Periodic magnetic geodesics on almost every energy level via variational methods *

I.A. Taimanov †

To V.V. Kozlov on the occasion of his 60th birthday

1 Introduction

In the article by using variational methods we prove that for almost every energy level "the principle of throwing out cycles" gives periodic magnetic geodesics on the critical levels defined by the "thrown out" cycles (Theorem 2).

In particular, Theorem 2 implies

Theorem 1 *Let M^n be a closed Riemannian manifold and A be a one-form on M^n. If there is a contractible closed curve $\gamma \subset M^n$ such that*

$$S_E(\gamma) = \int_\gamma \left(\sqrt{E g_{ik} \dot{x}^i \dot{x}^k} + A_i \dot{x}^i \right) dt < 0, \tag{1}$$

then either there is a contractible closed extremal of the functional S_E, either there is a sequence $\gamma_n, n = 1, \dots,$ of contractible closed extremals of the functionals S_{E_n} such that

$$\lim_{n\to\infty} E_n = E, \quad \lim_{n\to\infty} \mathrm{length}(\gamma_n) = \infty.$$

Theorem 1 is covered by other results obtained by using methods of symplectic geometry [6, 5] (see §3). However we prove it by using rather simple variational methods. Since the formulation of Theorem 2 needs some preliminaries we expose it in §2.

Now let us first explain what are magnetic geodesics.

Let M^n be a Riemannian manifold and F be a closed two-form on M^n. The motion of a charge in a magnetic field F is given by the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0, \quad i = 1, \dots, n,$$

*The work was supported by RFBR (grant 09-01-12130-ofi-m) and Max Planck Institute for Mathematics in Bonn.

†Institute of Mathematics, 630090 Novosibirsk, Russia; e-mail: taimanov@math.nsc.ru.

for the following Lagrangian function

$$\mathcal{L}^\alpha(x,\dot{x}) = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k + A_i^\alpha \dot{x}^i \tag{2}$$

where $A^\alpha = d^{-1}F$ are a one-form defined in a domain U_α such that the restriction F_{U_α} of F onto this domain is an exact form. Therewith the Riemannian metric g_{ik} is used to evaluate the kinetic energy of the charge.

If F is exact we have a globally defined Lagrangian function. Otherwise only the Euler–Lagrange equations are globally defined because the one-form A, the vector potential of the magnetic field, enters them via F and the trajectories of a charge are extremals of a multivalued functional $W^\alpha = \int \mathcal{L}^\alpha dt$ for which the variational derivative δW^α is globally defined. The analysis of this picture was done by Novikov [14, 15] who started the investigation of variational problems for such functionals on the spaces of closed curves looking for periodic trajectories of charges.

The flow corresponding to the Lagrangian (2) is called the *magnetic geodesic flow.*

The energy

$$E = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$$

is preserved along trajectories and on a given energy level the magnetic geodesic flow is trajectory equivalent to the flow defined by another Lagrangian function which is homogeneous in velocities:

$$\mathcal{L}^\alpha_E = \sqrt{E g_{ik}\dot{x}^i\dot{x}^k} + A_i^\alpha \dot{x}^i, \qquad E = \text{const.}$$

The corresponding functionals

$$S^\alpha_E(\gamma) = \int_\gamma \left(\sqrt{E g_{ik}\dot{x}^i\dot{x}^k} + A_i^\alpha \dot{x}^i\right) dt$$

are defined on the spaces of non-parameterized closed curves on which they are additive:

$$S(\gamma_1 \cup \gamma_2) = S(\gamma_1) + S(\gamma_2), \tag{3}$$

where the curve $\gamma_1 \cup \gamma_2$ consists in two curves γ_1 and γ_2 which are passed successively.

We thank A. Bahri for helpful discussions.

