

Weights and t -structures: in general triangulated categories, for 1-motives, mixed motives, and for mixed Hodge complexes and modules

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Abstract

We define and study *transversal* weight and t -structures (for triangulated categories). Our results axiomatize and describe in detail the relations between the *Chow weight structure* for Voevodsky's motives w_{Chow} (introduced in a preceding paper), the (conjectural) motivic t -structure, and the conjectural weight filtration for them. This picture becomes non-conjectural when restricted to the derived categories of Deligne's 1-motives (over a smooth base) and of Artin-Tate motives over number fields; a weight structure transversal to the canonical t -structure also exist for the Beilinson's $D_{\tilde{H}p}^b$ (the derived category of graded polarizable mixed Hodge complexes) and for the derived category of (Saito's) mixed Hodge modules.

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Introduction

It is widely believed that the triangulated category DM_{gm} of (geometric) Voevodsky's motives (over a perfect field k ; all the motives that we will consider in this paper will have rational coefficients) possesses a certain motivic t -structure t_{MM} . Its heart should be the category MM of mixed motives, that should have a *weight filtration* whose factors yield certain semi-simple abelian subcategories $MM_i \subset MM$ of pure motives of weight i ; the objects of MM_i should be shifts of certain Chow motives (note that in [Voe00] an embedding $Chow \rightarrow DM_{gm}$ was constructed) by $[-i]$. Since the existence of t_{MM} is very far from being known, people tried to find a candidate for the weight filtration for DM_{gm} ; this was to be a filtration by triangulated subcategories that would restrict to the weight filtration for MM . This activity was not really successful (in the general case); this is no surprise since (for example) the weight filtration for the motif of a smooth projective variety should correspond to its *Chow-Kunneth decomposition*.

An alternative method for defining (certain) weights in DM_{gm} was proposed and successfully implemented in [Bon10a]. To this end *weight structures* were defined. This notion is a natural important counterpart of t -structures; somewhat similarly to t -structures, weight structures for a triangulated $\underline{\mathcal{C}}$ are defined in terms of $\underline{\mathcal{C}}^{w \leq 0}, \underline{\mathcal{C}}^{w \geq 0} \subset Obj \underline{\mathcal{C}}$. The Chow weight structure w_{Chow} (defined in §6 of *ibid.*) certainly does not yield a weight filtration for DM_{gm} in the sense described above (since $DM_{gm}^{w_{Chow} \leq 0}$ and $DM_{gm}^{w_{Chow} \geq 0}$ are not stable with respect to shifts). Yet it allows us to define certain (Chow)-weight filtrations and (Chow)-weight spectral sequences for any cohomology of motives; for singular and étale cohomology those are isomorphic to the 'classical' ones (that should also have an expression in terms of weight filtration), and this should also be true for the 'mixed motivic' cohomology given by t_{MM} (see Remark 2.4.3 of *ibid.* and §2.2 below). Also note here: the Chow weight structure for Voevodsky's motives over a base scheme S (introduced in [Heb10] and [Bon10b]), is closely related with the weights for mixed complexes of sheaves introduced in §5.1.8 of [BBD82] (see §§3.4–3.6 of [Bon10b] for more detail), and with weights of mixed Hodge

complexes (see §2.3 below; we prove a very precise statement of this sort in the case when S is the spectrum of a field $k \subset \mathbb{C}$).

The goal of the current paper is to axiomatize and describe in detail the (conjectural) relations between w_{Chow} , t_{MM} , and the weight filtration for DM_{gm} . To this end we introduce the notion of *transversal weight* and *t-structures*. This is no surprise that this notion has several non-conjectural (and interesting) examples; this includes the derived categories of Deligne’s 1-motives (over a smooth base) and of Artin-Tate motives over number fields; the derived category of (Saito’s) mixed Hodge modules, and the Beilinson’s derived category of graded polarizable mixed Hodge complexes. Certain results of [B-VK10] were very useful for studying these examples.

We prove several equivalent conditions for existence of transversal weight and *t-structures* for a triangulated \underline{C} . One of them is the existence of a *strongly semi-orthogonal generating* system of semi-simple abelian subcategories $\underline{A}_i \subset \underline{C}$ (\underline{A}_i are the factors of the ‘weight filtration’ of the heart of t). We prove that any object of \underline{Hw} (the heart of w) splits into a sum of objects of $\underline{A}_i[i]$ (this should be a generalization of the Chow-Kunneth decomposition of motives of smooth projective varieties). This is a strong restriction on w ; it demonstrates that the notion of transversal structures is quite distinct from the notion of *adjacent weight* and *t-structures* (introduced in §4.4 of [Bon10a]).

In a subsequent paper the notion of transversal structures will allow us to apply a certain ‘gluing’ argument, that reduces the existence of a ‘nice’ motivic *t-structure* for Voevodsky’s motives over a base scheme S (as considered in [CiD09]) to the case of motives over algebraically closed fields.

Now we list the contents of the paper.

In the first section we prove the equivalence of six definitions of transversal weight and *t-structures*. This yields several relations between *t-structures*, weight structures, weight filtrations, and *semi-orthogonal* generators \underline{A}_i for this situation. We don’t recall the (general) theory of weight structures in this paper; so an interested reader should consult [Bon10a] for it (note that at the end of Notation of this paper there is a certain index of definitions); see also [Bon09s].

We start the second section by noting that the results of [B-VK10] yield a general statement for the existence of a weight structure that is transversal to the canonical *t-structure* for $D^b(\underline{A})$ if \underline{A} admits a certain ‘weight filtration’ (with semi-simple ‘factors’). We use this result in the construction of all our main examples (of transversal weight and *t-structures*; yet cf. Remark 2.2(1)). Applying even more results of [B-VK10], we deduce the existence of a weight structure that is transversal to the canonical (i.e. ‘motivic’) *t-structure* on the derived category $D^b(\mathcal{M}_1)$ of Deligne’s 1-motives. Then we

prove that all the functors relating $D^b(\mathcal{M}_1)$ with DM_{gm}^{eff} respect the weight structure constructed (this should also be true for the corresponding motivic t -structures if one can define t_{MM} for DM_{gm}^{eff}). Next we describe the conjectural relations between various 'structures' for $DM_{gm}^{eff} \subset DM_{gm}$. Lastly, we verify that on the derived category of mixed Hodge modules over a complex variety X , and on the Beilinson's derived category $D_{\mathbb{H}_p}^b$ of graded polarizable mixed Hodge complexes (over a base field $k \subset \mathbb{C}$) there exist weight structures transversal to the canonical t -structures; the singular realization functor $DM_{gm}(k) \rightarrow D_{\mathbb{H}_p}^b(k)$ respects the corresponding weight structures.

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Notation. \underline{C} below will always denote some triangulated category. t will denote a bounded t -structure for \underline{C} , and w will be a bounded weight structure for it (the theory of weight structures was studied in detail in [Bon10a]; see also [Bon09s]).

For $X \in \text{Obj}\underline{C}$, $i \in \mathbb{Z}$, we will consider the following distinguished triangles:

$$\tau_{\leq i}X \rightarrow X \rightarrow \tau_{\geq i+1}X \quad (1)$$

and

$$w_{\geq i+1}X \rightarrow X \rightarrow w_{\leq i}X \quad (2)$$

that come from t - and weight decompositions of $X[i]$ shifted by $[-i]$, respectively (i.e. $\tau_{\leq i}X \in \underline{C}^{t \leq i}$, $\tau_{\geq i+1}X \in \underline{C}^{t \geq i+1}$, $w_{\geq i+1}X \in \underline{C}^{w \geq i+1}$, $w_{\leq i}X \in \underline{C}^{w \leq i}$; see Remark 1.2.2 of [Bon10a]).

$X^{\tau=i} \in \underline{C}^{t=0}$ will denote the i -th cohomology of X with respect to t i.e. the cone of the corresponding morphism $\tau_{\leq -1}(X[i]) \rightarrow \tau_{\leq 0}(X[i])$; $\tau_{=i}X = X^{\tau=i}[-i]$; \underline{Ht} will denote the heart of t .

