A_{1[1]})^{\otimes \ell})$  is defined using cup products, hence it is  $\mathfrak{S}_{\ell}$ -equivariant. Thus, to prove the lemma, we have to study the map  $h_{\ell} : (A_{1[1]})^{\otimes \ell} \rightarrow (A_1^{\otimes \ell})_{[1]}$ .

Recall that  $h_{\ell}$  is built by iterated uses of [22, Prop 2.7]. Thus, if we define the graded object  $p(A_1, \dots, A_1) = \bigoplus_{i_1, \dots, i_{\ell}} \bigotimes_{s=1}^{\ell} A_1^{i_s p}$  with the component  $\bigotimes_{s=1}^{\ell} A_1^{i_s p}$  in degree  $2(\sum i_s)$ , we have well defined inclusions of  $p(A_1, \dots, A_1)$  into the complexes  $(A_{1[1]})^{\otimes \ell}$  and  $(A_1^{\otimes \ell})_{[1]}$ . Moreover  $h_{\ell}$  fits into a commutative diagram:

$$\begin{array}{ccc} (A_{1[1]})^{\otimes \ell} & \xrightarrow{h_{\ell}} & (A_1^{\otimes \ell})_{[1]} \\ (a) \uparrow & & \uparrow (b) \\ p(A_1, \dots, A_1) & \xlongequal{\quad} & p(A_1, \dots, A_1). \end{array}$$

Let  $\mathfrak{S}_{\ell}$  act on  $p(A_1, \dots, A_1)$  by permuting the factors of the tensor product, on  $(A_{1[1]})^{\otimes \ell}$  by permuting the factors of the tensor product with a Koszul

sign, and on  $(A_1^{\otimes \ell})_{[1]}$  by permuting the factors of the tensor product  $A_1^{\otimes \ell}$  (without sign). Then the map (b) is equivariant, and the map (a) is also equivariant since  $p(A_1, \dots, A_1)$  is concentrated in even degrees. The map  $h_\ell$  is *not* equivariant. However, by definition the equivariant map  $H_{\mathcal{P}}^*(gl^{(1)})^{\otimes \ell} \rightarrow H_{\mathcal{P}}^0((A_1)_{[1]})^{\otimes \ell}$  factors through  $H_{\mathcal{P}}^0(p(A_1, \dots, A_1))$  so that postcomposition of this map by  $H_{\mathcal{P}}^0(h_\ell)$  (ie: the map  $\vartheta_\ell$ ) is in fact equivariant.  $\square$

**Notation 4.4** By lemmas 4.2 and 4.3, for all  $\ell \geq 1$ , the map  $\vartheta_\ell$  induces a map  $\Gamma^\ell(H_{\mathcal{P}}^*(gl^{(1)})) \rightarrow \mathcal{Z}_\ell^{\text{even}}$ . We denote by  $\psi_\ell$  the composite

$$\psi_\ell := \Gamma^\ell(H_{\mathcal{P}}^*(gl^{(1)})) \rightarrow \mathcal{Z}_\ell^{\text{even}} \rightarrow H_{\mathcal{P}}^*(\Gamma^\ell(gl^{(1)})) .$$

**Lemma 4.5** *The map  $\psi_1$  equals the identity map.*

**Proof** For  $\ell = 1$ ,  $\vartheta_1$  is just the map  $H_{\mathcal{P}}^*(gl^{(1)}) \rightarrow \text{Hom}(\Gamma^p(gl), A_1)_{[1]}$  which sends the generator  $e_1(i)$  of  $H_{\mathcal{P}}^{2i}(gl^{(1)})$  to the cycle  $z_i$  representing this generator. Moreover, by definition of  $z_i$ , the map  $\mathcal{Z}_1^{\text{even}} \rightarrow H_{\mathcal{P}}^*(gl^{(1)})$  sends  $z_i$  to  $e_1(i)$ . Thus, for all  $i$ ,  $\psi_1$  sends  $e_1(i)$  to itself.  $\square$

## 4.2 Proof of theorem 3.1(3)

Let  $\mathcal{P}_k$  be the strict polynomial functor category and let  $\mathcal{TP}_k$  be the twist compatible subcategory [22, Def 3.9]. Before proving theorem 3.1(3), we need to study further properties of the functor  $A : Ch^{\geq 0}(\mathcal{TP}_k) \rightarrow \text{bi-}Ch^{\geq 0}(\mathcal{P}_k(1, 1))$  [22, Def. 3.17]. Recall that  $A$  is defined as the composite of the following three functors:

1. The ‘Troesch coresolution functor’ [22, Prop 3.13]

$$T : Ch^{\geq 0}(\mathcal{TP}_k) \rightarrow p\text{-}Ch^{\geq 0}(Ch^{\geq 0}(\mathcal{P}_k)) ,$$

2. The contraction functor

$$-_{[1]} : p\text{-}Ch^{\geq 0}(Ch^{\geq 0}(\mathcal{P}_k)) \rightarrow Ch^{\geq 0}(Ch^{\geq 0}(\mathcal{P}_k)) ,$$

3. Precomposition by the bifunctor  $gl$

$$- \circ gl : Ch^{\geq 0}(Ch^{\geq 0}(\mathcal{P}_k)) \rightarrow Ch^{\geq 0}(Ch^{\geq 0}(\mathcal{P}_k(1, 1))) = \text{bi-}Ch^{\geq 0}(\mathcal{P}_k(1, 1)) .$$

All the categories coming into play in the definition of  $A$  are equipped with tensor products (cf. notations and sign conventions 4.1). The functors  $T$  and  $- \circ gl$  commute with tensor products, but  $-_{[1]}$  does not. As a result, if  $F, G$  are homogeneous strict polynomial functors of respective degree  $f, g$  and with respective twist compatible coresolutions  $J_F, J_G$ , we have two (in general non isomorphic)  $H_{\mathcal{P}}^*$ -acyclic coresolutions of the tensor product  $F \otimes G$  at our disposal:

$$\mathrm{Tot}(A(J_F)) \otimes \mathrm{Tot}(A(J_G)) \quad \text{and} \quad \mathrm{Tot}(A(J_F \otimes J_G)).$$

Now the problem is the following. On the one hand, cycles representing cup products of classes in the cohomology of  $F$  and  $G$  are easily identified using the first complex. Indeed, by [22, Prop 3.3], the cup product

$$H_{\mathcal{P}}^*(F(gl^{(1)})) \otimes H_{\mathcal{P}}^*(G(gl^{(1)})) \rightarrow H_{\mathcal{P}}^*(F(gl^{(1)}) \otimes G(gl^{(1)}))$$

is defined at the cochain level by sending cocycles  $x$  and  $y$  respectively in  $\mathrm{Hom}(\Gamma^{pf}(gl), \mathrm{Tot}(A(J_F)))$  and  $\mathrm{Hom}(\Gamma^{pg}(gl), \mathrm{Tot}(A(J_G)))$  to the cocycle

$$x \cup y := (x \otimes y) \circ \Delta_{pf,pg} \in \mathrm{Hom}(\Gamma^{p(f+g)}(gl), \mathrm{Tot}(A(J_F)) \otimes \mathrm{Tot}(A(J_G))),$$

where  $\Delta_{pf,pg}$  is the diagonal map  $\Gamma^{p(f+g)}(gl) \rightarrow \Gamma^{pf}(gl) \otimes \Gamma^{pg}(gl)$ . But on the other hand, by functoriality of  $A$ , if  $E \in \mathcal{P}_k$  then the effect of a morphism  $E \rightarrow F \otimes G$  is easily computed in  $H_{\mathcal{P}}^0(\mathrm{Tot}(A(J_F \otimes J_G)))$ . So we want to be able to identify cup products in  $H_{\mathcal{P}}^0(\mathrm{Tot}(A(J_F \otimes J_G)))$  rather than in  $H_{\mathcal{P}}^0(\mathrm{Tot}(A(J_F)) \otimes \mathrm{Tot}(A(J_G)))$ . This is the purpose of next lemma.

