An algebraic proof of Bogomolov-Tian-Todorov theorem

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Abstract. We give a completely algebraic proof of the Bogomolov-Tian-Todorov theorem. More precisely, we shall prove that if X is a smooth projective variety with trivial canonical bundle defined over an algebraically closed field of characteristic 0, then the L_∞-algebra governing infinitesimal deformations of X is quasi-isomorphic to an abelian differential graded Lie algebra.

Contents

Introduction

Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic 0, with tangent sheaf Θ_X. Given an affine open cover $\mathcal{U} = \{U_i\}$ of X, we can consider the Čech complex $\check{C}(\mathcal{U}, \Theta_X)$. By classical deformation theory [**Kod86, Se06**], the group $H^1(\mathcal{U}, \Theta_X)$ classifies first order deformations of X, while $H^2(\mathcal{U}, \Theta_X)$ is an obstruction space for X.

Moreover, as a consequence of the results contained in [**Hin97, HiS97a, HiS97b, FMM08**], there exists a canonical sequence of higher brackets on $\check{C}(\mathcal{U}, \Theta_X)$, defining an L_∞ structure and governing deformations of X over local Artinian $\mathbb{K}$-algebras, via Maurer-Cartan equation. When $\mathbb{K} = \mathbb{C}$, such L_∞ structure is canonically quasi-isomorphic to the Kodaira-Spencer differential graded Lie algebra of X, whose Maurer-Cartan equation corresponds to the integrability condition of almost complex structures [**GM90**].

Assume now that X has trivial canonical bundle, the well known Bogomolov-Tian-Todorov (BTT) theorem states that X has unobstructed deformations. This was first proved by Bogolomov in [**Bo78**] in the particular case of complex hamiltonian manifolds; then, Tian [**Ti87**] and Todorov [**To89**] proved independently the theorem for compact

Kähler manifolds with trivial canonical bundle. Their proofs are transcendental and make a deep use of the underlying differentiable structure as well of the $\partial\overline{\partial}$-Lemma.

More algebraic proofs of BTT theorem, based on T^1-lifting theorem and degeneration of the Hodge spectral sequence, were given in [**Ra92**] for $\mathbb{K} = \mathbb{C}$ and in [**Kaw92, FM99**] for any $\mathbb{K}$ as above.

For $\mathbb{K} = \mathbb{C}$, the BTT theorem is also a consequence of the stronger result [**GM90, Ma04b**] that the Kodaira-Spencer differential graded Lie algebra of X is quasi-abelian, i.e., quasi-isomorphic to an abelian differential graded Lie algebra. This result, also proved with transcendental methods, is important for applications to Mirror symmetry [**BK98**].

The main result of this paper is to give a purely algebraic proof of the quasi-abelianity of the L_∞-algebra $\check{C}(\mathcal{U}, \Theta_X)$ when X is projective with trivial canonical bundle. This is achieved using the degeneration of the Hodge-de Rham spectral sequence, proved algebraically by Faltings, Deligne and Illusie [**Fa88, DeIl87**], and the L_∞ description of the period map [**FiMa08**].

More precisely, we prove the following theorems.

Theorem (A)**.** *Let X be a smooth projective variety of dimension n defined over an algebraically closed field of characteristic 0. If the contraction map*

$$H^*(\Theta_X) \xrightarrow{\boldsymbol{i}} \operatorname{Hom}^*(H^*(\Omega^n_X), H^*(\Omega^{n-1}_X))$$

is injective, then for every affine open cover $\mathcal{U}$ of X, the L_∞-algebra $\check{C}(\mathcal{U}, \Theta_X)$ is quasi-abelian.

Theorem (B)**.** *Let X be a smooth projective variety defined over an algebraically closed field of characteristic 0. If the canonical bundle of X is trivial or torsion, then the L_∞-algebra $\check{C}(\mathcal{U}, \Theta_X)$ is quasi-abelian.*

The paper goes as follows: the first section is intended for the non expert reader and is devoted to recall the basic notions of differential graded Lie algebras, L_∞-algebras and their role in deformation theory.

In Section 2, we review the construction of the Thom-Whitney complex associated with a semicosimplicial complex.

In Section 3, following [**FMM08**], we introduce semicosimplicial differential graded Lie algebras, the associated Thom-Whitney DGLAs and the L_∞ structure on the associated total complexes. We also investigate some properties of mapping cones associated with a morphism of DGLAs.

In Sections 4, we collect some technical results about Cartan homotopies and contractions.

In Section 5, we give the definition of the deformation functor $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)$, associated with a semicosimplicial Lie algebra $\mathfrak{g}^\Delta$, introduced essentially in [**Hin97, Pr03**] and described in more detailed way in [**FMM08**]. Moreover, following [**FMM08**], we prove, in a complete algebraic way, that the infinitesimal deformations of a smooth variety X, defined over a field of characteristic 0, are controlled by the L_∞-algebra $\check{C}(\mathcal{U}, \Theta_X)$, where $\mathcal{U}$ is an open affine cover of X.

Section 6 is devoted to the algebraic proof of the previous Theorems A and B together with some applications to deformation theory.

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1. Review of DGLAs and L_∞-algebras

Let $\mathbb{K}$ be a fixed algebraically closed field of characteristic zero. A *differential graded vector space* is a pair (V,d), where $V=\oplus_i V^i$ is a $\mathbb{Z}$-graded vector space and $d\colon V^i\to V^{i+1}$ is a differential of degree $+1$. For every integer n, we define a new differential graded vector space $V[n]$ by setting

$$V[n]^i=V^{n+i},\qquad d_{V[n]}=(-1)^n d_V.$$

A *differential graded Lie algebra* (DGLA for short) is the data of a differential graded vector space (L,d) together with a bilinear map $[-,-]\colon L\times L\to L$ (called bracket) of degree 0 such that:

(1) (graded skewsymmetry) $[a,b]=-(-1)^{\deg(a)\deg(b)}[b,a]$.
(2) (graded Jacobi identity) $[a,[b,c]]=[[a,b],c]+(-1)^{\deg(a)\deg(b)}[b,[a,c]]$.
(3) (graded Leibniz rule) $d[a,b]=[da,b]+(-1)^{\deg(a)}[a,db]$.

In particular, the Leibniz rule implies that the bracket of a DGLA L induces a structure of graded Lie algebra on its cohomology $H^*(L)=\oplus_i H^i(L)$.

Example 1.1. Let (V,d_V) be a differential graded vector space and $\operatorname{Hom}^i(V,V)$ the space of morphisms $V\to V$ of degree i. Then, $\operatorname{Hom}^*(V,V)=\bigoplus_i \operatorname{Hom}^i(V,V)$ is a DGLA with bracket

$$[f,g]=fg-(-1)^{\deg(f)\deg(g)}gf,$$

and differential d given by

$$d(f)=[d_V,f]=d_Vf-(-1)^{\deg(f)}fd_V.$$

For later use, we point out that there exists a natural isomorphism

$$H^*(\operatorname{Hom}^*(V,V))\xrightarrow{\simeq}\operatorname{Hom}^*(H^*(V),H^*(V)).$$

A morphism of differential graded Lie algebras $\varphi\colon L\to M$ is a linear map that preserves degrees and commutes with brackets and differentials. A *quasi-isomorphism* of DGLAs is a morphism that induces an isomorphism in cohomology. Two DGLAs L and M are said to be *quasi-isomorphic* if they are equivalent under the equivalence relation generated by: $L\sim M$ if there exists a quasi-isomorphism $\phi: L\to M$.

Next, denote by **Set** the category of sets (in a fixed universe) and by $\mathbf{Art}=\mathbf{Art}_{\mathbb{K}}$ the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Unless otherwise specified, for every objects $A\in\mathbf{Art}$, we denote by $\mathfrak{m}_A$ its maximal ideal. Given a DGLA L, we define the Maurer-Cartan functor $\operatorname{MC}_L\colon\mathbf{Art}\to\mathbf{Set}$ by setting [**Ma99**]:

$$\operatorname{MC}_L(A)=\left\{x\in L^1\otimes\mathfrak{m}_A \mid dx+\frac{1}{2}[x,x]=0\right\},$$

where the DGLA structure on $L\otimes\mathfrak{m}_A$ is the natural extension of the DGLA structure on L. The gauge action $*:\exp(L^0\otimes\mathfrak{m}_A)\times\operatorname{MC}_L(A)\longrightarrow\operatorname{MC}_L(A)$ may be defined by the explicit formula

$$e^a*x:=x+\sum_{n\geq 0}\frac{[a,-]^n}{(n+1)!}([a,x]-da).$$

The deformation functor $\operatorname{Def}_L : \mathbf{Art} \longrightarrow \mathbf{Set}$ associated to a DGLA L is:

$$\operatorname{Def}_L(A) = \frac{\operatorname{MC}_L(A)}{\text{gauge}} = \frac{\{x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0\}}{\exp(L^0 \otimes \mathfrak{m}_A)}.$$

Remark 1.2. Every morphism of DGLAs induces a natural transformation of the associated deformation functors. A basic result asserts that if L and M are quasi-isomorphic DGLAs, then the associated functor Def_L and Def_M are isomorphic [**SS79, GM88, GM90**], [**Ma99**, Corollary 3.2], [**Ma04b**, Corollary 5.52].

