Integrable pseudopotentials related to generalized hypergeometric functions

A.V. Odesskii [1,2], **V.V. Sokolov** [1]

[1] L.D. Landau Institute for Theoretical Physics (Russia)
[2] University of Manchester (UK)

Abstract
We construct integrable pseudopotentials with an arbitrary number of fields in terms of generalized hypergeometric functions. These pseudopotentials yield some integrable (2+1)-dimensional hydrodynamic type systems. In two particular cases these systems are equivalent to integrable scalar 3-dimensional equations of second order. An interesting class of integrable (1+1)-dimensional hydrodynamic type systems is also generated by our pseudopotentials.

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Address: L.D. Landau Institute for Theoretical Physics of Russian Academy of Sciences, Kosygina 2, 119334, Moscow, Russia

E-mail: alexander.odesskii@manchester.ac.uk, odesskii@itp.ac.ru, sokolov@itp.ac.ru

Contents

1 Introduction

The main object of integrability theory is the Lax equation

$$L_t = [L,\, A]. \tag{1.1}$$

Here A and L are operators depending on functions $u_1, ..., u_n$ and (1.1) is equivalent to a system of nonlinear differential equations for u_i. For the KP-hierarchy and its different reductions A is a linear differential operator $A = \sum r_i \partial_x^i$, whose coefficients r_i are differential polynomials in $u_1, ..., u_n$. The L-operator could be a differential operator or a more complicated object like a ratio of two differential operators or a formal (non-commutative) Laurent series with respect to ∂_x^{-1}.

The dispersionless analog of (1.1) has the following form

$$L_t = \{L, A\}, \tag{1.2}$$

where $\{L, A\} = A_p L_x - A_x L_p$. As usual, the commutator in (1.1) is replaced by the Poisson bracket and the non-commutative variable ∂_x by the commutative "spectral" parameter p. The transformation $L(x,t,p) \to p(x,t,L)$ reduces (1.2) to the following conservative form

$$p_t = A(p, u_1, \ldots, u_n)_x, \tag{1.3}$$

where L plays the role of a parameter. The latter equation can be rewritten as

$$\psi_t = A(\psi_x, u_1, \ldots, u_n), \tag{1.4}$$

where $p = \psi_x$.

Equations (1.4) can be chosen as a basis, on which a theory of integrable 3-dimensional dispersionless PDEs can be built. Most such equations can be written in the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,t_1} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{j,t_2} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{j,t_3} = 0, \qquad i = 1, ..., l, \tag{1.5}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$. All known integrable systems (1.5) admit the so-called pseudopotential representation

$$\psi_{t_2} = A(\psi_{t_1}, \mathbf{u}), \qquad \psi_{t_3} = B(\psi_{t_1}, \mathbf{u}), \tag{1.6}$$

by means of a pair of equations (1.4) whose the compatibility conditions $\psi_{t_2 t_3} = \psi_{t_3 t_2}$ are equivalent to (1.5). The functions $A,\ B$ are called pseudopotentials. Such a pseudopotential representation is a dispersionless version of the zero curvature representation, which is a basic notion in the integrability theory of solitonic equations (see [1]) .

One of the interesting and attractive features of the theory of integrable dispersionless equations is that the dependence of the pseudopotentials $A(p, u_1, \ldots, u_n)$ on p can be much more complicated then in the solitonic case. For instance, in [2, 3] some important examples of pseudopotentials A were found related to the Whitham averaging procedure for integrable dispersion PDEs and to the Frobenious manifolds. For these examples the p-dependence is determined by an algebraic curve of arbitrary genus g. In the paper [4] a certain class of pseudopotentials with movable singularities was described. Some of the pseudopotentials constructed in [4] are written in terms of degenerate hypergeometric functions.

In the paper [5] a wide class of pseudopotentials $A(p, u_1, ..., u_n)$ related to rational algebraic curves was constructed. These pseudopotentials were written in the following parametric form:

$$A = F_1(\xi, u_1, ..., u_n), \qquad p = F_2(\xi, u_1, ..., u_n),$$

where the ξ-dependence of the functions F_i is defined by the ODE

$$F_{i,\xi} = \phi_i(\xi, u_1, ..., u_n) \cdot \xi^{-s_1}(\xi - 1)^{-s_2}(\xi - u_1)^{-s_3}...(\xi - u_n)^{-s_{n+2}}. \tag{1.7}$$

Here $s_1, ..., s_{n+2}$ are arbitrary constants and ϕ_i are polynomials in ξ of degree n. The dependence of ϕ_i on $u_1, \dots, u_n$ was described in terms of solutions of some overdetermined linear system of PDEs with rational coefficients.

In this paper we generalize this result and construct new classes of pseudopotentials $A_{n,k}(p, u_1, ..., u_n)$ whose p-dependence is given by (1.7), where $\phi_i(\xi)$ are polynomials in ξ of degree $n - k$, $k = 0, ..., n - 1$. We call the corresponding functions $A_{n,k}$ *pseudopotentials of defect k*. The pseudopotentials of defect 0 are just pseudopotentials from [5] written in a different form.

We describe the pseudopotentials of defect k in terms of linearly independent solutions of the following system of linear PDEs with rational coefficients

$$\frac{\partial^2 h}{\partial u_i \partial u_j} = \frac{s_i}{u_i - u_j} \cdot \frac{\partial h}{\partial u_j} + \frac{s_j}{u_j - u_i} \cdot \frac{\partial h}{\partial u_i}, \quad i, j = 1, ..., n, \qquad i \neq j, \tag{1.8}$$

and

$$\begin{gathered} \frac{\partial^2 h}{\partial u_i \partial u_i} = -\Big(1 + \sum_{j=1}^{n+2} s_j\Big) \frac{s_i}{u_i(u_i - 1)}\, h + \frac{s_i}{u_i(u_i - 1)} \sum_{j \neq i}^{n} \frac{u_j(u_j - 1)}{u_j - u_i} \cdot \frac{\partial h}{\partial u_j} + \\ \Big(\sum_{j \neq i}^{n} \frac{s_j}{u_i - u_j} + \frac{s_i + s_{n+1}}{u_i} + \frac{s_i + s_{n+2}}{u_i - 1}\Big) \frac{\partial h}{\partial u_i} \end{gathered} \tag{1.9}$$

for one unknown function $h(u_1, \dots, u_n)$. If $n = 1$, then we have no equations (1.8) and the single equation (1.9) coincides with the standard hypergeometric equation,

$$u(u - 1)\, h(u)'' + [(\alpha + \beta + 1)\, u - \gamma]\, h(u)' + \alpha\beta\, h(u) = 0,$$

where $s_1 = -\alpha$, $s_2 = \alpha - \gamma$, $s_3 = \gamma - \beta - 1$. Notice, that hypergeometric functions already appeared in connection with dispersionless PDEs (see, for example [6, 4, 7]). For arbitrary

n the system (1.8), (1.9) can be solved in terms of *generalized hypergeometric functions* (see [8, 9]).

Note that the pseudopotential $A_{n,k}$ is written in terms of $k+2$ linearly independent solutions of the system (1.8), (1.9) and therefore the group GL_{k+2} acts on the set of such pseudopotentials. If $k=0$, then this is just the usual action of GL_2 on the space of independent variables $x,\ t$ in the equation (1.4). In the case $k>0$ the action of larger group GL_{k+2} is still to be explained. For a particular class of 3-dimensional equations the existence of such a group of symmetries was pointed out in [10] (see also [11]). It is known [12, 10] that knowledge of the symmetry group GL_n allows us to linearize systems of ODEs and PDEs.

The paper is organized as follows.

In Section 2 we describe some properties of system (1.8), (1.9) and its solutions needed for our purposes. Most of these properties are well known to experts.

In Section 3 we rewrite formulas of the paper [5] in terms of generalized hypergeometric functions. In our paper the pseudopotentials constructed in [5] are called pseudopotentials of defect 0. A couple of such pseudopotentials defines a system of the form (1.5) with $l=n$. These systems are rewritten in terms of generalized hypergeometric functions in Section 3. We also prove that each of these systems admits $n+1$ conservation laws of hydrodynamic type.

In Section 4, for any n and $k>0$ we construct *pseudopotentials of defect* k. A couple of such pseudopotentials defines a system of the form (1.5) with $l=n+k$. These systems are also constructed in Section 4. The particular cases $n=3,\ k=1$ and $n=5,\ k=3$ yield integrable equations of the form

$$\sum_{i,j} P_{i,j}(z_{t_1}, z_{t_2}, z_{t_3})\, z_{t_i,t_j} = 0, \qquad i,j = 1,2,3, \tag{1.10}$$

and

$$Q(z_{t_1,t_1}, z_{t_1,t_2}, z_{t_1,t_3}, z_{t_2,t_2}, z_{t_2,t_3}, z_{t_3,t_3}) = 0. \tag{1.11}$$

A classification of all integrable equations (1.10) and (1.11) was presented in [11] and in [13], correspondingly. Our integrable equations give generic solutions of these classification problems.

In Sections 5 we construct and study a certain class of integrable (1+1)-dimensional hydrodynamic type systems of the form

$$r^i_t = v^i(r^1,...,r^N) r^i_x, \qquad i=1,2,...,N. \tag{1.12}$$

These systems are defined by an universal overdetermined compatible system of PDEs of the Gibbons-Tsarev type [14, 15] for some functions $w(r^1,...,r^N)$, $\xi_1(r^1,...,r^N)$, ..., $\xi_N(r^1,...,r^N)$. This system has the following form

$$\partial_i \xi_j = \frac{\xi_j(\xi_j - 1)}{\xi_i - \xi_j}\partial_i w, \quad \partial_{ij} w = \frac{2\xi_i\xi_j - \xi_i - \xi_j}{(\xi_i-\xi_j)^2}\partial_i w \partial_j w, \qquad i,j=1,...,N, \quad i\neq j. \tag{1.13}$$

The only velocities $v^i(r^1,...,r^N)$ in (1.12) depend on $n,\ k$. They are described by $k+2$ linearly independent solutions of the linear system (1.8), (1.9) (see Section 5, Theorem 3). One has to substitute functions $u_1 = u_1(r^1, ..., r^N), ..., u_n = u_n(r^1, ..., r^N)$ for the arguments of these solutions. The functions u_i are also universal. They are defined by the following system of PDEs

$$\partial_i u_j = \frac{u_j(u_j - 1)\partial_i w}{\xi_i - u_j}, \quad i = 1, ..., N, \quad j = 1, ..., n. \tag{1.14}$$

It is easy to verify that the system (1.13), (1.14) is consistent. Therefore our (1+1)-dimensional systems (1.12) admit a local parametrization by $2N$ functions of one variable.