2 The principle of throwing out cycles

The topological problems which arise when we look for some analogs of the Morse inequalities for S^α_E were analyzed by Novikov [14, 15]. We did not consider here the case of multivalued functionals and in the sequel we assume that the magnetic flow is exact, i.e., there exists a globally defined one-form A such that

$$F = dA.$$

We have a functional

$$S_E(\gamma) = \int_\gamma \left(\sqrt{Eg_{ik}\dot{x}^i\dot{x}^k} + A_i\dot{x}^i\right) dt \tag{4}$$

which is defined, as we already mentioned above, on the space $\Omega^+(M^n)$ of non-parameterized closed curves in M^n. Indeed, since the Lagrangian is homogeneous of degree one in velocity, the value of S does not depend on the parameterization.

If

$$\sqrt{Eg_{ik}\dot{x}^i\dot{x}^k} + A_i\dot{x}^i > 0 \quad \text{for } \dot{x} \neq 0,$$

then the functional S is always positive outside one-point curves and the number of its extremals can be estimated from below by using the classical Morse theory. For instance, if all extremals are nondegenerate (in the Morse sense) we have the simplest version of the Morse inequalities:

$$m_k \geq \operatorname{rank} H_k(\Omega^+(M^n), M^n; \mathbb{Q})$$

where m_i is the number of extremals of index k, $k = 0, 1, \ldots$, and $M^n \subset \Omega^+(M^n)$ is the set formed by one-point curves. In general we have the sum of the k-th type numbers of extremals in the left-hand side (see the recent survey [25]).

Assume now that (1) holds. Then we have to use other topological reasonings to control critical levels of the functional S. They were introduced by Novikov [14, 15] as "the principle of throwing out cycles" which is as follows:

- *Let $w \in H_k(M^n)$ be a nontrivial cycle. We say that it is "thrown out" into the set $\{S < 0\}$ if there is a continuous map*

  $$F : P \times [0,1] \to \Omega^+(M^n)$$

  such that P is a polyhedron, the map $F_0 : P \times 0 \to \Omega^+(M^n)$ is a map into one-point curves which realizes the cycle w, i.e. $F_0(P) \subset M^n$ and $F_{0*}([P]) = w$, and $F(P \times 1) \subset \{S < 0\}$. If such a map exists then it generates a nontrivial cycle in

  $$H_{k+1}(\Omega^+(M^n), \{S \leq 0\}).$$

For $k = 0$ this argument reduces to a version of the mountain-pass lemma, however it is stronger because it takes care about all cycles and gives rise to critical levels with $k > 1$.

In [21] we proved that

- *if there is a closed contractible curve γ such that $S(\gamma) < 0$ then the whole manifold M^n is thrown out into $\{S < 0\}$ which implies the following inequalities:*

  $$\operatorname{rank} H_{k+1}(\Omega^+(M^n), M^n) \geq \operatorname{rank} H_k(M^n). \tag{5}$$

In fact the construction of such "throwing out" works for any functional $S = \int \mathcal{L} dt$ which satisfy the addition property (3).

If the existence of a critical level implies the existence of a critical point with the corresponding index on this level, we may put m_{k+1} into the left-hand side of (5). However this is not known for the functionals of the type (4). The usual reasonings by Morse which he did apply for the length functional [13] does not work in this situation (see the discussion in [22]): the gradient deformations for the functional S may increase the lengths of deformed curves and a priori the gradient deformation may not converge to critical points.

To every cycle $[z] \in H_k(M^n)$ there corresponds a critical level $c(w)$ of the functional S which is defined as follows:

$$c(w) = \inf_F \max_{x \in P \times [0,1]} S(x) \tag{6}$$

where the infimum is taken over all throwings out $F : P \times [0,1] \to \Omega^+(M^n)$ of the cycle w. Moreover for sufficiently small $\varepsilon > 0$ we have

$$H_{k+1}(\{S \leq c(w) + \varepsilon\}, \{S \leq c(w) - \varepsilon\}) \neq 0$$

which implies (5). These arguments are standard for the Morse theory.