**Lemma 4.6** *Let  $F, G$  be homogeneous strict polynomial functors of degree  $f, g$  which admit twist compatible coresolutions  $J_F$  and  $J_G$ . Let  $i, j, \ell, m$  be nonnegative integers, and let*

$$x_{i,2j} \in \mathrm{Hom}_{\mathcal{P}_{pf}^{pf}}(\Gamma^{pf}(gl), A(J_F)) \quad , \quad y_{\ell,2m} \in \mathrm{Hom}_{\mathcal{P}_{pg}^{pg}}(\Gamma^{pg}(gl), A(J_G))$$

be homogeneous cocycles of respective bidegrees  $(i, 2j)$  and  $(\ell, 2m)$ .

1. *The object  $A(J_F)^{i,2j} \otimes A(J_G)^{\ell,2m}$  appears once and only once in the bicomplex  $A(J_F \otimes J_G)$ . It appears in bidegree  $(i + \ell, 2j + 2m)$ . In particular, the formula*

$$(x_{i,2j} \otimes y_{\ell,2m}) \circ \Delta_{pf,pg} \in \mathrm{Hom}(\Gamma^{p(f+g)}(gl), A(J_F)^{i,2j} \otimes A(J_G)^{\ell,2m})$$

*defines a homogeneous element of bidegree  $(i + \ell, 2j + 2m)$  in the bicomplex  $H_{\mathcal{P}}^0(A(J_F \otimes J_G))$ .*

2. The element  $(x_{i,2j} \otimes y_{\ell,2m}) \circ \Delta_{pf,pg}$  is actually a cocycle, and represents the cup product  $[x_{i,2j}] \cup [y_{\ell,2m}]$  in  $H_{\mathcal{P}}^0(\text{Tot}(A(J_F \otimes J_G)))$ .

**Proof** Since  $T$  commutes with tensor products [22, Prop 3.13], the bicomplex  $A(J_F \otimes J_G)$  is naturally isomorphic to the precomposition by  $gl$  of the bicomplex  $(T(J_F) \otimes T(J_G))_{[1]}$ , while  $A(J_F) \otimes A(J_G)$  equals the precomposition by  $gl$  of the bicomplex  $T(J_F)_{[1]} \otimes T(J_G)_{[1]}$ . Recall that in the identification of  $Ch^{\geq 0}(Ch^{\geq 0}(\mathcal{P}_k(1, 1)))$  and  $bi-Ch^{\geq 0}(\mathcal{P}_k(1, 1))$ , the  $j$ -th object of a complex of complexes  $C^\bullet$  corresponds to the  $j$ -th row of the bicomplex  $C^{\bullet, \bullet}$  (that is the elements of bidegree  $(*, j)$ ). So the first statement simply follows from [22, Lemma 2.2]. Furthermore, by [22, Prop 2.4] there is a map of bicomplexes

$$A(J_F) \otimes A(J_G) \rightarrow A(J_F \otimes J_G)$$

which is the identity on  $A(J_F)^{i,2j} \otimes A(J_G)^{\ell,2m}$ . Applying the functor  $\text{Tot}$  we obtain a map of  $H_{\mathcal{P}}^*$ -acyclic coresolutions

$$\theta : \text{Tot}(A(J_F)) \otimes \text{Tot}(A(J_G)) \simeq \text{Tot}(A(J_F) \otimes A(J_G)) \rightarrow \text{Tot}(A(J_F \otimes J_G))$$

over the identity map of  $F(gl^{(1)}) \otimes G(gl^{(1)})$ , and whose restriction to  $A(J_F)^{i,2j} \otimes A(J_G)^{\ell,2m}$  equals the identity. (more specifically, this equality holds up to a  $(-1)^{2j\ell} = 1$  sign coming from the sign in the formula  $\text{Tot}(C \otimes D) \simeq \text{Tot}(C) \otimes \text{Tot}(D)$ ).

By definition of the cup product,  $(x_{i,2j} \otimes y_{\ell,2m}) \circ \Delta_{pf,pg}$  is a cocycle representing  $[x_{i,2j}] \cup [y_{\ell,2m}]$  in  $H_{\mathcal{P}}^0(\text{Tot}(A(J_F)) \otimes \text{Tot}(A(J_G)))$ . Applying  $\theta$  we obtain that  $(x_{i,2j} \otimes y_{\ell,2m}) \circ \Delta_{pf,pg}$  is a cocycle in  $H_{\mathcal{P}}^0(\text{Tot}(A(J_F \otimes J_G)))$ , representing the same cup product.  $\square$

We now turn to the specific situation of theorem 3.1(3), that is  $F = \Gamma^\ell$  and  $G = \Gamma^m$ . We first determine explicit maps between the bicomplexes  $A(J_\ell \otimes J_m)$  and  $A(J_{\ell+m})$ , which lift the multiplication  $\Gamma^\ell(gl^{(1)}) \otimes \Gamma^m(gl^{(1)}) \rightarrow \Gamma^{\ell+m}(gl^{(1)})$  and the diagonal  $\Gamma^{\ell+m}(gl^{(1)}) \rightarrow \Gamma^\ell(gl^{(1)}) \otimes \Gamma^m(gl^{(1)})$ . To do this, we first need new information about the twist compatible coresolutions  $J_\ell$  from [22, Prop 3.21].

**Lemma 4.7** *Let  $\ell, m$  be positive integers.*

1. The multiplication  $\Gamma^\ell \otimes \Gamma^m \rightarrow \Gamma^{\ell+m}$  lifts to a twist compatible chain map  $J_\ell \otimes J_m \rightarrow J_{\ell+m}$ . This chain map is given in degree 0 by the shuffle product  $(\otimes^\ell) \otimes (\otimes^m) = J_\ell^0 \otimes J_m^0 \rightarrow J_{\ell+m}^0 = \otimes^{\ell+m}$  which sends a tensor  $\otimes_{i=1}^{m+\ell} x_i$  to the sum  $\sum_{\sigma \in Sh(\ell, m)} \otimes_{i=1}^{m+\ell} x_{\sigma^{-1}(i)}$ .

2. The diagonal  $\Gamma^{\ell+m} \rightarrow \Gamma^\ell \otimes \Gamma^m$  lifts to a twist compatible chain map  $J_{\ell+m} \rightarrow J_\ell \otimes J_m$ . This chain map equals the identity map in degree 0.