Next, we briefly recall the definition of an L_∞ structures on a graded vector space V. For a more detailed description of such structures we refer to [**SS79, LS93, LM95, Ma02, Fu03, Kon03, Get04, FiMa07**] and [**Ma04b**, Chapter IX].

Let V be a graded vector space: we denote by $\bigodot^n V$ its graded symmetric n-th power. Given $v_1, \ldots, v_n$ homogeneous elements of V, for every permutation σ we have

$$v_1 \odot \cdots \odot v_n = \epsilon(\sigma; v_1, \ldots, v_n)\ v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)},$$

where $\epsilon(\sigma; v_1, \ldots, v_n)$ is the Koszul sign. When the sequence $v_1, \ldots, v_n$ is clear from the context, we simply write $\epsilon(\sigma)$ instead of $\epsilon(\sigma; v_1, \ldots, v_n)$.

Definition 1.3. Denote by Σ_n the group of permutations of the set $\{1, 2, \ldots, n\}$. The set of *unshuffles* of type $(p, n-p)$ is the subset $S(p, n-p) \subset \Sigma_n$ of permutations σ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(n)$.

Definition 1.4. An L_∞ structure on a graded vector space V is a sequence $\{q_k\}_{k\geq 1}$ of linear maps $q_k \in \operatorname{Hom}^1(\odot^k(V[1]), V[1])$ such that the map

$$Q : \bigoplus_{n\geq 1} \bigodot^n V[1] \to \bigoplus_{n\geq 1} \bigodot^n V[1],$$

defined as

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)},$$

is a codifferential on the reduced symmetric graded coalgebra $\bigoplus_{n\geq 1} \bigodot^n V[1]$: in other words, the datum $(V, q_1, q_2, q_3, \ldots)$ is called an L_∞-algebra if $QQ = 0$.

If $(V, q_1, q_2, q_3, \ldots)$ is an L_∞-algebra, then $q_1 q_1 = 0$ and therefore $(V[1], q_1)$ is a differential graded vector space. The relation between DGLAs and L_∞-algebras is given by the following example.

Example 1.5 ([**Qui69**])**.** Let $(L, d, [\ ,\ ])$ be a differential graded Lie algebra and define:

$$q_1 = -d : L[1] \to L[1],$$

$$q_2 \in \operatorname{Hom}^1(\odot^2(L[1]), L[1]), \qquad q_2(v \odot w) = (-1)^{\deg(v)}[v, w],$$

and $q_k = 0$ for every $k \geq 3$. Then $(L, q_1, q_2, 0, \ldots)$ is an L_∞-algebra.

An L_∞-*morphism* $(V, q_1, q_2, \ldots) \to (W, p_1, p_2, \ldots)$ of L_∞-algebras is a sequence $f_\infty = \{f_n\}$ of degree zero linear maps

$$f_n : \bigodot^n V[1] \to W[1], \qquad n \geq 1,$$

such that the (unique) morphism of graded coalgebras

$$F:\bigoplus_{n\geq 1}\bigodot^{n} V[1]\to\bigoplus_{n\geq 1}\bigodot^{n} W[1],$$

lifting $\sum_n f_n:\bigoplus_{n\geq 1}\bigodot^n V[1]\to W[1]$, commutes with the codifferentials. This condition implies that the *linear part* $f_1: V[1]\to W[1]$ of an L_∞-morphism $f_\infty:(V,q_1,q_2,\ldots)\to(W,p_1,p_2,\ldots)$ satisfies the condition $f_1\circ q_1=p_1\circ f_1$, and therefore f_1 is a map of differential complexes $(V[1],q_1)\to(W[1],p_1)$.

An L_∞-morphism $f_\infty=\{f_n\}$ is said to be linear if $f_n=0$, for every $n\geq 2$. Notice that an L_∞-morphism between two DGLAs is linear if and only if it is a morphism of differential graded Lie algebras.

A *quasi-isomorphism* of L_∞-algebra is an L_∞-morphism, whose linear part is a quasi-isomorphism of complexes. Two L_∞-algebras are *quasi-isomorphic* if they are equivalent under the equivalence relation generated by the relation: $L\sim M$ if there exists a quasi-isomorphism $\phi: L\to M$.

Definition 1.6. An L_∞-algebra $(V,q_1,q_2,\ldots)$ is called *abelian* if $q_i=0$ for every $i\geq 2$. An L_∞-algebra is called *quasi-abelian* if it is quasi-isomorphic to an abelian L_∞-algebra.

Given an L_∞-algebra $(V,q_1,q_2,\ldots)$ and a differential graded commutative algebra A, there exists a natural L_∞ structure on the tensor product $V\otimes A$. The *Maurer-Cartan functor* MC_V *associated with the L_∞-algebra V* is the functor [**SS79, Fu03, Kon03**]:

$$\mathrm{MC}_V:\mathbf{Art}\to\mathbf{Set}$$

$$\mathrm{MC}_V(A)=\left\{\gamma\in V[1]^0\otimes\mathfrak{m}_A\;\middle|\;\sum_{j\geq 1}\frac{q_j(\gamma^{\odot j})}{j!}=0\right\}.$$

Two elements x and $y\in\mathrm{MC}_V(A)$ are *homotopy equivalent* if there exists $g(s)\in\mathrm{MC}_{V\otimes\mathbb{K}[s,ds]}(A)$ such that $g(0)=x$ and $g(1)=y$. Then, the *deformation functor* Def_V associated with the L_∞-algebra V is

$$\mathrm{Def}_V:\mathbf{Art}\to\mathbf{Set},\qquad \mathrm{Def}_V(A)=\frac{\mathrm{MC}_V(A)}{\text{homotopy}}.$$

Example 1.7. Given an abelian L_∞-algebra $(V,q_1,0,0,\ldots)$, for every $A\in\mathbf{Art}$ there exists a canonical isomorphism

$$\mathrm{Def}_V(A)=H^0(V[1],q_1)\otimes\mathfrak{m}_A.$$

Remark 1.8. If L is a DGLA, then the deformation functor associated with L, viewed as an L_∞-algebra, is isomorphic to the previous one (Maurer-Cartan modulo gauge equivalence) [**SS79, Ma02, Fu03, Kon03**].

Remark 1.9. As for the DGLAs, every morphism of L_∞-algebras induces a natural transformation of the associated deformation functors. If two L_∞-algebras are quasi-isomorphic, then there exists an isomorphism between the associated deformation functors [**Fu03, Kon03**], [**Ma04b**, Corollary IX.22]. In particular, the deformation functor of a quasi-abelian L_∞-algebra is unobstructed.

Lemma 1.10. *Let $(V,q_1,q_2,\ldots)$, $(W,r_1,r_2,\ldots)$ be L_∞-algebras with W quasi-abelian and $f_\infty\colon V\to W$ an L_∞-morphism. If f_1 is injective in cohomology, then V is quasi-abelian.*

PROOF. According to homotopy classification of L_∞-algebras (see e.g. [**Kon03**]), if H is the cohomology of the complex (W, r_1), there exists an L_∞ structure on H and a surjective quasi-isomorphism of L_∞-algebras $p_\infty\colon W \to H$. The L_∞ structure on H depends, up to isomorphism, by the quasi-isomorphism class of W and therefore every bracket on H is trivial.