Now the existence of a critical point on the critical level is derived as follows: let us take a gradient-type deformation for the functional S which decreases its values and apply it to the set $\{S \leq c(w) + \varepsilon\}$. Since it can not be deformed into $\{S \leq c(w) - \varepsilon\}$ due to topological reasons, this deformation has either to suspend on a critical point (in a generic situation of index $k+1$), either there is a subset which realizes a nontrivial cycle and its deformation diverges to "infinity" (in the finite-dimensional Morse theory on compact manifolds this is impossible due to compactness however in the non-compact and, in particular, infinite-dimensional case this results in the existence of "a critical point at infinity" [2]).

If the Palais–Smale condition [17] [1] holds it rules out the latter case and this implies that on every critical level there is a critical point. Until recently this condition or some of its reasonable replacements is not established for the functionals of type (4).

However we show that "the principle of throwing out cycles" gives periodic magnetic geodesics on almost every energy level as follows:

Theorem 2 *Let M^n be a closed Riemannian manifold and A be a one-form on M^n and let there is a contractible closed curve $\gamma \subset M^n$ such that*

$$S_E(\gamma) = \int_\gamma \left(\sqrt{E g_{ik} \dot{x}^i \dot{x}^k} + A_i \dot{x}^i \right) dt < 0.$$

[1] It is said that a C^1-functional S *meets this condition in* $\{a \leq S \leq b\}$, $-\infty \leq a \leq b \leq +\infty$, *if for any sequence* x_n *such that* $|S(x_n)|$ *is bounded from above,* $S(x_n) \in [2, b]$, *and* $\lim_{n\to\infty} |\mathrm{grad}\, S(x_n)| = 0$ *there is a convergent subsequence.*

Then for every nontrivial cycle $w \in H_*(M^n)$ either there is a contractible closed extremal γ_0 of the functional S_E with $S_E(\gamma_0) = c(w)$, either there is a sequence $\gamma_n, n = 1, \dots,$ of contractible closed extremals of the functionals S_{E_n} such that

$$\lim_{n\to\infty} E_n = E, \quad \lim_{n\to\infty} S_E(\gamma_n) = c(w), \quad \lim_{n\to\infty} \text{length}(\gamma_n) = \infty.$$

Theorem 1 follows from Theorem 2 immediately.

3 On the existence of periodic magnetic geodesics

The study of the periodic problem for magnetic geodesics was initiated by Novikov [14, 15]. The principle of throwing out cycles gives the necessary critical levels in the situation where the classical Morse theory does not work. However the difficulties with the Palais–Smale type conditions as we mentioned above hinder a straightforward derivation of the existence of critical points on these levels. Until recently the application of the principle of throwing out cycles to proving the existence of periodic magnetic geodesics was confirmed in two cases:

1) for magnetic geodesics on a two-torus with everywhere positive magnetic field $F > 0$ [9]. In these case the magnetic field is exact only on the universal covering of the torus;

2) for exact magnetic fields which everywhere meet the condition

$$\min_{|\xi|=1} \left\{ \text{Ric}(\xi,\xi) - \sum_\alpha (\nabla_{e_\alpha} F)(e_\alpha, \xi) \right\} > 0 \tag{7}$$

where Ric is the Ricci curvature of the Riemannian metric, ξ is a tangent vector to M^n, and $\{e_\alpha\}$ is the orthonormal basis in the corresponding tangent space [3]. This result is valid for a manifold of any dimension.

There is another result obtained by variational methods in the situation of strong magnetic fields, when there are throwing out of cycles [16, 22, 23]. However it gives locally minimal periodic magnetic geodesics and reads:

- *given a strong magnetic field on a closed oriented two-manifold there is a non-self-intersecting periodic magnetic geodesic which is a local minimum of the functional S.*

An exact field is called strong if there is a two-dimensional submanifold N^2 with boundary such that

$$\text{length}\,(\partial N^2) + \int_{N^2} F < 0.$$

There is a similar definition of strong non-exact magnetic fields [22].

