Cohomology of classical algebraic groups from the functorial viewpoint

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April 18, 2011

Abstract

We prove that extension groups in strict polynomial functor categories compute the rational cohomology of classical algebraic groups. This result was previously known only for general linear groups. We give several applications to the study of classical algebraic groups, such as a cohomological stabilization property, the injectivity of external cup products, and the existence of Hopf algebra structures on the (stable) cohomology of a classical algebraic group with coefficients in a Hopf algebra. Our result also opens the way to new explicit cohomology computations. We give an example inspired by recent computations of Djament and Vespa.

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1 Introduction

Over the past fifteen years, the relations between functor categories and the cohomology of the algebraic general linear group GL_n have been successfully used to prove cohomological finite generation conjectures [10, 20], and they have also proved very useful to perform explicit cohomology computations [9, 3, 8]. The first purpose of this paper is to extend these relations to other classical algebraic groups. More specifically, we prove that if G is a symplectic group, an orthogonal group, a general linear group, or more generally a finite product of these groups, then Ext-groups in a suitable functor category compute the cohomology of G. The second purpose of this paper is to illustrate some advantages of the functorial point of view. In particular, we obtain new cohomological results for classical algebraic groups, whose proofs do not seem to belong to the usual algebraic group setting.

The cohomology we treat here is the cohomology of algebraic groups of [12], which was introduced by Hochschild (it is often called 'rational cohomology' to emphasize that it arises from rational representations). The functors which play a role in the algebraic group setting are the 'strict polynomial functors' of Friedlander and Suslin [10], and their multivariable analogues. Our results are the algebraic counterpart of recent results of Djament and Vespa [7] about the finite groups $O_{n,n}(\mathbb{F}_q)$, $Sp_n(\mathbb{F}_q)$. However, the methods required for algebraic groups are very different from those needed for finite groups. The cohomological stabilization property illustrates this difference vividly: in the algebraic group setting, it is an immediate consequence of the link between extension groups in functor categories and cohomology of algebraic groups, while in the finite group setting these two results are independent.

What follows is a synopsis of the results of the paper.

Relating functor categories to the cohomology of classical groups

In section 3, we establish the link between Ext-groups in strict polynomial functor categories and rational cohomology of general linear, orthogonal and symplectic groups. For example, we prove:

Theorem (3.17, the symplectic case)**.** *Let $\Bbbk$ be a commutative ring, and let n be a positive integer. For any $F \in \mathcal{P}$ we have a $*$-graded map, natural in F:*

$$\phi_{Sp_n,F} : \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\Lambda^2), F) \to H^*_{\mathrm{rat}}(Sp_n, F_n) \; .$$

The map $\phi_{Sp_n,F}$ is compatible with cup products:

$$\phi_{Sp_n,F\otimes F'}(x \cup y) = \phi_{Sp_n,F}(x) \cup \phi_{Sp_n,F'}(y) \; .$$

Moreover, $\phi_{Sp_n,F}$ is an isomorphism whenever $2n \geq \deg(F)$.

Here '$\mathcal{P}$' refers to the category of strict polynomial functors of Friedlander and Suslin. So if $\mathcal{V}_\Bbbk$ is the category of finitely generated projective $\Bbbk$-modules, objects of $\mathcal{P}$ are functors $F : \mathcal{V}_\Bbbk \to \mathcal{V}_\Bbbk$ with an additional 'polynomial structure' which ensures that the image $F(V)$ of a rational G-module V is a rational representation of the algebraic group G. The rational Sp_n-module F_n is obtained by evaluating F on the dual of the standard representation $\Bbbk^{2n}$ of Sp_n. The cup product on the left comes from the usual coalgebra structure on the divided powers $\Gamma^\star(\Lambda^2)$.

Our method is based on classical invariant theory [6]. The proofs for orthogonal, symplectic and general linear groups are analogous. For the orthogonal and symplectic groups the results are new. In the general linear case, we obtain a new treatment (and a generalization over a commutative ring $\Bbbk$) of previously known results: [10, Cor 3.13], [8, Thm 1.5] and [19, Thm 1.3].

In section 4, we use Künneth formulas to extend these results when G_n is a finite product of general linear, orthogonal and/or symplectic groups. In that case, one has to consider the category $\mathcal{P}_G$ of strict polynomial functors F 'adapted to G_n', that is with a number of variables taking into account the number of factors in the product G_n. Evaluation of F on specific representations of the factors of G_n yield a rational G_n-module F_n and we have:

Theorem (4.5)**.** *Let $\Bbbk$ be a commutative ring, let n be a positive integer and let G_n be a finite product of the algebraic groups (over $\Bbbk$) GL_n, Sp_n and $O_{n,n}$. For any $F \in \mathcal{P}_G$ we have a $*$-graded map, natural in F, which is compatible with cup products:*

$$\phi_{G_n,F} : \mathrm{Ext}^*_{\mathcal{P}_G}(\Gamma^\star(F_G), F) \to H^*_{\mathrm{rat}}(G_n, F_n)$$

Assume that $2n$ is greater or equal to the degree of F. If one of the factors of G_n equals $O_{n,n}$, assume furthermore that 2 is invertible in $\Bbbk$. Then $\phi_{G_n,F}$ is an isomorphism.

Some applications of the functorial viewpoint in algebraic group cohomology

As a first application, we deduce from theorem 4.5 a cohomological stabilization property.

Corollary (4.6)**.** *Let $\Bbbk$ be a commutative ring, let n be a positive integer and let G_n be a finite product of copies of GL_n, Sp_n or $O_{n,n}$. Let $F \in \mathcal{P}_G$ be a degree d functor adapted to G_n. Let n, m be two positive integers such that $2m \geq 2n \geq d$. If the orthogonal group appears as one of the factors of G_n, assume furthermore that 2 is invertible in $\Bbbk$. Then we have an isomorphism*

$$\phi_{n,m} : H^*_{\mathrm{rat}}(G_m, F_m) \xrightarrow{\simeq} H^*_{\mathrm{rat}}(G_n, F_n) \ .$$

We shall denote by $H^*_{\mathrm{rat}}(G_\infty, F_\infty)$ the stable value of $H^*_{\mathrm{rat}}(G_n, F_n)$ (though this stable value is obtained for relatively small values of n).

As a second application we obtain a striking injectivity property for cup products. In general, if G is an algebraic group and if $c \in H^*_{\mathrm{rat}}(G, M)$ and $c' \in H^*_{\mathrm{rat}}(G, N)$ are nontrivial cohomology classes, their (external) cup product $c \cup c' \in H^*_{\mathrm{rat}}(G, M \otimes N)$ may very well be zero. For example, if $\Bbbk$ is a field of odd characteristic and G_a is the additive group, then the cohomology algebra $H^*_{\mathrm{rat}}(G_a, \Bbbk)$ is [5] a free commutative graded algebra with generators $(x_i)_{i\geq 0}$ of degree 2 and generators $(\lambda_i)_{i\geq 0}$ of degree one. Since the multiplication $\Bbbk \otimes \Bbbk \to \Bbbk$ is an isomorphism, it is not hard to build pairs of non trivial classes (α, β) whose external cup product $\alpha \cup \beta$ is zero. This cancellation phenomenon does not occur in (stable) cohomology of classical groups over a field.

Corollary (6.2)**.** *Let $\Bbbk$ be a field. Let G_n be a product of copies of the groups GL_n, Sp_n or $O_{n,n}$, and let F_1, F_2 be two functors of degree d_1, d_2 adapted to G_n. If $O_{n,n}$ is a factor in G_n, assume that $\Bbbk$ has odd characteristic. For all n such that $2n \geq d_1 + d_2$, the cup product induces a injection:*

$$H^*_{\mathrm{rat}}(G_n, (F_1)_n) \otimes H^*_{\mathrm{rat}}(G_n, (F_2)_n) \hookrightarrow H^*_{\mathrm{rat}}(G_n, (F_1)_n \otimes (F_2)_n) \ .$$

This results partially explains some non-vanishing phenomena, like [20, Lemma 4.13]. It follows from a more general result, namely the existence of external coproducts in the stable cohomology of classical groups.

Theorem (6.1)**.** *Let $\Bbbk$ be a field. Let G_n be a product of copies of the groups GL_n, Sp_n or $O_{n,n}$, and let F_1, F_2 be strict polynomial functors adapted to G_n. If $O_{n,n}$ is a factor in G_n, assume that $\Bbbk$ has odd characteristic. The stable rational cohomology of G_n is equipped with a coproduct:*

$$H^*_{\mathrm{rat}}(G_\infty, (F_1 \otimes F_2)_\infty) \to H^*_{\mathrm{rat}}(G_\infty, F_{1\,\infty}) \otimes H^*_{\mathrm{rat}}(G_\infty, F_{2\,\infty}) \ .$$

Together with the usual cup product (cf. §2.4), they endow $H^*_{\mathrm{rat}}(G_\infty, -)$ with the structure of a graded Hopf monoidal functor (cf. definition 5.2).

Moreover, the cup product is a section of the coproduct.

The construction of the external coproduct uses the sum-diagonal adjunction, a feature which is specific to functor categories. Some hints that such coproducts exist were given in [9], where the authors built Hopf algebra structures on some specific extension groups in functor categories (when all the functors in play are 'Hopf exponential functors'). We build the external coproducts in section 5, where we make a more general attempt to classify the Hopf monoidal structures that may arise for extension groups in functor categories.