**Proof** The reduced bar construction yields a functor from the category of Commutative Differential Graded Augmented algebras over  $k$  to the category of Commutative Differential Graded bialgebras over  $k$ , see [13], resp. [7], for the algebra, resp. coalgebra, structure (this bialgebra structure is actually a Hopf algebra structure but we don't need this fact).

The category of strict polynomial functors splits as a direct sum of subcategories of homogeneous functors. Taking the  $(m + \ell)$  polynomial degree part of the multiplication (resp. comultiplication) of  $\overline{B}(\overline{B}(S^*(-)))$  we obtain chain maps  $\bigoplus J_i^\bullet \otimes J_j^\bullet \rightarrow J_{\ell+m}^\bullet$  and  $J_{\ell+m}^\bullet \rightarrow \bigoplus J_i^\bullet \otimes J_j^\bullet$  (the sums are taken over all nonnegative integers  $i, j$  such that  $i + j = \ell + m$ ). The bialgebra structure of  $\overline{B}(\overline{B}(S^*(-)))$  is defined using only the *algebra* structure of  $S^*$ . But the multiplication of  $S^*$  is a twist compatible map and the twist compatible category is additive and stable under tensor products [22, Lemma 3.8 and 3.10]. So the chain maps are twist compatible.

Next, we identify the chain maps in degree 0. We begin with the map  $J_\ell^0 \otimes J_m^0 \rightarrow J_{\ell+m}^0$  induced by the multiplication of the bar construction. By [22, Lemma 3.18], for all  $i \geq 1$  we have  $J_i^0 = \overline{B}_1(S^*(-))^{\otimes i} = \otimes^i$ . The product  $\overline{B}(\overline{B}(S^*(-)))^{\otimes 2} \rightarrow \overline{B}(\overline{B}(S^*(-)))$  is given by the shuffle product formula [13, p.313], more precisely it sends the tensor  $\otimes_{i=1}^{m+\ell} x_i$  to the sum  $\sum_{\sigma \in Sh(\ell, m)} \otimes_{i=1}^{m+\ell} x_{\sigma^{-1}(i)}$ . The signs in this shuffle product are all positive since the  $x_i$  are elements of degree  $1 + 1 = 2$  in the chain complex  $\overline{B}_\bullet(\overline{B}(S^*(-)))$ . The identification of the map  $J_{m+\ell}^0 \rightarrow J_m^0 \otimes J_\ell^0$  induced by the diagonal is simpler. The coproduct in  $\overline{B}(\overline{B}(S^*(-)))$  is given by the deconcatenation formula [7, p.268]:  $\Delta[x_1 | \dots | x_{\ell+m}] = \sum_{i=0}^{m+\ell} [x_1 | \dots | x_i] \otimes [x_{i+1} | \dots | x_{m+\ell}]$ . Thus, the map  $J_{m+\ell}^0 \rightarrow J_m^0 \otimes J_\ell^0$  sends the tensor product  $\otimes_{i=1}^{m+\ell} x_i$  to itself.

Finally, with the description of the chain maps in degree 0, one easily checks that  $J_\ell \otimes J_m \rightarrow J_{\ell+m}$ , resp.  $J_{\ell+m} \rightarrow J_\ell \otimes J_m$ , lifts the multiplication  $\Gamma^\ell \otimes \Gamma^m \rightarrow \Gamma^{\ell+m}$ , resp. the comultiplication  $\Gamma^{\ell+m} \rightarrow \Gamma^\ell \otimes \Gamma^m$ . (In fact, this actually proves that the quasi isomorphism  $\Gamma^* \rightarrow \overline{B}(\overline{B}(S^*(-)))$  is a Hopf algebra morphism)  $\square$

Applying the functor  $A$ , we obtain:

**Lemma 4.8** *Let  $\ell, m$  be positive integers.*



1. The multiplication  $\Gamma^\ell(\mathfrak{gl}^{(1)}) \otimes \Gamma^m(\mathfrak{gl}^{(1)}) \rightarrow \Gamma^{\ell+m}(\mathfrak{gl}^{(1)})$  lifts to a map of bicomplexes  $A(J_\ell \otimes J_m) \rightarrow A(J_{\ell+m})$ . The restriction of this map to the columns of index 0 equals

$$A(J_\ell \otimes J_m)^{0,\bullet} = (A_1^{\otimes \ell+m})_{[1]} \xrightarrow{sh_{[1]}} (A_1^{\otimes \ell+m})_{[1]} = A(J_{\ell+m})^{0,\bullet}$$

where  $sh$  is the unsigned shuffle map, which sends a tensor  $\otimes_{i=1}^{m+\ell} x_i$  to the sum  $\sum_{\sigma \in Sh(\ell, m)} \otimes_{i=1}^{m+\ell} x_{\sigma^{-1}(i)}$ .

2. The diagonal  $\Gamma^{\ell+m}(\mathfrak{gl}^{(1)}) \rightarrow \Gamma^\ell(\mathfrak{gl}^{(1)}) \otimes \Gamma^m(\mathfrak{gl}^{(1)})$  lifts to a twist compatible chain map  $A(J_{\ell+m}) \rightarrow A(J_\ell \otimes J_m)$ . The restriction of this map to the columns of index 0 equals the identity map of  $(A_1^{\otimes \ell+m})_{[1]}$ .

Next we identify cycles representing the cup products  $\psi_\ell(x) \cup \psi_m(y)$  in the bicomplex  $H_{\mathcal{P}}^0(A(J_\ell \otimes J_m))$ .

**Lemma 4.9** *Let  $x \in \Gamma^\ell H_{\mathcal{P}}^*(\mathfrak{gl}^{(1)})$  and  $y \in \Gamma^m H_{\mathcal{P}}^*(\mathfrak{gl}^{(1)})$  be classes of homogeneous degrees  $2d$  and  $2e$ . Then  $\vartheta_{\ell+m}(x \otimes y)$  is a cocycle of bidegree  $(0, 2d)$  in the bicomplex  $H_{\mathcal{P}}^0(A(J_\ell \otimes J_m))$ . Moreover, it represents the cup product  $\psi_\ell(x) \cup \psi_m(y) \in H_{\mathcal{P}}^*(\Gamma^\ell(\mathfrak{gl}^{(1)}) \otimes \Gamma^m(\mathfrak{gl}^{(1)}))$ .*

**Proof** By definition,  $\psi_\ell(x)$  is represented by the homogeneous cocycle  $\vartheta_\ell(x)$  of bidegree  $(0, 2d)$  in the bicomplex  $\text{Hom}(\Gamma^{p^\ell}(\mathfrak{gl}), A(J_\ell))$  (and similarly for  $\psi_m(y)$ ). Then, by lemma 4.6,  $\psi_\ell(x) \cup \psi_m(y)$  is represented by the cocycle  $(\vartheta_\ell(x) \otimes \vartheta_m(y)) \circ \Delta_{\ell p, m p}$  in the bicomplex  $\text{Hom}(\Gamma^{p(\ell+m)}(\mathfrak{gl}), A(J_\ell \otimes J_m))$ . Now if  $x = \otimes_{s=1}^\ell (e(i_s))$  and  $y = \otimes_{s=\ell+1}^m (e(i_s))$ , we compute that  $\vartheta_{\ell+m}(x \otimes y)$  and  $(\vartheta_\ell(x) \otimes \vartheta_m(y)) \circ \Delta_{\ell p, m p}$  both equal the element  $(\otimes_{i=1}^{\ell+m} z_i) \circ \Delta_{p, \dots, p}$ , where  $\Delta_{p, \dots, p}$  is the diagonal  $\Gamma^{p(\ell+m)}(\mathfrak{gl}) \rightarrow \Gamma^p(\mathfrak{gl})^{\otimes \ell+m}$ .  $\square$