So until recently it was possible either to use the special features of two-manifolds, either to clarify the Palais–Smale condition under certain conditions (see (7) to derive the existence of periodic magnetic geodesics by variational methods.

There is another approach to proving the existence of such geodesics by methods of symplectic geometry and it was initiated by Arnold [1] and Kozlov [12]. This approach was far-developed and we refer to the recent article [8] which, in particular, contains an extensive list of references. However we have to stress two results obtained by symplectic and dynamical methods:

1) in [4] it was proved that every exact magnetic field on a closed two-manifold possesses periodic orbits on all energy levels. [2]

The proof splits into three cases:

(a) $E > E_0$ where E_0 is some constant, the Mané strict critical level. In this case we have, roughly speaking, the geodesic flow of a Finsler metric and may apply the classical Morse theory;

(b) $E < E_0$. As it is established in [4] this is exactly the case of strong magnetic fields which is covered by results from [16, 22];

(c) $E = E_0$. In this case the existence of periodic orbits was established in [4] by methods coming from dynamical systems.

Moreover we see that due to [4] in the two-dimensional case the case when the principle of throwing out cycles gives us necessary critical levels and the case when the classical Morse theory works are separated exactly by a single energy level;

2) in [5] by using symplectic and dynamical methods as well as variational methods applied to the (free time) action functional which differs from S it was proved for exact magnetic fields that on an energy level which belong to a total Lebesgue measure subset of $(0, \infty)$ there is a periodic magnetic geodesic. This extends the earlier result where that was established for energy levels belonging to a total measure subset of $(0, d)$ with d is some small constant [6]. [3]

Recently two other approaches were introduced: in [11] it was proposed to use the geometric flows for pseudogradient deformations of closed curves for finding closed magnetic geodesics and a completely new approach to study of closed magnetic geodesics on surfaces was introduced in [19].

The approach used in the present article originates in the comments from [24, pp. 192–193]. For proof we use the approximation technique from [3] because the functional S is even not a C^1-functional. Although we discussed in [24, pp. 192–193] that such an approach can be used for establishing the existence of periodic magnetic geodesics on energy levels from a total measure subset we did not manage to do that here by such a mild technique which we use.

[2] Until recently it is the only dimension for which the existence of a periodic magnetic geodesic on every energy level in an exact magnetic field is established. In the same dimension there is an example of a non-exact magnetic field which does not possess periodic magnetic geodesics on certain (in the example just on one) energy levels [7].

[3] We have to mention the previous results by Polterovich, Kerman, and Macarini who established the existence of periodic magnetic geodesics for sequences of arbitrary small energy levels provided that the magnetic field is weakly exact or is given by a symplectic form (see references in [4].

4 Proof of Theorem 2

For brevity we denote

$$S(\gamma) = S_E(\gamma) = \int_\gamma \sqrt{E g_{ik}\dot{x}^i\dot{x}^k}\, dt.$$

Let us consider the family of functionals

$$S_{\varepsilon,\tau}(\gamma) = \int_\gamma \left(\varepsilon|\dot{x}|^2 + |\dot{x}|^{1+\tau} + A_i\dot{x}^i\right) dt, \qquad \varepsilon, \tau \geq 0,$$

where

$$\gamma \in H(S^1, M^n),$$

i.e. γ lies in the Hilbert manifold formed by H^1-maps

$$\gamma : [0,1] \to M^n,$$

such that $\gamma(0) = \gamma(1)$ [10].