$D \subset \text{Obj}\underline{C}$ will be called extension-stable if for any distinguished triangle $A \rightarrow B \rightarrow C$ in \underline{C} we have: $A, C \in D \implies B \in D$. Note that $\underline{C}^{t \leq i}$, $\underline{C}^{t \geq i}$, $\underline{C}^{t=i} = \underline{C}^{t \geq i} \cap \underline{C}^{t \leq i}$, $\underline{C}^{w \geq i}$, $\underline{C}^{w \leq i}$, $\underline{C}^{[i,j]} = \underline{C}^{w \geq i} \cap \underline{C}^{w \leq j}$, and $\underline{C}^{w=i} = \underline{C}^{[i,i]}$ are extension-stable for any t, w and any $i \leq j \in \mathbb{Z}$.

For a subcategory $H \subset \underline{C}$ we will call the smallest extension-stable subcategory of \underline{C} containing H the *envelope* of H (in \underline{C}).

For $D, E \subset \text{Obj}\underline{C}$ we will write $D \perp E$ if $\underline{C}(X, Y) = \{0\}$ for all $X \in D$, $Y \in E$.

For $B \subset \underline{C}$ we will call the subcategory of \underline{C} whose objects are all retracts of objects of B (in \underline{C}) the *Karoubi-closure* of B in \underline{C} .

For a class of objects $C_i \in \text{Obj}\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i ; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle C : C \in \text{Obj}D \rangle$.

\underline{A} will always be an abelian category; \underline{A}_i (for i running through all integral numbers) will always be additive, and will often be abelian semi-simple.

k will be our perfect base field (sometimes it will be contained in or equal to the field of complex numbers).

1 Transversal weight and t -structures: the general case

1.1 Auxiliary statements; weight filtrations for triangulated categories

We will need the following easy homological algebra statements.

Lemma 1.1. *1. Let $T : X \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} X[1]$ and $T' : X' \xrightarrow{a'} A' \xrightarrow{f'} B' \xrightarrow{b'} X'[1]$ be distinguished triangles.*

Let $B \perp A'[1]$. Then for any morphism $g : X \rightarrow X'$ there exist $h : A \rightarrow A'$ and $i : B \rightarrow B'$ completing g to a morphism of triangles $T \rightarrow T'$.

Moreover, if $B \perp A'$, then g and h are unique.

2. In particular, for any $i \in \mathbb{Z}$, $X, Y \in \text{Obj} \underline{\mathcal{C}}$, any $g \in \underline{\mathcal{C}}(X, X')$ could be completed to a morphism of distinguished triangles

$$\begin{array}{ccccc} w_{\geq i+1}X & \longrightarrow & X & \longrightarrow & w_{\leq i}X \\ \downarrow & & \downarrow g & & \downarrow \\ w_{\geq i+1}Y & \longrightarrow & Y & \longrightarrow & w_{\leq i}Y \end{array} \quad (3)$$

3. If $D \perp E$ ($D, E \subset \text{Obj} \underline{\mathcal{C}}$), then the same is true for their envelopes.

4. Let $D, E \subset \text{Obj} \underline{\mathcal{C}}$ be extension-stable, $D \perp (E \cup E[1])$. For some $F \subset \text{Obj} \underline{\mathcal{C}}$ suppose that for any $X \in F$ there exists a distinguished triangle $Y \rightarrow X \rightarrow Z$ with $Y \in D$, $Z \in E$. Then such a distinguished triangle also exists for any X belonging to the envelope of F (in $\underline{\mathcal{C}}$).

5. For any $i \leq j \in \mathbb{Z}$ we have: $\underline{\mathcal{C}}^{[i,j]}$ is the envelope of $\cup_{i \leq l \leq j} \underline{\mathcal{C}}^{w=l}$ in $\underline{\mathcal{C}}$.

Proof. 1. This is Lemma 1.4.1 of [Bon10a]; it follows immediately from Proposition 1.1.9 of [BBD82].

2. Follows immediately from assertion 1; cf. Lemma 1.5.1 of [Bon10a].

3. Very easy; note that for any $X \in \text{Obj} \underline{\mathcal{C}}$ the (corepresentable) functor $\underline{\mathcal{C}}(X, -)$ is homological, whereas $\underline{\mathcal{C}}(-, X)$ is cohomological.

4. See Remark 1.5.5 of [Bon10a] or Proposition 1.8 of [Heb10].

5. Easy from Proposition 1.5.6(2) of [Bon10a].

□

Below we will need a certain class of 'nice' weight decompositions.

Definition 1.2. For some \underline{C}, t, w we will say that a distinguished triangle (2) (for some i, X) is *nice* if $w_{\geq i+1}X, X, w_{\leq i}X \in \underline{C}^{t=0}$.

We will also say that this distinguished triangle is a *nice decomposition* of X (for the corresponding i), and that the morphism $w_{\geq i+1}X \rightarrow X$ extends to a nice decomposition.

Now we formulate a simple implication of Lemma 1.1 (we will use a very easy case of it below, and a somewhat more complicated one in a subsequent paper).

Lemma 1.3. *We fix some \underline{C}, w, t, i ; suppose that for a certain $N \subset \underline{C}^{t=0}$ a nice decomposition exists for any $X \in N$. Consider $N' \subset \underline{C}^{t=0}$ being the smallest subclass containing N that satisfies the following condition: if $A, C \in N'$,*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a complex (i.e. $g \circ f = 0$), f is monomorphic, g is epimorphic, $\text{Ker } g / \text{Im } f \in N'$, then $B \in N'$. Then a nice decomposition exists for any $X \in N'$ (and the same i).

Proof. It suffices to note that N' is exactly the smallest extension-stable subcategory of \underline{C} containing N , and apply Lemma 1.1(4). \square

Next we study certain ('weight') filtrations of triangulated categories.

Definition 1.4. 1. We will say that a family $\{\underline{A}_i\}$, $\underline{A}_i \subset \underline{C}$, $i \in \mathbb{Z}$ is *semi-orthogonal* if $\underline{A}_i \perp \underline{A}_j[s]$ for any $i, j, s \in \mathbb{Z}$ such that $s < 0$, or $s > j - i$.

We will say that $\{\underline{A}_i\}$ are *strongly semi-orthogonal* if we also have $\underline{A}_i \perp \underline{A}_j$ for any $i < j$ (and so, for any $i \neq j$).

2. We will say that $\{\underline{A}_i\}$ is *generating* (in \underline{C}) if $\langle \cup_i \text{Obj } \underline{A}_i \rangle = \underline{C}$.

Now we prove that a semi-orthogonal generating family yields a certain ('weight') filtration for \underline{C} (cf. Definition E17.1 of [B-VK10]), and study its properties. Note that our 'weights' differ from those in loc.cit. by sign. This is necessary to make the notation below compatible with that for weight structures.

Lemma 1.5. *Let $i < j \leq n \in \mathbb{Z}$.*

I Suppose that $\{\underline{A}_s \subset \underline{C}\}$ is a semi-orthogonal family; denote $\langle \underline{A}_s \rangle$ by \underline{C}_s (for any $s \in \mathbb{Z}$).

Then $\underline{C}_j \perp \underline{C}_i$.

II Let $\underline{C}_l \subset \underline{C}$ for $l \in \mathbb{Z}$ be triangulated; suppose that $\underline{C}_l \perp \underline{C}_m$ if $l > m$.

For any $r \leq q \in \mathbb{Z}$ denote $\langle \cup_{r \leq s \leq q} \text{Obj} \underline{\mathcal{C}}_s \rangle$ by $\underline{\mathcal{C}}_{[r,q]}$, and denote $\langle \cup_{s \geq r} \text{Obj} \underline{\mathcal{C}}_s \rangle$ by $\underline{\mathcal{C}}_{\geq r}$.

Then the following statements are fulfilled.