For some very special values of parameters s_i in (1.8), (1.9) our systems (1.12) are related to the Whitham hierarchies [2], to the Frobenious manifolds [3, 16], and to the associativity equation [3, 16].

In Section 6 we recall the definition of hydrodynamic reductions. According to [17], the existence of sufficiently many hydrodynamic reductions can be chosen as a definition of the integrability of the systems (1.5). We also recall the definition of integrable pseudopotentials (see [7]). We introduce the notion of compatible pseudopotentials and notice that each pair of them gives a system (1.5) that admits both a pseudopotential representation and sufficiently many hydrodynamic reductions. We show that the (1+1)-dimensional hydrodynamic type systems found in Section 5 are hydrodynamic reductions of our pseudopotentials $A_{n,k}$. This implies that these pseudopotentials and the corresponding 3-dimensional systems are integrable in the sense of the definitions mentioned above (see Theorem 4).

2 Generalized hypergeometric functions

The following statements can be verified straightforwardly.

Proposition 1. The system of linear equations (1.8), (1.9) is compatible for any constants $s_1, \ldots, s_{n+2}$. The dimension of the linear space $\mathcal{H}$ of solutions of the system (1.8), (1.9) is equal to $n + 1$. ■

We call elements of $\mathcal{H}$ *generalized hypergeometric functions.*

Proposition 2. The system (1.8), (1.9) is equivalent to the following system

$$Q_i(u_1 \frac{\partial}{\partial u_1}, ..., u_n \frac{\partial}{\partial u_n}) u_i^{-1} h = P_i(u_1 \frac{\partial}{\partial u_1}, ..., u_n \frac{\partial}{\partial u_n}) h, \quad i = 1, ..., n \tag{2.15}$$

where

$$Q_i(k_1, ..., k_n) = (k_1 + ... + k_n - s_1 - ... - s_{n+1})(k_i + 1),$$
$$P_i(k_1, ..., k_n) = (k_1 + ... + k_n - 1 - s_1 - ... - s_{n+2})(k_i - s_i).$$

■

Recall that a system of the form (2.15) is called a hypergeometric system [9]. It can be solved in terms of the so-called Horn series [9].

Example 1. The system (2.15) and hence (1.8), (1.9) has a unique solution holomorphic at the point $\mathbf{0} = (0, \dots, 0)$ such that $h(\mathbf{0}) = 1$. The derivatives of this solution at $\mathbf{0}$ is given by

$$h^{(k_1,\dots,k_n)}(\mathbf{0}) = \frac{\prod_{j=0}^{k_1+\dots+k_n-1}(1-j+s_{n+2}+r)}{\prod_{j=0}^{k_1+\dots+k_n-1}(-j+r)} \prod_{j=1}^{n}\prod_{i=0}^{k_j-1}(i-s_j),$$

where

$$r = \sum_{i=1}^{n+1} s_i.$$

Let us denote the solution described in this example by $F(s_1, \dots, s_{n+2},\, u_1, \dots, u_n)$. For brevity, we also will use the notation $F(s_1, \dots, s_{n+2})$.

Proposition 3. The function $F(s_1, \dots, s_{n+2})$ admits the following integral representation

$$F(s_1, \dots, s_{n+2},\, u_1, \dots, u_n) = C \int_0^1 t^{-2-r-s_{n+2}}(1-t)^{s_{n+2}}(1-tu_1)^{s_1}\cdots(1-tu_n)^{s_n}\, dt,$$

where

$$C = \frac{\Gamma(-r)}{\Gamma(1+s_{n+2})\Gamma(-1-r-s_{n+2})}.$$

■

It is well-known that for the standard hypergeometric equation there exist the Laplace transformations shifting the parameters by 1. Analogies of such transformations for the system (1.8), (1.9) are given by

Proposition 4. The following identities hold:

$$\frac{\partial F(s_1, \dots, s_i, \dots s_{n+2})}{\partial u_i} = -\frac{s_i(1+r+s_{n+2})}{r}\, F(s_1, \dots, s_i-1, \dots s_{n+2}), \qquad i \le n,$$

$$L_1\Big(F(s_1, \dots, s_{n+1}, s_{n+2})\Big) = \frac{s_{n+1}(1+r+s_{n+2})}{r}\, F(s_1, \dots, s_{n+1}-1, s_{n+2}),$$

$$L_2\Big(F(s_1, \dots, s_{n+1}, s_{n+2})\Big) = (1+r+s_{n+2})\, F(s_1, \dots, s_{n+1}, s_{n+2}-1),$$

where

$$L_1 = \sum_{j=1}^{n}(1-u_j)\frac{\partial}{\partial u_j} + (1+r+s_{n+2}), \qquad L_2 = -\sum_{j=1}^{n} u_j\frac{\partial}{\partial u_j} + (1+r+s_{n+2}),$$

and

$$M_i\Big(F(s_1, \dots, s_i, \dots s_{n+2})\Big) = (1+r)\, F(s_1, \dots, s_i+1, \dots s_{n+2}), \qquad i \le n,$$

$$M_{n+1}\Big(F(s_1, \dots, s_{n+1}, s_{n+2})\Big) = (1+r)\, F(s_1, \dots, s_{n+1}+1, s_{n+2}),$$

$$M_{n+2}\Big(F(s_1,\dots,s_{n+1},s_{n+2})\Big) = -(1+s_{n+2})\,F(s_1,\dots,s_{n+1},s_{n+2}+1),$$

where

$$M_i = \sum_{j=1}^{n} u_j(u_j-1)\frac{\partial}{\partial u_j} - \sum_{j=1}^{n} s_j u_j - (2+r+s_{n+2})u_i + (1+r), \qquad i \le n,$$

$$M_{n+1} = \sum_{j=1}^{n} u_j(u_j-1)\frac{\partial}{\partial u_j} - \sum_{j=1}^{n} s_j u_j + (1+r),$$

$$M_{n+2} = \sum_{j=1}^{n} u_j(u_j-1)\frac{\partial}{\partial u_j} - \sum_{j=1}^{n} s_j u_j - (1+s_{n+2}).$$

Furthermore, let $\mathcal{H}_{s_1,\dots,s_{n+2}}$ be the space of solutions of the system (1.8), (1.9). We have

$$\frac{\partial}{\partial u_i}\mathcal{H}_{s_1,\dots,s_{n+2}} \subset \mathcal{H}_{s_1,\dots,s_i-1,\dots,s_{n+2}}, \quad L_1\mathcal{H}_{s_1,\dots,s_{n+2}} \subset \mathcal{H}_{s_1,\dots,s_{n+1}-1,s_{n+2}},$$

$$L_2\mathcal{H}_{s_1,\dots,s_{n+2}} \subset \mathcal{H}_{s_1,\dots,S_{n+2}-1}, \qquad M_i\mathcal{H}_{s_1,\dots,s_{n+2}} \subset \mathcal{H}_{s_1,\dots,s_i+1,\dots,s_{n+2}}.$$

■

Proposition 5. Let $\mathcal{H} = \mathcal{H}_{s_1,\dots,s_{n+2}}$ and $\widetilde{\mathcal{H}} = \mathcal{H}_{s_1,\dots,s_n,0,s_{n+1},s_{n+2}}$. Then $\widetilde{\mathcal{H}}$ is spanned by $\mathcal{H}$ and the function

$$Z(u_1,...,u_n,u_{n+1}) = \int_0^{u_{n+1}} (t-u_1)^{s_1}\cdots(t-u_n)^{s_n} t^{s_{n+1}}(t-1)^{s_{n+2}}dt. \tag{2.16}$$

Moreover, the space $\mathcal{H}_{s_1,\dots,s_n,0,\dots,0,s_{n+1},s_{n+2}}$ (m zeros) is spanned by $\mathcal{H}$ and $Z(u_1,...,u_n,u_{n+1})$, $Z(u_1,...,u_n,u_{n+2}),...,Z(u_1,...,u_n,u_{n+m})$. ■

3 Pseudopotentials of defect 0

Most results of this Section was obtained in a different form in the paper [5].

For any generalized hypergeometric function $g\in\mathcal{H}$ we put

$$S_n(g,\xi) = \sum_{1\le i\le n} u_i(u_i-1)(\xi-u_1)...\hat{i}...(\xi-u_n)\,g_{u_i} + (1+\sum_{1\le i\le n+2} s_i)(\xi-u_1)...(\xi-u_n)\,g. \tag{3.17}$$

Here $g_{u_i} = \dfrac{\partial g}{\partial u_i}$. It is clear that $S_n(g,\xi)$ is a polynomial of degree n in ξ.

Example 2. In the simplest case $n=1$ we have

$$S_1(g,\xi) = u(u-1)g_u + (1+s_1+s_2+s_3)(\xi-u)g$$

where $u = u_1$.

We need the following property of the polynomial $S_n(g, \xi)$:

Lemma 1. For any $1 \le m \le n$ the following identity is valid

$$u_m(u_m-1)(u_m-u_1)...\hat{m}...(u_m-u_n)\frac{S_n(g,\xi)_{u_m}+\frac{(s_m+1)S_n(g,\xi)}{\xi-u_m}}{S_n(g,u_m)} =$$

$$\xi(\xi-1)(\xi-u_1)...\hat{m}...(\xi-u_n)\Big(\frac{s_1}{\xi-u_1}+...+\frac{s_m+1}{\xi-u_m}+...+\frac{s_n}{\xi-u_n}+\frac{s_{n+1}}{\xi}+\frac{s_{n+2}}{\xi-1}\Big).$$

■

Define $P_n(g, \xi)$ by the formula

$$P_n(g,\xi) = \int_0^\xi S_n(g,\xi)(\xi-u_1)^{-s_1-1}...(\xi-u_n)^{-s_n-1}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}-1}d\xi \qquad (3.18)$$

if $\operatorname{Re} s_{n+1} < -1$ and as the analytic continuation of this expression otherwise.