For simplicity, we denote by $L(\gamma)$ the length of γ:

$$L(\gamma) = \int_\gamma |\dot{x}|\, dt.$$

In [3] it was proved that

(a) the functional $S_{\varepsilon,\tau}$ is a C^1-functional on $H(S^1, M^n)$ and it meets the Palais–Smale condition for $\varepsilon, \tau > 0$;

(b) the extremals of $S_{\varepsilon,\tau}$ meet the equation

$$\frac{\partial\gamma^i}{\partial t} + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = \frac{g^{ik}F_{kj}\dot{\gamma}^j}{2\varepsilon + (1+\tau)|\dot{\gamma}|^{\tau-1}}, \tag{8}$$

or shortly

$$\frac{D\dot{\gamma}^i}{\partial t} = \frac{g^{ik}F_{kj}\dot{\gamma}^j}{2\varepsilon + (1+\tau)|\dot{\gamma}|^{\tau-1}},$$

and the extremals are arc-length parameterized;

(c) the subset $M^n \subset H(S^1, M^n)$ formed by one-point curves is a manifold of local minima of the functional $S_{\varepsilon,\tau}$ for $0 \leq \tau < 1$;

(d) given a sequence $\gamma_{\varepsilon_n,\tau_n}$ of extremals of S_{ε_n,τ_n} such that their lengths are bounded by constants K_0 and K_1:

$$0 < K_0 \leq L(\gamma_{\varepsilon_n,\tau_n}) \leq K_1 < \infty,$$

and

$$\lim_{n\to\infty}(\varepsilon_n, \tau_n) = (\varepsilon_0, \tau_0),$$

there is a subsequence γ_k which converges in $H(S^1, M^n)$ to an extremal γ_∞ of the functional S_{ε_0,τ_0}. [4]

The Hölder inequality implies that

$$L^m(\gamma) \le \int_\gamma |\dot{x}|^m \, dt$$

for any real $m > 0$ and the equality is achieved only on arc-length parameterized curves: $|\dot{x}| = \text{const}$. This implies

$$S_{\varepsilon,\tau} \ge \varepsilon L^2 + L^{1+\tau} + \int A_i \dot{x}^i dt \ge S.$$

Let there is a contractible closed curve γ such that

$$S(\gamma) < 0.$$

Let $w \in H_k(M^n)$ be a cycle which is thrown out into $\{S < 0\}$ by the map $F : P \times [0,1] \to \Omega^+(M^n)$. By continuity this map also defines the throwings out of w into $\{S_{\varepsilon,\tau} < 0\}$ for sufficiently small ε and τ. Hence we have the critical level

$$c(w) > 0$$

of S and the corresponding critical levels $c_{\varepsilon,\tau}(w)$ of $S_{\varepsilon,\tau}$. The Hölder inequality implies

$$c_{\varepsilon,\tau}(w) \ge c_{\varepsilon',\tau'}(w) \quad \text{for } \varepsilon \ge \varepsilon',\ \tau \ge \tau'.$$

It is also clear from the continuity that

for any given $w \in H_*(M^n)$ and $\beta > 0$ there exist $\varepsilon(w,\beta)$ and $\tau(w,\beta)$ such that

$$c_{\varepsilon,\tau}(w) \le c(w) + \beta \quad \text{for } \varepsilon < \varepsilon(w,\beta), \tau < \tau(w,\beta).$$

Now let us take a smooth non-decreasing function $f : \mathbb{R} \to [0,1]$ such that

$$f(x) = 0 \ \ x \le \tfrac{c(w)}{20}, \quad f(x) = 1 \ \ x \ge \tfrac{c(w)}{10}, \quad 0 < f(x) < 1 \ \ \text{otherwise},$$

and instead of $S_{\varepsilon,\tau}$ we consider the C^1-functional

$$F_{w,\varepsilon,\tau}(\gamma) = f(S_{0,\tau}(\gamma)) \cdot S_{\varepsilon,\tau}(\gamma).$$

The following statements are clear

(I) for sufficiently small ε and τ the throwing out of the cycle w determines a critical level

$$\widetilde{c}_{\varepsilon,\tau}(w)$$

which meets the inequality

$$c_{\varepsilon,\tau}(w) \le \widetilde{c}_{\varepsilon,\tau}(w);$$

[4] In [3] this was formulated for $(\varepsilon_0, \tau_0) = (0,0)$ however the proof works in the general situation.