Replacing W with H and f_∞ with $p_\infty f_\infty$ it is not restrictive to assume $r_i = 0$ for every i. There exist a graded vector space K and a morphism of graded vector spaces $\beta\colon W[1] \to K[1]$ such that the composition

$$(V[1], q_1) \xrightarrow{f_1} (W[1], 0) \xrightarrow{\beta} (K[1], 0)$$

is a quasi-isomorphism of complexes. Then, β is a linear L_∞-morphism and the composition

$$(V, q_1, q_2, \dots) \xrightarrow{f_\infty} (W, 0, 0, \dots) \xrightarrow{\beta} (K, 0, 0, \dots)$$

is a quasi-isomorphism of L_∞-algebras. □

2. The Thom-Whitney complex

Let $\mathbf{\Delta}_{\text{mon}}$ be the category whose objects are the finite ordinal sets $[n]=\{0, 1, \dots, n\}$, $n = 0, 1, \dots$, and whose morphisms are order-preserving injective maps among them. Every morphism in $\mathbf{\Delta}_{\text{mon}}$, different from the identity, is a finite composition of *coface* morphisms:

$$\partial_k\colon [i-1] \to [i], \qquad \partial_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } k \le p \end{cases}, \qquad k = 0, \dots, i.$$

The relations about compositions of them are generated by

$$\partial_l \partial_k = \partial_{k+1} \partial_l\,, \qquad \text{for every } l \le k.$$

According to [**EZ50, We94**], a *semicosimplicial* object in a category $\mathbf{C}$ is a covariant functor $A^\Delta\colon \mathbf{\Delta}_{\text{mon}} \to \mathbf{C}$. Equivalently, a semicosimplicial object A^Δ is a diagram in $\mathbf{C}$:

$$A_0 \rightrightarrows A_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} A_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots ,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i+1$ morphisms

$$\partial_k\colon A_{i-1} \to A_i, \qquad k = 0, \dots, i,$$

such that $\partial_l \partial_k = \partial_{k+1} \partial_l$, for any $l \le k$.

Given a semicosimplicial differential graded vector space

$$V^\Delta: \qquad V_0 \rightrightarrows V_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} V_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots ,$$

the graded vector space $\bigoplus_{n\ge 0} V_n[-n]$ has two differentials

$$d = \sum_n (-1)^n d_n, \qquad \text{where} \quad d_n \text{ is the differential of } V_n,$$

and

$$\partial = \sum_i (-1)^i \partial_i, \qquad \text{where} \quad \partial_i \text{ are the coface maps.}$$

More explicitly, if $v \in V_n^i$, then the degree of v is $i+n$ and

$$d(v) = (-1)^n d_n(v) \in V_n^{i+1}, \qquad \partial(v) = \partial_0(v) - \partial_1(v) + \cdots + (-1)^{n+1} \partial_{n+1}(v) \in V_{n+1}^i.$$

Since $d\partial + \partial d = 0$, we define $\operatorname{Tot}(V^\Delta)$ as the graded vector space $\bigoplus_{n\ge 0} V_n[-n]$, endowed with the differential $d + \partial$.

Example 2.1. Let $\mathcal{U} = \{U_i\}$ be an affine open cover of a smooth variety X, defined over an algebraically closed field of characteristic 0; denote by Θ_X the tangent sheaf of X. Then, we can define the Čech semicosimplicial Lie algebra $\Theta_X(\mathcal{U})$ as the semicosimplicial Lie algebra

$$\Theta_X(\mathcal{U}): \quad \prod_i \Theta_X(U_i) \rightrightarrows \prod_{i<j} \Theta_X(U_{ij}) \rightrightarrows\!\!\!\rightarrow \prod_{i<j<k} \Theta_X(U_{ijk}) \rightrightarrows\!\!\!\rightrightarrows \cdots,$$

where the coface maps $\partial_h\colon \prod_{i_0<\cdots<i_{k-1}} \Theta_X(U_{i_0\cdots i_{k-1}}) \to \prod_{i_0<\cdots<i_k} \Theta_X(U_{i_0\cdots i_k})$ are given by

$$\partial_h(x)_{i_0\ldots i_k} = x_{i_0\ldots\widehat{i_h}\ldots i_k|U_{i_0\cdots i_k}}, \qquad \text{for } h = 0,\ldots,k.$$

Since every Lie algebra is, in particular, a differential graded vector space (concentrated in degree 0), it makes sense to consider the total complex $\operatorname{Tot}(\Theta_X(\mathcal{U}))$, which coincides with the Čech complex $\check{C}(\mathcal{U},\Theta_X)$.

Example 2.2. Let Ω^*_X be the algebraic de Rham complex of a smooth variety X of dimension n:

$$\Omega^*_X\colon \qquad 0 \to \mathcal{O}_X = \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \to 0.$$

Given an affine open cover $\mathcal{U} = \{U_i\}$ of X, we can define a semicosimplicial differential graded vector space

$$\Omega^*_X(\mathcal{U}): \quad \prod_i \Omega^*_X(U_i) \rightrightarrows \prod_{i<j} \Omega^*_X(U_{ij}) \rightrightarrows\!\!\!\rightarrow \prod_{i<j<k} \Omega^*_X(U_{ijk}) \rightrightarrows\!\!\!\rightrightarrows \cdots,$$

where $\prod_{i_0<\cdots<i_k} \Omega^*_X(U_{i_0\cdots i_k})$ is the complex

$$\prod_{i_0<i_1<\cdots<i_k} \mathcal{O}_X(U_{i_0\cdots i_k}) \xrightarrow{d} \prod_{i_0<i_1<\cdots<i_k} \Omega^1_X(U_{i_0\cdots i_k}) \xrightarrow{d} \cdots \prod_{i_0<\cdots<i_n} \Omega^n_X(U_{i_0\cdots i_n}).$$

Here the total complex $\operatorname{Tot}(\Omega^*_X(\mathcal{U}))$ is the Čech complex $\check{C}(\mathcal{U},\Omega^*_X)$ of Ω^*_X, with respect to the affine cover $\mathcal{U}$.

Let V^Δ be a semicosimplicial differential graded vector space and $(A_{PL})_n$ the differential graded commutative algebra of polynomial differential forms on the standard n-simplex $\{(t_0,\ldots,t_n) \in \mathbb{K}^{n+1} \mid \sum t_i = 1\}$ [**FHT01**]:

$$(A_{PL})_n = \frac{\mathbb{K}[t_0,\ldots,t_n,dt_0,\ldots,dt_n]}{(1-\sum t_i, \sum dt_i)}.$$

For every n,m the tensor product $V_n \otimes (A_{PL})_m$ is a differential graded vector space and then also $\prod_n V_n \otimes (A_{PL})_n$ is a differential graded vector space.

Denoting by

$$\delta^k\colon (A_{PL})_n \to (A_{PL})_{n-1}, \quad \delta^k(t_i) = \begin{cases} t_i & \text{if } 0 \le i < k \\ 0 & \text{if } i = k \\ t_{i-1} & \text{if } k < i \end{cases}, \qquad k = 0,\ldots,n,$$

the face maps, for every $0 \le k \le n$, there are well-defined morphisms of differential graded vector spaces

$$Id \otimes \delta^k\colon V_n \otimes (A_{PL})_n \to V_n \otimes (A_{PL})_{n-1},$$

$$\partial_k \otimes Id\colon V_{n-1} \otimes (A_{PL})_{n-1} \to V_n \otimes (A_{PL})_{n-1}.$$

The Thom-Whitney differential graded vector space $\mathrm{Tot}_{TW}(V^\Delta)$ of V^Δ is the differential graded subvector space of $\prod_n V_n \otimes (A_{PL})_n$, whose elements are the sequences $(x_n)_{n\in\mathbb{N}}$ satisfying the equations

$$(Id \otimes \delta^k)x_n = (\partial_k \otimes Id)x_{n-1}, \quad \text{for every} \;\; 0 \leq k \leq n.$$

In [**Whi57**], Whitney noted that the integration maps

$$\int_{\Delta^n} \otimes \mathrm{Id}\colon (A_{PL})_n \otimes V_n \to \mathbb{C}[n] \otimes V_n = V_n[n]$$

give a quasi-isomorphism of differential graded vector spaces

$$I\colon (\mathrm{Tot}_{TW}(V^\Delta), d_{TW}) \to (\mathrm{Tot}(V^\Delta), d_{\mathrm{Tot}}).$$

Moreover, there exist an explicit injective quasi-isomorphism of differential graded vector spaces

$$E\colon \mathrm{Tot}(V^\Delta) \to \mathrm{Tot}_{TW}(V^\Delta)$$

and an explicit homotopy

$$h\colon \mathrm{Tot}_{TW}(V^\Delta) \to \mathrm{Tot}_{TW}(V^\Delta)[-1],$$

such that

$$IE = \mathrm{Id}_{\mathrm{Tot}(V^\Delta)}; \qquad EI - \mathrm{Id}_{\mathrm{Tot}_{TW}(V^\Delta)} = hd_{TW} + d_{TW}h.$$

Moreover, the morphisms I, E, h are functorial and commute with morphisms of semicosimplicial differential graded vector spaces. For more details and explicit description of E and h, we refer to [**Dup76, Dup78, NaA87, Get04, FMM08, CG08**].