1. For any $X \in \text{Obj} \underline{\mathcal{C}}_{[i,n]}$ there exists a distinguished triangle

$$X_1 \rightarrow X \rightarrow X_2 \quad (4)$$

such that $X_1 \in \text{Obj} \underline{\mathcal{C}}_{[j,n]}$, $X_2 \in \text{Obj} \underline{\mathcal{C}}_{[i,j-1]}$. More generally, for $X \in \underline{\mathcal{C}}_{\geq i}$ one can find (4) with $X_1 \in \text{Obj} \underline{\mathcal{C}}_{\geq j}$

Besides, this triangle is canonical and functorial in X (in both cases).

2. The embedding $\underline{\mathcal{C}}_{[i,j-1]} \rightarrow \underline{\mathcal{C}}_{\geq i}$ possesses an exact left adjoint $a_{i,j}$; the 'kernel' of $a_{i,j}$ is exactly $\underline{\mathcal{C}}_{\geq j}$.

3. Suppose that $\{\underline{\mathcal{C}}_i\}$ are generating. Then the embedding $\underline{\mathcal{C}}_{\geq i} \rightarrow \underline{\mathcal{C}}$ possesses an exact right adjoint b_j .

Proof. I If $i > l$, then $\underline{A}_i[r] \perp \underline{A}_l$ for any $r \in \mathbb{Z}$ (by Definition 1.4). Now the result is immediate from Lemma 1.1(3).

II 1. Since w is bounded, we have $\underline{\mathcal{C}}_{\geq i} = \cup_{m \geq i} \underline{\mathcal{C}}_{[i,m]}$; hence it suffices to verify the existence of (4) for $X \in \underline{\mathcal{C}}_{[j,n]}$.

We have a 'trivial' example of (4) if $X \in \underline{\mathcal{C}}^l$ for any $i \leq l \leq n$. Hence the existence of (4) in general is immediate from Lemma 1.1(4). Now, any morphism $X \rightarrow X'$ could be uniquely extended to a morphism of the corresponding triangles by Lemma 1.1(1). Hence we obtain the functoriality of (4).

2,3: Immediate from assertion II1; cf. Proposition E.15.1 of [B-VK10]. \square

1.2 Transversal weight and t -structures: equivalent definitions and their consequences

Theorem 1.6. *The following conditions are equivalent.*

(i) *There exists a strongly semi-orthogonal generating family $\{\underline{A}_i\}$ in $\underline{\mathcal{C}}$ such that all of \underline{A}_i are abelian semi-simple.*

(ii) *There exists a semi-orthogonal family $\{\underline{A}_i\}$ in $\underline{\mathcal{C}}$ such that for $\underline{\mathcal{C}}^{t \leq 0}$ (resp. $\underline{\mathcal{C}}^{t \geq 0}$) being the envelope of $\cup_{i \in \mathbb{Z}, j \geq 0} \underline{A}_i[j]$ (resp. of $\cup_{i \in \mathbb{Z}, j \leq 0} \underline{A}_i[j]$) we have: $(\underline{\mathcal{C}}^{t \leq 0}, \underline{\mathcal{C}}^{t \geq 0})$ yield a t -structure for $\underline{\mathcal{C}}$.*

(ii') *There exists a semi-orthogonal family $\{\underline{A}_i\}$ in $\underline{\mathcal{C}}$ such that the envelope of $\cup_{i \in \mathbb{Z}} \underline{A}_i$ yields the heart of a certain t .*

(iii) *There exist a t and a semi-orthogonal family $\{\underline{A}_i \subset \underline{H}t\}$ that satisfy the following condition: for each $X \in \underline{\mathcal{C}}^{t=0}$ there exists an exhaustive separated decreasing filtration by subobjects $W_{\geq i}X$, $i \in \mathbb{Z}$, such that $W_{\geq i}X/W_{\geq i+1}X$ belongs to $\text{Obj} \underline{A}_i$ for all $i \in \mathbb{Z}$.*

(iii') The filtration (of any $X \in \underline{C}^{t=0}$) described above exists and is Ht-functorially determined by X .

(iv) There are t, w for \underline{C} such that for any $X \in \underline{C}^{t=0}$, $i \in \mathbb{Z}$, there exists a nice decomposition (see Definition 1.2).

(iv') Nice decompositions exist, and they are also Ht-functorial in X (if we fix i); the corresponding functors $X \mapsto w_{\geq i+1}X$ and $X \mapsto w_{\leq i}X$ are exact (on Ht).

(v) There are t, w for \underline{C} such that for any $X \in \underline{C}^{t=0}$, $i \in \mathbb{Z}$, there exists a choice of $w_{\geq i+1}X$ such that the morphism $\text{Im}((w_{\geq i+1}X)^{\tau=0} \rightarrow X) \rightarrow X$ extends to a nice decomposition of X .

(v') For t, w and any $X, i, w_{\geq i+1}X$ (as above) the morphism $\text{Im}(w_{\geq i+1}X)^{\tau=0} \rightarrow X) \rightarrow X$ extends to a nice decomposition of X .

Proof. Certainly, (ii') implies (ii) (since t is bounded), (iii') implies (iii), (iv') implies (iv), and (v') implies (v).

(i) \implies (ii).

Semi-orthogonality yields that the 'generators' of $\underline{C}^{t \leq 0}[1]$ are orthogonal to those of $\underline{C}^{t \geq 0}$; hence Lemma 1.1(3) yields: $\underline{C}^{t \leq 0}[1] \perp \underline{C}^{t \geq 0}$.

It remains to verify the existence of t -decompositions.

For any $i \in \mathbb{Z}$, since \underline{A}_i is semi-simple, we have $\underline{C}_i \cong \bigoplus_{j \in \mathbb{Z}} \underline{A}_i[j]$. Hence any object of \underline{C}_i admits a t -decomposition $X \cong \tau_{\leq 0}X \bigoplus \tau_{\geq 1}X$ whose components also belong to \underline{A}_i .

Now, it suffices to verify: if for some $j > i$ any object of $\underline{C}_{[i+1, j]}$ admits a t -decomposition inside $\underline{C}_{[i+1, j]}$, then a similar statement holds for any $X \in \text{Obj} \underline{C}_{[i, j]}$.

Lemma 1.5(III1) yields the existence of a distinguished triangle $X_1 \rightarrow X \rightarrow X_2 \xrightarrow{g} X_1[1]$ such that $X_1 \in \text{Obj} \underline{C}_{[i+1, j]}$, $X_2 \in \text{Obj} \underline{C}_i$.

Now we argue as in the proof of Lemma 1.5.4 of [Bon10a]. We can complete g to a morphism of distinguished triangles

$$\begin{array}{ccccc} \tau_{\leq 0}X_2 & \longrightarrow & X_2 & \longrightarrow & \tau_{\geq 1}X_2 \\ \downarrow & & \downarrow g & & \downarrow \\ (\tau_{\leq 0}X_1)[1] & \longrightarrow & X_1[1] & \longrightarrow & (\tau_{\geq 1}X_1)[1] \end{array} \quad (5)$$

Indeed, by Lemma 1.1(1) it suffices to verify that $\tau_{\leq 0}X_2 \perp (\tau_{\geq 1}X_1)[1]$; the latter easily follows from the strong semi-orthogonality of $\{\underline{A}_s\}$ (see Lemma 1.1(3)).

Moreover, we can complete g to the following diagram (starting from the

left hand side square of (5), and using Proposition 1.1.11 of [BBD82]):

$$\begin{array}{ccccc}
\tau_{\leq 0}X_2 & \longrightarrow & X_2 & \longrightarrow & \tau_{\geq 1}X_2 \\
\downarrow & & \downarrow^g & & \downarrow \\
(\tau_{\leq 0}X_1)[1] & \longrightarrow & X_1[1] & \longrightarrow & (\tau_{\geq 1}X_1)[1] \\
\downarrow & & \downarrow & & \downarrow \\
Y[1] & \longrightarrow & X[1] & \longrightarrow & Z[1] \\
\downarrow & & \downarrow & & \downarrow \\
(\tau_{\leq 0}X_2)[1] & \longrightarrow & X_2[1] & \longrightarrow & (\tau_{\geq 1}X_2)[1]
\end{array} \tag{6}$$

such that all rows and columns are distinguished triangles, and all squares are commutative. Therefore the extension-stability of $\underline{C}^{t \leq 0}$ yields that it contains Y ; the extension-stability of $\underline{C}^{t \geq 1}$ yields that it contains Z ; hence $Y \rightarrow X \rightarrow Z$ is the t -decomposition desired.