As a consequence of theorem 6.1, we also obtain Hopf algebra structures (without antipode) on rational cohomology of classical groups (compare [9, lemma 1.11]):

Corollary (6.4)**.** *Let $\Bbbk$ be a field. Let G_n be a product of copies of the groups GL_n, Sp_n or $O_{n,n}$, and let A^* be an n-graded strict polynomial functor adapted to G_n, endowed with the structure of a Hopf algebra. If $O_{n,n}$ is a factor in G_n, assume that $\Bbbk$ has odd characteristic. The usual cup product $H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)^{\otimes 2} \to H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)$ may be supplemented with a coproduct $H^*_{\mathrm{rat}}(G_\infty, A^*_\infty) \to H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)^{\otimes 2}$ which endow $H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)$ with the structure of a $(1+n)$-graded Hopf algebra.*

Such Hopf algebra structures offer a nice framework in which we can reformulate some previously known cohomological computations, such as the existence of the universal classes of [20, Thm 4.1], cf. corollary 6.5.

Finally, Ext-computations in strict polynomial functor categories is a classical subject. Many results and computational techniques are already available. So by expressing rational cohomology of orthogonal and symplectic groups as extension in $\mathcal{P}$, we open the way to new cohomology computations. To illustrate this fact, we give one example, which may be proved by the method of Djament and Vespa [7, §4.2] and the computations of [9]:

Theorem (6.6)**.** *Let $\Bbbk$ be a field of odd characteristic. Let r be a nonnegative integer. Let $S^*(I^{(r)})$ denote the symmetric algebra over the r-th Frobenius twist (with $S^d(I^{(r)})$ placed in degree $2d$) and let $\Lambda^*(I^{(r)})$ denote the exterior powers of the r-th Frobenius twist (with $\Lambda^d(I^{(r)})$ placed in degree d).*

(i) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(O_{\infty,\infty}, S^\star(I^{(r)})_\infty)$ is a symmetric Hopf algebra on generators e_m of bidegree $(2m, 4)$ for $0 \le m < p^r$.

(ii) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(Sp_\infty, S^\star(I^{(r)})_\infty)$ is trivial.

(iii) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(O_{\infty,\infty}, \Lambda^\star(I^{(r)})_\infty)$ is trivial.

(iv) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(Sp_\infty, \Lambda^\star(I^{(r)})_\infty)$ is a divided power Hopf algebra on generators e_m of bidegree $(2m, 2)$ for $0 \le m < p^r$.

2 Review of functor categories and group cohomology

2.1 Notations

If $\Bbbk$ is a commutative ring, we denote by $\mathcal{V}_\Bbbk$ the category of finitely generated projective $\Bbbk$-modules. The symbol '${}^\vee$' means $\Bbbk$-linear duality: $V^\vee := \mathrm{Hom}_\Bbbk(V, \Bbbk)$.

Let $V \in \mathcal{V}_\Bbbk$. For all $d \geq 0$, we denote by $\Gamma^d(V)$ the d-th divided power of V, that is the invariants $(V^{\otimes d})^{\mathfrak{S}_d}$ where $\mathfrak{S}_d$ acts by permuting the factors of the tensor product (for $d = 0$, we let $\Gamma^0(V) = \Bbbk$). We also denote by $S^d(V)$, resp. $\Lambda^d(V)$ the d-th symmetric, resp. exterior, power of V. Let $A^* = S^*, \Lambda^*$ or Γ^*. Then A^* satisfies an 'exponential isomorphism' natural in V, W and associative in the obvious sense: $A^*(V \oplus W) \simeq A^*(V) \otimes A^*(W)$. Let δ_2 be the diagonal $V \to V \oplus V$, $x \mapsto (x, x)$, and let Σ_2 be the sum $V \oplus V \to V$, $(x, y) \mapsto x + y$. 'The' graded Hopf algebra structure on the divided powers $\Gamma^*(V)$ (without further specification) means the following. We consider $\Gamma^d(V)$ in degree $2d$, the unit is $\Bbbk = \Gamma^0(V) \hookrightarrow \Gamma^*(V)$, the counit is $\Gamma^*(V) \twoheadrightarrow \Gamma^0(V) = \Bbbk$, the multiplication and the comultiplication are:

$$\Gamma^*(V)^{\otimes 2} \simeq \Gamma^*(V \oplus V) \xrightarrow{\Gamma^*(\Sigma_2)} \Gamma^*(V)\ ,\ \Gamma^*(V) \xrightarrow{\Gamma^*(\delta_2)} \Gamma^*(V \oplus V) \simeq \Gamma^*(V)^{\otimes 2}\ .$$

2.2 Strict polynomial functors

Let $\Bbbk$ be a commutative ring and let $\mathcal{A}$ be a finite product of the categories $\mathcal{V}_\Bbbk$ and $\mathcal{V}_\Bbbk^{\mathrm{op}}$ (the 'op' stands for the opposite category). We recall here the basic definitions and properties of the category of strict polynomial functors from $\mathcal{A}$ to $\mathcal{V}_\Bbbk$. The case $\mathcal{A} = \mathcal{V}_\Bbbk$ was introduced in [10] over a field and in [18] over an arbitrary commutative ring, the case $\mathcal{A} = \mathcal{V}_\Bbbk^{\mathrm{op}} \times \mathcal{V}_\Bbbk$ corresponds to the category strict polynomial bifunctors, contravariant in the first variable and covariant in the second one, used in [8]. The definitions and the proofs generalize immediately when $\mathcal{A}$ is a more general product.

Basic definitions

A strict polynomial functor F from $\mathcal{A}$ to $\mathcal{V}_\Bbbk$ is the following collection of data: for each $X \in \mathcal{A}$, an element $F(X) \in \mathcal{V}_\Bbbk$ and for each X, Y in $\mathcal{A}$ a polynomial $F_{X,Y} \in S^*(\mathrm{Hom}_\mathcal{A}(X,Y)^\vee) \otimes \mathrm{Hom}_\Bbbk(F(X), F(Y))$. These polynomials must satisfy two conditions: (1) $F_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$, and (2) the *polynomials* $(f, g) \mapsto F_{X,Y}(f) \circ F_{Y,Z}(g)$ and $(f, g) \mapsto F_{X,Z}(f \circ g)$ are equal. Natural transformations between strict polynomial functors F, G are linear maps $\phi_X : F(X) \to G(X)$ such that the *polynomials* $f \mapsto G_{X,Y}(f) \circ \phi_X$ and $f \mapsto \phi_Y \circ F_{X,Y}(f)$ are equal. Examples of strict polynomial functors are $\mathrm{Hom}_\mathcal{A}(X, -)$, the divided powers $\Gamma^d(\mathrm{Hom}_\mathcal{A}(X, -))$ or the symmetric powers $S^d(\mathrm{Hom}_\mathcal{A}(X, -))$. If G is an affine algebraic group acting rationally on a

$\Bbbk$-module V and if $F : \mathcal{V}_\Bbbk \to \mathcal{V}_\Bbbk$ is a strict polynomial functor, $F(V)$ is a rational G-module ($g \in G$ acts on $F(V)$ by $v \mapsto F(g)(v)$). More generally:

Lemma 2.1. *Assume $\mathcal{A} = (\mathcal{V}_\Bbbk^{\mathrm{op}})^{\times k} \times (\mathcal{V}_\Bbbk)^{\times \ell}$. Let $(G_i)_{1\le i\le k+\ell}$ be algebraic groups over $\Bbbk$, let $(V_i)_{1\le i\le k}$ be right G_i-modules and $(V_i)_{k+1\le i\le k+\ell}$ be left G_i-modules. Evaluation on $(V_1,\dots,V_n)$ yields a functor from the category of strict polynomial functors with source $\mathcal{A}$ to the category of rational $\prod G_i$-modules.*

A strict polynomial functor F is homogeneous of degree d if all the polynomials $F_{X,Y}$ are homogeneous of degree d. It is of finite degree if the family of the degrees of the $F_{X,Y}$ is bounded. We denote by $\mathcal{P}_\mathcal{A}$ the category of strict polynomial functors of finite degree with source $\mathcal{A}$. Then the category $\mathcal{P}_\mathcal{A}$ splits as the direct sum of its full subcategories $\mathcal{P}_{d,\mathcal{A}}$ of homogeneous functors of degree d:

$$\mathcal{P}_\mathcal{A} = \textstyle\bigoplus_{d\ge 0} \mathcal{P}_{d,\mathcal{A}}\ .$$

There is an equivalence of categories $\mathcal{P}_{0,\mathcal{A}} \simeq \mathcal{V}_\Bbbk$ induced by $F \mapsto F(0,\dots,0)$.

Remark 2.2. If $\mathcal{A} = (\mathcal{V}_\Bbbk^{\mathrm{op}})^{\times k} \times (\mathcal{V}_\Bbbk)^{\times \ell}$, we could refine the splitting by introducing multidegrees. Then the category $\mathcal{P}_{d,\mathcal{A}}$ would split as the direct sum of its full subcategories of homogeneous functors of multidegree $(d_1,\dots,d_{k+\ell})$, with $\sum d_i = d$. For sake of simplicity, we don't use multidegrees. Thus the term 'degree' always refers to the total degree of the functors.

Another presentation of strict polynomial functors

We have defined strict polynomial functors as functors from $\mathcal{A}$ to $\mathcal{V}_\Bbbk$ endowed with an additional structure (polynomials). Equivalently, one can define degree d homogeneous strict polynomial functors as $\Bbbk$-linear functors from a $\Bbbk$-linear category $\Gamma^d\mathcal{A}$ to $\mathcal{V}_\Bbbk$ (cf. [16] where T. Pirashvili credits Bousfield for this presentation). In this presentation, the polynomial structure is encoded in the source category $\Gamma^d\mathcal{A}$, and strict polynomial functors are genuine $\Bbbk$-linear functors, which may make some statements clearer.