(II) for any $\beta > 0$ there exist $\varepsilon_1(w,\beta)$ and $\tau_1(w,\beta)$ such that

$$\widetilde{c}_{\varepsilon,\tau}(w) \leq c(w) + \beta \quad \text{for } \varepsilon < \varepsilon_1(w,\beta), \tau < \tau_1(w,\beta); \tag{9}$$

(III) the functional $F_{w,\varepsilon,\tau}$ meets the Palais–Smale condition in the closed domain $\{F_{w,\varepsilon,\tau} \geq \frac{c(w)}{10}\}$.

Let us fix some $\beta > 0$. The Morse theory [5] applied to the functionals $F_{w,\varepsilon,\tau}$ implies for

$$\varepsilon < \varepsilon_1(w,\beta), \quad \tau < \tau_1(w,\beta)$$

the existence of extremals $\gamma_{\varepsilon,\tau}$ with

$$F_{w,\varepsilon,\tau}(\gamma_{\varepsilon,\tau}) = \widetilde{c}_{\varepsilon,\tau},$$

$$0 \leq \frac{c(w)}{10} \leq L(\gamma_{\varepsilon,\tau}) \leq \sqrt{\frac{c(w)+\beta}{\varepsilon}}. \tag{10}$$

The latter inequalities follow from (9) and the definition of $F_{w,\varepsilon,\tau}$ (in fact, these functionals were defined exactly for achieving these inequalities). The curves $\gamma_{\varepsilon,\tau}$ are also extremals of $S_{\varepsilon,\tau}$ and hence by fixing $\varepsilon > 0$ we take a subsequence $\gamma_{\varepsilon,\tau_n}$ which converges as $\tau_n \to 0$ to a closed curve γ_ε which the extremal of $S_{\varepsilon,0}$ meeting the inequality (10). By (8), these curves γ_ε satisfy the equation

$$\frac{1}{|\dot\gamma_\varepsilon|}\frac{D\dot\gamma^i_\varepsilon}{\partial t} = \frac{g^{ik}F_{kj}\dot\gamma^j_\varepsilon}{2\varepsilon|\dot\gamma_\varepsilon|+1}. \tag{11}$$

However, since $\gamma_\varepsilon : [0,1] \to M^n$ is arc-length parameterized, we have

$$l(\varepsilon) := |\dot\gamma_\varepsilon| = L(\gamma_\varepsilon).$$

By (10), we have

$$l(\varepsilon) \leq \sqrt{\frac{c(w)+\beta}{\varepsilon}}.$$

and therefore

$$\varepsilon\, l(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Now at least one of two cases holds:

1) there is a sequence γ_{ε_n} with uniformly bounded lengths: $l(\varepsilon_n) \leq K = \text{const} < \infty$. In this case it contains a subsequence which converges to a periodic magnetic geodesic (on the given energy level E);

2) there is a sequence ε_n such that

$$\nu_n = \varepsilon_n l(\varepsilon_n), \quad \lim_{n\to\infty} l(\varepsilon_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \nu_n = 0.$$

In these case, by (11), $\gamma_n = \gamma_{\varepsilon_n}$ are periodic magnetic geodesics on the energy levels $E_n = E(1+2\nu_n)$, $\lim_{n\to\infty} S_E(\gamma_n) = c(w)$, and $\lim_{n\to\infty} \text{length}(\gamma_n) = \infty$.

This proves Theorem 2.

---

[5] Since the considered functionals are only C^1, in this case we have to consider pseudogradient flows instead of the gradient flows [18] (see also [20]).

References

[1] Arnold, V.I.: First steps in symplectic topology. Russian Math. Surveys **41**:6 (1986), 1–21.

[2] Bahri, A.: Critical points at infinity in some variational problems. Pitman Research Notes Math. **182**, Longman House, Harlow, 1989.