Proposition 6. The expression

$$\frac{P_n(g,\xi)_{u_m}}{S_n(g,u_m)} \qquad (3.19)$$

does not depend on g. More precisely,

$$u_m(u_m-1)(u_m-u_1)...\hat{m}...(u_m-u_n)\frac{P_n(g,\xi)_{u_m}}{S_n(g,u_m)} =$$
$$-(\xi-u_1)^{-s_1}...(\xi-u_m)^{-s_m-1}...(\xi-u_n)^{-s_n}\xi^{-s_{n+1}}(\xi-1)^{-s_{n+2}}. \qquad (3.20)$$

Proof. The derivative of (3.19) with respect to ξ is equal to

$$\frac{S_n(g,\xi)_{u_m}+\frac{(s_m+1)S_n(g,\xi)}{\xi-u_m}}{S_n(g,u_m)}(\xi-u_1)^{-s_1-1}...(\xi-u_n)^{-s_n-1}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}-1}.$$

Lemma 1 implies that this derivative does not depend on g. Since the value of (3.19) at $\xi = 0$ is equal to zero, expression (3.19) itself does not depend of g. Identity (3.20) also follows from Lemma 1. ■

Let $g_1,\ g_0$ be linearly independent elements of $\mathcal{H}$. A pseudopotential $A_n(p, u_1, ..., u_n)$ defined in a parametric form by

$$A_n = P_n(g_1,\xi), \qquad p = P_n(g_0,\xi) \qquad (3.21)$$

is called *pseudopotential of defect 0.*

Relations (3.21) mean that to find $A_n(p, u_1, ..., u_n)$, we have to express ξ from the second equation and substitute the result into the first equation.

Let $g_0, g_1, ..., g_n \in \mathcal{H}$ be a basis in $\mathcal{H}$. Define pseudopotentials $B_\alpha(p, u_1, ..., u_n)$ of defect 0, where $\alpha = 1, ..., n$, by

$$B_\alpha = P_n(g_\alpha, \xi), \qquad p = P_n(g_0, \xi), \qquad \alpha = 1, ..., n.$$

Suppose that $u_1, ..., u_n$ are functions of $t_0 = x,\ t_1, ..., t_n$.

Theorem 1. The compatibility conditions $\psi_{t_\alpha t_\beta} = \psi_{t_\beta t_\alpha}$ for the system

$$\psi_{t_\alpha} = B_\alpha(\psi_x, u_1, ..., u_n), \qquad \alpha = 1, ..., n. \tag{3.22}$$

are equivalent to the following system of PDEs for $u_1, ..., u_n$:

$$\sum_{1\le i\le n, i\ne j} (g_{q,u_j} g_{r,u_i} - g_{r,u_j} g_{q,u_i}) \frac{u_j(u_j-1)u_{i,t_s} - u_i(u_i-1)u_{j,t_s}}{u_j - u_i} + \sigma\cdot(g_q g_{r,u_j} - g_r g_{q,u_j})u_{j,t_s} +$$

$$\sum_{1\le i\le n, i\ne j} (g_{r,u_j} g_{s,u_i} - g_{s,u_j} g_{r,u_i}) \frac{u_j(u_j-1)u_{i,t_q} - u_i(u_i-1)u_{j,t_q}}{u_j - u_i} + \sigma\cdot(g_r g_{s,u_j} - g_s g_{r,u_j})u_{j,t_q} + \tag{3.23}$$

$$\sum_{1\le i\le n, i\ne j} (g_{s,u_j} g_{q,u_i} - g_{q,u_j} g_{s,u_i}) \frac{u_j(u_j-1)u_{i,t_r} - u_i(u_i-1)u_{j,t_r}}{u_j - u_i} + \sigma\cdot(g_s g_{q,u_j} - g_q g_{s,u_j})u_{j,t_r} = 0,$$

where $j = 1, ..., n,\ \sigma = 1 + s_1 + ... + s_{n+2}$. Here q, r, s run from 0 to n and $t_0 = x$.

Proof. If B_α are given in a parametric form

$$B_\alpha = f_\alpha(\xi, u_1, ..., u_n), \qquad p = f_0(\xi, u_1, ..., u_n),$$

then the compatibility conditions for (3.22) is equivalent to

$$\sum_{i=1}^{n} \Big((f_{q,\xi} f_{r,u_i} - f_{r,\xi} f_{q,u_i}) u_{i,t_s} + (f_{r,\xi} f_{s,u_i} - f_{s,\xi} f_{r,u_i}) u_{i,t_q} + (f_{s,\xi} f_{q,u_i} - f_{q,\xi} f_{s,u_i}) u_{i,t_r} \Big) = 0. \tag{3.24}$$

Taking into account (3.18), (3.20), we get

$$f_{q,\xi} f_{r,u_i} - f_{r,\xi} f_{q,u_i} =$$

$$\Big(S_n(g_q,\xi)P_n(g_r,\xi)_{u_i} - S_n(g_r,\xi)P_n(g_q,\xi)_{u_i}\Big)(\xi-u_1)^{-s_1-1}\cdots(\xi-u_n)^{-s_n-1}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}-1}$$

$$= \frac{S_n(g_q,\xi)S_n(g_r,u_i) - S_n(g_r,\xi)S_n(g_q,u_i)}{(\xi-u_i)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)} \cdot T = \frac{S_n(g_q,\xi)g_{r,u_i} - S_n(g_r,\xi)g_{q,u_i}}{\xi - u_i} \cdot T.$$

Here

$$T = -(\xi-u_1)^{-2s_1-1}...(\xi-u_n)^{-2s_n-1}\xi^{-2s_{n+1}-1}(\xi-1)^{-2s_{n+2}-1}$$

does not depend on i. Using the above formula for $f_{q,\xi}f_{r,u_i} - f_{r,\xi}f_{q,u_i}$ and similar formulas for $f_{r,\xi}f_{s,u_i} - f_{s,\xi}f_{r,u_i},\ f_{s,\xi}f_{q,u_i} - f_{q,\xi}f_{s,u_i}$, we can rewrite (3.24) as follows:

$$\sum_{1\le i\le n}\Big(\frac{S_n(g_q,\xi)g_{r,u_i} - S_n(g_r,\xi)g_{q,u_i}}{\xi - u_i}u_{i,t_s} + \frac{S_n(g_r,\xi)g_{s,u_i} - S_n(g_s,\xi)g_{r,u_i}}{\xi - u_i}u_{i,t_q} +$$

$$\frac{S_n(g_s,\xi)g_{q,u_i} - S_n(g_q,\xi)g_{s,u_i}}{\xi - u_i}u_{i,t_r}\Big) = 0.$$

It follows from (3.17) that the left hand side is a polynomial of degree $n-1$ in ξ. To conclude the proof, it remains to evaluate this polynomial at $\xi = u_1, ..., u_n$. ■

Remark 1. Given t_1, t_2, t_3, Theorem 1 yields a 3-dimensional system of the form (1.5) with $l = n$ equations possessing pseudopotential representation.

Remark 2. A system of PDEs for $u_1, ..., u_n$, which is equivalent to compatibility conditions for equations of the form (3.24), was called in [2] a *Whitham hierarchy*. In the paper [2] I.M. Krichever constructed some Whitham hierarchies related to algebraic curves of arbitrary genus g. The hierarchy corresponding to $g = 0$ is equivalent to one described by Theorem 1 if $s_1 = \ldots = s_{n+2} = 0$. In this case the vector space $\mathcal{H}$ is spanned by $1, u_1, u_2, \ldots, u_n$.

Proposition 7. The system (3.23) possesses $n+1$ hydrodynamic type conservation laws.

Proof. Let $\widehat{\mathcal{H}} = \mathcal{H}_{-2s_1,...,-2s_n,-2s_{n+1}-1,-2s_{n+2}-1}$ be the space of generalized hypergeometric functions defined by (1.8), (1.9) with $\hat{s}_i = -2s_i$ for $i = 1, ..., n$ and $\hat{s}_i = -2s_i - 1$ for $i = n+1,\ n+2$. Let $Z \in \widehat{\mathcal{H}}$ be an arbitrary element in $\widehat{\mathcal{H}}$. Denote by X_j the left hand side of (3.23). Define functions A_i, B_i, C_i by

$$\sum_{i=1}^{n}(A_i u_{i,t_q} + B_i u_{i,t_r} + C_i u_{i,t_s}) = \sum_{j=1}^{n}\frac{1}{s_j}Z_{u_j}X_j.$$

One can check that $(A_i)_{u_j} = (A_j)_{u_i},\ (B_i)_{u_j} = (B_j)_{u_i},\ (C_i)_{u_j} = (C_j)_{u_i}$. Therefore $A_i = A_{u_i}$, $B_i = B_{u_i}$, $C_i = C_{u_i}$ for some functions $A,\ B,\ C$ and we have

$$A_{t_q} + B_{t_r} + C_{t_s} = 0.$$

Since $\dim \widehat{\mathcal{H}} = n+1$, we obtain $n+1$ conservation laws of the hydrodynamic type.

4 Pseudopotentials of defect $k > 0$

In this section we construct a new class of pseudopotentials. We call them *pseudopotentials of defect* k. To define pseudopotentials of defect k, we fix k linearly independent generalized hypergeometric functions $h_1, ..., h_k \in \mathcal{H}$. For any $g \in \mathcal{H}$ define $S_{n,k}(g,\xi)$ by the formula

$$S_{n,k}(g,\xi) = \frac{1}{\Delta}\sum_{1\le i\le n-k+1} u_i(u_i - 1)(\xi - u_1)...\hat{i}...(\xi - u_{n-k+1})\Delta_i(g). \qquad (4.25)$$

Here

$$\Delta = \det\begin{pmatrix} h_1 & ... & h_k \\ h_{1,u_{n-k+2}} & ... & h_{k,u_{n-k+2}} \\ .... & ... & .... \\ h_{1,u_n} & ... & h_{k,u_n} \end{pmatrix}, \qquad \Delta_i(g) = \det\begin{pmatrix} g & h_1 & ... & h_k \\ g_{u_i} & h_{1,u_i} & ... & h_{k,u_i} \\ g_{u_{n-k+2}} & h_{1,u_{n-k+2}} & ... & h_{k,u_{n-k+2}} \\ .... & ... & ... & .... \\ g_{u_n} & h_{1,u_n} & ... & h_{k,u_n} \end{pmatrix}.$$

It is clear that $S_{n,k}(g,\xi)$ is a polynomial in ξ of degree $n-k$. Notice that $S_{n,k}(h_1,\xi) = ... = S_{n,k}(h_k,\xi) = 0$. It is easy to see that linear transformations $h_i \to c_{i1}h_1 + ... + c_{ik}h_k$, $g \to g + d_1h_1 + ... + d_kh_k$ with constant coefficients c_{ij}, d_i do not change $S_{n,k}(g,\xi)$.