We are now ready to prove theorem 3.1(3). We begin with the commutativity of the diagram involving the multiplication. Let  $x \in \Gamma^\ell H_{\mathcal{P}}^*(\mathfrak{gl}^{(1)})$  and  $y \in \Gamma^m H_{\mathcal{P}}^*(\mathfrak{gl}^{(1)})$  be homogeneous elements of respective degrees  $2d$  and  $2e$ . By lemmas 4.8 and 4.9,  $m_{\ell, m}(\psi_\ell(x) \cup \psi_m(y))$  is represented by the cocycle

$$\sum_{\sigma \in Sh(\ell, m)} \sigma \cdot \vartheta_{\ell+m}(x \otimes y)$$

of bidegree  $(0, 2d+2e)$  in the bicomplex  $H_{\mathcal{P}}^0(A(J_{\ell+m}))$ . By definition of  $\psi_{\ell+m}$ ,  $\psi_{\ell+m}(m_{\ell, m}(x \otimes y))$  is represented by the cocycle

$$\vartheta_{\ell+m} \left( \sum_{\sigma \in Sh(\ell, m)} \sigma \cdot (x \otimes y) \right)$$

in the same bicomplex. Since  $\vartheta_{\ell+m}$  is equivariant (lemma 4.3), these two cocycles are equal. Hence, the diagram involving the multiplication is commutative. The diagram involving the comultiplication commutes for a similar reason: if  $x \in \Gamma^{\ell+m} H_{\mathcal{P}}^*(gl^{(1)})$ , the cohomology classes  $(\psi_{\ell} \cup \psi_m)(\Delta_{\ell,m}(x))$  and  $\Delta_{\ell,m*}(\psi_{\ell+m}(x))$  are both represented by the cycle  $\vartheta_{\ell+m}(x)$ . This concludes the proof of theorem 3.1(3).

### 4.3 Proof of theorem 3.1(2)

To prove theorem 3.1(2), it suffices to prove for all  $n \geq p$  the injectivity of the composite:

$$\Gamma^{\ell} H_{\mathcal{P}}^*(gl^{(1)}) \xrightarrow{\psi_{\ell}} H_{\mathcal{P}}^*(\Gamma^{\ell}(gl^{(1)})) \xrightarrow{\phi_{\Gamma^{\ell}(gl^{(1)})}, n} H^*(GL_{n,k}, \Gamma^{\ell}(\mathfrak{gl}_n^{(1)})) \xrightarrow{\Delta_{1, \dots, 1}^*} H^*(GL_{n,k}, \mathfrak{gl}_n^{(1)\otimes \ell}).$$

By naturality of the maps  $\phi_{\Gamma^{\ell}(gl^{(1)})}, n$  [22, Th 1.3] and by the compatibility of the  $\phi_i$  with diagonals and cup products given in theorem 3.1(3), this composite equals the composite:

$$\Gamma^{\ell} H_{\mathcal{P}}^*(gl^{(1)}) \hookrightarrow H_{\mathcal{P}}^*(gl^{(1)})^{\otimes \ell} \rightarrow H^*(GL_{n,k}, \mathfrak{gl}_n^{(1)})^{\otimes \ell} \xrightarrow{\cup} H^*(GL_{n,k}, \mathfrak{gl}_n^{(1)\otimes \ell})$$

Thus, the proof of theorem 3.1(2) follows from:

**Lemma 4.10** *Let  $k$  be a field of characteristic  $p > 0$  and let  $j \geq 1$  be an integer. For all  $n \geq p$ , the following map is injective:*

$$\begin{aligned} \cup_{i=1}^j \phi_{gl^{(1)}, n} : H_{\mathcal{P}}^*(gl^{(1)})^{\otimes j} &\rightarrow H^*(GL_{n,k}, (\mathfrak{gl}_n^{(1)})^{\otimes j}) \\ \otimes_{i=1}^j c_i &\mapsto \cup_{i=1}^j \phi_{gl^{(1)}, n}(c_i) \end{aligned}$$

**Proof** We prove this lemma by reducing our cohomology classes to an infinitesimal one parameter subgroup  $\mathbb{G}_{a1}$  of  $GL_{n,k}$ , as it is done in [20]. Since  $n \geq p$ , we can find a  $p$ -nilpotent matrix  $\alpha \in \mathfrak{gl}_n$ . Using this matrix, we define an embedding  $\mathbb{G}_{a1} \rightarrow \mathbb{G}_a \xrightarrow{exp\alpha} GL_{n,k}$ . For all  $\ell$ , this embedding makes the  $GL_{n,k}$ -module  $\mathfrak{gl}_n^{(1)\otimes \ell}$  into a trivial  $\mathbb{G}_{a1}$ -module. Thus there is an isomorphism  $H^*(\mathbb{G}_{a1}, \mathfrak{gl}_n^{(1)\otimes \ell}) \simeq H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes \ell}$ . The algebra  $H^*(\mathbb{G}_{a1}, k)$  is computed in [5]. In particular,  $H^{even}(\mathbb{G}_{a1}, k) = k[x_1]$  is a polynomial algebra on one generator  $x_1$  of degree 2. Let's thrash out the compatibility of this isomorphism with the cup product. If  $x_1^{\ell} \otimes \beta_{\ell}$  and  $x_1^m \otimes \beta_m$  are classes

in  $H^*(\mathbb{G}_{a_1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes \ell}$ , resp.  $H^*(\mathbb{G}_{a_1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes m}$ , their cup product is the class  $x_1^{\ell+m} \otimes (\beta_\ell \otimes \beta_m)$  in  $H^*(\mathbb{G}_{a_1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes \ell+m}$ .