We recall the definition of $\Gamma^d\mathcal{A}$. Let $d \ge 0$, and let $\mathcal{A}$ be a finite product of copies of $\mathcal{V}_k$ or its opposite category. The objects of $\Gamma^d\mathcal{A}$ are the same as the objects of $\mathcal{A}$, and the sets of morphisms are the $\Bbbk$-modules $\mathrm{Hom}_{\Gamma^d\mathcal{A}}(X,Y) := \Gamma^d(\mathrm{Hom}_\mathcal{A}(X,Y))$. The identity of X equals $\mathrm{Id}_X^{\otimes d}$. Let's define the composition. If $U, V \in \mathcal{V}_\Bbbk$, the group $\mathfrak{S}_d \times \mathfrak{S}_d$ acts by permuting the factors of the tensor product $U^{\otimes d} \otimes V^{\otimes d}$. The diagonal inclusion $\mathfrak{S}_d \simeq \Delta\mathfrak{S}_d \subset \mathfrak{S}_d \times \mathfrak{S}_d$ induces a morphism $j_d : \Gamma^d(U)\otimes\Gamma^d(V) \to \Gamma^d(U\otimes V)$. The composition in $\Gamma^d\mathcal{A}$ is defined as the composite:

$$\begin{aligned}\Gamma^d(\mathrm{Hom}_\mathcal{A}(X,Y)) \otimes \Gamma^d(\mathrm{Hom}_\mathcal{A}(Y,Z)) &\xrightarrow{j_d} \Gamma^d(\mathrm{Hom}_\mathcal{A}(X,Y) \otimes \mathrm{Hom}_\mathcal{A}(Y,Z)) \\ &\to \Gamma^d(\mathrm{Hom}_\mathcal{A}(X,Z))\ ,\end{aligned}$$

where the last map is induced by the composition in $\mathcal{A}$.

The following key lemma (compare [10, Lemma 2.8 and proof of Prop. 2.9]) induces the existence of projective resolutions, and will also have an important role in our computations.

Lemma 2.3 (key lemma)**.** *Let $d \geq 0$. Let $Y = (Y_i) \in \mathcal{A}$ be a tuple of free $\Bbbk$-modules, such that each Y_i has rank greater or equal to d. Then for all $X, Z \in \mathcal{A}$ the composition in $\Gamma^d\mathcal{A}$ induces an epimorphism:*

$$\Gamma^d(\mathrm{Hom}_{\mathcal{A}}(X,Y)) \otimes \Gamma^d(\mathrm{Hom}_{\mathcal{A}}(Y,Z)) \twoheadrightarrow \Gamma^d(\mathrm{Hom}_{\mathcal{A}}(X,Z))$$

Proof. Using the exponential isomorphism for the divided power algebra, one reduces to the case where $\mathcal{A} = \mathcal{V}_\Bbbk$. By naturality, one reduces furthermore to the case where X, Y are free $\Bbbk$-modules.

If $I = (d_1, \dots, d_n)$ is a tuple of positive integers such that $\sum d_i = d$, we denote by $\mathfrak{S}_I$ the subgroup $\prod \mathfrak{S}_{d_i} \subset \mathfrak{S}_d$. If V is a free $\Bbbk$-module with basis (b_i), and if $b_{i_1}, \dots, b_{i_n}$ are distinct elements of the basis we let:

$$(b_{i_1}, \dots, b_{i_n}, I) := \sum_{\sigma \in \mathfrak{S}_d/\mathfrak{S}_I} \sigma.(\underbrace{b_{i_1} \otimes \cdots \otimes b_{i_1}}_{d_1 \text{ factors}} \otimes \cdots \otimes \underbrace{b_{i_n} \otimes \cdots \otimes b_{i_n}}_{d_n \text{ factors}}) \ .$$

Such elements form a basis of $(V^{\otimes d})^{\mathfrak{S}_d}$. Now we may choose basis $(e^{Y,X}(j,i))$, $(e^{Z,Y}(k,j))$ and $(e^{Z,X}(k,i))$ of $\mathrm{Hom}_\Bbbk(X,Y)$, $\mathrm{Hom}_\Bbbk(Y,Z)$ and $\mathrm{Hom}_\Bbbk(X,Z)$ respectively, such that $e^{Z,Y}(k,j_1) \circ e^{Y,X}(j_2,i) = e^{Z,X}(k,i)$ if $j_1 = j_2$, and 0 in the other cases.

To prove surjectivity, it suffices to show that for all tuple $I = (d_1, \dots, d_n)$ and all n-tuple of distinct elements $(e^{Z,X}(k_s,i_s))_{1 \leq s \leq n}$, the map induced by the composition hits $(e^{Z,X}(k_1,i_1), \dots, e^{Z,X}(k_n,i_n), I) \in (\mathrm{Hom}_\Bbbk(X,Z)^{\otimes d})^{\mathfrak{S}_d}$. To do this, we use that $\mathrm{rk} Y \geq d \geq n$. Thus we may choose *distinct* indices $j_1, \dots, j_n$ and form the element:

$$(e^{Y,X}(j_1,i_1), \dots, e^{Y,X}(j_n,i_n), I) \otimes (e^{Z,Y}(k_1,j_1), \dots, e^{Z,Y}(k_n,j_n), I) \ .$$

The map induced by the composition in $\Gamma^d\mathcal{V}_\Bbbk$ sends this element to $(e^{Z,X}(k_1,i_1), \dots, e^{Z,X}(k_n,i_n), I)$ and we are done. $\square$

Homological algebra

Kernels, cokernels, products or sums of strict polynomial functors are computed in the target category, so that categories of strict polynomial functor inherit the structure of $\mathcal{V}_\Bbbk$. Thus, if $\Bbbk$ is a field, $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{d,\mathcal{A}}$ are abelian categories. This is no longer the case over an arbitrary commutative ring. Nonetheless, they are exact category in the sense of Quillen [17], with admissible exact sequences being the sequences $0 \to F' \to F \to F'' \to 0$ which are exact after evaluation on every object X. The theory of extensions in exact categories is very similar to the abelian one. One minor change is that Ext-groups are defined in terms of 'admissible' extensions (ie: Yoneda composites of admissible short exact sequences), so that we must use 'admissible' projective or injective resolutions to compute them (See also [2] for a recent exposition).

The standard projectives are the functors: $P^d_X := \Gamma^d(\mathrm{Hom}_{\mathcal{A}}(X,-))$, for all $X \in \mathcal{A}$. They satisfy a Yoneda isomorphism, natural in X, F:

$$\mathrm{Hom}_{P_{d,\mathcal{A}}}(P^d_X, F) \simeq F(X)\ , \quad f \mapsto f_X(\mathrm{Id}_X^{\otimes d})\ .$$

If F is homogeneous of degree d and if $X = (X_1, \dots, X_n) \in \mathcal{A}$ is a tuple of free $\Bbbk$-modules such that each X_i has a rank greater or equal to d, lemma 2.3 implies that the canonical map $F(X) \otimes P^d_X \to F$ is an epimorphism. Since every epimorphism is admissible (ie: they admit a kernel in $\mathcal{P}_{d,\mathcal{A}}$) this shows that F has an admissible projective resolution by finite sums of standard projectives.

If $F \in \mathcal{P}_{d,\mathcal{A}}$, then $F^\vee : V \mapsto F(V)^\vee$ is a degree d homogeneous strict polynomial functor with source the opposite category $\mathcal{A}^{\mathrm{op}}$, and we have a natural isomorphism: $\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(F, G^\vee) \simeq \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^{\mathrm{op}}}}(G, F^\vee)$. By this duality, the functors $I^d_X := S^d(\mathrm{Hom}_{\mathcal{A}}(X,-)) = (\Gamma^d(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(X,-)))^\vee$ are injective. We call them 'standard injectives'. They satisfy a Yoneda isomorphism, natural in F, X:

$$\mathrm{Hom}_{\mathcal{P}_{d,\mathcal{A}}}(F, I^d_X) \simeq F(X)^\vee\ , \quad f \mapsto f^\vee_X(\mathrm{Id}_X^{\otimes d})\ ,$$

and each $F \in \mathcal{P}_{d,\mathcal{A}}$ has an admissible injective resolution by direct sums of standard injectives. In particular the injectives of $\mathcal{P}_{d,\mathcal{A}}$ are direct summands of finite sums of standard injectives and we have:

Lemma 2.4. *Assume $\mathcal{A} = (\mathcal{V}_\Bbbk^{\mathrm{op}})^{\times k} \times (\mathcal{V}_\Bbbk)^{\times \ell}$. Let $d \geq 0$. Then for all tuple $(i_1, \dots, i_{k+\ell})$ of positive integers, the functor*

$$I^d_{i_1,\dots,i_{k+\ell}} : (V_1, \dots, V_{k+\ell}) \mapsto S^d\left(\bigoplus_{s=1}^k (V_s^\vee)^{\oplus i_s} \oplus \bigoplus_{t=k+1}^{k+\ell} V_t^{\oplus i_t}\right)$$

is an injective of $\mathcal{P}_{d,\mathcal{A}}$. Moreover the injectives of $\mathcal{P}_{d,\mathcal{A}}$ are direct summands of finite sums of such functors.