Let V^Δ and W^Δ be two semicosimplicial graded vector spaces, with degeneracy maps $\partial_{k,n}$ and $\partial'_{k,n}$, respectively. Then, we define the tensor product $(V \otimes W)^\Delta$ as the semicosimplicial graded vector space, such that $(V\otimes W)^\Delta_n = V_n \otimes W_n$ and the degeneracy maps are defined on each factor, i.e., $\partial_{k,n} = \partial'_{k,n} \otimes \partial''_{k,n} : (V \otimes W)^\Delta_{n-1} \to (V \otimes W)^\Delta_n$.

Remark 2.3. The Thom-Withney construction is compatible with tensor product, i.e., there exists a natural transformation

$$\Phi\colon \; \mathrm{Tot}_{TW}(V^\Delta) \otimes \mathrm{Tot}_{TW}(W^\Delta) \to \mathrm{Tot}_{TW}((V \otimes W)^\Delta)$$

Indeed, we have to prove that $x \otimes y \in \mathrm{Tot}_{TW}((V \otimes W)^\Delta)$, for every pair of sequences $x = (x_n)_{n\in\mathbb{N}} \in \mathrm{Tot}_{TW}(V^\Delta)$ and $y = (y_n)_{n\in\mathbb{N}} \in \mathrm{Tot}_{TW}(W^\Delta)$.

We have

$$\partial_k \otimes Id(x_n \otimes y_n) = (\partial_k \otimes Id(x_n)) \otimes (\partial_k \otimes Id(y_n)),$$

and,

$$Id \otimes \delta^k(x_{n+1} \otimes y_{n+1}) = (Id \otimes \delta^k(x_{n+1})) \otimes (Id \otimes \delta^k(y_{n+1})).$$

It is sufficient to observe that the right parts of the above two equation are the same for every $k \leq n$.

In particular, every bilinear map of semicosimplicial graded vector spaces $V^\Delta \times W^\Delta \to Z^\Delta$ induces a bilinear map $\mathrm{Tot}_{TW}(V^\Delta) \times \mathrm{Tot}_{TW}(W^\Delta) \to \mathrm{Tot}_{TW}(Z^\Delta)$.

3. Semicosimplicial differential graded Lie algebras and mapping cones

Let

$$\mathfrak{g}^{\Delta}: \quad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathfrak{g}_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots,$$

be a *semicosimplicial differential graded Lie algebra.* Every $\mathfrak{g}_i$ is a DGLA and so, in particular, a differential graded vector space, thus we can consider the total complex $\operatorname{Tot}(\mathfrak{g}^{\Delta})$.

Example 3.1. Every morphism $\chi\colon L \to M$ of differential graded Lie algebras can be interpreted as the semicosimplicial DGLA

$$\chi^{\Delta}: \quad L \rightrightarrows M \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} 0\,, \qquad \partial_0 = \chi,\ \partial_1 = 0,$$

and the total complex $\operatorname{Tot}(\chi^{\Delta})$ coincides with the mapping cone of χ, i.e.,

$$\operatorname{Tot}(\chi^{\Delta})^i = L^i \oplus M^{i-1}, \qquad d(l,m) = (dl, \chi(l) - dm).$$

Even in the case of L and M Lie algebras, it is not possible to define a canonical bracket on the mapping cone, making $\operatorname{Tot}(\chi^{\Delta})$ a DGLA and the projection $\operatorname{Tot}(\chi^{\Delta}) \to L$ a morphism of DGLAs. To see this it is sufficient to consider $L = M$ the Lie subalgebra of $sl(2,\mathbb{K})$ generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and χ equal to the identity. If $\operatorname{Tot}(\chi^{\Delta})$ is a DGLA then, by functoriality, for every $x \in L$ the subspace generated by $x, \chi(x)$ is a subalgebra and, for every $\lambda \in \mathbb{K}$, the linear map $B \mapsto B$, $A \mapsto \lambda A$ is an automorphism of DGLA. It is an easy exercise to prove that these properties imply the failure of Jacobi identity.

Even if the complex $\operatorname{Tot}(\mathfrak{g}^{\Delta})$ has no natural DGLA structure, it can be endowed with a canonical L_∞ structure by homological perturbation theory [**FiMa07, FMM08**].

Indeed, as in the previous section, we can consider the Thom-Whitney construction also for semicosimplicial differential graded Lie algebras: it is evident that in such case we have $\operatorname{Tot}_{TW}(V^{\Delta})$ a differential graded lie algebra. In this case, the morphisms I, E, h are functorial and commute with morphisms of semicosimplicial DGLAs. Moreover, these morphisms I, E, h can be used to apply the following basic result about L_∞-algebras, dating back to Kadeishvili's work on the cohomology of A_∞ algebras [**Kad82**]; see also [**HK91**].

Theorem 3.2 (Homotopy transfer). *Let $(V, q_1, q_2, q_3, \dots)$ be an L_∞-algebra and (C, δ) a differential graded vector space. Assume we have two morphisms of complexes*

$$\pi : (V[1], q_1) \to (C[1], \delta_{[1]}), \qquad \imath_1 : (C[1], \delta_{[1]}) \to (V[1], q_1)$$

and a linear map $h \in \operatorname{Hom}^{-1}(V[1], V[1])$ such that $hq_1 + q_1 h = \imath_1\pi - Id$.

Then, there exist a canonical L_∞-algebra structure $(C, \langle\,\rangle_1, \langle\,\rangle_2, \dots)$ on C extending its differential complex structure, and a canonical L_∞-morphism $\imath_\infty\colon (C, \langle\,\rangle_1, \langle\,\rangle_2, \dots) \to (V, q_1, q_2, q_3, \dots)$ extending $\imath_1$. In particular, if $\imath_1$ is an injective quasi-isomorphism of complexes, then also $\imath_\infty$ is an injective quasi-isomorphism of L_∞-algebras.

Proof. See [**FiMa07**] and references therein; explicit formulas for the quasi-isomorphism $\imath_\infty$ and the brackets $\langle\,\rangle_n$ have been described by Merkulov in [**Me99**]; then, it has been remarked by Kontsevich and Soibelman in [**KS00, KS01**] (see also [**Fu03**]) that Merkulov's formulas can be nicely written as summations over rooted trees. □

Corollary 3.3 ([**FiMa07, FMM08**]). *There exists a canonical L_∞-algebra structure $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ on the differential graded vector space $\mathrm{Tot}(\mathfrak{g}^\Delta)$, together with an injective quasi-isomorphism $E_\infty\colon \widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)\to \mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$.*

Proof. It is sufficient to apply Theorem 3.2 to the morphisms I,E,h, in order to define a canonical L_∞-algebra structure $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ on $\mathrm{Tot}(\mathfrak{g}^\Delta)$, and an injective quasi-isomorphism $E_\infty\colon \widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)\to \mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ extending E.

Notice that E_∞ induces an isomorphism of functors $\mathrm{Def}_{\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)}\xrightarrow{\simeq}\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$. □

Let $\chi\colon L\to M$ be a morphism of differential graded Lie algebras over a field $\mathbb{K}$ of characteristic 0. We have already seen, in Example 3.1, that χ can be interpreted as the semicosimplicial DGLA

$$\chi^\Delta:\qquad L \underset{\chi}{\overset{0}{\rightrightarrows}} M \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 0\cdots,$$

and therefore we have a canonical L_∞ structure on the total complex

$$\mathrm{Tot}(\chi^\Delta)=\bigoplus_i \mathrm{Tot}(\chi^\Delta)^i,\qquad \mathrm{Tot}(\chi^\Delta)^i=L^i\oplus M^{i-1}.$$

The brackets

$$\mu_n\colon \bigwedge^n \mathrm{Tot}(\chi^\Delta)\to \mathrm{Tot}(\chi^\Delta)[2-n],\qquad n\geq 1,$$

have been explicitly described in [**FiMa07**]. Namely, one has

$$\mu_1(l,m)=(dl,\chi(l)-dm),\qquad l\in L, m\in M,$$

$$\mu_2((l_1,m_1)\wedge(l_2,m_2))=\left([l_1,l_2],\frac{1}{2}[m_1,\chi(l_2)]+\frac{(-1)^{\deg(l_1)}}{2}[\chi(l_1),m_2]\right)$$

and for $n\geq 3$

$$\mu_n((l_1,m_1)\wedge\cdots\wedge(l_n,m_n))=\pm\frac{B_{n-1}}{(n-1)!}\sum_{\sigma\in S_n}\varepsilon(\sigma)[m_{\sigma(1)},[\cdots,[m_{\sigma(n-1)},\chi(l_{\sigma(n)})]\cdots]].$$

Here the B_n's are the Bernoulli numbers, ε is the Koszul sign and we refer to [**FiMa07**] for the exact determination of the overall $\pm$ sign in the formulas (it will not be needed in the present paper).