We recall that  $H_{\mathcal{P}}^*(gl^{(1)})$  is a graded module with basis the classes  $e_1(i)$  of degree  $2i$  for  $0 \leq i < p$ . By [20, Th 4.9], the composite

$$H_{\mathcal{P}}^*(gl^{(1)}) \xrightarrow[\simeq]{\phi_{gl^{(1)},n}} H^*(GL_{n,k}, \mathfrak{gl}_n^{(1)}) \rightarrow H^*(\mathbb{G}_{a_1}, \mathfrak{gl}_n^{(1)}) \simeq H^*(\mathbb{G}_{a_1}, k) \otimes \mathfrak{gl}_n^{(1)}$$

sends  $e_1(i)$  to the class  $x_1^i \otimes (\alpha^{(1)})^i \in H^*(\mathbb{G}_{a_1}, k) \otimes \mathfrak{gl}_n^{(1)}$ . Since restriction to  $\mathbb{G}_{a_1}$  is compatible with cup products, the composite

$$H_{\mathcal{P}}^*(gl^{(1)})^{\otimes j} \rightarrow H^*(GL_{n,k}, \mathfrak{gl}_n^{(1)\otimes j}) \rightarrow H^*(\mathbb{G}_{a_1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes j}$$

sends the tensor product  $\otimes_{\ell=1}^j e(i_\ell)$  to the class  $x_1^{\sum_{\ell=1}^j i_\ell} \otimes (\otimes_{\ell=1}^j (\alpha^{(1)})^{i_\ell})$ . As a result, this composite sends the basis  $(\otimes_{i=\ell}^j e(i_\ell))_{(i_1, \dots, i_\ell)}$  into the linearly independent family  $(x_1^m \otimes (\otimes_{\ell=1}^j (\alpha^{(1)})^{i_\ell}))_{(m, i_1, \dots, i_\ell)}$ . Hence the map  $\cup_{i=1}^j \phi_{gl^{(1)},n}$  is injective.  $\square$

## Part II

# The second proof

## 5 The starting point

Many notations are as in [23]. We work over a field  $k$  of positive characteristic  $p$ . Fix an integer  $n$ , with  $n > 1$ . The important case is when  $n$  is large. So if one finds this convenient, one may take  $n \geq p$ . We wish to draw conclusions from the following result.

**Theorem 5.1 (Lifted universal cohomology classes [22])** *There are cohomology classes  $c[m]$  so that*

1.  $c[1] \in H^2(GL_{n,k}, \mathfrak{gl}_n^{(1)})$  is nonzero,
2. For  $m \geq 1$  the class  $c[m] \in H^{2m}(GL_{n,k}, \Gamma^m(\mathfrak{gl}_n^{(1)}))$  lifts  $c[1] \cup \dots \cup c[1] \in H^{2m}(GL_{n,k}, \otimes^m(\mathfrak{gl}_n^{(1)}))$ .

**Remark 5.2** We really have in mind that  $c[1]$  is the Witt vector class of [23, section 4], which is certainly nonzero. The computation of  $H^2(GL_{n,k}, \mathfrak{gl}_n^{(1)})$  is easy, using [5, Corollary (3.2)]. One finds that  $H^2(GL_{n,k}, \mathfrak{gl}_n^{(1)})$  is one dimensional. Thus any nonzero  $c[1]$  is up to scaling equal to the Witt vector class.

## 6 Using the classes

We write  $G$  for  $GL_{n,k}$ , the algebraic group  $GL_n$  over  $k$ . Sometimes it is instructive to restrict to  $SL_n$  or other reductive subgroups of  $GL_n$ . We leave this to the the reader.

### 6.1 Other universal classes

We recall some constructions from [23]. If  $M$  is a finite dimensional vector space over  $k$  and  $r \geq 1$ , we have a natural homomorphism between symmetric algebras  $S^*(M^{\#(r)}) \rightarrow S^*(M^{\#(1)})$  induced by the map  $M^{\#(r)} \rightarrow S^{p^{r-1}}(M^{\#(1)})$  which raises an element to the power  $p^{r-1}$ . It is a map of bialgebras. Dually we have the bialgebra map  $\pi^{r-1} : \Gamma^*(M^{(1)}) \rightarrow \Gamma^*(M^{(r)})$  whose kernel is the ideal generated by  $\Gamma^1(M^{(1)})$  through  $\Gamma^{p^{r-1}-1}(M^{(1)})$ . So  $\pi^{r-1}$  maps  $\Gamma^{p^{r-1}a}(M^{(1)})$  onto  $\Gamma^a(M^{(r)})$ .

**Notation 6.1** We now introduce analogues of the classes  $e_r$  and  $e_r^{(j)}$  of Friedlander and Suslin [9, Theorem 1.2, Remark 1.2.2]. We write  $\pi_*^{r-1}(c[ap^{r-1}]) \in H^{2ap^{r-1}}(G, \Gamma^a(\mathfrak{gl}_n^{(r)}))$  as  $c_r[a]$ . Next we get  $c_r[a]^{(j)} \in H^{2ap^{r-1}}(G, \Gamma^a(\mathfrak{gl}_n^{(r+j)}))$  by Frobenius twist. As in [9] a notation like  $S^*(M(i))$  means the symmetric algebra  $S^*(M)$ , but graded, with  $M$  placed in degree  $i$ .

Here is the analogue of [23, Lemma 4.7].

**Lemma 6.2** *The  $c_i[a]^{(r-i)}$  enjoy the following properties ( $r \geq i \geq 1$ )*

1. *There is a homomorphism of graded algebras  $S^*(\mathfrak{gl}_n^{\#(r)}(2p^{i-1})) \rightarrow H^{2p^{i-1}*}(G_r, k)$  given on  $\mathfrak{gl}_n^{\#(r)}(2p^{i-1}) = H^0(G_r, \mathfrak{gl}_n^{\#(r)})$  by cup product with the restriction of  $c_i[1]^{(r-i)}$  to  $G_r$ . If  $i = 1$ , then it is given on  $S^a(\mathfrak{gl}_n^{\#(r)}(2)) = H^0(G_r, S^a(\mathfrak{gl}_n^{\#(r)}))$  by cup product with the restriction of  $c[a]^{(r-1)}$  to  $G_r$ .*

2. For  $r \geq 1$  the restriction of  $c_r[1]$  to  $H^{2p^{r-1}}(G_1, \mathfrak{gl}_n^{(r)})$  is nontrivial, so that  $c_r[1]$  may serve as the universal class  $e_r$  in [9, Thm 1.2].

**Proof** 1. When  $M$  is a  $G$ -module, one has a commutative diagram

$$\begin{array}{ccc} \Gamma^m M \otimes \bigotimes^m M^\# & \rightarrow & \bigotimes^m M \otimes \bigotimes^m M^\# \\ \downarrow & & \downarrow \\ \Gamma^m M \otimes S^m M^\# & \rightarrow & k \end{array}$$