Examples

We finish the presentation by giving ingredients to build examples. First, the tensor product yields a functor $\mathcal{P}_{d,\mathcal{A}} \times \mathcal{P}_{d',\mathcal{A}} \to \mathcal{P}_{d+d',\mathcal{A}}$. Let $\mathcal{P}_d$ be the category of degree d homogeneous strict polynomial functors of with source $\mathcal{V}_\Bbbk$. If $F \in \mathcal{P}_d$ and $G \in \mathcal{P}_{d',\mathcal{A}}$, composition of polynomials endow $X \mapsto F(G(X))$ with the structure of a strict polynomial functor. In that way we obtain a functor $\mathcal{P}_d \times \mathcal{P}_{d',\mathcal{A}} \to \mathcal{P}_{dd',\mathcal{A}}$. We can get numerous new examples by combining these two methods with the following basic examples. The divided powers Γ^d, the symmetric powers S^d, the exterior powers Λ^d

and the tensor products $\otimes^d$ are objects of $\mathcal{P}_d$ (and more generally, so are the Schur functors S_λ associated with a partition λ of weight d). The natural transformations $\otimes^d \to \otimes^d$ induced by permuting the factors are morphisms in $\mathcal{P}_d$, as well as the multiplication $A^{d-i} \otimes A^i \to A^d$ and the comultiplication $A^d \to A^{d-i} \otimes A^i$ if $A^* = S^*, \Gamma^*, \Lambda^*$. Finally, the exponential isomorphisms $A^*(V \oplus W) \simeq A^*(V) \otimes A^*(W)$ are morphisms of $\mathcal{P}_{\mathcal{V}_\Bbbk \times \mathcal{V}_\Bbbk}$.

2.3 Functor cohomology and cup products

Let E^* be an n-graded functor in $\mathcal{P}_\mathcal{A}$. We call 'functor cohomology' the extension groups

$$\mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, -) = \bigoplus_{j,i_1,\dots,i_n} \mathrm{Ext}^j_{\mathcal{P}_\mathcal{A}}(E^{i_1,\dots,i_n}, -) \ .$$

If $F, G \in \mathcal{P}_\mathcal{A}$, we denote by $F \otimes G$ their tensor product $X \mapsto F(X) \otimes G(X)$. This yields a biexact functor: $\mathcal{P}_\mathcal{A} \times \mathcal{P}_\mathcal{A} \to \mathcal{P}_\mathcal{A}$. Moreover if $F \hookrightarrow F_0 \to \cdots \to F_n \twoheadrightarrow E$ and $F' \hookrightarrow F'_0 \to \cdots \to F'_m \twoheadrightarrow E'$ are two admissible extensions, their 'cross product':

$$F \otimes F' \hookrightarrow F_0 \otimes F'_0 \to \cdots \to (F_n \otimes E' \oplus E \otimes F'_m) \twoheadrightarrow E \otimes E'$$

is once again an admissible extension (It is an exact sequence by the Künneth theorem, to prove that it is admissible, one just needs to see that the kernels of its differentials have projective values. To do this, use its exactness and that for all $X \in \mathcal{A}$, $E(X) \otimes E'(X)$ is a projective $\Bbbk$-module). In this way, we obtain an associative cross product in extension groups:

$$\times : \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E, F) \otimes \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E', F') \to \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E \otimes E', F \otimes F') \ .$$

Assume now that E^* has an n-graded coalgebra structure: we have an n-graded coproduct $\Delta_E : E^* \to E^* \otimes E^*$ and an augmentation $\epsilon_E : E^* \to \Bbbk$, where $\Bbbk$ is considered as a functor of degree $(0, \dots, 0)$. Then we may define an external cup product

$$\begin{array}{rccc} \cup : & \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, F) \otimes \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, F') & \to & \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, F \otimes F') \\ & c \otimes c' & \mapsto & \Delta_E^*(c \times c') \end{array} ,$$

and a unit $\Bbbk = \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(\Bbbk, \Bbbk) \xrightarrow{\epsilon_E^*} \mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, \Bbbk)$, which satisfy an associativity and a unit axiom. These axioms may be summarized by saying that $\mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, -)$ is a (multigraded) monoidal functor [15, XI.2].

2.4 Cohomology of algebraic groups and cup products

Let $\Bbbk$ be a commutative ring and let G be a flat algebraic group over $\Bbbk$ (ie: G is a group scheme represented by a $\Bbbk$-flat finitely generated Hopf algebra $\Bbbk[G]$). Then the category of rational G-modules is an abelian category with enough injectives. The rational cohomology of G with coefficients in a G-module M is defined as the extension groups $H^*_{\mathrm{rat}}(G,M) = \mathrm{Ext}^*_{G\text{-mod}}(\Bbbk, M)$ ($\Bbbk$ is the trivial G-module).

These extension groups may be computed [12, 4.14-4.16] as the homology of the Hochschild complex $C^\bullet(G,M)$ with $M\otimes\Bbbk[G]^{\otimes i}$ in degree i. Interpreting $C^i(G,M)$ as the set of functions $G^{\times i}\to M$, the external cup product

$$H^*_{\mathrm{rat}}(G,M)\otimes H^*_{\mathrm{rat}}(G,N)\to H^*_{\mathrm{rat}}(G,M\otimes N)$$

is defined at the chain level by sending $u\in C^r(G,M)$ and $v\in C^s(G,M)$ to

$$(u\cup v)(g_1,\dots,g_{r+s}) := u(g_1,\dots,g_r)\otimes{}^{g_1\cdots g_r}\,v(g_{r+1},\dots,g_{r+s})\;,$$

where ${}^g m$ denotes the image of $m\in M$ under the action of $g\in G$. If $M=N=R$ is an algebra with a rational G-action, then the composite

$$C^\bullet(G,R)\otimes C^\bullet(G,R)\to C^\bullet(G,R\otimes R)\xrightarrow{C^\bullet(G,m_R)} C^\bullet(G,R)$$

is the internal cup product of [20, Section 6.3], which makes $H^*_{\mathrm{rat}}(G,R)$ into a graded algebra.

Another construction of cup products

Now we want to give another construction of external cup products, in terms of cross products of extensions, as we did for functor cohomology. Over a field $\Bbbk$, this is an easy job: (i) the two constructions coincide in degree 0, and (ii) a δ-functor argument [14, XII, proof of thm 10.4] shows that the two constructions coincide in all degrees. Over an arbitrary ring, exactness of tensor products fails, so the cross product of two extensions does not always make sense. We have a weaker statement, proved by *ad hoc* methods.

Lemma 2.5. *Let G be a flat algebraic group over a commutative ring $\Bbbk$ and let M,M' be two $\Bbbk$-flat G-modules. Assume that the classes $c\in H^r(G,M)$ and $c'\in H^s(G,M')$ are represented by extensions $M\hookrightarrow M_0\to\cdots\to M_r\to\Bbbk$ and $M'\hookrightarrow M'_0\to\cdots\to M'_s\to\Bbbk$ whose objects are $\Bbbk$-flat. Then the cross product is an exact sequence:*

$$M\otimes M'\hookrightarrow M_0\otimes M'_0\to\cdots\to(M_r\otimes\Bbbk\oplus\Bbbk\otimes M'_s)\twoheadrightarrow\Bbbk\otimes\Bbbk\;.$$

Its pullback by the diagonal $\Delta:\Bbbk\simeq\Bbbk\otimes\Bbbk$, $1\mapsto 1\otimes 1$ represents the external cup product $c\cup c'\in H^{r+s}_{\mathrm{rat}}(G,M\otimes M')$.

Proof. **Step 1.** Consider the algebra $\Bbbk[G]$ with G acting by left translation. Then $C^\bullet := C^\bullet(G,\Bbbk[G])$ is a differential graded algebra with an action of G [20, Section 6.3]. By [12, Part I, Chap 4, sections 4.14 to 4.16], the complex $C^\bullet$ is *homotopy equivalent* to $\Bbbk$ concentrated in degree 0. Thus, for all G-modules M,M', the multiplication of $C^\bullet$ induces a G-equivariant morphism of acyclic resolutions over $\mathrm{Id}_{M\otimes M'}$: $M\otimes C^\bullet\otimes M'\otimes C^\bullet\to M\otimes M'\otimes C^\bullet$.

Now $(M \otimes C^\bullet)^G = \mathrm{Hom}_G(\Bbbk, M \otimes C^\bullet)$ equals the Hochschild complex $C^\bullet(G, M)$. As a result, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_G(\Bbbk, M \otimes C^\bullet) \otimes \mathrm{Hom}_G(\Bbbk, M' \otimes C^\bullet) \arrow[r, equal] \arrow[d, "f\otimes g \mapsto f \otimes g"] & C^\bullet(G, M) \otimes C^\bullet(G, M') \arrow[d, "\cup"] \\
\mathrm{Hom}_G(\Bbbk \otimes \Bbbk, M \otimes C^\bullet \otimes M' \otimes C^\bullet) \arrow[d, "-\circ\Delta"] & C^\bullet(G, M \otimes N) \arrow[d, equal] \\
\mathrm{Hom}_G(\Bbbk, M \otimes C^\bullet \otimes M' \otimes C^\bullet) \arrow[r] & \mathrm{Hom}_G(\Bbbk, M \otimes M' \otimes C^\bullet)\ .
\end{tikzcd}$$

We deduce that if c and c' are cohomology classes represented by cycles $f \in \mathrm{Hom}_G(\Bbbk, M \otimes C^\bullet)$ and $f' \in \mathrm{Hom}_G(\Bbbk, M' \otimes C^\bullet)$, the cup product $c \cup c'$ is represented by $(f \otimes f') \circ \Delta \in \mathrm{Hom}_G(\Bbbk, M \otimes C^\bullet \otimes M' \otimes C^\bullet)$.

Step 2. Each cycle $f \in \mathrm{Hom}_G(\Bbbk, M \otimes C^i)$ defines an extension $E(f)$: $M \hookrightarrow M \otimes C^0 \to \cdots \to M \otimes C^{i-2} \to N^{i-1} \twoheadrightarrow \Bbbk$, where N^{i-1} is the subset of all $x \in M \otimes C^{i-1}$ such that $(\mathrm{Id}_M \otimes \partial)(x)$ is a multiple of $f(1)$.