Example 3. In the simplest case $n=2, \ k=1$ we have

$$S_{2,1}(g,\xi) = u_1(u_1-1)(\xi-u_2)\frac{gh_{1,u_1} - g_{u_1}h_1}{h_1} + u_2(u_2-1)(\xi-u_1)\frac{gh_{1,u_2} - g_{u_2}h_1}{h_1}.$$

Lemma 2. If $1 \le m < n-k+2$, then the following identity is valid:

$$u_m(u_m-1)(u_m-u_1)...\hat{m}...(u_m-u_n)\frac{S_{n,k}(g,\xi)_{u_m} + \frac{(s_m+1)S_{n,k}(g,\xi)}{\xi-u_m}}{S_{n,k}(g,u_m)} =$$

$$-(u_m-u_{n-k+2})...(u_m-u_n)\frac{1}{\Delta}\sum_{1\le i\le n-k+1} u_i(u_i-1)(\xi-u_1)...\hat{i}...(\xi-u_{n-k+1})\widetilde{\Delta_i}+$$

$$\frac{1}{\Delta}\sum_{n-k+2\le i\le n, 1\le j\le n-k+1} (u_m-u_{n-k+2})...\hat{i}...(u_m-u_n)s_iu_j(u_j-1)(\xi-u_1)...\hat{j}...(\xi-u_{n-k+1})\widetilde{\Delta_{i,j}}+$$

$$\frac{(s_m+1)u_m(u_m-1)(u_m-u_{n-k+2})...(u_m-u_n)(\xi-u_1)...\hat{m}...(\xi-u_{n-k+1})}{\xi-u_m}+$$

$$(u_m-u_{n-k+2})...(u_m-u_n)\sum_{1\le i\le n-k+1, i\ne m} s_iu_i(u_i-1)\prod_{1\le j\le n-k+1, j\ne i,m}(\xi-u_j)+$$

$$(u_m-u_{n-k+2})...(u_m-u_n)(\xi-u_1)...\hat{m}...(\xi-u_{n-k+1})(\sum_{1\le i\le n-k+1}(u_m+u_i-1)s_i + 2u_m - 1)+$$

$$u_m(u_m-1)(\xi-u_1)...\hat{m}...(\xi-u_{n-k+1})\sum_{n-k+2\le i\le n}(u_m-u_{n-k+2})...\hat{i}...(u_m-u_n)s_i+$$

$$(u_m-u_{n-k+2})...(u_m-u_n)(\xi-u_1)...\hat{m}...(\xi-u_{n-k+1})((u_m-1)s_{n+1} + u_ms_{n+2}).$$

If $n-k+2 \le m$, then

$$u_m(u_m-1)(u_m-u_1)...\hat{m}...(u_m-u_n)\frac{S_{n,k}(g)(\xi)_{u_m} + \frac{s_mS_{n,k}(g,\xi)}{\xi-u_m}}{S_{n,k}(g,u_m)} =$$

$$\frac{1}{\Delta}\sum_{n-k+2\le i\le n, 1\le j\le n-k+1} (u_m-u_{n-k+2})...\hat{i}...(u_m-u_n)s_iu_j(u_j-1)(\xi-u_1)...\hat{j}...(\xi-u_{n-k+1})\widetilde{\Delta_{i,j}}+$$

$$\frac{s_m u_m(u_m-1)(u_m-u_{n-k+2})...\hat{m}...(u_m-u_n)(\xi-u_1)...(\xi-u_{n-k+1})}{\xi-u_m}.$$

Here

$$\widetilde{\Delta_i}=\det\begin{pmatrix} h_{1,u_i} & ... & h_{k,u_i} \\ h_{1,u_{n-k+2}} & ... & h_{k,u_{n-k+2}} \\ .... & ... & .... \\ h_{1,u_n} & ... & h_{k,u_n} \end{pmatrix}$$

and $\widetilde{\Delta_{i,j}}$ is obtained from Δ by replacing the row $(h_{1,u_i},...,h_{k,u_i})$ by $(h_{1,u_j},...,h_{k,u_j})$. ■

Define functions $P_{n,k}(g,\xi)$ by

$$P_{n,k}(g,\xi)= \tag{4.26}$$

$$\int_0^\xi S_{n,k}(g,\xi)(\xi-u_1)^{-s_1-1}...(\xi-u_{n-k+1})^{-s_{n-k+1}-1}(\xi-u_{n-k+2})^{-s_{n-k+2}}...(\xi-u_n)^{-s_n}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}-1}d\xi$$

if $\mathrm{Re}\, s_{n+1}<-1$, and as the analytic continuation of this expression otherwise.

Proposition 8. The expression

$$\frac{P_{n,k}(g,\xi)_{u_m}}{S_{n,k}(g,u_m)} \tag{4.27}$$

does not depend on g. Moreover, we have

$$\sum_{1\le i\le k+1} u_{m_i}(u_{m_i}-1)(u_{m_i}-u_1)...\widehat{m_1,...,m}_{k+1}...(u_{m_i}-u_n)\frac{P_{n,k}(g,\xi)_{u_{m_i}}}{S_{n,k}(g,u_{m_i})}= \tag{4.28}$$

$$-\frac{(\xi-u_1)^{-s_1}...(\xi-u_{n-k+1})^{-s_{n-k+1}}(\xi-u_{n-k+2})^{-s_{n-k+2}+1}...(\xi-u_n)^{-s_n+1}\xi^{-s_{n+1}}(\xi-1)^{-s_{n+2}}}{(\xi-u_{m_1})...(\xi-u_{m_{k+1}})}.$$

Proof. The derivative of expression (4.27) with respect to ξ is equal to

$$\frac{S_{n,k}(g,\xi)_{u_m}+\frac{(s_m+1)S_{n,k}(g,\xi)}{\xi-u_m}}{S_{n,k}(g,u_m)}(\xi-u_1)^{-s_1-1}\dots$$

$$(\xi-u_{n-k+1})^{-s_{n-k+1}-1}(\xi-u_{n-k+2})^{-s_{n-k+2}}...(\xi-u_n)^{-s_n}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}-1}$$

for $1\le m<n-k+2$ and is equal to

$$\frac{S_{n,k}(g,\xi)_{u_m}+\frac{s_m S_{n,k}(g,\xi)}{\xi-u_m}}{S_{n,k}(g,u_m)}(\xi-u_1)^{-s_1-1}\dots$$

$$(\xi-u_{n-k+1})^{-s_{n-k+1}-1}(\xi-u_{n-k+2})^{-s_{n-k+2}}...(\xi-u_n)^{-s_n}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}-1}$$

otherwise. Lemma 2 implies that this derivative does not depend on g. Moreover, the value of the expression (4.27) at $\xi=0$ is equal to zero. Therefore the expression (4.27) itself does not depend on g. The proof of (4.28) is similar. ■

Let $g_1,\ g_2 \in \mathcal{H}$. Assume that $g_1, g_2, h_1, ..., h_k$ are linearly independent. Define pseudopotential $A_{n,k}(p, u_1, ..., u_n)$ in parametric form by

$$A_{n,k} = P_{n,k}(g_1, \xi), \qquad p = P_{n,k}(g_2, \xi). \tag{4.29}$$

To construct $A_{n,k}(p, u_1, ..., u_n)$, we find ξ from the second equation and substitute into the first one. The pseudopotential $A_{n,k}(p, u_1, ..., u_n)$ is called *pseudopotential of defect k*.

Theorem 2. Let $g_0, g_1, ..., g_{n-k}, h_1, ..., h_k \in \mathcal{H}$ be a basis in $\mathcal{H}$ and $B_\alpha,\ \alpha = 1, ..., n-k$ are defined by

$$B_\alpha = P_{n,k}(g_\alpha, \xi), \qquad p = P_{n,k}(g_0, \xi), \qquad \alpha = 1, ..., n-k.$$

Then the compatibility conditions for (3.22) are equivalent to the following system of PDEs for $u_1, ..., u_n$:

$$\begin{aligned}
&\sum_{1\le i\le n-k, i\ne j} \Big(\Delta_j(g_q)\Delta_i(g_r) - \Delta_j(g_r)\Delta_i(g_q)\Big) \frac{u_j(u_j-1)u_{i,t_s} - u_i(u_i-1)u_{j,t_s}}{u_j - u_i} + \\
&\sum_{1\le i\le n-k, i\ne j} \Big(\Delta_j(g_r)\Delta_i(g_s) - \Delta_j(g_s)\Delta_i(g_r)\Big) \frac{u_j(u_j-1)u_{i,t_q} - u_i(u_i-1)u_{j,t_q}}{u_j - u_i} + \\
&\sum_{1\le i\le n-k, i\ne j} \Big(\Delta_j(g_s)\Delta_i(g_q) - \Delta_j(g_q)\Delta_i(g_s)\Big) \frac{u_j(u_j-1)u_{i,t_r} - u_i(u_i-1)u_{j,t_r}}{u_j - u_i} = 0,
\end{aligned} \tag{4.30}$$

where $j = 1, ..., n-k$ and

$$\sum_{i=1}^{n-k+1} \Delta_i(g_r) u_{i,t_s} = \sum_{i=1}^{n-k+1} \Delta_i(g_s) u_{i,t_r}, \tag{4.31}$$

$$\sum_{i=1}^{n-k+1} \Delta_i(g_r) \frac{u_m(u_m-1)u_{i,t_s} - u_i(u_i-1)u_{m,t_s}}{u_m - u_i} = \sum_{i=1}^{n-k+1} \Delta_i(g_s) \frac{u_m(u_m-1)u_{i,t_r} - u_i(u_i-1)u_{m,t_r}}{u_m - u_i}, \tag{4.32}$$

where $m = n-k+2, ..., n$. Here q, r, s run from 0 to n and $t_0 = x$.