Proposition 3.4. *In the notation above, assume that:*

(1) $\chi\colon L\to M$ *is injective,*
(2) $\chi\colon H^*(L)\to H^*(M)$ *is injective.*

Then, the L_∞-algebra $\widetilde{\mathrm{Tot}}(\chi^\Delta)$ is quasi-abelian.

Proof. Since $H^0(\mathrm{Hom}^*(L,L))=\mathrm{Hom}^0(H^*(L),H^*(L))$ and $H^0(\mathrm{Hom}^*(M,L))=\mathrm{Hom}^0(H^*(M),H^*(L))$, the surjective morphism

$$\mathrm{Hom}^*(M,L)\to\mathrm{Hom}^*(L,L),\qquad \phi\mapsto\phi\chi,$$

induces surjective maps

$$H^0(\mathrm{Hom}^*(M,L))\to H^0(\mathrm{Hom}^*(L,L)),\qquad Z^0(\mathrm{Hom}^*(M,L))\to Z^0(\mathrm{Hom}^*(L,L))$$

and then the identity on L can be lifted to a morphism $\pi\colon M\to L$ of differential graded vector spaces. Denoting $V=\ker(\pi)$, we have a direct sum decomposition of differential graded vector spaces $M=\chi(L)\oplus V$.

Denote by d the differential on M, by assumption $d(V) \subset V$ and the inclusion $V[-1] \hookrightarrow \mathrm{Tot}(\chi^{\Delta})$ is an injective quasi-isomorphism. Let H be a graded vector space, endowed with trivial differential, and $g\colon H \to V[-1]$ an injective morphism of differential graded vector spaces, inducing an isomorphism $H \xrightarrow{\sim} H^*(V[-1])$. The linear map

$$f\colon H \to \widetilde{\mathrm{Tot}}(\chi^{\Delta})), \qquad f(h) = (0, g(h)),$$

is annihilated by every bracket of $\widetilde{\mathrm{Tot}}(\chi^{\Delta})$ and then it is an injective quasi-isomorphism of L_∞-algebras. □

Example 3.5. Let W be a differential graded vector space and let $U \subset W$ be a differential graded subspace. Assume that the induced morphism $H^*(U) \to H^*(W)$ is injective, then the injective morphism of DGLAs

$$\chi\colon \{f \in \mathrm{Hom}^*(W, W) \mid f(U) \subset U\} \to \mathrm{Hom}^*(W, W)$$

satisfies the hypothesis of Proposition 3.4. In fact, the same argument used above shows that there exists a direct sum decomposition of differential graded vector spaces $W = U \oplus V$. Next, consider the subspace

$$K = \{f \in \mathrm{Hom}^*(W, W) \mid f(W) \subset V,\ f(V) = 0\}.$$

It is straightforward to check that K is a complementary subcomplex of the image of χ inside $\mathrm{Hom}^*(W, W)$.

Remark 3.6. Let W be a differential graded vector space and let $U \subset W$ be a differential graded subspace. It is showed in [**FiMa06**], that the deformation functor associated with the morphism of DGLAs

$$\chi\colon \{f \in \mathrm{Hom}^*(W, W) \mid f(U) \subset U\} \to \mathrm{Hom}^*(W, W),$$

i.e., the deformation functor associated to the L_∞-algebra $\widetilde{\mathrm{Tot}}(\chi^{\Delta})$, has a natural interpretation as the local structure of the derived Grassmannian of W at the point U. Moreover, Proposition 3.4 implies that the derived Grassmannian of W is smooth at the points corresponding to subspaces U such that $H^*(U) \to H^*(W)$ is injective.

4. Semicosimplicial Cartan homotopies

The abstract notion of Cartan homotopy has been introduced in [**FiMa06, FiMa08**] as a powerful tool for the construction of L_∞ morphisms.

Definition 4.1. Let L and M be two differential graded Lie algebras. A linear map of degree -1

$$\boldsymbol{i}\colon L \to M$$

is called a *Cartan homotopy* if, for every $a, b \in L$, we have:

$$\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, d_M \boldsymbol{i}_b + \boldsymbol{i}_{d_L b}] \qquad \text{and} \qquad [\boldsymbol{i}_a, \boldsymbol{i}_b] = 0.$$

For every Cartan homotopy $\boldsymbol{i}$, it is convenient to consider the map

$$\boldsymbol{l}\colon L \to M, \qquad \boldsymbol{l}_a = d_M \boldsymbol{i}_a + \boldsymbol{i}_{d_L a}.$$

It is straightforward to check that $\boldsymbol{l}$ is a morphism of DGLAs and the conditions of Definition 4.1 become

$$\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, \boldsymbol{l}_b] \qquad \text{and} \qquad [\boldsymbol{i}_a, \boldsymbol{i}_b] = 0.$$

As a morphism of complexes, $\boldsymbol{l}$ is homotopic to 0 (with homotopy $\boldsymbol{i}$).

Example 4.2. Let X be a smooth algebraic variety, Θ_X the tangent sheaf and (Ω^*_X, d) the algebraic de Rham complex. Then, for every open subset $U \subset X$, the contraction

$$\Theta_X(U) \otimes \Omega^k_X(U) \xrightarrow{\ \lrcorner\ } \Omega^{k-1}_X(U)$$

induces a linear map of degree -1

$$\boldsymbol{i}\colon \Theta_X(U) \to \operatorname{Hom}^*(\Omega^*_X(U), \Omega^*_X(U)), \qquad \boldsymbol{i}_\xi(\omega) = \xi \lrcorner \omega$$

that is a Cartan homotopy. In fact, the differential of $\Theta_X(U)$ is trivial and the differential on the DGLA $\operatorname{Hom}^*(\Omega^*_X(U), \Omega^*_X(U))$ is $\phi \mapsto [d, \phi] = d\phi - (-1)^{\deg(\phi)}\phi d$. Therefore, since $\boldsymbol{i}_a$ has degree -1, we have that $\boldsymbol{l}_a = d\boldsymbol{i}_a + \boldsymbol{i}_a d$ is the Lie derivative and the above conditions reduce to the classical Cartan's homotopy formulas:

(1) $[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0$;
(2) $\boldsymbol{i}_{[a,b]} = \boldsymbol{l}_a \boldsymbol{i}_b - \boldsymbol{i}_b \boldsymbol{l}_a = [\boldsymbol{l}_a, \boldsymbol{i}_b] = [\boldsymbol{i}_a, \boldsymbol{l}_b]$,

The relation between Cartan's homotopy and L_∞ algebras is given by the following theorem.

Theorem 4.3 ([**FiMa08**, Corollary 3.7])**.** *Let $\chi\colon N \to M$ be a morphism of DGLAs and $\boldsymbol{i}\colon L \to M$ be a Cartan homotopy. Assume that $\phi\colon L \to N$ is a morphism of DGLAs, such that $\chi\phi = \boldsymbol{l} = d_M \boldsymbol{i} + \boldsymbol{i} d_L$. Then, the linear map*

$$\Phi\colon L \longrightarrow \widetilde{\operatorname{Tot}}(\chi^\Delta) \qquad \Phi(a) = (\phi(a), \boldsymbol{i}_a)$$

is a linear L_∞-morphism. In particular, if N is a subalgebra of M containing $\boldsymbol{l}(L)$ and χ is the inclusion, then the map

$$L \longrightarrow \widetilde{\operatorname{Tot}}(\chi^\Delta) \qquad a \mapsto (\boldsymbol{l}_a, \boldsymbol{i}_a)$$

is a linear L_∞-morphism.