(ii) \implies (iv). We take C_1 being the envelope of $\{\underline{A}_i[j], j \geq i, i \in \mathbb{Z}\}$ in \underline{C} , C_2 being the envelope of $\{\underline{A}_i[j], j \leq i\}$. Note that C_1 is the envelope of $\{H[j], j \geq 0\}$, C_2 is the envelope of $\{H[j], j \leq 0\}$, where $\text{Obj}H = \bigoplus_{i \in \mathbb{Z}} \text{Obj}\underline{A}_i[i]$. Besides, H is negative i.e. $H \perp H[j]$ for all $j > 0$.

Hence, as shown (in the proof of) Theorem 4.3.2(II) of [Bon10a], the Karoubi-closures of C_1, C_2 (in \underline{C}) yield a bounded weight structure w for \underline{C} (actually, this is a simple consequence of Lemma 1.1(3,4)). Moreover, the heart of this weight structure is the idempotent completion of H . Now, H is idempotent complete itself, since all \underline{A}_i are (note that $\underline{A}_i[i] \perp \underline{A}_j[j]$ for $j < i$, hence for $A = \bigoplus A_i[i]$, $A_i \in \underline{A}_i$, $s \in \underline{C}(A, A)$, s is nilpotent if all of its 'diagonal components' $s_{ii} \in \underline{C}(A_i, A_i)$ are 0). Therefore $H = \underline{H}w$. Then Lemma 1.1(5) implies that $\underline{C}^{w \leq 0} = C_1$ and $\underline{C}^{w \geq 0} = C_2$ (i.e. we don't need a Karoubi-closure here; here we use the fact that $\underline{C}^{w \leq 0} = \bigcup_{i \leq 0} \underline{C}^{[i, 0]}$ and $\underline{C}^{w \geq 0} = \bigcup_{i \geq 0} \underline{C}^{[0, i]}$ for a bounded w).

Now, by Lemma 1.3 it suffices to verify the existence of nice decompositions for those objects of $\underline{H}t$ that belong to one of \underline{A}_i ; this is obvious.

Obviously, (v) implies (iv). Now we verify that (iv) implies (v'). To this end it suffices to note: by Proposition 2.1.2(1) of [Bon10a], $\text{Im}((w_{\geq i+1}X)^{\tau=0} \rightarrow X)$ does not depend on the choice of $w_{\geq i+1}X$. Hence it suffices to consider the case when $w_{\geq i+1}X$ comes from a nice decomposition of X , and then the statement is obvious.

Now we verify that (iv) implies (iv') and (iii).

We set $\underline{A}_i = \underline{C}^{t=0} \cap \underline{C}^{w=i}$. The orthogonality properties of weight and t -structures immediately yield that \underline{A}_i are semi-orthogonal.

Now we prove (iii). Since all terms of nice decompositions belong to $\underline{C}^{t=0}$, it yields a short exact sequence in \underline{Ht} . In particular, the morphism $w_{i+1}X \rightarrow X$ is monomorphic in \underline{Ht} .

Now suppose that $X \in \underline{C}^{w \geq i}$. Then we have $w_{\leq i}X \in \underline{C}^{w=i}$; see Proposition 1.3.3(6) of [Bon10a]. Hence, $w_{\leq i}X$ belongs to \underline{A}_i for any nice (2).

Loc.cit. also yields: if $X \in \underline{C}^{w \leq j}$, $j > i$, then any choice of $w_{\geq i+1}X$ belongs to $\underline{C}^{w \leq j}$. Hence for $X \in \underline{C}^{[r,s]} \cap \underline{C}^{t=0}$, $r \leq s \in \mathbb{Z}$, one can take $W_{\geq l}X = X$ for $l \leq r$, then by induction starting from $i = r$ up to $i = s - 1$ take a choice of $W_{\geq i+1}X$ coming from a nice decomposition of $W_{\geq i}X$, and set $W_{\geq l} = 0$ for $l > s$; this filtration would satisfy the conditions of (iii).

Now we verify (iv'). Any morphism in \underline{C} could be extended to a morphism of (any choices of) weight decompositions by Lemma 1.1(2). Moreover, this extension is unique in our case by parts 1 and 3 of loc.cit. (here we apply the orthogonality statement proved above). Hence we obtain that nice choices of (3) (for a fixed i) yield a functor (here we take $X \in \underline{C}^{t=0}$).

Now, Lemma 1.5.4 of [Bon10a] yields that for any distinguished triangle $A \rightarrow B \rightarrow C$ in \underline{C} , any triangles (2) for A, C could be completed to a diagram

$$\begin{array}{ccccc}
w_{\geq i+1}A & \longrightarrow & A & \longrightarrow & w_{\leq i}A \\
\downarrow & & \downarrow g & & \downarrow \\
w_{\geq i+1}B & \longrightarrow & B & \longrightarrow & w_{\leq i}B \\
\downarrow & & \downarrow & & \downarrow \\
w_{\geq i+1}C & \longrightarrow & C & \longrightarrow & w_{\leq i}C
\end{array} \tag{7}$$

all of whose rows and columns are distinguished triangles (and the middle row is given by some choice of (2) for B). Applying this fact for $A, B, C \in \underline{C}^{t=0}$ and nice decompositions of A, C , we obtain that the middle row is a nice decomposition of B (since \underline{Ht} is extension-stable in \underline{C}). Then the exactness of columns (in \underline{Ht}) concludes the proof of (iv').

Next we note that (ii) along with (iii) implies (ii'). Indeed, the envelope of \underline{A}_i obviously lies in \underline{Ht} , whereas (iii) yields that this inclusion is an equality.

It remains to verify that (iii) implies (iii') and (i).

To this end first we verify that (iii) implies (iv). Obviously, the family $\{\underline{A}_i\}$ is generating (since t is bounded). We consider $C_1, C_2 \subset \underline{C}$ introduced in the proof (ii) \implies (iv). As we have already noted above, the Karoubi-closures of C_1 and C_2 in \underline{C} yield a weight structure for \underline{C} . Hence the distinguished triangles coming from the short exact sequences $0 \rightarrow W_{\geq i+1}X \rightarrow X \rightarrow X/W_{\geq i+1}X \rightarrow 0$ yield (2). We obtain that (iii) implies (iv); hence (iii) also yields (iv').

Obviously, (iv') implies (iii'). Also, (iv') yields that $\underline{A}_i \perp \underline{A}_j$ for $j > i$ by the (duals of) Remark E7.8 and Proposition E7.4(4) of [B-VK10] (cf. Proposition 2.1 below).

So, it remains to prove that \underline{A}_i are abelian semi-simple. We verify that (for a fixed $i \in \mathbb{Z}$) the classes $Obj \underline{C}_i \cap \underline{C}^{t \leq 0}$ and $Obj \underline{C}_i \cap \underline{C}^{t \geq 0}$ yield a t -structure for \underline{C}_i (i.e. that t could be restricted to \underline{C}_i).