Take  $M = \mathfrak{gl}_n^{(1)}$ . There is a homomorphism of algebras  $\bigotimes^*(\mathfrak{gl}_n^{(1)}) \rightarrow H^{2*}(G_1, k)$  given on  $\mathfrak{gl}_n^{(1)}$  by cup product with  $c[1]$ . (We do not mention obvious restrictions to subgroups like  $G_1$  any more.) On  $\bigotimes^m(\mathfrak{gl}_n^{(1)})$  it is given by cup product with  $c[1] \cup \dots \cup c[1]$ , so by Theorem 5.1 it is also given by cup product with  $c[m]$ , using the pairing  $\Gamma^m M \otimes \bigotimes^m M^\# \rightarrow k$ . As this pairing factors through  $\Gamma^m M \otimes S^m M^\#$  we get that the induced algebra map  $S^*(\mathfrak{gl}_n^{(1)}) \rightarrow H^{2*}(G_1, k)$  is given by cup product with  $c[m]$  on  $S^m(\mathfrak{gl}_n^{(1)})$ . If we compose with the algebra map  $S^*(\mathfrak{gl}_n^{(i)}) \rightarrow S^{p^{i-1}*}(\mathfrak{gl}_n^{(1)})$  we get an algebra map  $\psi : S^*(\mathfrak{gl}_n^{(i)}) \rightarrow H^{2p^{i-1}*}(G_1, k)$  given on  $\mathfrak{gl}_n^{(i)}$  by cup product with  $c[p^{i-1}]$ , using the pairing  $\mathfrak{gl}_n^{(i)} \otimes \Gamma^{p^{i-1}}(\mathfrak{gl}_n^{(1)}) \rightarrow k$ . This pairing factors through  $\text{id} \otimes \pi^{i-1} : \mathfrak{gl}_n^{(i)} \otimes \Gamma^{p^{i-1}}(\mathfrak{gl}_n^{(1)}) \rightarrow \mathfrak{gl}_n^{(i)} \otimes \mathfrak{gl}_n^{(i)}$ , so the homomorphism  $\psi$  is given on  $\mathfrak{gl}_n^{(i)}$  by cup product with  $\pi_*^{i-1} c[p^{i-1}] = c_i[1]$ . We can lift it to an algebra map  $S^*(\mathfrak{gl}_n^{(i)}) \rightarrow H^{2p^{i-1}*}(G_i, k)$  by simply still using cup product with  $c_i[1]$  on  $\mathfrak{gl}_n^{(i)}$ . Pull back along the  $(r-i)$ -th Frobenius homomorphism  $G_r \rightarrow G_i$  and you get an algebra map  $\psi^{(r-i)} : S^*(\mathfrak{gl}_n^{(r)}) \rightarrow H^{2p^{i-1}*}(G_r, k)$ , given on  $\mathfrak{gl}_n^{(r)}$  by cup product with  $c_i[1]^{(r-i)}$ . If  $i = 1$ , pull back the cup product with  $c[m]$  on  $S^m(\mathfrak{gl}_n^{(1)})$  to a cup product with  $c[m]^{(r-1)}$  on  $S^m(\mathfrak{gl}_n^{(r)}(2))$ . This then describes the homomorphism  $\psi^{(r-1)}$  degree-wise.

2. In fact if we restrict  $c_r[1]$  as in remark [23, 4.1] to  $H^{2p^{r-1}}(\mathbb{G}_{a1}, (\mathfrak{gl}_n^{(r)})_{p^{r\alpha}}) = H^{2p^{r-1}}(\mathbb{G}_{a1}, k) \otimes (\mathfrak{gl}_n^{(r)})_{p^{r\alpha}}$ , then even that restriction is nontrivial. That is because the Witt vector class generates the polynomial ring  $H^{\text{even}}(\mathbb{G}_{a1}, k)$ , see [12, I 4.26]. And at this level  $\Gamma^m \hookrightarrow \bigotimes^m$  gives an isomorphism, showing that  $c[m]$  restricts to the  $m$ -th power of the polynomial generator.  $\square$

## 6.2 Noetherian homomorphisms

Let  $A$  be a commutative  $k$ -algebra. The cohomology algebra  $H^*(G, A)$  is then graded commutative, so we must also consider graded commutative algebras.

**Definition 6.3** If  $f : A \rightarrow B$  is a homomorphism of graded commutative  $k$ -algebras, we call  $f$  *noetherian* if  $f$  makes  $B$  into a noetherian left  $A$  module.

**Remark 6.4** In algebraic geometry a noetherian homomorphism between finitely generated commutative  $k$ -algebras is called a *finite* morphism. With our terminology we wish to stress the importance of chain conditions in our arguments.

**Lemma 6.5** *The composite of noetherian homomorphisms is noetherian.*

**Proof** If  $A \rightarrow B$  and  $B \rightarrow C$  are noetherian, view  $C$  as a quotient of the module  $B^r$  for some  $r$ . □

**Lemma 6.6** *If the composite of  $A \rightarrow B$  and  $B \rightarrow C$  is noetherian, so is  $B \rightarrow C$ .*

**Proof** View  $B$ -submodules of  $C$  as  $A$ -modules. □

**Remark 6.7** In this lemma  $A \rightarrow C$  and  $B \rightarrow C$  must be homomorphisms, but  $A \rightarrow B$  could be just a map.

**Lemma 6.8** *Suppose  $B$  is finitely generated as a graded commutative  $k$ -algebra. Then  $f : A \rightarrow B$  is noetherian if and only if  $B^{\text{even}}$  is integral over  $f(A^{\text{even}})$ .*

**Proof** The map  $B^{\text{even}} \rightarrow B$  is noetherian. So if  $B^{\text{even}}$  is integral over  $f(A^{\text{even}})$ , then  $f$  is noetherian. Conversely, if  $f$  is noetherian and  $b \in B^{\text{even}}$ , then for some  $r$  one must have  $b^r \in \sum_{i < r} f(A)b^i$ . But then in fact  $b^r \in \sum_{i < r} f(A^{\text{even}})b^i$ . □

In particular one has

**Lemma 6.9** *Suppose  $B$  is a finitely generated commutative  $k$ -algebra. Let  $n > 1$  and let  $A$  be a subalgebra of  $B$  containing  $x^n$  for every  $x \in B$ . Then  $A \hookrightarrow B$  is noetherian and  $A$  is also finitely generated.*

**Proof** We follow Emmy Noether [17]. Indeed  $B$  is integral over  $A$ . Take finitely many generators  $b_i$  of  $B$  and let  $C$  be the subalgebra generated by the  $b_j^n$ . Then  $A$  is a  $C$ -submodule of  $B$ , hence finitely generated.  $\square$

Invariant theory tells

**Lemma 6.10** [11, Thm. 16.9] *Let  $f : A \rightarrow B$  be a noetherian homomorphism of finitely generated graded commutative  $k$ -algebras with rational  $G$  action. Then  $A^G \rightarrow B^G$  is noetherian.*  $\square$

**Lemma 6.11** *Let  $f : A \rightarrow B$  be a noetherian homomorphism of finitely generated graded commutative  $k$ -algebras with rational  $G_r$  action. Then  $H^*(G_r, A) \rightarrow H^*(G_r, B)$  is noetherian.*

**Proof** Take  $C = H^0(G_r, A^{\text{even}})$  or its subalgebra generated by the  $p^r$ -th powers in  $A^{\text{even}}$ . Then apply [9, Theorem 1.5, Remark 1.5.1].  $\square$

We will need a minor variation on a theorem of Friedlander and Suslin.