Proof. Straightforward consequence of the explicit description of the L_∞ structure of $\widetilde{\operatorname{Tot}}(\chi^\Delta)$. ☐

Remark 4.4. It is plain from definition that Cartan homotopies are stable under composition with morphisms of DGLAs. More precisely, if $f\colon L' \to L$ and $g\colon M \to M'$ are morphisms of differential graded Lie algebras and $\boldsymbol{i}\colon L \to M$ is a Cartan homotopy, then also $g\boldsymbol{i}f\colon L' \to M'$ is a Cartan homotopy.

Lemma 4.5. *Let $\boldsymbol{i}\colon L \to M$ be a Cartan homotopy and A be a differential graded commutative algebra. Then the map*

$$\boldsymbol{i} \otimes \operatorname{Id}\colon L \otimes A \to M \otimes A$$
$$(x \otimes a) \mapsto \boldsymbol{i}_x \otimes a$$

is a Cartan homotopy.

Proof. By definition, for any $x \otimes a$ and $y \otimes b \in L \otimes A$, we have

$$[(\boldsymbol{i} \otimes \operatorname{Id})_{x\otimes a}, (\boldsymbol{i} \otimes \operatorname{Id})_{y\otimes b}] = [\boldsymbol{i}_x \otimes a, \boldsymbol{i}_y \otimes b] = (-1)^{\deg(a)(\deg(y)-1)}[\boldsymbol{i}_x, \boldsymbol{i}_y] \otimes ab = 0.$$

Moreover, denoting $\boldsymbol{l} = d_M \boldsymbol{i} + \boldsymbol{i} d_L$ and $\tilde{\boldsymbol{l}} = d_{M\otimes A}(\boldsymbol{i} \otimes Id) + (\boldsymbol{i} \otimes Id) d_{L\otimes A}$ we have

$$\tilde{\boldsymbol{l}}_{x\otimes a} = d_{M\otimes A}(\boldsymbol{i}_x \otimes a) + (\boldsymbol{i} \otimes Id)(d_L x \otimes a + (-1)^{\deg(x)} x \otimes d_A a) =$$

$$= d_M \boldsymbol{i}_x \otimes a - (-1)^{\deg(x)} x \otimes d_A a + \boldsymbol{i}_{d_L x} \otimes a + (-1)^{\deg(x)} x \otimes d_A a = \boldsymbol{l}_x \otimes a.$$

Thus, for any $x \otimes a$ and $y \otimes b \in L \otimes A$, we get

$$(\boldsymbol{i} \otimes \operatorname{Id})_{[x\otimes a, y\otimes b]} = (\boldsymbol{i} \otimes \operatorname{Id})_{(-1)^{\deg(a)\deg(y)}[x,y]\otimes ab} = (-1)^{\deg(a)\deg(y)} \boldsymbol{i}_{[x,y]} \otimes ab =$$

$$(-1)^{\deg(a)\deg(y)}[\boldsymbol{i}_x,\boldsymbol{l}_y]\otimes ab=[\boldsymbol{i}_x\otimes a,\boldsymbol{l}_y\otimes b]=[(\boldsymbol{i}\otimes \mathrm{Id})_{x\otimes a},\tilde{\boldsymbol{l}}_{y\otimes b}].$$

□

Definition 4.6. Let L be a differential graded Lie algebra and V a differential graded vector space. A bilinear map

$$L\times V\xrightarrow{\ \lrcorner\ }V$$

of degree -1 is called a *contraction* if the induced map

$$\boldsymbol{i}\colon L\to \mathrm{Hom}^*(V,V),\qquad \boldsymbol{i}_l(v)=l\lrcorner v,$$

is a Cartan Homotopy.

The notion of contraction is stable under scalar extensions, more precisely:

Lemma 4.7. *Let V be a differential graded vector space and*

$$L\times V\xrightarrow{\ \lrcorner\ }V$$

a contraction. Then, for every differential graded commutative algebra A, the natural extension

$$(L\otimes A)\times(V\otimes A)\xrightarrow{\ \lrcorner\ }(V\otimes A)\qquad (l\otimes a)\lrcorner(v\otimes b)=(-1)^{\deg(a)\deg(v)}l\lrcorner v\otimes ab,$$

is a contraction.

Proof. According to Remark 4.4, Lemma 4.5 and Definition 4.6, it is sufficient to prove that the natural map

$$\alpha\colon\ \mathrm{Hom}^*(V,V)\otimes A\to \mathrm{Hom}^*(V\otimes A,V\otimes A),\quad \alpha(\phi\otimes a)(v\otimes b)=(-1)^{\deg(a)\deg(v)}\phi(v)\otimes ab,$$

is a morphism of DGLAs. This is completely straightforward and it is left to the reader.

□

The notions of Cartan homotopy and contraction extend naturally to the semicosimplicial setting. Here, we consider only the case of contractions.

Definition 4.8. Let $\mathfrak{g}^\Delta$ be a semicosimplicial DGLA and V^Δ a semicosimplicial differential graded vector space. A semicosimplicial contraction

$$\mathfrak{g}^\Delta\times V^\Delta\xrightarrow{\lrcorner}V^\Delta,$$

is a sequence of contractions $\mathfrak{g}_n\times V_n\xrightarrow{\lrcorner}V_n$, $n\geq 0$, commuting with coface maps, i.e., $\partial_k(l\lrcorner v)=\partial_k(l)\lrcorner\partial_k(v)$, for every k.

Proposition 4.9. *Every semicosimplicial contraction*

$$\mathfrak{g}^\Delta\times V^\Delta\xrightarrow{\lrcorner}V^\Delta$$

extends naturally to a contraction

$$\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)\times \mathrm{Tot}_{TW}(V^\Delta)\xrightarrow{\lrcorner}\mathrm{Tot}_{TW}(V^\Delta).$$

Proof. By definition, for every n, we have a Cartan homotopy

$$\boldsymbol{i}:\mathfrak{g}_n\to \mathrm{Hom}^*(V_n,V_n),$$

and so, by Lemma 4.7, a Cartan homotopy

$$\boldsymbol{i}\colon \mathfrak{g}_n\otimes(A_{PL})_n\to \mathrm{Hom}^*(V_n\otimes(A_{PL})_n,V_n\otimes(A_{PL})_n).$$

Therefore, it is enough to prove that $\boldsymbol{i}_x(y)\in \mathrm{Tot}_{TW}(V^\Delta)$, for every pair of sequences $x=(x_n)_{n\in\mathbb{N}}\in \mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ and $y=(y_n)_{n\in\mathbb{N}}\in \mathrm{Tot}_{TW}(V^\Delta)$; it follows from Remark 2.3.

□

5. Semicosimplicial Lie algebras and deformations of smooth varieties

Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra, then we can apply the construction of Section 3 in order to construct the Thom-Whitney DGLA $\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)$, the L_∞-algebra $\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)$ and their associated (and isomorphic) deformation functors $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)} \simeq \operatorname{Def}_{\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)}$.

Beyond this way, there is another natural, and more geometric, way to define a deformation functor, see [**Pr03**, Definitions 1.4 and 1.6] and [**FMM08**, Section 3]. More precisely, if $\mathfrak{g}^\Delta$ is a semicosimplicial Lie algebra, then we denote

$$Z^1_{sc}(\exp \mathfrak{g}^\Delta)\colon \mathbf{Art} \to \mathbf{Set}$$

as

$$Z^1_{sc}(\exp \mathfrak{g}^\Delta)(A) = \{x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A \mid e^{\partial_0(x)} e^{-\partial_1(x)} e^{\partial_2(x)} = 1\},$$

and

$$H^1_{sc}(\exp \mathfrak{g}^\Delta)\colon \mathbf{Art} \to \mathbf{Set}$$

such that

$$H^1_{sc}(\exp \mathfrak{g}^\Delta)(A) = Z^1_{sc}(\exp \mathfrak{g}^\Delta)(A)/\sim,$$

where $x \sim y$ if and only if there exists $a \in \mathfrak{g}_0 \otimes \mathfrak{m}_A$, such that $e^{-\partial_1(a)} e^x e^{\partial_0(a)} = e^y$.

Example 5.1. Let $\mathcal{L}$ be a sheaf of Lie algebras on a paracompact topological space X, and $\mathcal{U}$ an open covering of X; it is naturally defined the Čech semicosimplicial Lie algebra $\mathcal{L}(\mathcal{U})$

$$\mathcal{L}(\mathcal{U}): \qquad \prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i<j} \mathcal{L}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i<j<k} \mathcal{L}(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots,$$

and, for every $A \in \mathbf{Art}$, the set $H^1_{sc}(\exp \mathcal{L}(\mathcal{U}))(A)$ is exactly the cohomology set $H^1(\mathcal{U}, \exp(\mathcal{L} \otimes \mathfrak{m}_A))$ [**Hir78**].