Now we note that for any $i \leq j \in \mathbb{Z}$, $X \in \underline{C}_{>i}$ (see Lemma 1.5(II)) the distinguished triangle $W_{\geq j} X \rightarrow X \rightarrow X/W_{\geq j} X$ (as considered above) is simultaneously a choice of (4). It easily follows that all b_j and $a_{i,j}$ respect \underline{Ht} . Hence they also respect t -decompositions; see Lemma E19.1 of [B-VK10]. Hence applying $a_{[i+1,i]} \circ b_i$ to the t -decomposition of $X \in \underline{C}_i$ (see (1)) we obtain that its components belong to \underline{C}_i . We also obtain that the heart of this t -structure is $a_{[i+1,i]} \circ b_i(\underline{Ht}) = \underline{A}_i$. Hence \underline{A}_i is an abelian category, and short exact sequences in it yield distinguished triangles in \underline{C} . Then \underline{A}_i is abelian semi-simple, since $\underline{A}_i \perp \underline{A}_i[1]$ by semi-orthogonality. \square

Definition 1.7. If w, t satisfy the (equivalent) conditions of the theorem, we will say that t is transversal to w .

Remark 1.8. 1. One could (try to) modify the conditions of the Theorem so that they would be related also in the case when w and t are not necessarily bounded. Yet to this end one would require some technical restrictions on \underline{C} (cf. Theorems 4.3.2 and 4.5.3 of [Bon10a]).

2. More details on the relations between w, t , and $\{\underline{A}_i\}$ are contained in the proof of the Theorem. In particular, note that $\underline{C}^{w=0} = \bigoplus Obj \underline{A}_i[i]$ (though we don't have an isomorphism of the corresponding categories, since there could be non-zero morphisms from $\underline{A}_i[i]$ into $\underline{A}_j[j]$ for $j > i$). Besides, \underline{Ht} possesses a separated exhaustive filtration with semi-simple factors $\underline{A}_i = \underline{Ht} \cap \underline{C}^{w=i}$.

3. Condition (i) of the Theorem is self-dual. It follows: if w, t are transversal for \underline{C} , then the structures w^{op}, t^{op} for the opposite category \underline{C}^{op} are transversal also. The latter structures are defined as follows: $\underline{C}^{op, w^{op} \leq 0} = \underline{C}^{w \geq 0}$ and $\underline{C}^{op, w^{op} \geq 0} = \underline{C}^{w \leq 0}$; $\underline{C}^{op, t^{op} \leq 0} = \underline{C}^{t \geq 0}$ and $\underline{C}^{op, t^{op} \geq 0} = \underline{C}^{t \leq 0}$ (cf. Remark 1.1.2(1) of [Bon10a]).

Besides, for any $i, j \in \mathbb{Z}$ the structures $w[i], t[j]$ are also transversal; here $\underline{C}^{w[i] \leq 0} = \underline{C}^{w \leq i}$ and $\underline{C}^{w[i] \geq 0} = \underline{C}^{w \geq i}$; $\underline{C}^{t[j] \leq 0} = \underline{C}^{t \leq j}$ and $\underline{C}^{t[j] \geq 0} = \underline{C}^{t \geq j}$.

4. Proposition 2.1.2(1) of [Bon10a] also yields (cf. the proof of the Theorem) that for **any** t, w the correspondence $X \mapsto \text{Im}((w_{\geq i+1} X)^{\tau=0} \rightarrow X)$ yields a functor in $\underline{Ht} \rightarrow \underline{Ht}$; this is also true for $X \mapsto \text{Coker}((w_{\geq i+1} X)^{\tau=0} \rightarrow X)$. So, in order to verify that t, w are transversal it suffices to verify that

these functors take their values in $\underline{C}^{w \geq i+1}$ and $\underline{C}^{w \leq i}$, respectively (for all $i \in \mathbb{Z}$).

5. Alternatively, one could describe $W_{\geq i}X$ (given by part (iii') of the Theorem) as $\text{Im}((w_{\geq i-1}X)^{\tau=0} \rightarrow (w_{\geq i}X)^{\tau=0})$; so $W_{\geq i}X$ is given by the corresponding *virtual t -truncation* of $M \mapsto M^{\tau=0}$; see §2.5 of [Bon10a].

Most of the following results were also (essentially) verified in the process of proving Theorem 1.6.

Proposition 1.9. *Let t be transversal to w , $i \in \mathbb{Z}$. Then the following statements are fulfilled.*

I1. *The functors $X \mapsto \tau_{\leq i}X$ and $X \mapsto \tau_{\geq i}X$ map $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ into themselves.*

2. *$X \in \underline{C}^{w \leq i}$ (resp. $X \in \underline{C}^{w \geq i}$) whenever for any $j \in \mathbb{Z}$ we have $W_{\geq i-j}(X^{\tau=j}) = 0$ (resp. $W_{\geq i-j}(X^{\tau=j}) = X^{\tau=j}$; here $W_{\geq i}(-)$ is the filtration given by condition (iii') of Theorem 1.6).*

III1. *The functor $X \mapsto W_{\geq i}X$ (from \underline{Ht} into \underline{Ht}) given by condition (iii') of the Theorem, is right adjoint to the embedding $\underline{C}^{w \geq i} \cap \underline{Ht} \rightarrow \underline{Ht}$; it is exact.*

2. *For $X \in \underline{C}^{t=0}$ denote $X/W_{\geq i+1}X$ by $W_{\leq i}X$. Then the functor $W_{\leq i}(-)$ is left adjoint to the embedding $\underline{C}^{w \leq i} \cap \underline{Ht} \rightarrow \underline{Ht}$; it is exact.*

III 1. *The functors $Gr_i : X \mapsto W_{\leq i}(W_{\geq i}X)$ and $Gr'_i : X \mapsto W_{\geq i}(W_{\leq i}X)$ (defined as the compositions of the functors from assertion II) are canonically isomorphic exact projections of \underline{Ht} onto \underline{A}_i . Moreover, $Gr_i(X) \cong W_{\geq i}X/W_{\geq i+1}X$.*

2. *For $X \in \underline{C}^{t=0}$ we have: $X \in \underline{C}^{w \leq i}$ (resp. $X \in \underline{C}^{w \geq i}$) whenever $Gr_j(X) = 0$ for all $j > i$ (resp. for all $j < i$).*

Proof. I1. Since t is bounded, it suffices to verify a similar statement for the functors $\tau_{=j}$ for all $j \in \mathbb{Z}$ (since the functors mentioned in the assertions could be obtained from these functors via 'extensions').

Lemma 1.1(5) allows reducing the latter statement to its analogue for $\underline{C}^{w=0}$ (and all $j \in \mathbb{Z}$). Indeed, note that for any $l \in \mathbb{Z}$ the functor $W_{\geq l}$ (see condition (iv') of Theorem 1.6) is idempotent and exact; hence the class of objects of $\underline{A}_{\geq l} = W_{\geq l}\underline{Ht}$ contains all subobjects and factor-objects of its elements (in \underline{Ht}). Hence the long exact sequences coming from applying $\tau_{=j}$ to the \underline{C} -extensions given by Lemma 1.1(5) yields the reduction in question (by induction; here we also use Remark 1.8(3)).

Lastly, by Remark 1.8(2) we have $\underline{C}^{w=0} = \bigoplus_{j \in \mathbb{Z}} \text{Obj} \underline{A}_j[j]$; the result follows immediately.

2. By the previous assertion, it suffices to verify the statement for $X \in \underline{C}^{t=j}$. Then the fact is immediate from the statement that 'nice' filtrations of $X[j]$ yields its nice decompositions.

III1. As noted in the proof of Theorem 1.6, this functor is the restriction to \underline{Ht} of the functor b_i that is right adjoint to the embedding $\underline{C}_{>i} \rightarrow \underline{C}$. The result follows immediately.

2. Dual to the previous assertion (see Remark 1.8(3)).

III All of these assertions are easy consequences of the existence of a 'weight filtration' for \underline{Ht} (see Definition E7.2 of [B-VK10]).

The functors Gr_i and Gr'_i are exact as compositions of exact functors. They obviously take their values in \underline{A}_i and are identical on it; hence they are idempotent.