**Theorem 6.12** *Let  $r \geq 1$ . Let  $S \subset G_r$  be an infinitesimal group scheme over  $k$  of height at most  $r$ . Let further  $C$  be a finitely generated commutative  $k$ -algebra (considered as a trivial  $S$ -module) and let  $M$  be a noetherian  $C$ -module on which  $S$  acts by  $C$ -linear transformations. Then  $H^*(S, M)$  is a noetherian module over the algebra  $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes C$ , with the map given as suggested by Lemma 6.2.*

**Corollary 6.13** *The restriction map  $H^*(G_r, C) \rightarrow H^*(G_{r-1}, C)$  is noetherian.*

**Proof** Take  $S = G_{r-1}$  and note that the map  $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes C \rightarrow H^*(G_{r-1}, C)$  factors through  $H^*(G_r, C)$ .  $\square$

**Proof of the theorem** The key difference with [9, Theorem 1.5, Remark 1.5.1] is that we do not require the height of  $S$  to be  $r$ . (As  $S \subset G_r$  the fact that its height is at most  $r$  is automatic.) Thus, to start their inductive argument, we must also check the obvious case where  $r = 1$  and  $S$  is the trivial group. The rest of the proof goes through without change.  $\square$

**Remark 6.14** If  $S$  has height  $s$ , then the map  $(\mathfrak{gl}_n^{(r)})^\#(2p^{i-1}) \rightarrow H^{2p^{i-1}}(S, k)$  is trivial for  $r - i \geq s$ .

### 6.3 Cup products on the cochain level

As we will need a differential graded algebra structure on Hochschild–Serre spectral sequences, we now expand the discussion of the Hochschild complex in [12, I 5.14]. Let  $L$  be an affine algebraic group scheme over the field  $k$ ,  $N$  a normal subgroup scheme,  $R$  a commutative  $k$ -algebra on which  $L$  acts rationally by algebra automorphisms. We have a Hochschild complex  $C^*(L, R)$  with  $R \otimes k[L]^{\otimes i}$  in degree  $i$ . Define a cup product on  $C^*(L, R)$  as follows. If  $u \in C^r(L, R)$  and  $v \in C^s(L, R)$ , then  $u \cup v$  is defined in simplified notation by

$$(u \cup v)(g_1, \dots, g_{r+s}) = u(g_1, \dots, g_r) \cdot^{g_1 \cdots g_r} v(g_{r+1}, \dots, g_{r+s}),$$

where  ${}^g r$  denotes the image of  $r \in R$  under the action of  $g$ . One checks that

**Lemma 6.15** *With this cup product  $C^*(L, R)$  is a differential graded algebra.*

In particular, taking for  $R$  the algebra  $k[L]$  with  $L$  acting by right translation, we get the differential graded algebra  $C^*(L, k[L])$ , quasi-isomorphic to  $k$ . And the action by left translation on  $k[L]$  is by  $L$ -module isomorphisms, so this makes  $C^*(L, k[L])$  into a differential graded algebra with  $L$  action. It consists of injective  $L$ -modules in every degree. We write  $C^*(L)$  for this differential graded algebra with  $L$  action. One has  $C^i(L) = k[L]^{\otimes i+1}$  and this is our elaboration of [12, I 4.15 (1)].

The Hochschild–Serre spectral sequence

$$E_2^{r,s} = H^r(L/N, H^s(N, R)) \Rightarrow H^{r+s}(L, R)$$

can now be based on the double complex  $(C^*(L/N) \otimes (C^*(L) \otimes R)^N)^{L/N}$ . The tensor product over  $k$  of two differential graded algebras is again a differential graded algebra and the spectral sequence inherits differential graded algebra structures [2, 3.9] from such structures on  $C^*(L) \otimes R$ ,  $(C^*(L) \otimes R)^N$ ,  $C^*(L/N) \otimes (C^*(L) \otimes R)^N$ .



## 6.4 Hitting invariant classes

We now come to the main result of this section, which is the counterpart of [23, Cor. 4.8]. It does not seem to follow from the cohomological finite generation conjecture, but we will show it implies the conjecture.

**Theorem 6.16** *Let  $r \geq 1$ . Further let  $A$  be a finitely generated commutative  $k$ -algebra with  $G$  action. Then  $H^{\text{even}}(G, A) \rightarrow H^0(G, H^*(G_r, A))$  is noetherian.*

**Remark 6.17** Recall that  $H^0(G, H^*(G_r, A))$  is finitely generated as a  $k$ -algebra, by [9] and invariant theory.

### Proof

1. If  $M$  is a  $G$ -module on which  $G_r$  acts trivially, then  $H^0(G, M)$  and  $H^0(G/G_r, M)$  denote the same subspace of  $M$ . We may thus switch between these variants.

2. We argue by induction on  $r$ . Put  $C = H^0(G_r, A)$ . Then  $C$  contains the elements of  $A$  raised to the power  $p^r$ , so  $C$  is also a finitely generated algebra and  $A$  is a noetherian module over it.

3. Let  $r = 1$ . This case is the same as in [23]. By [9, Thm 1.5]  $H^*(G_1, A)$  is a noetherian module over the finitely generated algebra

$$R = S^*((\mathfrak{g}_n^{(1)})^\#(2)) \otimes C.$$

Then, by invariant theory [11, Thm. 16.9],  $H^0(G, H^*(G_1, A))$  is a noetherian module over the finitely generated algebra  $H^0(G, R)$ . By lemma 6.2 we may take the algebra homomorphism  $R \rightarrow H^*(G_1, A)$  of [9] to be based on cup product with our  $c[a] = c[a]^{(0)}$  on the summand  $S^a((\mathfrak{g}_n^{(1)})^\#(2)) \otimes C$ . But then the map  $H^0(G, R) \rightarrow H^*(G_1, A)$  factors, as a linear map, through  $H^{\text{even}}(G, A)$ . This settles the case  $r = 1$  by 6.7.

4. Now let the level  $r$  be greater than 1. We are going to follow the analysis in [9, section 1] to peel off one level at a time. Heuristically, in the tensor product of Theorem 6.12 we treat one factor at a time. That is the main difference with the argument in [23].

Thus, consider the Hochschild–Serre spectral sequence  $E_2^{ij}(C) = H^i(G_r/G_{r-1}, H^j(G_{r-1}, C)) \Rightarrow H^{i+j}(G_r, C)$ . We first wish to argue that this spectral sequence stops, meaning that  $E_s^{**}(C) = E_\infty^{**}(C)$  for some finite  $s$ . This is proved in [9, section 1] for a very similar spectral sequence. So we

imitate the argument. We need to apply [9, Lemma 1.6] and its proof. We use  $H^{\text{even}}(G_r, C)$  for the  $A$  of that lemma and  $S^*((\mathfrak{gl}_n^{(r)})^\#(2))$  for its  $B$ . We map  $H^{\text{even}}(G_r, C)$  in the obvious way to the abutment  $H^*(G_r, C)$  and for  $B \rightarrow E_2^{*0}(k) = H^*(G_r/G_{r-1}, k) = H^*(G_1, k)^{(r-1)}$  we use the  $(r-1)$ -st Frobenius twist of the map  $S^*((\mathfrak{gl}_n^{(1)})^\#(2)) \rightarrow H^*(G_1, k)$  of Lemma 6.2. So we use the class  $c[a]^{(r-1)}$  on  $S^a((\mathfrak{gl}_n^{(r)})^\#(2))$ . By Corollary 6.13 and Lemma 6.8 the restriction map  $H^{\text{even}}(G_r, C) \rightarrow H^*(G_{r-1}, C)$  is noetherian and by Theorem 6.12, compare [9, Cor 1.8], it follows that the  $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$  module  $E_2^{**}(C) = H^*(G_1, H^*(G_{r-1}, C)^{(1-r)})^{(r-1)}$  is noetherian, so the spectral sequence stops, say at  $E_s^{**}(C)$ . Note also that the image of  $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$  in  $E_2^{**}(C)$  consists of permanent cycles.