The relation between the above functors is given by the following theorem.

Theorem 5.2. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial Lie algebra. Then, for every $A \in \mathbf{Art}$,*

$$\operatorname{MC}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)}(A) = Z^1_{sc}(\exp \mathfrak{g}^\Delta)(A),$$

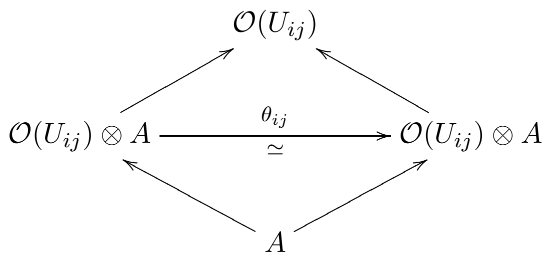
as subsets of $\mathfrak{g}_1 \otimes \mathfrak{m}_A$. Moreover, we have natural isomorphisms of deformation functors

$$\operatorname{Def}_{\operatorname{Tot}_{TW}(\mathfrak{g}^\Delta)} \simeq \operatorname{Def}_{\widetilde{\operatorname{Tot}}(\mathfrak{g}^\Delta)} \xrightarrow{\simeq} H^1_{sc}(\exp \mathfrak{g}^\Delta),$$

Proof. For the proof, we refer to [**FMM08**]. □

Next, assume that X is a smooth algebraic variety over a field $\mathbb{K}$ of characteristic 0, with tangent sheaf Θ_X, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of X.

Since every infinitesimal deformation of a smooth affine scheme is trivial [**Se06**, Lemma II.1.3], every infinitesimal deformation X_A of X over $\operatorname{Spec}(A)$ is obtained by gluing the trivial deformations $U_i \times \operatorname{Spec}(A)$ along the double intersections U_{ij}, and therefore it is determined by the sequence $\{\theta_{ij}\}_{i<j}$ of automorphisms of sheaves of A-algebras



satisfying the cocycle condition

$$\theta_{jk}\theta_{ik}^{-1}\theta_{ij} = \mathrm{Id}_{\mathcal{O}(U_{ijk})\otimes A}, \qquad \forall\ i<j<k\in I. \tag{1}$$

Since we are in characteristic zero, we can take the logarithms and write $\theta_{ij} = e^{d_{ij}}$, where $d_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$. Therefore, the Equation (1) is equivalent to

$$e^{d_{jk}}e^{-d_{ik}}e^{d_{ij}} = 1 \in \exp(\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A), \qquad \forall\ i<j<k\in I.$$

Next, let X'_A be another deformation of X over $\mathrm{Spec}(A)$, defined by the cocycle θ'_{ij}. To give an isomorphism of deformations $X'_A \simeq X_A$ is the same to give, for every i, an automorphism α_i of $\mathcal{O}(U_i)\otimes A$ such that $\theta_{ij} = {\alpha_i}^{-1}{\theta'_{ij}}^{-1}\alpha_j$, for every $i<j$. Taking again logarithms, we can write $\alpha_i = e^{a_i}$, with $a_i \in \Theta_X(U_i)\otimes\mathfrak{m}_A$, and so $e^{-a_i}e^{d'_{ij}}e^{a_j} = e^{d_{ij}}$.

Theorem 5.3. *Let $\mathcal{U}$ be an affine open cover of a smooth algebraic variety X defined over an algebraically closed field of characteristic 0. Denoting by Def_X the functor of infinitesimal deformations of X, there exist isomorphisms of functors*

$$\mathrm{Def}_X \cong H^1_{sc}(\exp\Theta_X(\mathcal{U})) \cong \mathrm{Def}_{\mathrm{Tot}_{TW}(\Theta_X(\mathcal{U}))} \cong \mathrm{Def}_{\widetilde{\mathrm{Tot}}(\Theta_X(\mathcal{U}))},$$

where $\Theta_X(\mathcal{U})$ is the semicosimplicial Lie algebra defined in Example 2.1.

Proof. By Theorem 5.2, it is sufficient to prove $\mathrm{Def}_X \cong H^1_{sc}(\exp\Theta_X(\mathcal{U}))$. By definition,

$$Z^1_{sc}(\Theta_X(\mathcal{U}))(A) = \{\{x_{ij}\} \in \prod_{i<j}\Theta_X(U_{ij})\otimes\mathfrak{m}_A \mid e^{x_{jk}}e^{-x_{ik}}e^{x_{ij}} = 1\ \forall\ i<j<k\},$$

for each $A \in \mathbf{Art}$. Moreover, given $x = \{x_{ij}\}$ and $y = \{y_{ij}\} \in \prod_{i<j}\Theta_X(U_{ij})\otimes\mathfrak{m}_A$, we have $x \sim y$ if and only if there exists $a = \{a_i\} \in \prod_i \Theta_X(U_i)\otimes\mathfrak{m}_A$ such that $e^{-a_j}e^{x_{ij}}e^{a_i} = e^{y_{ij}}$ for all $i<j$. □

Remark 5.4. Note that, when $\mathbb{K} = \mathbb{C}$, the L_∞-algebra $\widetilde{\mathrm{Tot}}(\Theta_X(\mathcal{U}))$ is quasi-isomorphic to the Kodaira-Spencer differential graded Lie algebra of X.

6. Proof of the main theorem

In this section, we use the results developed before to give a complete algebraic proof of the following theorem.

Theorem 6.1. *Let X be a smooth projective variety of dimension n, defined over an algebraically closed field of characteristic 0. If the contraction map*

$$H^*(\Theta_X) \xrightarrow{\boldsymbol{i}} \mathrm{Hom}^*(H^*(\Omega^n_X), H^*(\Omega^{n-1}_X))$$

is injective, then, for every affine open cover $\mathcal{U}$ of X, the DGLA $\mathrm{Tot}_{TW}(\Theta_X(\mathcal{U}))$ is quasi-abelian.

Proof. According to Lemma 1.10, it is sufficient to prove that there exist a quasi abelian L_∞-algebra H and a morphism $\mathrm{Tot}_{TW}(\Theta_X(\mathcal{U})) \to H$ that is injective in cohomology.

Let n be the dimension of X and denote by Ω^*_X the algebraic de Rham complex. For every $i \leq n$, let $\check{C}(\mathcal{U},\Omega^i_X)$ be the Čech complex of the coherent sheaf Ω^i_X, with respect to the affine cover $\mathcal{U}$, and $\check{C}(\mathcal{U},\Omega^*_X)$ the total complex of the semicosimplicial differential graded vector space $\Omega^*_X(\mathcal{U})$ (Example 2.2). Notice that

$$\check{C}(\mathcal{U},\Omega^*_X)^i = \bigoplus_{a+b=i}\check{C}(\mathcal{U},\Omega^a_X)^b.$$

and $\check{C}(\mathcal{U},\Omega^n_X)$ is a subcomplex of $\check{C}(\mathcal{U},\Omega^*_X)$.

Then, we have a commutative diagram of complexes with horizontal quasi-isomorphisms:

$$\begin{array}{ccc}\check{C}(\mathcal{U},\Omega^n_X)=\operatorname{Tot}(\Omega^n_X(\mathcal{U})) & \xrightarrow{E} & \operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U}))\\ \downarrow & & \downarrow\\ \check{C}(\mathcal{U},\Omega^*_X)=\operatorname{Tot}(\Omega^*_X(\mathcal{U})) & \xrightarrow{E} & \operatorname{Tot}_{TW}(\Omega^*_X(\mathcal{U})).\end{array}$$

Since $\mathbb{K}$ has characteristic 0 and X is smooth and proper, the Hodge spectral sequence degenerates at E_1 (we refer to [**Fa88, DeIl87**] for a purely algebraic proof of this fact). Therefore, we have injective maps

$$H^*(X,\Omega^n_X)=H^*(\check{C}(\mathcal{U},\Omega^n_X))\hookrightarrow H^*(\check{C}(\mathcal{U},\Omega^*_X))=H^*_{DR}(X/\mathbb{K}).$$

$$H^*(X,\Omega^{n-1}_X)=H^*(\check{C}(\mathcal{U},\Omega^{n-1}_X))\hookrightarrow H^*\left(\frac{\check{C}(\mathcal{U},\Omega^*_X)}{\check{C}(\mathcal{U},\Omega^n_X)}\right).$$

Thus, the natural inclusions of complexes

$$\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U}))\to\operatorname{Tot}_{TW}(\Omega^*_X(\mathcal{U})),$$

$$\operatorname{Tot}_{TW}(\Omega^{n-1}_X(\mathcal{U}))\to\frac{\operatorname{Tot}_{TW}(\Omega^*_X(\mathcal{U}))}{\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U}))},$$

are injective in cohomology.