Now, we can compute Gr_i using the following functorial short exact sequence

$$0 \rightarrow W_{\geq i+1}X \rightarrow W_{\geq i}X \rightarrow Gr_iX \rightarrow 0; \quad (8)$$

we also consider its dual

$$0 \rightarrow Gr'_iX \rightarrow W_{\leq i}X \rightarrow W_{\leq i-1}X \rightarrow 0. \quad (9)$$

We obtain that both of Gr_i and Gr'_i kill $\underline{Ht} \cap \underline{C}^{w \geq i+1}$ and $\underline{Ht} \cap \underline{C}^{w \leq i-1}$. Since any object of \underline{Ht} could be presented as an extension of an object of \underline{A}_i by that of $\underline{Ht} \cap \underline{C}^{w \geq i+1}$ and $\underline{Ht} \cap \underline{C}^{w \leq i-1}$, we obtain that $Gr_i \cong Gr'_i$, whereas (8) yields the last statement in assertion III1.

Next, (8) yields that $Gr_iX = 0$ whenever $W_{\geq i+1}X \cong W_{\geq i}X$; (9) also yields that this is equivalent to $W_{\leq i}X \cong W_{\leq i-1}X$. So, we obtain assertion III2. □

Remark 1.10. 1. So, we obtain that $\tau_{\leq i}$ and $\tau_{> i}$ preserve $\underline{C}^{w=0}$.

This statement is somewhat weaker than the transversality of weight and t -structures (in contrast to condition (iv) of Theorem 1.6 where t and w are 'permuted'), since it does not imply the semi-simplicity of the corresponding \underline{A}_i . For example, let \underline{A} be a non-semi-simple abelian category such that any object of \underline{A} has finite projective dimension; then $\underline{C} = D^b(\underline{A}) \cong K^b(Proj \underline{A})$. Then we can consider the 'stupid' weight structure on \underline{C} whose heart is $Proj \underline{A}$. Certainly, \underline{Hw} is preserved by the truncations with respect to the canonical t -structure for \underline{C} . Yet if we put $\underline{A}_0 = Proj \underline{A}$ and all other $\underline{A}_i = 0$, non-projective objects of $\underline{Ht} \cong \underline{A}$ wouldn't have filtration by objects of \underline{A}_i (i.e. by projective ones).

One could also consider the direct sum of some (shifted) examples of this sort in order to get several non-zero \underline{A}_i .

2. For $\underline{A}_{>l}$ as in the proof of assertion I1, and $\underline{A}_{<l-1}$ being the categorical kernel of $W_{\geq l}(-) : \underline{Ht} \rightarrow \underline{Ht}$ we can re-formulate assertion I2 as follows: $X \in \underline{C}^{w \leq 0}$ (resp. $X \in \underline{C}^{w \geq 0}$) whenever $X^{\tau=i} \in \underline{A}_{\leq -i}$ (resp. $X^{\tau=i} \in \underline{A}_{\geq -i}$) for all $i \in \mathbb{Z}$ (for $X \in Obj \underline{C}$).

This statement corresponds to the definition of weights for mixed Hodge complexes (by Deligne) and for complexes of mixed Hodge modules (by Saito); see §2.3 below. For Artin-Tate motives over a number field this result was established in Theorem 3.8 of [Wil09]. Note here that the 'usual' convention for the 'signs of weights' is opposite to ours.

3. Let $H : \underline{C} \rightarrow \underline{A}$, \underline{A} is an abelian category, be either homological or cohomological (i.e. it is either covariant or contravariant, and converts distinguished triangles into long exact sequences) and $X \in \text{Obj}\underline{C}$. In §2.3–2.4 of [Bon10a] a certain *weight spectral sequence* $T(H, X)$ was constructed. For a homological H we have $E_1^{pq}(T) = H(X^p[q])$ for some $X^p \in \underline{C}^{w=0}$ (that could be expressed in terms of w -truncations), and T converges to $H(X[p+q])$; the formulas for a cohomological H are similar. T is (canonical and) functorial in X starting from E_2 ; it is also functorial in H .

Now, it is easily seen that $T(H, X)$ degenerates at E_2 for any H that could be presented as $F((-)^{\tau=i})$, where $F : \underline{Ht} \rightarrow \underline{A}$ is an exact functor (either a covariant or a contravariant one). Indeed, the functoriality of T in H yields: it suffices to verify this statement for $H = (-)^{\tau=i}$. In this case we have $E_1^{pq}T(H, X) \in \underline{A}_{-q-i}$ for any $p, q \in \mathbb{Z}$; hence the same is true for E_r^{pq} for any $r \geq 1$. Hence the boundary morphisms of $E_i(T)$ for $i \geq 2$ vanish, since their sources and targets necessarily belong to distinct \underline{A}_i .

In particular, this argument would 'explain' the degeneration of the (Deligne's) weight spectral sequence for singular and étale cohomology of motives (with rational coefficients) if we knew the existence of the motivic t -structure transversal to w_{chow} ; cf. Remark 2.4.3 of *ibid.*

2 Examples

A result from [B-VK10] yields: certain conditions on \underline{A} ensure for $D^b(\underline{A})$ the existence of a weight structure transversal to the canonical t -structure. We will use this statement for all our (main) examples.

Proposition 2.1. *Suppose that $1_{\underline{A}}$ is equipped with a separated exhaustive decreasing system of exact subfunctors $W_{\geq i}$. For all $i \in \mathbb{Z}$ denote the categorical kernel of the restriction of $W_{\geq i+1}$ to the image of $W_{\geq i}$ by \underline{A}_i .*

Suppose that all \underline{A}_i are semi-simple. Then \underline{A}_i yield a strongly semi-orthogonal generating system in $D^b(\underline{A})$, and the corresponding weight structure is transversal to the canonical t -structure for $D^b(\underline{A})$.

Proof. $\{\underline{A}_i\}$ are obviously generating. The orthogonality statements required are immediate by applying duality to Remark E7.8, Lemma E7.5, and Proposition E7.4(4) of *ibid.* \square

Remark 2.2. 1. There is a funny way to make a series of new examples of transversal weight and t -structures from one given example.

Suppose that a category $\underline{\mathcal{C}}$ with a bounded weight structure w admits a *differential graded enhancement* (see Definition 6.1.2 of [Bon10a]; note that one can easily find an enhancement for $D^b(\underline{\mathcal{A}})$ for any small $\underline{\mathcal{A}}$, since a localization of an enhanceable category is enhanceable). Then for any $N \geq 0$ there exist a triangulated category $\underline{\mathcal{C}}_N$ and an exact *truncation functor* $t_N : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}_N$ such that: $\underline{\mathcal{C}} = \langle t_N(\underline{\mathcal{C}}^{w=0}) \rangle$; and $\underline{\mathcal{C}}_N(t_N(A), t_N(B))$ for $A \in \underline{\mathcal{C}}^{w=0}$, $B \in \underline{\mathcal{C}}^{w=i}$ ($i \in \mathbb{Z}$) is zero for $i > N$ and $i < 0$, and is isomorphic via t_N to $\underline{\mathcal{C}}(A, B)$ for $0 \leq i \leq N$ (see Remark 6.2.2 and §6.3 of [Bon10a]). In particular, for $N = 0$ we obtain the *strong weight complex* functor $t : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}_0 \cong K^b(\underline{H}w)$; see loc.cit.

We obtain: for a strongly semi-orthogonal generating family of semi-simple $\{\underline{\mathcal{A}}_i \subset \underline{\mathcal{C}}\}$ (and the corresponding w) the family $\{t_N(\underline{\mathcal{A}}_i)\}$ is also strongly semi-orthogonal and generating in $\underline{\mathcal{C}}_N$ (since for any $i, j, s \in \mathbb{Z}$, $X \in \text{Obj} \underline{\mathcal{A}}_i$, $Y \in \text{Obj} \underline{\mathcal{A}}_j$ the group $\underline{\mathcal{C}}_N(t_N(X), t_N(Y)[s])$ is either 0, or is isomorphic to $\underline{\mathcal{C}}(X, Y[s])$); moreover, $t_N(\underline{\mathcal{A}}_i)$ are semi-simple.