We claim that $E(f)$ is not only exact, but also homotopy equivalent to the zero complex. Indeed, let $\widetilde{C}^\bullet$ denote the complex $\Bbbk \hookrightarrow C^0 \to C^1 \to \cdots$ (that is, $\widetilde{C}^i = C^i$ for $i \geq 0$ and $C^{-1} = \Bbbk$). Then $\widetilde{C}^\bullet$, hence $M \otimes \widetilde{C}^\bullet$, is homotopy equivalent to the zero complex. If $s^n : M \otimes \widetilde{C}^n \to M \otimes \widetilde{C}^{n-1}$, $n \geq 0$ is the homotopy between 0 and the identity map, then the formula: $s^{k'} = s^k$ for $k < i$ and $s^{i'} = s^i \circ f$ defines a homotopy between zero and the identity map for $E(f)$.

Step 3. Now we turn to cross product of extensions. One easily shows that if $E : M \hookrightarrow \cdots \twoheadrightarrow \Bbbk$ and $E' : M' \hookrightarrow \cdots \twoheadrightarrow \Bbbk$ are two extensions, and if *one of the two* is either $\Bbbk$-flat or homotopy equivalent to the zero complex, then their cross product $E \times E'$ is an exact sequence. We derive two consequences from this: (1) $E(f) \times E(f')$ is an extension, and $\Delta^*(E(f) \times E(f'))$ represents the cohomology class $[(f \otimes f') \circ \Delta] = [f] \cup [f']$ (cf. step 1 for this equality). (2) If E, E' are $\Bbbk$-flat extensions equivalent to $E(f)$ and $E(f')$ then $\Delta^*(E \times E')$ is equivalent to $\Delta^*(E(f) \times E(f'))$. Putting (1) and (2) together, we conclude the proof. $\square$

3 Rational cohomology of classical groups via strict polynomial functor cohomology

In this section, $\Bbbk$ is a commutative ring. We show that the rational cohomology of the general linear groups GL_n, the symplectic groups Sp_n and the orthogonal groups $O_{n,n}$ with coefficients in functorial representations may be computed as functor cohomology. To be more specific, for $G = GL_n$, the rational cohomology is related to extensions in the category $\mathcal{P}(1,1)$ of functors with source $V_\Bbbk^{\mathrm{op}} \times \mathcal{V}_\Bbbk$ (ie: $\mathcal{P}(1,1)$ is the category of strict polynomial bifunctors, contravariant in the first variable and covariant in the second one [8]). For the orthogonal and symplectic case, the cohomology is related to extensions in the category $\mathcal{P}$ of Friedlander and Suslin [10] (ie: the category of functors with source $\mathcal{V}_{\Bbbk}$).

Let us outline the proof. Let $G_n = Sp_n$, $O_{n,n}$ or GL_n. Set $\mathcal{A} = \mathcal{V}_{\Bbbk}$, or $V_{\Bbbk}^{\mathrm{op}} \times \mathcal{V}_{\Bbbk}$ in the general linear case. To each $F \in \mathcal{P}_{\mathcal{A}}$, we may associate a rational representation F_n of G_n. In that way, we obtain a δ-functor: $F \mapsto H^*_{\mathrm{rat}}(G_n, F_n)$ (that is, a nonnegatively graded functor, sending admissible short exact sequences in $\mathcal{P}_{\mathcal{A}}$ to long exact sequences in $\Bbbk$-mod, cf [11]).

On the other hand, we associate to G_n a 'characteristic functor' $F_G \in \mathcal{P}_{\mathcal{A}}$. To be more specific, for Sp_n, resp. $O_{n,n}$, resp. GL_n, we take $F_G = \Lambda^2$, resp. S^2, resp. $gl(-,-) = \mathrm{Hom}_{\Bbbk}(-,-)$ (the characteristic functors Λ^2 and S^2 appear in the context of finite groups in [7, Thm 3.21] and gl appears in [8, Thm 1.5]). Taking the divided powers of F_G, one obtains a δ-functor $F \mapsto \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}}}(\Gamma^\star(F_G), F)$, which is by definition universal (ie: it vanishes on the injectives in positive $*$-degree).

Now we wish to compare these two $*$-graded δ-functors (We don't take the gradation of the divided power algebra into account) by the well-known elementary lemma [11]:

Lemma 3.1. *Let K^*, H^* be universal δ-functors and let $\phi^* : K^* \to H^*$ be a morphism of δ-functors. If ϕ^0 is an isomorphism, then for all $i \geq 0$, ϕ^i is an isomorphism.*

This is done in four steps.

Step 1: We build a morphism of δ-functors:

$$\phi_{G_n,-} : \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}}}(\Gamma^\star(F_G), -) \to H^*_{\mathrm{rat}}(G_n, -_n)\ .$$

Moreover, we check that $\phi_{G_n,-}$ is compatible with cup products. To be more specific, the cup product on the right is the usual cup product in rational cohomology (cf. §2.4), and the cup product on the left is induced (cf. §2.3) by the coalgebra structure on $\Gamma^\star(F_G)$ (cf. §2.1).

Step 2: We prove that $F \mapsto H^*_{\mathrm{rat}}(G_n, F_n)$ is universal. This step involves good filtrations of G_n-modules.

Step 3: We prove that the degree zero map $\phi^0_{G_n,F}$ is injective if $2n$ is greater than the degree of F. This step relies on an explicit functor computation.

Step 4: We prove that the degree zero map $\phi^0_{G_n,F}$ is an isomorphism if $2n$ is greater than the degree of F. The surjectivity is proved via classical invariant theory.

Let us now give the details.

3.1 General linear groups

Let $\Bbbk$ be a commutative ring, and let $\mathcal{P}(1,1)$ be the category of strict polynomial functors with source $\mathcal{V}_{\Bbbk}^{\mathrm{op}} \times \mathcal{V}_{\Bbbk}$. For any $F \in \mathcal{P}(1,1)$, F_n denotes the rational representation of GL_n with underlying $\Bbbk$-module $F(\Bbbk^n, \Bbbk^n)$, and with action of $g \in GL_n$ given by $F(g^{-1}, g)$. In particular, for $gl(-,-) := \mathrm{Hom}_{\Bbbk}(-,-)$, one recovers the adjoint representation gl_n of GL_n. Since $\mathrm{Id}_{\Bbbk^n} \in gl_n$ is invariant under the action of GL_n, for all $d \geq 0$ we have an equivariant map:

$$\iota^d : \Bbbk \to \Gamma^d(gl_n)\,, \quad \lambda \mapsto \lambda \mathrm{Id}_{\Bbbk^n}^{\otimes d}\,.$$

Step 1: construction of $\phi_{GL_n,F}$. Since F splits naturally as a direct sum of homogeneous bifunctors, it suffices to do the construction for a homogeneous bifunctor F. The bifunctors $\Gamma^d(gl)$ are homogeneous of degree $2d$. As a consequence, if F is homogeneous of odd degree, then $\mathrm{Ext}^*_{\mathcal{P}(1,1)}(\Gamma^\star(gl), F) = 0$ and we define $\phi_{GL_n,F}$ as the zero map. If F is homogeneous of even degree $2d$, a class $x \in \mathrm{Ext}^j_{\mathcal{P}(1,1)}(\Gamma^\star(gl), F)$ is represented by an admissible extension

$$0 \to F \to F^0 \to \cdots \to F^{j-1} \to \Gamma^d(gl) \to 0\,.$$

We define $\phi_{GL_n,F}(x) \in H^j_{\mathrm{rat}}(GL_n, F_n) = \mathrm{Ext}^*_{GL_n\text{-mod}}(\Bbbk, F_n)$ as the class of the extension obtained by evaluation on $(\Bbbk^n, \Bbbk^n)$ and pullback along ι^d:

$$\iota^{d\,*}\left(0 \to F_n \to F_n^0 \to \cdots \to F_n^{j-1} \to \Gamma^d(gl_n) \to 0\right)\,.$$

Lemma 3.2 (Completion of Step 1)**.** *For all $n \geq 0$, the map $\phi_{GL_n,-}$: $\mathrm{Ext}^*_{\mathcal{P}(1,1)}(\Gamma^\star(gl), -) \to H^*_{\mathrm{rat}}(GL_n, -_n)$ is a map of δ-functors. Moreover it is compatible with cup products: $\phi_{GL_n,F\otimes F'}(x \cup y) = \phi_{GL_n,F}(x) \cup \phi_{GL_n,F'}(y)$.*

Proof. Straightforward, except for the compatibility with cup products, which we now give in detail. Since a bifunctor splits naturally as a direct sum of homogeneous bifunctors, it suffices to prove the compatibility for homogeneous F, F'. Furthermore, one easily reduces to the case where F and F' have even degrees $2d$ and $2d'$. Let E and E' be two admissible exact sequences representing classes $x \in \mathrm{Ext}^i_{\mathcal{P}(1,1)}(\Gamma^d(gl), F)$ and $y \in \mathrm{Ext}^j_{\mathcal{P}(1,1)}(\Gamma^{d'}(gl), F')$. Since E and E' are admissible, their kernels are bifunctors with projective values. As a result, evaluation on $(\Bbbk^n, \Bbbk^n)$ and pullback by $\iota^d, \iota^{d'}$ yield $\Bbbk$-*projective* extensions $\iota^{d\,*}(E_n), \iota^{d'\,*}(E'_n)$. By lemma 2.5, the cohomology class $\phi_{GL_n,F}(x) \cup \phi_{GL_n,F'}(y)$ is represented by the pullback of the cross product $\iota^{d\,*}(E_n) \times \iota^{d'\,*}(E'_n)$ by the diagonal $\Bbbk \to \Bbbk \otimes \Bbbk$. Now the diagonals of $\Bbbk$ and $\Gamma^*(gl)$ induce a commutative diagram

$$\begin{tikzcd}
\Bbbk \arrow[r, "\Delta"] \arrow[d, "\iota^{d+d'}"] & \Bbbk \otimes \Bbbk \arrow[d, "\iota^d \otimes \iota^{d'}"] \\
\Gamma^{d+d'}(gl_n) \arrow[r, "\Delta"] & \Gamma^d(gl_n) \otimes \Gamma^{d'}(gl_n)\,.
\end{tikzcd}$$

Thus $\iota^{d+d'*}((E\cup E')_n)$ equals $\iota^{d\,*}(E_n)\cup\iota^{d'\,*}(E'_n)$ and we are done. $\square$

Step 2: $F\mapsto H^*_{\mathrm{rat}}(GL_n,F_n)$ is a universal δ-functor. Now we prove that $H^{>0}_{\mathrm{rat}}(GL_n,F_n)$ vanishes when F is an injective functor of $\mathcal{P}(1,1)$.