Proof. We have to explicitly calculate the coefficients in (3.24). Using (4.26), (4.27), we find that

$$f_{q,\xi} f_{r,u_i} - f_{r,\xi} f_{q,u_i} = \Big(S_{n,k}(g_q, \xi) P_{n,k}(g_r, \xi)_{u_i} - S_{n,k}(g_r, \xi) P_{n,k}(g_q, \xi)_{u_i}\Big) \cdot T =$$

$$\Big(S_{n,k}(g_q, \xi) S_{n,k}(g_r, u_i) - S_{n,k}(g_r, \xi) S_{n,k}(g_q, u_i)\Big) \cdot \frac{P_{n,k}(g_q, \xi)_{u_i}}{S_{n,k}(g_q, u_i))} \cdot T.$$

Similar formulas are valid for $f_{r,\xi} f_{s,u_i} - f_{s,\xi} f_{r,u_i},\ f_{s,\xi} f_{q,u_i} - f_{q,\xi} f_{s,u_i}$. Here

$$T = (\xi - u_1)^{-s_1-1} ... (\xi - u_{n-k+1})^{-s_{n-k+1}-1} (\xi - u_{n-k+2})^{-s_{n-k+2}} ... (\xi - u_n)^{-s_n} \xi^{-s_{n+1}-1} (\xi - 1)^{-s_{n+2}-1}$$

does not depend on i. Using (4.28), we can express $\frac{P_{n,k}(g_q,\xi)_{u_i}}{S_{n,k}(g_q,u_i)}$, $i=1,...,n-k$ in terms of $\frac{P_{n,k}(g_q,\xi)_{u_m}}{S_{n,k}(g_q,u_m)}$, $m=n-k+1,...,n$, which are linearly independent as functions of ξ. Substituting these into (3.24), we obtain

$$\sum_{i=1}^{n-k}\Big(\frac{S_{n,k}(g_q,\xi)S_{n,k}(g_r,u_i)-S_{n,k}(g_r,\xi)S_{n,k}(g_q,u_i)}{(\xi-u_i)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_{n-k})}u_{i,t_s}+$$
$$\frac{S_{n,k}(g_r,\xi)S_{n,k}(g_s,u_i)-S_{n,k}(g_s,\xi)S_{n,k}(g_r,u_i)}{(\xi-u_i)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_{n-k})}u_{i,t_q}+$$
$$\frac{S_{n,k}(g_s,\xi)S_{n,k}(g_q,u_i)-S_{n,k}(g_q,\xi)S_{n,k}(g_s,u_i)}{(\xi-u_i)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_{n-k})}u_{i,t_r}\Big)=0, \tag{4.33}$$

$$\sum_{i=1}^{n-k}\Big(\frac{S_{n,k}(g_q,\xi)S_{n,k}(g_r,u_i)-S_{n,k}(g_r,\xi)S_{n,k}(g_q,u_i)}{(u_i-u_m)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_{n-k})}u_{i,t_s}+$$
$$\frac{S_{n,k}(g_r,\xi)S_{n,k}(g_s,u_i)-S_{n,k}(g_s,\xi)S_{n,k}(g_r,u_i)}{(u_i-u_m)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_{n-k})}u_{i,t_q}+$$
$$\frac{S_{n,k}(g_s,\xi)S_{n,k}(g_q,u_i)-S_{n,k}(g_q,\xi)S_{n,k}(g_s,u_i)}{(u_i-u_m)\cdot u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_{n-k})}u_{i,t_r}\Big)+$$
$$\frac{S_{n,k}(g_q,\xi)S_{n,k}(g_r,u_m)-S_{n,k}(g_r,\xi)S_{n,k}(g_q,u_m)}{u_m(u_m-1)(u_m-u_1)...(u_m-u_{n-k})}u_{m,t_s}+$$
$$\frac{S_{n,k}(g_r,\xi)S_{n,k}(g_s,u_m)-S_{n,k}(g_s,\xi)S_{n,k}(g_r,u_m)}{u_m(u_m-1)(u_m-u_1)...(u_m-u_{n-k})}u_{m,t_q}+$$
$$\frac{S_{n,k}(g_s,\xi)S_{n,k}(g_q,u_m)-S_{n,k}(g_q,\xi)S_{n,k}(g_s,u_m)}{u_m(u_m-1)(u_m-u_1)...(u_m-u_{n-k})}u_{m,t_r}=0, \tag{4.34}$$

where $m=n-k+1,...,n$. One can check straightforwardly that (4.34) is equivalent to (4.31) for $m=n-k+1$ and to (4.32) for $m=n-k+2,...,n$. Notice that the left hand side of equation (4.33) is a polynomial in ξ of degree $n-k-1$. Evaluating this polynomial at $\xi=u_j,\ j=1,...,n-k$ we obtain (4.30). ■

Remark 3. Given t_1,t_2,t_3, Theorem 2 yields a 3-dimensional system of the form (1.5) with $l=n+k$ equations possessing pseudopotential representation. Indeed, the formulas (4.31), (4.32) give $3k$ linearly independent equations if $q,r,s=1,2,3$. The formula (4.30) gives $n-k$ equations. On the other hand, one can construct exactly k linear combinations of equations (4.31), (4.32) with $q,r,s=1,2,3$ such that derivatives of $u_i,\ i=n-k+1,...,n$ cancel out. Moreover, these linear combinations belong to the span of equations (4.30). Therefore, there exist $(n-k)+3k-k=n+k$ linearly independent equations.

Remark 4. In (4.30), (4.31), (4.32) we have to assume $n\geq k+2$. Indeed, for $n=k+1$ we cannot construct more then one pseudopotential and therefore there is no any system of the form (1.5) associated with this case. However the corresponding pseudopotential generates interesting integrable (1+1)-dimensional systems of hydrodynamic type (see Section 5). Probably these pseudopotentials for $k = 0, 1, ...$ are also related to some infinite integrable chains of the Benney type [18, 19].

The system (4.30)-(4.32) possesses many conservation laws of the hydrodynamic type. In particular, the following statement can be verified by a straightforward calculation.

Proposition 9. For any $r \neq s = 0, 1, ..., n$ there exist k conservation laws for the system (4.30)-(4.32) of the form:

$$\left(\frac{\Delta(g_r, h_1, ...\hat{i}...h_k)}{\Delta(h_1, ..., h_k)} \right)_{t_s} = \left(\frac{\Delta(g_s, h_1, ...\hat{i}...h_k)}{\Delta(h_1, ..., h_k)} \right)_{t_r}, \tag{4.35}$$

where $i = 1, ..., k$. Here

$$\Delta(f_1, ..., f_k) = \det \begin{pmatrix} f_1 & ... & f_k \\ h_{1,u_{n-k+2}} & ... & f_{k,u_{n-k+2}} \\ .... & ... & .... \\ f_{1,u_n} & ... & f_{k,u_n} \end{pmatrix}.$$

Proposition 9 allows us to define functions $z_1, ..., z_k$ such that

$$\frac{\Delta(g_r, h_1, ...\hat{i}...h_k)}{\Delta(h_1, ..., h_k)} = z_{i,t_r} \tag{4.36}$$

for all $i = 1, ..., k$ and $r = 0, 1, ..., n$.

Suppose $n \geq 3k$; then the system of the form (1.5) obtained from (4.30), (4.31), (4.32) with $q, r, s = 1, 2, 3$ consists of $3k$ equations (4.31), (4.32) (they are equivalent to (4.35)) and $n - 2k$ equations of the form (4.30). Indeed, only $n - 2k$ equations (4.30) are linearly independent on (4.31), (4.32). Expressing $u_1, ..., u_{3k}$ in terms of $z_{i,t_1}, z_{i,t_2}, z_{i,t_3},\ i = 1, ..., k$ from (4.36) and substituting into $n - 2k$ equations of the form (4.30), we obtain a 3-dimensional system of $n - 2k$ equations for $n - 2k$ unknowns $z_1, ..., z_k, u_{3k+1}, ...u_n$. This is a quasi-linear system of the second order with respect to z_i and of the first order with respect to u_j, whose coefficients depend on $z_{i,t_1}, z_{i,t_2}, z_{i,t_3},\ i = 1, ..., k$ and $u_{3k+1}, ...u_n$. It is clear that the general solution of the system can be locally parameterized by $n - k$ functions in two variables.

In the case $2k \leq n < 3k$ the functions $z_{i,t_1}, z_{i,t_2}, z_{i,t_3},\ i = 1, ..., k$ are functionally dependent. We have $3k - n$ equations of the form

$$R_i(z_{1,t_1}, z_{1,t_2}, z_{1,t_3}, ..., z_{k,t_1}, z_{k,t_2}, z_{k,t_3}) = 0, \qquad i = 1, ..., 3k - n$$

and $n - 2k$ second order quasi-linear equations. Totally we have $(3k - n) + (n - 2k) = k$ equations for k unknowns $z_1, ..., z_k$. It is clear that the general solution of this system can be locally parameterized by $n - k$ functions in two variables.

Suppose $n < 2k$; then we have $n + k < 3k$, which means that $3k$ equations of the form (4.31), (4.32) are linearly dependent. Probably in this case the general solution of the system can also be locally parameterized by $n - k$ functions in two variables.

One of the most interesting cases is $n = 3k$, when we have a system of k quasi-linear second order equations for the functions $z_1, ..., z_k$. Consider the simplest case $k = 1$.