So, using any of the examples (of transversal t and w) described below, one obtains the existence of transversal w and t for all the corresponding $K^b(\underline{H}w)$ and also for their 'higher' analogues (i.e the corresponding $\underline{\mathcal{C}}_N$ for $N > 0$). In particular, the 'motivic' conjectures imply the existence a t -structure transversal to the 'stupid' weight structure (the latter is the 'simplest' weight structure with the heart = *Chow*, that corresponds to the stupid truncations of complexes; see §1.1 of [Bon10a]) for $K^b(\text{Chow})$; this is true unconditionally for the '1-motivic' part of this category (over any smooth variety S/k).

The author does not know whether $\underline{\mathcal{C}}_N$ for $\underline{\mathcal{C}} = D^b(\underline{\mathcal{A}})$ (as in the proposition) is isomorphic to the derived category of the corresponding $\underline{H}t_N$; in any case, this construction surely produces some 'new' examples from the ones that we describe below.

2. Below we will consider several 'motivic' and 'Hodge' examples of our setting. All the motives, Hodge structures, complexes, and modules, and connecting functors between them that we will consider below will have rational coefficients. This is because the results of this paper cannot be applied (directly) to motives with integral coefficients. Indeed, even the category of finitely generated abelian groups (the 'easiest' part of motives of weight zero) is not semi-simple.

Note also that people usually do not expect Voevodsky's motives with integral coefficients to possess a 'reasonable' motivic t -structure (see §3.4 and Proposition 4.3.8 of [Voe00]).

2.1 1-motives; the comparison with Voevodsky's motives (endowed with the Chow weight structure)

First we consider the triangulated category $D^b(\mathcal{M}_1)$ of 1-motives.

Proposition 2.3. *Let S be connected and regular essentially of finite type over k . Then the category $D^b(\mathcal{M}_1)(S)$ (the derived category of Deligne's 1-motives over S ; see Appendix C.12 of [B-VK10]) is equipped with a weight structure w_1 that is transversal to the canonical t -structure for it.*

Proof. By Proposition C12.1 of *ibid.*, the category $\mathcal{M}_1 = \underline{A}$ satisfies the conditions of Proposition 2.1. \square

Next we set $S = \text{Spec } k$ and recall that the category $DM_{gm}^{eff} (\subset DM_{gm})$ of effective geometric Voevodsky's motives over k possesses a certain *Chow* weight structures whose heart is $Chow^{eff}$; see §6.6 of [Bon10a].

Proposition 2.4. *1. The embedding $T : D^b(\mathcal{M}_1) \rightarrow DM_{gm}^{eff}$ defined in Theorem 2.1.2 of [B-VK10] is weight-exact (i.e. $T(D^b(\mathcal{M}_1)^{w_1 \leq 0}) \subset DM_{gm}^{eff w_{Chow} \leq 0}$ and $T(D^b(\mathcal{M}_1)^{w_1 \geq 0}) \subset DM_{gm}^{eff w_{Chow} \geq 0}$).*

*2. The functor $LAlb : DM_{gm}^{eff} \rightarrow D^b(\mathcal{M}_1)$ introduced in Definition 5.2.1 of *ibid.* is weight-exact also, as well as $RPic$ (see Definition 5.3.1 of *ibid.*; since $RPic$ is contravariant, here weight-exactness means that $RPic(DM_{gm}^{eff w_{Chow} \leq 0}) \subset D^b(\mathcal{M}_1)^{w_1 \geq 0}$, and $RPic(DM_{gm}^{eff w_{Chow} \geq 0}) \subset D^b(\mathcal{M}_1)^{w_1 \leq 0}$).*

Proof. 1. Since w_1 is bounded, it suffices to verify that $T(D^b(\mathcal{M}_1)^{w_1=0}) \subset ObjChow^{eff}$.

To this end it suffices to prove (in the notation of Theorem 1.6) that $\underline{A}_i[i] \subset Chow^{eff}$. This is immediate from the description of \underline{A}_i , that could be immediately obtained from Definition C11.1 of [B-VK10] along with Lemma 16.1.1 of *ibid.*

2. It suffices to verify that $LAlb$ and $RPic$ map Chow motives into Chow ones.

Now, (for any $X \in ObjDM_{gm}^{eff}$) $LAlb(X)$ could be obtained from $L_i Alb(X)[i]$ (see Definition 8.1.1 of *ibid.*) via extensions (as usual for cohomology coming from a t -structure). Since the heart of a weight structure is always extension-stable, it suffices to verify that $L_i Alb[i]$ sends any smooth projective variety P/k into $Chow^{eff}$. We have $L_i Alb(P) = 0$ for $p \neq 0, 1, 2$. Moreover, Corollary 10.2.3 of *ibid.* immediately implies that $L_i Alb(P)[i]$ is a Chow motif for $i = 0, 2$. The case $i = 1$ is immediate from Lemma 16.1.1 of *ibid.*

The result for $RPic$ follows easily, since the functors are interchanged by Poincare duality (see Corollary 5.3.2 of *ibid.*); note that Poincare duality

maps Chow motives into Chow ones. Alternatively, one could apply Corollary 10.6.1 of *ibid.* (combined with Lemma 16.1.1 of *ibid.*).

□

Remark 2.5. 1. Very probably, an analogue of (at least) part 1 of the proposition is also fulfilled for motives over any S that is regular and essentially of finite type over k . Recall that a certain version of Voevodsky's S -motives (with rational coefficients) was thoroughly studied in [CiD09]; a Chow weight structure for this category was introduced in [Heb10] and in [Bon10b]. The main difficulty here is to construct a comparison functor.

2. As shown in §2 of [Bon10a], for any cohomology theory defined on $\underline{\mathcal{C}}$ a weight structure w for it yields certain *weight filtrations* (cf. Theorem 1.6(v,v') above), *weight spectral sequences* (cf. Remark 1.10(3)), and *virtual t -truncations*. The Proposition immediately yields the compatibility of all of these notions for $D^b(\mathcal{M}_1)$ and DM_{gm}^{eff} (with respect to T , LAlb, and RPic). Also, these comparison functors respect the weight complex functor (see §3 of *ibid.* and §6.3 of [Bon09]).

3. An analogue of Proposition 2.3 (along with Proposition 2.4(1)) for Artin-Tate motives over a number field was established in §3 of [Wil09].

2.2 Mixed motives: the conjectural picture

It is widely believed that Voevodsky's DM_{gm} (over a base scheme S) possesses the so-called motivic t -structure, whose heart $MM(S)$ (the category of *mixed motivic sheaves* over S) possesses a 'weight filtration' whose factors \underline{A}_i are abelian semi-simple. Moreover, at least in the case when S is the spectrum of a (perfect) field, people believe that any $\underline{A}_i[i]$ consists of Chow motives. It would follow (immediately, from condition (iii) of Theorem 1.6) that the Chow weight structure for $DM_{gm}(S)$ (that is known to exist unconditionally) is transversal to the motivic t -structure. Thus, the conjectures mentioned yield that the results of section 1 could be applied to this situation; we obtain some new (conjectural) information this way. Note that 'classically' people were looking only for the mixed motivic t -structure and for the 'pure' triangulated categories $\underline{\mathcal{C}}_i$ and other ingredients of Lemma 1.5 (the 'weight filtration'; cf. Definition E7.2 and Definition E17.1 of [B-VK10]). The notion of weight structure is new in this picture; it allows us to construct certain 'weights' for motives unconditionally.