A Chevalley group scheme over $\mathbb{Z}$ is a connected split reductive algebraic $\mathbb{Z}$-group. A Chevalley group scheme G over a commutative ring $\Bbbk$ is a group scheme obtained by base change from a Chevalley group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$: $G=(G_{\mathbb{Z}})_{\Bbbk}$. If we deal with Chevalley group schemes (such as GL_n, $SO_{n,n}$, Sp_n, etc.), cohomological vanishing over arbitrary ground rings $\Bbbk$ can often be reduced to the case where $\Bbbk$ is a field by the following standard lemma.

Lemma 3.3. *Let $G_{\mathbb{Z}}$ be a Chevalley group scheme over the integers, acting rationally on a free $\mathbb{Z}$-module M of finite type. Denote by $G_{\Bbbk}$ the group obtained from $G_{\mathbb{Z}}$ by base change. The following assertions are equivalent:*

(i) The cohomology groups $H^i_{\mathrm{rat}}(G_{\mathbb{Z}},M)$ are trivial for $i>0$.

(ii) For all field $\Bbbk$, $H^i_{\mathrm{rat}}(G_{\Bbbk},M\otimes\Bbbk)=0$ for $i>0$.

(iii) For all commutative ring $\Bbbk$, $H^i_{\mathrm{rat}}(G_{\Bbbk},M\otimes\Bbbk)=0$ for $i>0$.

Proof. $(iii)\Rightarrow(i)$ is trivial. $(i)\Rightarrow(iii)$ and $(i)\Rightarrow(ii)$ follow from the universal coefficient theorem [12, Part I, Chap 4, Prop 4.18]. So it remains to prove $(ii)\Rightarrow(i)$. By the universal coefficient theorem, (ii) implies that for all field $\Bbbk$, $H^i_{\mathrm{rat}}(G_{\mathbb{Z}},M)\otimes\Bbbk=0$. But $G_{\mathbb{Z}}$ is a Chevalley group scheme, so the cohomology groups $H^i_{\mathrm{rat}}(G_{\mathbb{Z}},M)$ are finitely generated by [12, Part II, Lemma B.5]. So the equality $H^i_{\mathrm{rat}}(G_{\mathbb{Z}},M)\otimes\Bbbk=0$ for all field $\Bbbk$ implies that $H^i_{\mathrm{rat}}(G_{\mathbb{Z}},M)=0$. $\square$

Lemma 3.4. *Let $\Bbbk$ be a commutative ring. Let J be an injective in the category $\mathcal{P}(1,1)$ of bifunctors defined over $\Bbbk$. Then $H^i_{\mathrm{rat}}(GL_n,J_n)=0$ if $i>0$. As a result, $F\mapsto H^*_{\mathrm{rat}}(GL_n,F_n)$ is a universal δ-functor.*

Proof. By lemma 2.4, it suffices to prove the vanishing on the injectives of the form $I^d_{k,\ell}:(V,W)\mapsto S^d((V^\vee)^{\oplus k}\oplus W^{\oplus\ell})$, for $k,\ell,d\geq 0$. The GL_n-module associated to $I^d_{k,\ell}$ by evaluation on $(\Bbbk^n,\Bbbk^n)$ is a direct summand of the polynomial algebra over the sum $(\Bbbk^n)^{\oplus k}\oplus(\Bbbk^{n\vee})^{\oplus\ell}$. Thus, it suffices to prove that for all integer k,ℓ, and for all commutative ring $\Bbbk$, we have $H^i_{\mathrm{rat}}(GL_n,S^*((\Bbbk^{n\vee})^{\oplus k}\oplus(\Bbbk^n)^{\oplus\ell}))=0$ for $i>0$.

By lemma 3.3, this statement reduces to the case where $\Bbbk$ is a field. In this latter case, $S^*((\Bbbk^{n\vee})^{\oplus k}\oplus(\Bbbk^n)^{\oplus\ell})$ has a good filtration [1, Section 4.9 p. 508]. In particular, the cohomology vanishes in positive degree. $\square$

Step 3: injectivity in degree 0.

Lemma 3.5. *Let $d \geq 0$, let $n \geq d$ and let $X = \Bbbk^n$. There is an epimorphism:*

$$\theta : P^{2d}_{(X,X)} \twoheadrightarrow \Gamma^d(gl).$$

Moreover, if we evaluate the bifunctors on (X,X), then $\theta_{(X,X)}$ sends $\mathrm{Id}_{(X,X)}{}^{\otimes 2d} \in P^{2d}_{(X,X)}(X,X)$ to $\mathrm{Id}_X{}^{\otimes d} \in \Gamma^d(\mathrm{Hom}(X,X))$.

Proof. The exponential isomorphism for the divided powers induce a epimorphism of $P^{2d}_{(X,X)}$ onto $\Gamma^d(\mathrm{Hom}(-,X)) \otimes \Gamma^d(\mathrm{Hom}(X,-))$. Moreover, if we evaluate on (X,X), this epimorphism sends $\mathrm{Id}^{\otimes 2d}_{(X,X)}$ to $\mathrm{Id}^{\otimes d}_{(X,X)} \otimes \mathrm{Id}^{\otimes d}_{(X,X)}$. If we postcompose this map by the map from $\Gamma^d(\mathrm{Hom}(-,X)) \otimes \Gamma^d(\mathrm{Hom}(X,-))$ to $\Gamma^d(gl)$ induced by composition in $\Gamma^d\mathcal{V}_\Bbbk$, then the resulting map sends $\mathrm{Id}^{\otimes 2d}_{(X,X)}$ to $\mathrm{Id}_X{}^{\otimes d}$, and is an epimorphism by lemma 2.3. □

Lemma 3.6 (Completion of Step 3)**.** *Let $F \in \mathcal{P}(1,1)$ be a bifunctor defined over a commutative ring $\Bbbk$. If $2n$ is greater than the total degree of F, then $\phi^0_{GL_n,F} : \mathrm{Hom}_{\mathcal{P}(1,1)}(\Gamma^*(gl), F) \to H^0_{\mathrm{rat}}(GL_n, F_n)$ is injective.*

Proof. Since F splits as a direct sum of homogeneous functors, we can restrict to the case of homogeneous functors. Moreover, if F is homogeneous of odd degree, then $\mathrm{Hom}_{\mathcal{P}(1,1)}(\Gamma^*(gl), F) = 0$ and $\phi^0_{GL_n,F}$ is injective. Now we assume that F is homogeneous of degree $2d$. Let $X = \Bbbk^n$, with $n \geq d$. By lemma 3.5, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}(1,1)}(P^{2d}_{(X,X)}, F) \arrow[r, "\simeq"] & F(X,X) \\
\mathrm{Hom}_{\mathcal{P}(1,1)}(\Gamma^d(gl), F) \arrow[u, "\mathrm{Hom}_{\mathcal{P}(1,1)}(\theta,F)"] \arrow[r, "\phi^0_{GL_n,F}"] & H^0_{\mathrm{rat}}(GL_n, F_n) \arrow[u, hook]
\end{tikzcd}.$$

The horizontal arrow is the Yoneda isomorphism. Since θ is an epimorphism, $\mathrm{Hom}(\theta, F)$ is injective. Thus, $\phi^0_{GL_n,F}$ is injective. □

Step 4: isomorphism in degree 0. Recall from lemma 2.4 that for all $k, \ell \geq 0$, $I^d_{k,\ell}$ denotes the d-th symmetric power of the bifunctor $(V,W) \mapsto (V^\vee)^{\oplus k} \oplus W^{\oplus \ell}$. The evaluation of $I^d_{k,\ell}$ on the pair $(\Bbbk^n, \Bbbk^n)$ equals the GL_n-module of homogeneous polynomials of degree d on the vector space $(\Bbbk^n)^{\oplus k} \oplus (\Bbbk^{n\vee})^{\oplus \ell}$. For $1 \leq i \leq k$ and $1 \leq j \leq \ell$ we denote by $(i|j)$ the contraction:

$$\begin{array}{cccc}
(i|j): & (\Bbbk^n)^{\oplus k} \oplus (\Bbbk^{n\vee})^{\oplus \ell} & \to & \Bbbk \\
& (v_1, \dots, v_k, f_1, \dots, f_\ell) & \mapsto & f_j(v_i)
\end{array}.$$

The contractions are homogeneous polynomials of degree two (invariant under the action of GL_n), hence elements of $(I^2_{k,\ell})_n$.

In fact, by [6, Theorem 3.1], these contractions generate the GL_n-invariant subalgebra of the algebra of polynomials over $(\Bbbk^n)^{\oplus k} \oplus (\Bbbk^{n\vee})^{\oplus \ell}$. We use this fact to prove surjectivity of the $\phi^0_{GL_n,F}$ below.