Example 4. In the case $n = 3$, $k = 1$ the formulas (4.30), (4.31) can be rewritten as follows. Let h_1, g_0, g_1, g_2 be linearly independent elements of $\mathcal{H}$. Denote by B_{ij} the cofactors of the matrix

$$\begin{pmatrix} h_1 & g_0 & g_1 & g_2 \\ h_{1,u_1} & g_{0,u_1} & g_{1,u_1} & g_{1,u_1} \\ h_{1,u_2} & g_{0,u_2} & g_{1,u_2} & g_{1,u_1} \\ h_{1,u_3} & g_{0,u_3} & g_{1,u_3} & g_{1,u_3} \end{pmatrix}.$$

Define vector fields V_i by

$$V_1 = B_{22}\,\frac{\partial}{\partial t_0} + B_{23}\,\frac{\partial}{\partial t_1} + B_{24}\,\frac{\partial}{\partial t_2},$$
$$V_2 = B_{32}\,\frac{\partial}{\partial t_0} + B_{33}\,\frac{\partial}{\partial t_1} + B_{34}\,\frac{\partial}{\partial t_2},$$
$$V_3 = B_{42}\,\frac{\partial}{\partial t_0} + B_{43}\,\frac{\partial}{\partial t_1} + B_{44}\,\frac{\partial}{\partial t_2}.$$

Then (4.31) is equivalent to

$$V_1(u_2) = V_2(u_1), \qquad V_2(u_3) = V_3(u_2), \qquad V_3(u_1) = V_1(u_3). \tag{4.37}$$

Relation (4.30) leads to one more equation

$$u_3(u_3-1)(u_1-u_2)V_1(u_2) + u_1(u_1-1)(u_2-u_3)V_2(u_3) + u_2(u_2-1)(u_3-u_1)V_3(u_1) = 0. \tag{4.38}$$

The conservation laws (4.35) have the form

$$\left(\frac{g_r}{h_1}\right)_{t_s} = \left(\frac{g_s}{h_1}\right)_{t_r}.$$

Introducing z such that $z_{t_r} = \frac{g_r}{h_1}$, we reduce (4.38) to a quasi-linear equation of the form

$$\sum_{i,j} P_{i,j}(z_{t_0}, z_{t_1}, z_{t_2})\, z_{t_i,t_j} = 0, \qquad i,j = 0,1,2. \tag{4.39}$$

In the paper [11] an inexplicit description of all integrable equations (4.39) was proposed. The equation constructed above corresponds to the generic case in this classification. Indeed, it depends on five essential parameters $s_1, ..., s_5$ which agrees with the results of [11].

For integer values of parameters s_i equations (1.8), (1.9) can be solved in elementary functions. This provides simple examples of equations (4.39) having pseudopotentials. In the most degenerate case $s_1 = \cdots = s_5 = 0$ one can choose $h_1 = 1$, $g_0 = u_1$, $g_1 = u_2$, $g_2 = u_3$. The corresponding equation (4.39) is given by

$$z_{t_2}(z_{t_2}-1)(z_{t_0}-z_{t_1})z_{t_0t_1} + z_{t_0}(z_{t_0}-1)(z_{t_1}-z_{t_2})z_{t_1t_2} + z_{t_1}(z_{t_1}-1)(z_{t_2}-z_{t_0})z_{t_2t_0} = 0.$$

More general examples of equations

$$P_1(z_{t_0}, z_{t_1}, z_{t_2})\, z_{t_0t_1} + P_2(z_{t_0}, z_{t_1}, z_{t_2})\, z_{t_1t_2} + P_3(z_{t_0}, z_{t_1}, z_{t_2})\, z_{t_2t_0} = 0 \tag{4.40}$$

correspond to $s_1 = s_2 = s_3 = 0$. In this case one can choose $h = 1$, $g_0 = f(u_1)$, $g_1 = f(u_2)$, $g_2 = f(u_3)$, where $f'(x) = x^{s_4}(x-1)^{s_5}$. In the new variables $\bar{u}_i = f(u_i)$ the system (4.37), (4.38) is equivalent to a single equation of the form (4.40). One of the results of the paper [11] is a complete classification of equations (4.40) possessing a pseudopotential representation. The above example seems to be the generic case in this classification. ■

The system (4.30)-(4.32) has conservation laws different from (4.35).

Conjecture. The system (4.30) - (4.32) possesses $n+1$ conservation laws of the general form

$$A_{t_q} + B_{t_r} + C_{t_s} = 0$$

additional to (4.35). This family of conservation laws can be parameterized by elements from $\widehat{\mathcal{H}} = \mathcal{H}_{-2s_1,...,-2s_n,-2s_{n+1}-1,-2s_{n+2}-1}$ (cf. Proposition 7). This conjecture is supported by some computer computations for small n and k.

Remark 5. Let us make in (4.26) a change of variables of the form

$$\xi \to \frac{a\xi + b}{c\xi + d}, \qquad u_1 \to \phi_1, ..., u_n \to \phi_n, \tag{4.41}$$

where a, b, c, d, $\phi_1, ..., \phi_n$ are arbitrary functions in $u_1, ..., u_n$. After that we get under the integral in (4.26) an expression of the form

$$S(\xi)(\xi-\rho_1)^{-s_1-1}...(\xi-\rho_{n-k+1})^{-s_{n-k+1}-1}(\xi-\rho_{n-k+2})^{-s_{n-k+2}}...(\xi-\rho_n)^{-s_n}(\xi-\rho_{n+1})^{-s_{n+1}-1}$$
$$\times(\xi-\rho_{n+2})^{-s_{n+2}-1}(\xi-\rho_{n+3})^{s_1+...+s_{n+2}+1},$$

where $S(\xi)$ is a polynomial in ξ of degree $n-k$ and $\rho_1, ..., \rho_{n+3}$ are functions of $u_1, ..., u_n$. Therefore the numbers

$$\{-s_1-1, ..., -s_{n-k+1}-1, -s_{n-k+2}, ..., -s_n, -s_{n+1}-1, -s_{n+2}-1, s_1+...+s_{n+2}+1\} \tag{4.42}$$

play a symmetric role in the constructed pseudopotentials $A_{n,k}$. Using transformations (4.41), one can choose any three of the functions $\rho_1, ..., \rho_{n+3}$ to be equal to $0,\ 1,\ \infty$ and the other n functions to be equal to $u_1, ..., u_n$ (cf. [5], Section 3). It would be interesting to study the degenerate cases when some of the functions ρ_i coincide (cf. [7], Section 5).

The most symmetric case is given by

$$s_1 = ... = s_{n-k+1} = s_{n+1} = s_{n+2} = -\frac{k+1}{n+3}, \quad s_{n-k+2} = ... = s_n = \frac{n-k+2}{n+3}.$$

In this case all numbers (4.42) are equal to $-\frac{n-k+2}{n+3}$. Possibly for $n=3,\ k=1$ these values of parameters correspond to pseudopotentials for integrable Lagrangians of the form $\mathcal{L}(u_x,u_y,u_z)$ [20, 10] whereas for $n=5,\ k=3$ they are related to the integrable Hirota type equations [13].

Example 5. Let $n=5$, $k=3$ and $s_1=s_2=s_3=s_6=s_7=-\frac{1}{2},\ s_4=s_5=\frac{1}{2}$. It turns out that there exists a basis $g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3$ in $\mathcal{H}$ such that

$$\begin{aligned} \Delta(g_1,h_1,h_2)&=\Delta(g_3,h_2,h_3),\\ \Delta(g_2,h_2,h_3)&=\Delta(g_1,h_3,h_1),\\ \Delta(g_3,h_3,h_1)&=\Delta(g_2,h_1,h_2). \end{aligned} \tag{4.43}$$

Indeed, the system (4.43) is a consequence of equations

$$\begin{aligned} g_1h_{1,u_4}-h_1g_{1,u_4}+g_2h_{2,u_4}-h_2g_{2,u_4}+g_3h_{3,u_4}-h_3g_{3,u_4}&=0,\\ g_1h_{1,u_5}-h_1g_{1,u_5}+g_2h_{2,u_5}-h_2g_{2,u_5}+g_3h_{3,u_5}-h_3g_{3,u_5}&=0,\\ g_{1,u_4}h_{1,u_5}-h_{1,u_4}g_{1,u_5}+g_{2,u_4}h_{2,u_5}-h_{2,u_4}g_{2,u_5}+g_{3,u_4}h_{3,u_5}-h_{3,u_4}g_{3,u_5}&=0. \end{aligned} \tag{4.44}$$

Consider the system consisting of equations (4.44) and all its first and second derivatives with respect to $u_1,...,u_5$. Note that differentiating (4.44), we eliminate second derivatives of h_i and g_i by (1.8), (1.9). One can check that this system is invariant with respect to the derivations by $u_1,...,u_5$. At a fixed generic point $u_1^0,...,u_5^0$ the system can be regaded as an algebraic variety for the values of $g_i,\ h_i$ and their first derivatives. It can be checked that this variety consists of several components and the maximal dimension of the component equals 24. Since the vector fields $\frac{\partial}{\partial u_i}$ are tangent to this variety, any its point considered as the initial data defines the solutions $g_i,\ h_i$ of (1.8), (1.9) such that the corresponding point belongs to the variety for any values of $u_1,...,u_5$. It is possible to check that there exists an algebraic component of dimension 21 of the variety such that the Wronskian of $g_i,\ h_i$ at $u_1^0,...,u_5^0$ is non-zero.

Proposition 9 and equations (4.43) allow us to define a function z such that

$$z_{t_1,t_1}=\frac{\Delta(g_1,h_2,h_3)}{\Delta(h_1,h_2,h_3)},\ z_{t_2,t_2}=\frac{\Delta(g_2,h_3,h_1)}{\Delta(h_1,h_2,h_3)},\ z_{t_3,t_3}=\frac{\Delta(g_3,h_1,h_2)}{\Delta(h_1,h_2,h_3)},$$

$$z_{t_1,t_2}=\frac{\Delta(g_2,h_2,h_3)}{\Delta(h_1,h_2,h_3)}=\frac{\Delta(g_1,h_3,h_1)}{\Delta(h_1,h_2,h_3)},\ z_{t_2,t_3}=\frac{\Delta(g_3,h_3,h_1)}{\Delta(h_1,h_2,h_3)}=\frac{\Delta(g_2,h_1,h_2)}{\Delta(h_1,h_2,h_3)},$$

$$z_{t_3,t_1}=\frac{\Delta(g_1,h_1,h_2)}{\Delta(h_1,h_2,h_3)}=\frac{\Delta(g_2,h_1,h_2)}{\Delta(h_1,h_2,h_3)}.$$

It terms of this function we can rewrite the system (4.30), (4.31), (4.32) as a single equation of the form (1.11). Integrable systems of this form were studied in [13]. The pseudopotentials considered above correspond to the generic integrable system of this form.

Remark 6. It is easy to see that the group SP_6 acts on the set of bases in $\mathcal{H}$ satisfying (4.44). This agrees with the result of [13] that this group acts on the set of integrable equations of the form (1.11).■

5 Integrable (1+1)-dimensional systems of hydrodynamic type

In this section we consider integrable (1+1)-dimensional hydrodynamic type systems (1.12) constructed in terms of generalized hypergeometric functions. These systems appear as the so-called hydrodynamic reductions of pseudopotentials $A_{n,k}$ (see the next Section). By integrability we mean the existence of infinite number of hydrodynamic commuting flows and conservation laws. It is known [21] that this is equivalent to the following relations for the velocities $v^i(r^1, ..., r^N)$:

$$\partial_j \frac{\partial_i v^k}{v^i - v^k} = \partial_i \frac{\partial_j v^k}{v^j - v^k}, \qquad i \neq j \neq k, \tag{5.45}$$

Here $\partial_\alpha = \frac{\partial}{\partial r^i}$, $\alpha = 1, \ldots, N$. The system (1.12) is called *semi-Hamiltonian* if conditions (5.45) hold.