[3] Bahri, A., and Taimanov, I.A.: Periodic orbits in magnetic fields and Ricci curvature of Lagrangian systems. Trans. Amer. Math. Soc. **350** (1998), 2697–2717.

[4] Contreras, G., Macarini, L., and Paternain, G.: Periodic orbits for exact magnetic flows on surfaces. Int. Math. Res. Not. 2004, no. 8, 361–387.

[5] Contreras, G.: The Palais-Smale condition on contact type energy levels for convex Lagrangian systems. Calc. Var. Partial Differential Equations **27** (2006), 321–395.

[6] Frauenfelder, U., and Schlenk, F.: Hamiltonian dynamics on convex symplectic manifolds. Israel J. Math. **159** (2007), 1–56.

[7] Ginzburg, V.L.: On the existence and non-existence of closed trajectories for some Hamiltonian flows. Math. Z. **223** (1996), 397–409.

[8] Ginzburg V., and Gürel, B.: Periodic orbits of twisted geodesic flows and the Weinstein-Moser theorem. Comment. Math. Helv. **l84** (2009), 865–907.

[9] Grinevich, P.G., and Novikov, S.P.: Nonselfintersecting magnetic orbits on the plane. Proof of the overthrowing of cycles principle. Topics in topology and mathematical physics, 59–82, Amer. Math. Soc. Transl. Ser. 2, 170, Amer. Math. Soc., Providence, RI, 1995.

[10] Klingenberg, W.: Lectures on closed geodesics. Grundlehren der Mathematischen Wissenschaften, **230**, Springer-Verlag, Berlin–New York, 1978.

[11] Koh, D.: On the evolution equation for magnetic geodesics. Calc. Var. Partial Differential Equations **36** (2009), 453–480.

[12] Kozlov, V.V.: Calculus of variations in the large and classical mechanics. Russian Math. Surveys **40**:2 (1985), 37–71.

[13] Morse, M.: The calculus of variations in the large. AMS, Providence, RI, 1934.

[14] Novikov, S.P.: Multivalued functions and functionals. An analogue of the Morse theory. Soviet Math. Dokl. **24** (1981), 222–226.

[15] Novikov, S.P.: The Hamiltonian formalism and a multivalued analogue of Morse theory. Russian Math. Surveys **37**:5 (1982), 1–56.

[16] Novikov, S.P., and Taimanov, I.A.: Periodic extremals of multivalued or not everywhere positive functionals. Soviet Math. Dokl. **29** (1984), 18–20.

[17] Palais, R. S., and Smale, S.: A generalized Morse theory. Bull. Amer. Math. Soc. **70** (1964), 165–172.

[18] Rabinowitz, P.H.: Variational methods for nonlinear eigenvalue problems. In: Eigenvalues of nonlinear problems (ed. Prodi, G.), C.I.M.E., Edizioni Cremonese, Roma (1975), 141-?195.

[19] Schneider, M.: Closed magnetic geodesics on S^2. arXiv:0808.4038, 2008.

[20] Struwe, M.: Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Fourth edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **34**. Springer-Verlag, Berlin, 2008.

[21] Taimanov, I.A.: The principle of throwing out cycles in Morse–Novikov theory. Soviet Math. Dokl. **27** (1983), 43–46.

[22] Taimanov, I.A.: Non-self-intersecting closed extremals of multivalued or not-everywhere-positive functionals. Math. USSR-Izv. **38** (1992), 359–374.

[23] Taimanov, I.A.: Closed non-self-intersecting extremals of multivalued functionals. Siberian Math. J. **33** (1992), 686–692.

[24] Taimanov, I.A.: Closed extremals on two-dimensional manifolds. Russian Math. Surveys **47**:2 (1992), 163–211.

[25] Taimanov, I.A.: The type numbers of closed geodesics. Regular and Chaotic Dynamics (to appear), also: arXiv:0912.5226.