According to Example 4.2, for every open subset $U\subset X$, the contraction of vector fields with differential forms defines a Cartan homotopy:

$$\boldsymbol{i}\colon\Theta_X(U)\to\operatorname{Hom}^*(\Omega^*_X(U),\Omega^*_X(U)),\qquad \boldsymbol{i}_\xi(\omega)=\xi\lrcorner\omega.$$

Since the contraction $\lrcorner$ commutes with restrictions to open subsets, we have a semicosimplicial contraction

$$\Theta_X(\mathcal{U})\times\Omega^*_X(\mathcal{U})\xrightarrow{\lrcorner}\Omega^*_X(\mathcal{U}),$$

and, by Proposition 4.9, this induces naturally a Cartan homotopy

$$\boldsymbol{i}\colon\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U}))\longrightarrow\operatorname{Hom}^*(\operatorname{Tot}_{TW}(\Omega^*(\mathcal{U})),\operatorname{Tot}_{TW}(\Omega^*(\mathcal{U}))).$$

Notice that, for every $\xi\in\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U}))$ and every i, we have

$$\boldsymbol{i}_\xi(\operatorname{Tot}_{TW}(\Omega^i(\mathcal{U})))\subset\operatorname{Tot}_{TW}(\Omega^{i-1}(\mathcal{U})),$$

$$\boldsymbol{l}_\xi(\operatorname{Tot}_{TW}(\Omega^i(\mathcal{U})))\subset\operatorname{Tot}_{TW}(\Omega^i(\mathcal{U})),\qquad \boldsymbol{l}_\xi=d\boldsymbol{i}_\xi+\boldsymbol{i}_{d\xi}.$$

Moreover, the assumption of the theorem, together with [**NaA87**, 3.1], implies that the map

$$\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U}))\xrightarrow{\boldsymbol{i}}\operatorname{Hom}^*\left(\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U})),\operatorname{Tot}_{TW}(\Omega^{n-1}_X(\mathcal{U}))\right)$$

is injective in cohomology.

Next, consider the differential graded Lie algebras

$$M=\operatorname{Hom}^*(\operatorname{Tot}_{TW}(\Omega^*_X(\mathcal{U})),\operatorname{Tot}_{TW}\Omega^*_X(\mathcal{U})),$$

$$L=\{f\in M\mid f(\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U})))\subset\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U}))\},$$

and let $\chi\colon L\to M$ be the inclusion. According to Example 3.5, the L_∞-algebras $\widetilde{\operatorname{Tot}}(\chi^\Delta)$ is quasi-abelian.

Moreover, we have $\boldsymbol{l}(\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U})))\subset L$ and so, by Theorem 4.3, there exists a linear L_∞-morphism

$$\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U}))\xrightarrow{(\boldsymbol{l},\boldsymbol{i})}\widetilde{\operatorname{Tot}}(\chi^\Delta),\quad x\mapsto(\boldsymbol{l}_x,\boldsymbol{i}_x).$$

Since the map χ in injective, its mapping cone $\operatorname{Tot}(\chi^{\Delta})$ is quasi-isomorphic to its cokernel

$$\operatorname{Coker}\chi = \operatorname{Hom}^*\left(\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U})), \frac{\operatorname{Tot}_{TW}(\Omega^*_X(\mathcal{U}))}{\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U}))}\right)$$

and we have a commutative diagram of complexes

$$\begin{array}{ccc}
\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U})) & \xrightarrow{(\boldsymbol{l},\boldsymbol{i})} & \operatorname{Tot}(\chi^{\Delta}) \\
\downarrow{\scriptstyle \boldsymbol{i}} & & \downarrow{\scriptstyle q-iso} \\
\operatorname{Hom}^*\left(\operatorname{Tot}_{TW}(\Omega^n_X(\mathcal{U})), \operatorname{Tot}_{TW}(\Omega^{n-1}_X(\mathcal{U}))\right) & \xrightarrow{\alpha} & \operatorname{Coker}\chi.
\end{array}$$

Since both $\boldsymbol{i}$ and α are injective in cohomology, also the L_∞-morphism $(\boldsymbol{l},\boldsymbol{i})$ is injective in cohomology.

□

Theorem 6.2. *Let $\mathcal{U} = \{U_i\}$ be an affine open cover of a smooth projective variety X defined over an algebraically closed field of characteristic 0. If the canonical bundle of X is trivial or torsion, then the DGLA $\operatorname{Tot}_{TW}(\Theta_X(\mathcal{U}))$ is quasi-abelian.*

Proof. Assume first that X has trivial canonical bundle. If n is the dimension of X, the cup product with a nontrivial section of the canonical bundle gives the isomorphisms $H^i(\Theta_X) \simeq H^i(\Omega^{n-1}_X)$ and the conclusion follows immediately from Theorem 6.1. If X has torsion canonical bundle we may consider the canonical cyclic cover $\pi\colon Y \to X$ and the affine open cover $\mathcal{V} = \{\pi^{-1}(U_i)\}$. Now the variety Y has trivial canonical bundle and then the L_∞-algebra $\widetilde{\operatorname{Tot}}(\Theta_Y(\mathcal{V}))$ is quasi-abelian. Since π is an unramified cover the natural injective map $\widetilde{\operatorname{Tot}}(\Theta_X(\mathcal{U})) \to \widetilde{\operatorname{Tot}}(\Theta_Y(\mathcal{V}))$ is also injective in cohomology and we conclude the proof by using the same argument of Theorem 6.1. □

Remark 6.3. When $\mathbb{K} = \mathbb{C}$, the previous theorems together with Remark 5.4, implies that the Kodaira-Spencer DGLA is quasi abelian, for a projective manifold with trivial or torsion canonical bundle.

Theorem 6.4. *Let X be a smooth projective variety of dimension n defined over an algebraically closed field of characteristic 0. Then, the obstructions to deformations of X are contained in the kernel of the contraction map*

$$H^2(\Theta_X) \xrightarrow{\boldsymbol{i}} \prod_p \operatorname{Hom}(H^p(\Omega^n_X), H^{p+2}(\Omega^{n-1}_X)).$$

Proof. We have seen in the proof of Theorem 6.1 that, for every affine open cover $\mathcal{U}$ of X, there exists an L_∞-morphism $\widetilde{\operatorname{Tot}}(\Theta_X(\mathcal{U})) \to \widetilde{\operatorname{Tot}}(\chi^{\Delta})$ and that $\widetilde{\operatorname{Tot}}(\chi^{\Delta})$ is quasi-abelian. Therefore, we are in the condition to apply the general strategy used in [**Ma04a, Ma09, Ia07**]: the deformation functor associated to $\widetilde{\operatorname{Tot}}(\chi^{\Delta})$ is unobstructed and the obstructions of $\operatorname{Def}_X \simeq \operatorname{Def}_{\widetilde{\operatorname{Tot}}(\Theta_X(\mathcal{U}))}$ are contained in the kernel of the obstruction map $H^2(\operatorname{Tot}(\Theta_X(\mathcal{U}))) \to H^2(\operatorname{Tot}(\chi^{\Delta}))$. □

Corollary 6.5. *Let X be a smooth projective variety defined over an algebraically closed field of characteristic 0. If the canonical bundle of X is trivial, then X has unobstructed deformations.*

Proof. The previous Corollary 6.2 implies that $\widetilde{\operatorname{Tot}}(\Theta_X(\mathcal{U}))$ is quasi-abelian and so $\operatorname{Def}_{\widetilde{\operatorname{Tot}}(\Theta_X(\mathcal{U}))}$ is smooth. By Theorem 5.3, $\operatorname{Def}_X \cong \operatorname{Def}_{\widetilde{\operatorname{Tot}}(\Theta_X(\mathcal{U}))}$. □

Remark 6.6. Transcendental proofs of the analogue of Theorem 6.4 for compact Kähler manifolds can be found in [**Ma04a, Cle05, Ma09**], while we refer to [**Ia07, Ma09**] for the proof that the T^1-lifting is definitely insufficient for proving Theorem 6.4.

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