The main geometrical object related to a semi-Hamiltonian system (1.12) is a diagonal metric g_{kk}, $k = 1, \ldots, N$, where

$$\frac{1}{2}\partial_i \log g_{kk} = \frac{\partial_i v^k}{v^i - v^k}, \qquad i \neq k. \tag{5.46}$$

In view of (5.45), the overdetermined system (5.46) is compatible and the function g_{kk} is defined up to arbitrary factor $\eta_k(r^k)$. The metric g_{kk} is called the *metric associated to (1.12).* It is known that two hydrodynamic type systems are compatible iff they possess a common associated metric [21].

A diagonal metric g_{kk} is called a *metric of Egorov type* if for any i, j

$$\partial_i g_{jj} = \partial_j g_{ii}. \tag{5.47}$$

Note that if a Egorov-type metric associated with a hydrodynamic-type system of the form (1.12) exists, then it is unique. For any Egorov's metric there exists a potential G such that $g_{ii} = \partial_i G$. Semi-Hamiltonian systems possessing associated metrics of Egorov type play important role in the theory of WDVV associativity equations and in the theory of Frobenious manifolds [3, 16, 22].

Let $w(r^1, ..., r^N)$, $\xi_1(r^1, ..., r^N), ..., \xi_N(r^1, ..., r^N)$ be a solution of (1.13). It can be easily verified that this system is in involution and therefore its solution admits a local parameterization by $2N$ functions of one variable. Let $u_1(r^1, ..., r^N), ..., u_n(r^1, ..., r^N)$ be a set of solutions of the system (1.14). It is easy to verify that this system is in involution and therefore has an one-parameter family of solutions for fixed ξ_i, w.

Consider the following system

$$r^i_t = \frac{S_{n,k}(g_1, \xi_i)}{S_{n,k}(g_2, \xi_i)} r^i_x, \tag{5.48}$$

where g_1, g_2 are linearly independent solutions of (1.8), (1.9), the polynomials $S_{n,k}$, $k > 0$ are defined by (4.25), and $S_{n,0} = S_n$ (see formula (3.17)).

Theorem 3. The system (5.48) is semi-Hamiltonian. The associated metric is given by

$$g_{ii} = S_n(g_2, \xi_i)^2 (\xi_i - u_1)^{-2s_1-2} \cdots (\xi_i - u_n)^{-2s_n-2} \xi_i^{-2s_{n+1}-1} (\xi_i - 1)^{-2s_{n+2}-1} \partial_i w$$

for $k = 0$, and by

$$g_{ii} = S_{n,k}(g_2, \xi_i)^2 (\xi_i - u_1)^{-2s_1-2} \cdots (\xi_i - u_{n-k+1})^{-2s_{n-k+1}-2} \times$$
$$(\xi_i - u_{n-k+2})^{-2s_{n-k+2}} \cdots (\xi_i - u_n)^{-2s_n} \xi_i^{-2s_{n+1}-1} (\xi_i - 1)^{-2s_{n+2}-1} \partial_i w$$

for $k > 0$.

Proof. Substituting the expression for the metric into (5.46), where v^i are specified by (5.48), one obtains the identity by virtue of (1.13), (1.14). ■

Remark 7. The system (5.48) does not possess the associated metric of the Egorov type in general. However, for very special values of the parameters s_i in (1.8), (1.9) there exists $g_2 \in \mathcal{H}$ such that the metric is of the Egorov type for all solutions of the system (1.13), (1.14). For instance, if the defect k equals zero, then this happens exactly in the following cases:

$s_i = 0$ for all i;

$s_l = -1$ for some l and $s_i = 0$ for $i \neq l$;

$s_l = -\frac{1}{2}$ for some l and $s_i = 0$ for $i \neq l$;

$s_j = s_l = -\frac{1}{2}$ for some $j \neq l$ and $s_i = 0$ for $i \neq j, i \neq l$.

Proposition 10. Suppose that a solution ξ_i, w of (1.13) and solutions $u_1, ..., u_n$ of (1.14) are fixed. Then the hydrodynamic type systems

$$r^i_{t_1} = \frac{S_{n,k}(g_1, \xi_i)}{S_{n,k}(g_3, \xi_i)} r^i_x, \qquad r^i_{t_2} = \frac{S_{n,k}(g_2, \xi_i)}{S_{n,k}(g_3, \xi_i)} r^i_x \tag{5.49}$$

are compatible for all $g_1,\ g_2$.

Proof. Indeed, the metric associated with (5.48) does not depend on g_2. Therefore the systems (5.49) has a common metric depending on g_3 and on solutions of (1.13), (1.14). ■

Remark 8. One can also construct some compatible systems of the form (5.49) using Proposition 5. Set $g_2 = Z(u_1, ..., u_n, u_{n+1})$ in (5.49). Here u_{n+1} is an arbitrary solution of (1.14) distinct from $u_1, ..., u_n$. It is clear that the flows (5.49) are compatible for such g_2 and any $g_1 \in \mathcal{H}$. Moreover, Proposition 5 implies that the flows (5.49) are compatible if we set $g_1 = Z(u_1, ..., u_n, u_{n+1})$, $g_2 = Z(u_1, ..., u_n, u_{n+2})$ for two arbitrary solutions $u_{n+1},\ u_{n+2}$ of (1.14).

All members of the hierarchy constructed in Proposition 10 possess a dispersionless Lax representation (1.2) with common $L(p, r^1, ..., r^N)$. Define a function $L(\xi, r^1, ..., r^N)$ by the following system

$$\partial_i L = \frac{\xi(\xi - 1)\partial_i w\, L_\xi}{\xi - \xi_i}, \qquad i = 1, ..., N. \tag{5.50}$$

Note that the system (5.50) is in involution and therefore the function L is defined uniquely up to inessential transformations $L \to \lambda(L)$. To find the function $L(p, r^1, ..., r^N)$ one has to express ξ in terms of p by (3.21) for $k = 0$ or by (4.29) for $k > 0$.

Proposition 11. Let $u_1, \ldots, u_n$ be arbitrary solution of (1.14). Then system (5.48) admits the dispersionless Lax representation (1.2), where $A = A_{n,k}$ is defined by (3.21) for $k = 0$ and by (4.29) for $k > 0$.

Proof. Substituting $A = A_{n,k}$ defined by (3.21) for $k = 0$ and by (4.29) for $k > 0$ into (1.2) and calculating L_t by virtue of (5.48) we arrive to the expression

$$\partial_i L = \frac{\partial_i P_{n,k}(g_2, \xi) \cdot S_{n,k}(g_1, \xi_i) - \partial_i P_{n,k}(g_1, \xi) \cdot S_{n,k}(g_2, \xi_i)}{P_{n,k}(g_2, \xi)_\xi \cdot S_{n,k}(g_1, \xi_i) - P_{n,k}(g_1, \xi)_\xi \cdot S_{n,k}(g_2, \xi_i)} L_\xi.$$

Taking into account the equation $P_{n,k}(g_i, \xi)_\xi = S_{n,k}(g_i, \xi)(\xi - u_1)^{-s_1-1} ... (\xi - u_{n-k+1})^{-s_{n-k+1}-1}(\xi - u_{n-k+2})^{-s_{n-k+2}} ... (\xi - u_n)^{-s_n} \xi^{-s_{n+1}-1}(\xi - 1)^{-s_{n+2}-1}$ and writing down $P_{n,k}(g_i, \xi)_{u_m}$ in terms of $P_{n,k}(g_1, \xi)_{u_{n-k+1}}, ..., P_{n,k}(g_1, \xi)_{u_n}$ by (4.27), (4.28), one can readily check this equation. ■

Let the function $\xi(L, r^1, ..., r^N)$ be inverse to $L(\xi, r^1, ..., r^N)$. It is easy to check that $u = \xi(L, r^1, ..., r^N)$, where L plays a role of arbitrary parameter, satisfies (1.14).

As usual, the Lax representation defines conserved densities, common for the whole hierarchy, by formula (1.3). Since our definition of $A_{n,k}$ is parametric, we can reformulate this fact as

Proposition 12. Suppose (5.48) is defined by solutions $u_1, \ldots, u_n$ of the system (1.14). Let U be any solution of (1.14). Then

$$\frac{\partial}{\partial t} P_{n,k}\Big(g_2(u_1, u_1, \ldots, u_n), U\Big) = \frac{\partial}{\partial x} P_{n,k}\Big(g_1(u_1, u_1, \ldots, u_n), U\Big)$$

is a conservation law for (5.48).

Since the generic solution U depends on a parameter, we have constructed an one-parametric family of common conservation laws for our hierarchy (5.48) of hydrodynamic type systems.

6 Hydrodynamic reductions and integrability

In this section we show that integrable (1+1)-dimensional systems constructed in Section 5 define hydrodynamic reductions for pseudopotentials and 3-dimensional systems from Sections 3 and 4.

Following [17, 15, 7], we give a definition of integrability for equations (1.2), (1.4) and (1.5) in terms of hydrodynamic reductions.

Suppose there exists a pair of compatible semi-Hamiltonian hydrodynamic-type systems of the form

$$r^i_{t_1} = v^i_1(r^1, ..., r^N) r^i_x, \qquad r^i_{t_2} = v^i_2(r^1, ..., r^N) r^i_x \tag{6.51}$$

and functions $u_i = u_i(r^1, ..., r^N)$ such that these functions satisfy (1.5) for any solution of (6.51). Then (6.51) is called *a hydrodynamic reduction* for (1.5).

Definition 1 [17]. A system of the form (1.5) is called *integrable* if equation (1.2) possesses sufficiently many hydrodynamic reductions for each $N \in \mathbb{N}$. "Sufficiently many" means that the set of hydrodynamic reductions can be locally parameterized by $2N$ functions of one variable. Note that due to gauge transformations $r^i \to \lambda_i(r^i)$ we have N essential functional parameters for hydrodynamic reductions.