In particular, for transversal t - and weight structures, $X \in \underline{\mathcal{C}}^{t=0}$, a nice decomposition of X (see condition (iv) of Theorem 1.6) yields a distinguished triangle of the type (4) (see condition (iii') of the Theorem). Now, for any (contravariant) cohomology theory $H : DM_{gm} \rightarrow \underline{\mathcal{A}}$ the i -th level

of the 'weight filtration' of $H^j(X) = H(X[-j])$ ($X \in \text{Obj}DM_{gm}$, $j \in \mathbb{Z}$) is 'classically' defined as $H^j(X_2)$, where X_2 is taken from (4). Now suppose that H factorizes through MM (i.e. it is the composition of the functor $M \mapsto M^{\tau_{MM}=0}$ with a contravariant exact functor; note that conjecturally one could factorize through MM all cohomology theories endowed with 'classical' weights); then $H^j(X)$ and its weight filtration coincides with those for $X' = \tau_{=-j}X$ (here we use the fact that the weight filtration functors given by Lemma 1.5 commute with t -truncations). Hence

$$H^j(X_2) \cong H^j(X'_2) \cong \text{Im}(H^j(w_{\leq i+j}X') \rightarrow H^j(X')) \cong \text{Im}(H^j(w_{\leq i+j}X) \rightarrow H^j(X)).$$

Now note that the last two terms do not depend on the choices of the corresponding weight decompositions (see Proposition 2.1.2 of [Bon10a]; this is also easy from Lemma 1.1(2)); hence one can define the weight filtration of $H^j(X)$ unconditionally (using the last term of the formula)!

One can also make this observation using a 'weight' filtration on \underline{A} (if it exists); see Remark 2.4.3 of [Bon10a]. Whereas this approach is somewhat 'cheating' for pure motives (since it usually gives no new information on their cohomology); yet it yields interesting results on mixed motives and their cohomology. Note here: for $X = \mathcal{M}(P)$, where P is smooth projective over k , the motives X_1, X_2 should come from a *Chow-Kunneth decomposition* of X ; so their construction is completely out of reach at the moment (in general).

2.3 Graded polarizable mixed Hodge complexes and Hodge modules; the Hodge realization

Proposition 2.6. *I Let X be a complex variety X .*

1. *There exists a weight structure w on the category $D^bMHM(X)$ (the derived category of mixed Hodge modules over X ; see [Sai89]) that is transversal to the canonical t -structure for it.*

2. *For this weight structure $D^bMHM(X)^{w \leq 0}$ is the class of complexes of mixed Hodge modules of weight ≥ 0 , and $D^bMHM(X)^{w \geq 0}$ is the class of complexes of mixed Hodge modules of weight ≤ 0 in the sense of Definition 1.6 of *ibid.**

II Let k be a subfield of the field of complex numbers.

1. *There exists a weight structure w_{Hodge} for the category $D_{\tilde{H}_p}^b(k)$ introduced in §3 of [Bei86], that is transversal to the canonical t -structure for it (given by Lemma 3.11 of *ibid.*).*

2. *The Hodge realization functor $H^{\text{sing}}(k) : DM_{gm}(k) \rightarrow D_{\tilde{H}_p}^b(k)$ (for example, the composition of the 'polarizable mixed realization' one constructed*

in §2.3 of [Hub00] with the natural functor 'forgetting all other realizations'; cf. §17.2 of [B-VK10]) is weight-exact with respect to these weight structures (i.e. $H^{sing}(DM_{gm}(k)^{w_{Chow} \leq 0}) \subset D_{\tilde{H}_p}^b(k)^{w_{Hodge} \geq 0}$ and $H^{sing}(DM_{gm}(k)^{w_{Chow} \geq 0}) \subset D_{\tilde{H}_p}^b(k)^{w_{Hodge} \leq 0}$).

Proof. I1. We should verify that $MHM(X)$ satisfies the conditions of Proposition 2.1. This is immediate from Propositions 1.5 and 1.9 of [Sai89].

2. Immediate from Definition 1.6 of [Sai89] along with Remark 1.10(2).

II1. By Lemma 3.11 of [Bei86], $D_{\tilde{H}_p}^b(k)$ is isomorphic to the bounded derived category of the abelian category of graded polarizable mixed Hodge structures (over k). Similarly to the proof of assertion I1, it remains to apply Proposition 2.1.

2. As in the proof of Proposition 2.4, we should verify that H^{sing} maps the heart of $w_{Chow}(k)$ into that of $w_{Hodge}(k)$. To this end it suffices to note that the i -th cohomology of a smooth projective P/\mathbb{C} is a pure (polarizable) Hodge structure of weight i (for $i \in \mathbb{Z}$). \square

Remark 2.7. 1. The weight filtration on $D_{\tilde{H}_p}^b$ obtained corresponds to the Deligne's definition of weights for mixed Hodge complexes (cf. assertion I2).

2. For a certain category of *Beilinson motives* over X (these are relative Voevodsky's motives with rational coefficients over X ; see [CiD09]) a certain Chow weight structure was introduced in [Heb10] and [Bon10b]). Very probably, this weight structure is compatible (similarly to assertion II2) with the one introduced in assertion I1 (and so with Saito's weights for complexes of mixed Hodge modules); note also that the 'functoriality' properties of the weights of the latter (see Proposition 1.7 of [Sai89]) are parallel to those for X -motives (as described in Theorem 3.7 [Heb10] and Theorem 2.2.1 in [Bon10b]). The problem is that (to the knowledge of the author) no 'mixed Hodge module' realization of X -motives is known to exist at the moment.

3. It does not seem very difficult to extend assertion II to Huber's category $D_{\mathcal{MRP}}$ of mixed realizations (introduced in §21 of [Hub95]), the corresponding cohomology functor (see the Remark after Corollary 2.3.4 of [Hub00]), and also to its étale cohomology analogue.

On the other hand, in finite characteristic one cannot prove that the étale cohomology of smooth projective varieties is polarizable in absence of the Hodge standard conjecture (for this case). Note here: without the polarizability restriction there will be 1-extensions inside \underline{A}_i for a single i (and so, $\underline{A}_i[i]$ cannot be contained in the heart of any weight structure).

References

- [B-VK10] L. Barbieri-Viale, B. Kahn, On the derived category of 1-motives, preprint <http://arxiv.org/abs/1009.1900>
- [BBD82] Beilinson A., Bernstein J., Deligne P., Faisceaux pervers// Astérisque 100, 1982, 5–171.
- [Bei86] Beilinson A. Notes on absolute Hodge cohomology, in: Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemporary Mathematics, 55. American Mathematical Society, 1986. pp. 35–68.
- [Bon10a] Bondarko M., Weight structures vs. t -structures; weight filtrations, spectral sequences, and complexes (for motives and in general)// J. of K-theory, v. 6, i. 03, pp. 387–504, 2010, see also <http://arxiv.org/abs/0704.4003>
- [Bon09] Bondarko M.V., Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura// J. of the Inst. of Math. of Jussieu, v.8 (2009), no. 1, 39–97, see also <http://arxiv.org/abs/math.AG/0601713>
- [Bon09s] Bondarko M.V., Weight structures and motives; comotives, coniveau and Chow-weight spectral sequences, and mixed complexes of sheaves: a survey, preprint, <http://arxiv.org/abs/0903.0091>
- [Bon10b] Bondarko M., Weights for Voevodsky’s motives over a base; relation with mixed complexes of sheaves, preprint, <http://arxiv.org/abs/1007.4543>
- [CiD09] Cisinski D., Deglise F., Triangulated categories of mixed motives, preprint, <http://arxiv.org/abs/0912.2110>
- [Heb10] Hébert D., Structures de poids à la Bondarko sur les motifs de Beilinson, preprint, <http://arxiv4.library.cornell.edu/abs/1007.0219>
- [Hub95] Huber A., Mixed Motives And Their Realization In Derived Categories, Springer, LNM 1604, 1995.
- [Hub00] Huber A. Realizations of Voevodsky’s motives// J. Alg. Geometry, 9, 2000, no. 4, 755–799.

- [Sai89] Saito M. Introduction to mixed Hodge modules (Actes du Colloque de Théorie de Hodge, Luminy, 1987)// Astérisque No. 179–180 (1989), 10, 145–162.
- [Voe00] Voevodsky V. Triangulated category of motives, in: Voevodsky V., Suslin A., and Friedlander E. Cycles, transfers and motivic homology theories, Annals of Mathematical studies, vol. 143, Princeton University Press, 2000, 188–238.
- [Wil09] Wildeshaus J., Notes on Artin-Tate motives, preprint, <http://www.math.uiuc.edu/K-theory/0918/>