Lemma 3.7. *For all $n \geq 1$ and all $k, \ell \geq 1$, the contractions lie in the image of $\phi^0_{GL_n, I^2_{k,\ell}}$.*

Proof. Let $\rho : gl(V, W) \simeq V^\vee \otimes W \hookrightarrow S^2(V^\vee \oplus W)$ be the map induced by the exponential isomorphism for S^2. Let $(e_i)_{1\leq i\leq n}$ be a basis of $\Bbbk^n$ and let $(e_i^\vee)_{1\leq i\leq n}$ be the dual basis. Then for $V = W = \Bbbk^n$, ρ sends $\mathrm{Id}_{\Bbbk^n} = \sum e_i^\vee \otimes e_i$ to $\sum (e_i^\vee, 0)(0, e_i)$ (we denote the elements of $\Bbbk^{n\vee} \oplus \Bbbk^n$ as pairs). This latter polynomial is nothing but the polynomial $\Bbbk^n \oplus \Bbbk^{n\vee} \to \Bbbk$, $(v, f) \mapsto f(v)$.

Now denote by $\iota_{i,j}$ the inclusion of $V^\vee \oplus W$ in the i-th and j-th term of $(V^\vee)^{\oplus k} \oplus W^{\oplus \ell}$. Then for all i, j, $\phi^0_{GL_n, I^2_{k,\ell}}$ sends $S^2(\iota_{i,j}) \circ \rho$ to $(i|j)$. □

Lemma 3.8. *For all $k, \ell, n \geq 1$ and all $d \geq 0$, $\phi^0_{GL_n,-}$ induces an epimorphism:*

$$\mathrm{Hom}_{\mathcal{P}(1,1)}(\Gamma^*(gl), I^d_{k,\ell}) \twoheadrightarrow H^0_{\mathrm{rat}}(GL_n, (I^d_{k,\ell})_n)\ .$$

Proof. By lemma 3.2, $\phi_{GL_n,-}$ is compatible with external cup products. In particular, if A^* is a graded bifunctor endowed with an algebra structure, we obtain an algebra morphism:

$$\phi^0_{GL_n, A^*} : \mathrm{Hom}_{\mathcal{P}(1,1)}(\Gamma(gl), A^*) \to H^0_{\mathrm{rat}}(GL_n, A^*_n)\ .$$

We apply this to $A^* = I^*_{k,\ell}$. By invariant theory [6, Theorem 3.1], $H^0_{\mathrm{rat}}(GL_n, (I^*_{k,\ell})_n)$ is generated by the contractions $(i|j)$. By lemma 3.7, the contractions are in the image of $\phi^0_{GL_n, I^*_{k,\ell}}$. This proves surjectivity. □

Lemma 3.9 (Completion of Step 4)**.** *Let $F \in \mathcal{P}(1,1)$ and let n be an integer such that $2n \geq \deg F$. Then $\phi^0_{GL_n,F}$ is an isomorphism.*

Proof. By lemma 2.4 and by left exactness of $F \mapsto \mathrm{Hom}(\Gamma^*(gl), F)$ and $F \mapsto H^0_{\mathrm{rat}}(GL_n, F_n)$, it suffices to prove the statement for the $I^d_{k,\ell}$, $k, \ell \geq 1$, $d \geq 0$. For these bifunctors, the isomorphism follows from lemmas 3.6 and 3.8. □

Theorem 3.10 (The GL_n case)**.** *Let $\Bbbk$ be a commutative ring, and let n be a positive integer. For all $F \in \mathcal{P}(1,1)$ we have a $*$-graded map, natural in F:*

$$\phi_{GL_n,F} : \mathrm{Ext}^*_{\mathcal{P}(1,1)}(\Gamma^\star(gl), F) \to H^*_{\mathrm{rat}}(GL_n, F_n)$$

The map $\phi_{GL_n,F}$ is compatible with cup products:

$$\phi_{GL_n, F\otimes F'}(x \cup y) = \phi_{GL_n,F}(x) \cup \phi_{GL_n,F'}(y)\ .$$

Moreover, $\phi_{G_n,F}$ is an isomorphism whenever $2n \geq \deg(F)$.

Proof. The first part of the theorem is given by lemma 3.2. It remains to prove the isomorphism. By homogeneity, it suffices to prove the isomorphism for homogeneous functors of degree $d \leq 2n$. To do this, we restrict $\phi_{GL_n,-}$ to the subcategory $\mathcal{P}_d(1,1)$ of homogeneous functors of degree d and we apply lemma 3.1. □

Remark 3.11. This theorem was already known over a positive characteristic field $\Bbbk$: a $\Bbbk$-linear isomorphism is built in [8, Thm 1.5], and compatibility with cup products is proved in [19, Thm 1.3]. However, our proof is new and extends the result to arbitrary commutative rings.

3.2 Symplectic groups

Let $\Bbbk$ be a commutative ring, and let $\mathcal{P}$ be the category of strict polynomial functors with source $\mathcal{V}_\Bbbk$. Let $(e_i)_{1\le i\le 2n}$ be a basis of $\Bbbk^{2n}$ and let $(e_i)^\vee_{1\le i\le 2n}$ be its dual basis. For all $n>0$ we denote by Sp_n the symplectic group, that is, the algebraic group of $2n\times 2n$ matrices preserving the skew-symmetric form: $\omega_n := \sum_{i=1}^n e_i^\vee \wedge e_{n+i}^\vee$. The standard representation of Sp_n is $\Bbbk^{2n}$ with left action given by matrix multiplication. For all functor $F\in\mathcal{P}$, we denote by F_n the rational Sp_n-module obtained by evaluating F on the dual $(\Bbbk^{2n})^\vee$ of the standard representation. In particular for $F=\Lambda^2$, Λ^2_n is the $\Bbbk$-module of skew-symmetric forms of degree 2. Since $\omega_n\in\Lambda^2_n$ is invariant under the action of Sp_n, we have for all $d\ge 0$ an equivariant map:

$$\iota^d : \Bbbk \to \Gamma^d(\Lambda^2_n)\,, \quad \lambda\mapsto \lambda\omega_n^{\otimes d}\ .$$

Step 1: construction of $\phi_{Sp_n,F}$. By homogeneity, it suffices to do the construction for a homogeneous functor F of degree $2d$. In that case, a class $x\in \mathrm{Ext}^j_{\mathcal{P}(1,1)}(\Gamma^*(\Lambda^2),F)$ is represented by an admissible extension

$$0\to F\to F^0\to\cdots\to F^{j-1}\to \Gamma^d(\Lambda^2)\to 0\ .$$

We define $\phi_{Sp_n,F}(x)\in H^j_{\mathrm{rat}}(Sp_n,F_n) = \mathrm{Ext}^*_{Sp_n\text{-mod}}(\Bbbk,F_n)$ as the class of the extension obtained by first evaluating on $(\Bbbk^{2n})^\vee$, and then taking the pullback along ι^d. The proof of the following lemma is analogous to the GL_n case.

Lemma 3.12 (Completion of Step 1)**.** *For all $n\ge 0$, the map $\phi_{Sp_n,-} : \mathrm{Ext}^\star_{\mathcal{P}}(\Gamma^*(\Lambda^2),-)\to H^\star_{\mathrm{rat}}(Sp_n,-_n)$ is a map of δ-functors. Moreover it is compatible with cup products: $\phi_{Sp_n,F\otimes F'}(x\cup y) = \phi_{Sp_n,F}(x)\cup\phi_{Sp_n,F'}(y)$.*

Lemma 3.13 (Step 2)**.** *Let $\Bbbk$ be a commutative ring. Let J be an injective in the category $\mathcal{P}$ of functors defined over $\Bbbk$. Then $H^i_{\mathrm{rat}}(Sp_n,J_n)=0$ if $i>0$. As a result, $F\mapsto H^*_{\mathrm{rat}}(Sp_n,F_n)$ is a universal δ-functor.*

Proof. By lemma 2.4, it suffices to prove the vanishing on the injectives $I^d_k : V\mapsto S^d(V^{\oplus k})$, for $k,d\ge 0$. As in the case of GL_n, it suffices to show the vanishing of $H^i_{\mathrm{rat}}(Sp_n, S^*((\Bbbk^{2n\vee})^{\oplus k}))$, $i>0$, when $\Bbbk$ is a field. Once again, this vanishing comes from the existence of a good filtration [1, Section 4.9 p. 508-509]. $\square$

Step 3: injectivity in degree 0. We need a variant of lemma 3.5.

Lemma 3.14. *Let $d \geq 0$, let $n \geq d$ and let X, X' be two copies of $\Bbbk^n$ with respective basis $(e_i)_{1\leq i\leq n}$ and $(e_i)_{n+1\leq i\leq 2n}$. There is an epimorphism*

$$\widetilde{\theta} : P^{2d}_{X\oplus X'} \twoheadrightarrow \Gamma^d(\otimes^2)\;.$$

Moreover, if we evaluate the functors on $X \oplus X'$, then $\widetilde{\theta}_{X\oplus X'}$ sends $\mathrm{Id}^{\otimes 2d}_{X\oplus X'}$ to $(\sum_{i=1}^n e_i \otimes e_{n+i})^{\otimes d}$.