Suppose there exists a semi-Hamiltonian hydrodynamic-type system (1.12) and functions $u_i = u_i(r^1, ..., r^N)$, $L = L(p, r^1, ..., r^N)$ such that these functions satisfy dispersionless Lax equation (1.2) for any solution $r^1(x,t), ..., r^N(x,t)$ of the system (1.12). Then (1.12) is called *a hydrodynamic reduction* for (1.2).

Definition 2 [7]. A dispersionless Lax equation (1.2) is called *integrable* if equation (1.2) possesses sufficiently many hydrodynamic reductions for each $N \in \mathbb{N}$.

We also call the corresponding pseudopotential $A(p, u_1, ..., u_n)$ integrable.

Example 6. Let us show that $A = \ln(p - u)$ is integrable. Let $w(r^1, ..., r^N)$, $p_i(r^1, ..., r^N)$, $i = 1, ..., N$ be an arbitrary solution of the following system (the so-called Gibbons-Tsarev system [14])

$$\partial_j \xi_i = \frac{\partial_j w}{\xi_j - \xi_i}, \qquad \partial_{ij} w = \frac{2\partial_i w \partial_j w}{(\xi_i - \xi_j)^2}, \qquad i, j = 1, ..., N, \quad i \neq j. \tag{6.52}$$

It is easy to verify that this system is in involution and therefore its general solution admits a local parameterizations by $2N$ functions of one variable. Define a function $L(p, r^1, ..., r^N)$ by the following system

$$\partial_i L = \frac{\partial_i w \, L_p}{p - \xi_i}, \qquad i = 1, ..., N. \tag{6.53}$$

This system is in involution and therefore defines the function L uniquely up to inessential transformations $L \to \lambda(L)$. Finally, let $u(r^1, ..., r^N)$ be a solution of the system

$$\partial_i u = \frac{\partial_i w}{\xi_i - u}, \quad i = 1, ..., N. \tag{6.54}$$

It is easy to check that the system (6.54) is in involution. It follows from (6.52), (6.53), (6.54) that the system

$$r^i_t = \frac{1}{\xi_i - u} r^i_x \tag{6.55}$$

is a hydrodynamic reduction of equation (1.2) with $A = \ln(p - u)$.

Remark 9. The standard form for the Gibbons-Tsarev system [15] related to hydrodynamic reductions is given by

$$\partial_i \xi_j = F(\xi_i, \xi_j, u_1, \dots, u_n) \, \partial_i u_n, \qquad \partial_i \partial_j u_n = H(\xi_i, \xi_j, u_1, \dots, u_n) \, \partial_i u_n \partial_j u_n, \qquad i \neq j$$

$$\partial_i u_l = G_l(\xi_i, u_1, \dots, u_n)\, \partial_i u_n, \qquad l < n.$$

Here $i, j = 1, ..., N$, $u_l(r^1, \dots, r^N)$ are the functions, which define the reduction, and $\xi_i(r^1, \dots, r^N)$ are some auxiliary functions. To bring (6.52), (6.54) to this form, one has to eliminate the additional unknown w. The result is given by

$$\partial_j \xi_i = \frac{\xi_i - u}{\xi_j - \xi_i} \partial_j u, \qquad \partial_i \partial_j u = \frac{\xi_i + \xi_j - 2u}{(\xi_i - \xi_j)^2}\, \partial_i u \partial_j u. \tag{6.56}$$

In this case $n = 1, u_1 = u$. There is the following generalization of (6.56) to the case of arbitrary polynomial $P(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and arbitrary n:

$$\frac{u_1 - \xi_i}{P(u_1)} \partial_i u_1 = ... = \frac{u_n - \xi_i}{P(u_n)} \partial_i u_n, \quad i = 1, ..., N,$$

$$\partial_{ij} u_n = \frac{K_2(\xi_i, \xi_j) u_n^2 + K_1(\xi_i, \xi_j) u_n + K_0(\xi_i, \xi_j)}{P(u_n)(\xi_i - \xi_j)^2}\, \partial_i u_n \partial_j u_n, \tag{6.57}$$

$$\partial_i \xi_j = \frac{P(\xi_j)(u_n - \xi_i)}{P(u_n)(\xi_i - \xi_j)} \partial_i u_n, \quad i, j = 1, ..., N, \quad i \neq j,$$

where

$$K_2(\xi_i, \xi_j) = 2a_3(\xi_i - \xi_j)^2,$$

$$K_1(\xi_i, \xi_j) = -a_3(\xi_i^2 \xi_j + \xi_i \xi_j^2) + a_2(\xi_i^2 + \xi_j^2 - 4\xi_i \xi_j) - a_1(\xi_i + \xi_j) - 2a_0,$$

$$K_0(\xi_i, \xi_j) = 2a_3 \xi_i^2 \xi_j^2 + a_2(\xi_i^2 \xi_j + \xi_i \xi_j^2) + a_1(\xi_i^2 + \xi_j^2) + a_0(\xi_i + \xi_j).$$

Using transformations of the form $u_i \to \dfrac{a u_i + b}{c u_i + d}$, $\xi_i \to \dfrac{a \xi_i + b}{c \xi_i + d}$, one can put the polynomial P to one of the canonical forms: $P(x) = x(x-1)$, $P(x) = x$, or $P(x) = 1$. If $P(x) = 1$, then (6.57) with $n = 1$ coincides with (6.56). Formulas (1.13), (1.14) are equivalent to (6.57), where $P(x) = x(x-1)$.

Definition 3. Two integrable pseudopotentials A_1, A_2 are called *compatible* if the system

$$L_{t_1} = \{L, A_1\}, \qquad L_{t_2} = \{L, A_2\}$$

possesses sufficiently many hydrodynamic reductions (6.51) for each $N \in \mathbb{N}$.

If A_1, A_2 are compatible, then $A = c_1 A_1 + c_2 A_2$ is integrable for any constants c_1, c_2. Indeed, the system

$$r_t^i = (c_1 v_1^i(\mathbf{r}) + c_2 v_2^i(\mathbf{r}))\, r_x^i$$

is a hydrodynamic reduction of (1.2).

Example 7. The functions $A_1 = \ln(p - u_1)$ and $A_2 = \ln(p - u_2)$ are compatible. Moreover, $A = c_1 \ln(p - u_1) + ... + c_n \ln(p - u_n)$ is integrable for any constants $c_1, ..., c_n$. Indeed, let w, p_i satisfy (6.52) and u_1, u_2 be two different solutions of (6.54). It is easy to check that the corresponding flows are compatible by virtue of (6.52), (6.53), (6.54).

Definition 4. By 3-dimensional system associated with compatible functions A_1, A_2 we mean the system of the form (1.5) equivalent to compatibility conditions for the system

$$\psi_{t_2} = A_1(\psi_{t_1}, u_1, ..., u_n), \qquad \psi_{t_3} = A_2(\psi_{t_1}, u_1, ..., u_n). \tag{6.58}$$

It is clear that any system associated with a pair of compatible functions possesses sufficiently many hydrodynamic reductions and therefore it is integrable in the sense of Definition 1.

Example 8. Let $A_1 = \ln(p-u)$ and $A_2 = \ln(p-v)$. The associated 3-dimensional system has the form

$$u_{t_3} = v_{t_2}, \quad v_{t_1} - vu_{t_3} = u_{t_1} - uv_{t_2}.$$

The following statement is a reformulation of Proposition 11.

Theorem 4. The system (5.48) is a hydrodynamic reduction of the pseudopotential $A_{n,k}$ defined by (3.21) if $k = 0$ and by (4.29) if $k > 0$. Recall that we use the notation $S_n \equiv S_{n,0}$, $A_n \equiv A_{n,0}$, $P_n \equiv P_{n,0}$.

Proposition 13. Suppose $g_1, g_2, g_3, h_1, ..., h_k \in \mathcal{H}$ are linearly independent. Define pseudopotentials A_1, A_2 by

$$A_1 = P_{n,k}(g_1, \xi), \qquad A_2 = P_{n,k}(g_2, \xi), \qquad p = P_{n,k}(g_3, \xi).$$

Then A_1 and A_2 are compatible.

Proof. Note that the system (1.13), (5.50) does not depend on g_1, g_2, g_3 and therefore we have a family of functions L, ξ_i, u_i which give hydrodynamic reduction of the form (5.48) for both A_1 and A_2. Moreover, according to Proposition 10 the flows

$$r^i_{t_1} = \frac{S_{n,k}(g_1, \xi_i)}{S_{n,k}(g_3, \xi_i)} r^i_x, \qquad r^i_{t_2} = \frac{S_{n,k}(g_2, \xi_i)}{S_{n,k}(g_3, \xi_i)} r^i_x$$

are compatible. ■

Remark 10. This result implies that 3-dimensional hydrodynamic type systems constructed in Sections 4, 5 possess sufficiently many hydrodynamic reductions.

Remark 11. Using proposition 5, one can construct compatible pseudopotentials depending on different number of u_i. Indeed, let $g_1, g_3, h_1, ..., h_k \in \mathcal{H}$ and $g_2 = Z(u_1, ..., u_n, u_{n+1})$. Then A_2 depends on $u_1, ..., u_n, u_{n+1}$ and A_1 depends on $u_1, ..., u_n$ only.

7 Conclusion

All known integrable pseudopotentials $A(p, u_1, ..., u_n)$ satisfy the property

$$P\left(\frac{A_{ppp}}{A_{pp}^2}, A_p\right) = 0,$$

where $P(x, y)$ is a polynomial in x, y with coefficients depending on $u_1, ..., u_n$. In this sense any pseudopotential A is associated with the algebraic curve $\mathcal{E} = \{(x, y) \in \mathbb{C}^2;\ P(x, y) = 0\}$. Moreover, compatible pseudopotentials are associated to isomorphic curves. If a 3-dimensional dispersionless system is constructed by two compatible pseudopotentials, then this curve is isomorphic to the so-called spectral curve (see [17]) of the system. In this paper we have constructed a wide class of integrable pseudopotentials associated with rational curves. We believe that all pseudopotentials associated with rational curves can be obtained as a limit from our pseudopotentials. We are going to describe all such limits in a separate paper.

It is known [2] that pseudopotentials associated with curves of higher genus also exist. It is likely that one can describe all pseudopotentials associated with the elliptic curve in a similar manner to the way we have done the rational case in this paper. We are going to consider this problem in the next paper.

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