**5.** As the spectral sequence is one of graded commutative differential graded algebras, the  $p$ -th power of an even cochain in a page passes to the next page. As the spectral sequence stops at page  $s$  one finds that for an  $x \in E_2^{\text{even, even}}(C)$  the power  $x^{p^s}$  is a permanent cycle. Let  $P$  be the algebra generated by permanent cycles in  $E_2^{\text{even, even}}(C)$ . Then  $P \rightarrow E_t^{ij}(C)$  is noetherian for  $2 \leq t \leq \infty$ . So  $P^G \rightarrow (E_\infty^{**}(C))^G$  is noetherian by Lemma 6.10.

**6.** By the inductive assumption  $H^0(G, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))))$  is noetherian over  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$ . By step 1 we may rewrite  $H^0(G, H^j(G_{r-1}, C \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2))))$  as  $H^0(G/G_{r-1}, H^j(G_{r-1}, C \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2))))$ . The latter description will be needed in the sequel. We may map  $H^0(G/G_{r-1}, H^j(G_{r-1}, C \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2))))$  by restriction to  $H^0(G_r/G_{r-1}, H^j(G_{r-1}, C) \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2)))$  and then to  $E_2^{2i, j}(C) = H^{2i}(G_r/G_{r-1}, H^j(G_{r-1}, C))$  by cup product with  $c[i]^{(r-1)}$ . So we now have a map from  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$  to  $E_2^{**}(C)$ . We will factor it further.

**7.** One checks that the map from  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$  to  $H^0(G_r/G_{r-1}, H^*(G_{r-1}, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$  of step 6 factors naturally through the algebra  $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$  of step 4. Moreover, as the algebra  $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$  acts on the full spectral sequence, we may make  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$  act on the full spectral sequence by way of that algebra.

**8.** The noetherian map  $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)) \rightarrow E_2^{**}(C)$  factors through  $H^0(G_r/G_{r-1}, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))))$ . But then  $H^0(G_r/G_{r-1}, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))) \rightarrow E_2^{**}(C)$  is noetherian by Lemma 6.6. So by Lemma 6.10 the map  $H^0(G/G_{r-1}, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))) \rightarrow (E_2^{**}(C))^G$  is noetherian. Combining with the inductive hypothesis, we learn that  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))) \rightarrow (E_2^{**}(C))^G$  is noetherian. It lands in  $P^G$ , because the map in step 4 lands in  $P$ . We

conclude that  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))) \rightarrow (E_\infty^{**}(C))^G$  is noetherian.

9. We filter  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$  by putting  $H^{\text{even}}(G, C \otimes S^t((\mathfrak{gl}_n^{(r)})^\#(2)))$  in  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))^{\geq j}$  for  $t \geq j$ . As in [9, section 1] the filtered algebra may be identified with its associated graded, and the map  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))) \rightarrow H^*(G_r, C)$  respects filtrations. Now we care about  $H^*(G_r, C)^G$  as a module for  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$ . To see that it is noetherian we may pass as in [9, section 1] to the associated graded, where one puts  $(H^*(G_r, C)^G)^{\geq j} = (H^*(G_r, C)^G) \cap H^*(G_r, C)^{\geq j}$ . This associated graded of  $H^*(G_r, C)^G$  is a submodule of  $(E_\infty^{**}(C))^G$ , containing the image of  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$ . So it is indeed noetherian. We conclude that  $H^*(G_r, C)^G$  is noetherian over  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))$ .

10. As in the case  $r = 1$  the map  $H^{\text{even}}(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))) \rightarrow H^*(G_r, C)^G$  factors, as a linear map, through  $H^{\text{even}}(G, C)$ , so  $H^{\text{even}}(G, C) \rightarrow H^*(G_r, C)^G$  is noetherian by 6.7. As  $H^*(G_r, C)^G \rightarrow H^*(G_r, A)^G$  is noetherian, the result follows. □

## 6.5 Cohomological finite generation

Now let  $A$  be a finitely generated commutative  $k$ -algebra with  $G$  action. We wish to show that  $H^*(G, A)$  is finitely generated, following the same path as in [23], but using improvements from [21]. As in [23] we denote by  $\mathcal{A}$  the coordinate ring of a flat family with general fiber  $A$  and special fiber  $\text{gr } A$  [10, Theorem 13]. Choosing  $r$  as in [23, Prop. 3.8] we have the spectral sequence

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \text{gr } A)) \Rightarrow H^{i+j}(G, \text{gr } A).$$

and  $R = H^*(G_r, \text{gr } A)^{(-r)}$  is a finite module over the algebra

$$\bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1})) \otimes \text{hull}_\nabla(\text{gr } A).$$

This algebra has a good filtration, and by the main result of [21] the ring  $R$  has finite good filtration dimension. In particular, there are only finitely many nonzero  $H^i(G, R)$ . Thus the same main result tells that  $E_2^{0*} \rightarrow E_2^{**}$  is noetherian. Now  $H^0(G/G_r, H^*(G_r, \mathcal{A})) \rightarrow H^0(G/G_r, H^*(G_r, \text{gr } A))$  is noetherian by [9] and lemma 6.10. And  $H^*(G, \mathcal{A}) \rightarrow H^0(G/G_r, H^*(G_r, \mathcal{A}))$

is noetherian by theorem 6.16, so another application of [9, Lemma 1.6] (with  $B = k$ ) shows that  $H^*(G, \mathcal{A}) \rightarrow H^*(G, \text{gr } A)$  is noetherian.

There is a map of spectral sequences from a totally degenerate spectral sequence

$$E(\mathcal{A}) : \quad E_1^{ij}(\mathcal{A}) = H^{i+j}(G, \text{gr}_{-i} \mathcal{A}) \Rightarrow H^{i+j}(G, \mathcal{A}),$$

with pages  $H^*(G, \mathcal{A})$ , to the spectral sequence

$$E(A) : \quad E_1^{ij}(A) = H^{i+j}(G, \text{gr}_{-i} A) \Rightarrow H^{i+j}(G, A).$$

This makes that  $H^*(G, \mathcal{A})$  acts on  $E(A)$  and the noetherian homomorphism  $H^*(G, \mathcal{A}) \rightarrow H^*(G, \text{gr } A)$  is used in standard fashion [24, slogan 3.9] to make the spectral sequence  $E(A)$  stop. It follows easily that  $H^*(G, A)$  is finitely generated. So far  $G$  was  $GL_{n,k}$ . As explained in some detail in [24] this case implies our Cohomological Finite Generation Conjecture (over fields.)

**Remark 6.18** The spectral sequence  $E(A)$  is based on filtering the Hochschild complex of  $A$ . As it lives in the second quadrant, the exposition of multiplicative structure in [2, 3.9] does not apply as stated. (In order to avoid convergence issues [2] uses a filtration that reaches a maximum.) But [4, Ch XV, Ex. 2] is sufficiently general to cover our case. Or see [14].

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