Proof. The exponential formula for the divided powers induce an epimorphism from $P^{2d}_{X\oplus X'}$ onto $\Gamma^d(\mathrm{Hom}_\Bbbk(X,-))\otimes\Gamma^d(\mathrm{Hom}_\Bbbk(X',-))$. If we evaluate the functors on $X\oplus X'$, this epimorphism sends $\mathrm{Id}^{\otimes 2d}_{X\oplus X'}$ to $\mathrm{in}_X^{\otimes d}\otimes\mathrm{in}_{X'}^{\otimes d}$, where $\mathrm{in}_X, \mathrm{in}_{X'}$ are the inclusions of X, X' into $X \oplus X'$. Now there is an isomorphism $X \to (X')^\vee$ which sends e_i to $e_{i+n}^\vee$ for all i, where $(e_{i+n}^\vee)$ is the dual basis. This induces an isomorphism from $\Gamma^d(\mathrm{Hom}_\Bbbk(X,-))\otimes\Gamma^d(\mathrm{Hom}_\Bbbk(X',-))$ to $\Gamma^d(\mathrm{Hom}_\Bbbk(-^\vee,X'))\otimes\Gamma^d(\mathrm{Hom}_\Bbbk(X',-))$, which sends (after evaluation on $X \oplus X'$) $\mathrm{in}_X^{\otimes d}\otimes\mathrm{in}_{X'}^{\otimes d}$ to $(\sum e_i\otimes e_{i+n})^{\otimes d}\otimes(\sum e^\vee_{i+n}\otimes e_{i+n})^{\otimes d}$. If we postcompose by the map from $\Gamma^d(\mathrm{Hom}_\Bbbk(-^\vee,X'))\otimes\Gamma^d(\mathrm{Hom}_\Bbbk(X',-))$ to $\Gamma^d(\otimes^2)$ induced by the composition in $\Gamma^d\mathcal{V}_\Bbbk$ then by lemma 2.3 we obtain the required epimorphism. □

Lemma 3.15 (Completion of Step 3)**.** *Let $F \in \mathcal{P}$ be a functor defined over a commutative ring $\Bbbk$. If $2n \geq \deg F$, then $\phi^0_{GL_n,F}$ is injective.*

Proof. Using lemma 3.14, we obtain an epimorphism $\widetilde{\theta} : P^{2d}_{(\Bbbk^{2n})^\vee} \twoheadrightarrow \Gamma^d(\Lambda^2)$ which sends $\mathrm{Id}^{\otimes 2d}_{(\Bbbk^{2n})^\vee}$ to $\omega_n^{\otimes d}$. Thus, $\phi^0_{GL_n,F}$ factorizes as the composite of the injection $\mathrm{Hom}_\mathcal{P}(\widetilde{\theta},F)$ and the Yoneda isomorphism $\mathrm{Hom}_\mathcal{P}(P^{2d}_{X\oplus X'},F)\simeq F(X\oplus X')$. Hence $\phi^0_{GL_n,F}$ is injective. □

Lemma 3.16 (Step 4)**.** *Let $F \in \mathcal{P}$ be a functor defined over a commutative ring $\Bbbk$. If $2n \geq \deg F$, then $\phi^0_{Sp_n,F}$ is an isomorphism.*

Proof. By lemma 2.4, and left exactness of $F \mapsto \mathrm{Hom}_\mathcal{P}(\Gamma^*(\Lambda),F)$ and $F \mapsto H^0_{\mathrm{rat}}(Sp_n,F_n)$, it suffices to prove the isomorphism for the functors of the form $I^d_k : V \mapsto S^d(V^{\oplus k})$, for $k \geq 1$ and $d \leq 2n$. Lemma 3.15 already gives injectivity. It remains to prove surjectivity. But $\phi^0_{Sp_n,I^*_k}$: $\mathrm{Hom}(\Gamma^*(\Lambda^2),I^*_k) \to H^0(Sp_n,(I^*_k)_n)$ is an algebra morphism, so we only have to prove that the generators of $H^0(Sp_n,(I^*_k)_n)$ lie in the image of $\phi^0_{I^*_k}$. Now $(I^*_k)_n$ is the polynomial algebra over k copies of the standard representation of Sp_n. Invariant theory gives [6, Thm 6.6] the generators of $H^0(Sp_n,(I^*_k)_n)$: they are homogeneous polynomials of degree two $(i|j) : (\Bbbk^{2n})^{\oplus k} \to \Bbbk$, $1 \leq i < j \leq k$, sending $(v_1,\dots,v_n)$ to $\omega_n(v_i,v_j)$. In particular, if $k = 1$ $H^0(Sp_n,(I^*_k)_n) = \Bbbk$ and the surjectivity of $\phi^0_{Sp_nI^2_1}$ is clear. So the proof will be completed if we show that the $(i|j)$ lie in the image of $\phi^0_{Sp_nI^2_k}$, for $k \geq 2$.

Let V, V' be two copies of $V \in \mathcal{V}_{\Bbbk}$. The exponential isomorphism for S^2 yields a monomorphism $V \otimes V' \hookrightarrow S^2(V \oplus V')$. Now if we take $V' = V$, and if we precompose by the inclusion $\Lambda^2(V) \to V^{\otimes 2}$, we get a natural transformation $\rho : \Lambda^2(V) \to S^2(V \oplus V)$. If $V = \Bbbk^{2n\vee}$, with basis $(e_i^\vee)_{1 \le i \le 2n}$, then ρ sends $e_i^\vee \wedge e_j^\vee$ to $(e_i^\vee, 0)(0, e_j^\vee) - (e_j^\vee, 0)(0, e_i^\vee)$ (we denote the elements of $\Bbbk^{2n\vee} \oplus \Bbbk^{2n\vee}$ as pairs). Thus, ρ sends ω_n to the sum $\sum_{i=1}^n (e_i^\vee, 0)(0, e_{i+n}^\vee) - \sum_{i=1}^n (e_{i+n}^\vee, 0)(0, e_i^\vee)$, which is nothing but the polynomial $\Bbbk^{2n} \oplus \Bbbk^{2n} \to \Bbbk$, $(x, y) \mapsto \omega_n(x, y)$. For $i < j$, we denote by $\iota_{i,j}$ the inclusion of $V \oplus V$ into the i-th and the j-th term of the sum $V^{\oplus k}$. Then ϕ_{Sp_n} send the natural transformation $S^2(\iota_{i,j}) \circ \rho$ to $(i|j)$ and we are done. □

Theorem 3.17 (The Sp_n case)**.** *Let $\Bbbk$ be a commutative ring, and let n be a positive integer. For all $F \in \mathcal{P}$ we have a $*$-graded map, natural in F:*

$$\phi_{Sp_n, F} : \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\Lambda^2), F) \to H^*_{\mathrm{rat}}(Sp_n, F_n)$$

The map $\phi_{Sp_n,F}$ is compatible with cup products:

$$\phi_{Sp_n, F \otimes F'}(x \cup y) = \phi_{Sp_n,F}(x) \cup \phi_{Sp_n,F'}(y) \ .$$

Moreover, $\phi_{Sp_n,F}$ is an isomorphism whenever $2n \ge \deg(F)$.

3.3 Orthogonal groups

Let $\Bbbk$ be a commutative ring, and let $\mathcal{P}$ be the category of strict polynomial functors with source $\mathcal{V}_{\Bbbk}$. Let $(e_i)_{1 \le i \le 2n}$ be a basis of $\Bbbk^{2n}$ and let $(e_i)^\vee_{1 \le i \le 2n}$ be its dual basis. For all $n > 0$ we denote by $O_{n,n}$ the algebraic group of $2n \times 2n$ matrices preserving the quadratic form $q_n := \sum_{i=1}^n e_i^\vee e_{n+i}^\vee$. The standard representation of $O_{n,n}$ is $\Bbbk^{2n}$ with left action given by matrix multiplication. For all functor $F \in \mathcal{P}$, we denote by F_n the rational $O_{n,n}$-module obtained by evaluating F on the dual $(\Bbbk^{2n})^\vee$ of the standard representation. In particular for $F = S^2$, S^2_n is the $\Bbbk$-module of polynomials of degree 2 over $\Bbbk^{2n}$. Since $q_n \in S^2_n$ is invariant under the action of $O_{n,n}$, we have for all $d \ge 0$ an equivariant map:

$$\iota^d : \Bbbk \to \Gamma^d(S^2_n) \qquad \lambda \mapsto \lambda q_n^{\otimes d} \ .$$

The case of the orthogonal group is analogous to the case of the symplectic group, except for a restriction on the characteristic of the commutative ring $\Bbbk$ which is needed in step 2 only.

Step 1: construction of $\phi_{O_{n,n},F}$. We follow rigorously the symplectic case. If F is homogeneous of degree $2d$, a class $x \in \mathrm{Ext}^j(\Gamma^*(S^2), F)$ is represented by an extension $0 \to F \to \dots \Gamma^d(S^2) \to 0$. We define $\phi_{O_{n,n},F}(x)$ as the class of the extension obtained by evaluation on $(\Bbbk^{2n})^\vee$ and pullback along ι^d. We have:

Lemma 3.18 (Completion of Step 1)**.** *For all $n \geq 0$, the map $\phi_{O_{n,n},-}$: $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(S^2), -) \to H^*_{\mathrm{rat}}(O_{n,n}, -_n)$ is a map of δ-functors. Moreover it is compatible with cup products: $\phi_{O_{n,n},F\otimes F'}(x \cup y) = \phi_{O_{n,n},F}(x) \cup \phi_{O_{n,n},F'}(y)$.*

Step 2: $F \mapsto H^*_{\mathrm{rat}}(O_{n,n}, F_n)$ is a universal δ-functor. We want to prove that $H^*_{\mathrm{rat}}(O_{n,n}, F_n)$ vanishes in positive cohomological degree when F is an injective of $\mathcal{P}$. But the case of the orthogonal group is slightly different from the general linear and symplectic cases. Define $SO_{n,n}$ as the kernel of the Dickson invariant, or equivalently as the kernel of the determinant if 2 is invertible in $\Bbbk$ (see [13, p. 348] or [4] for details). Then we have an extension of group schemes:

$$SO_{n,n} \triangleleft O_{n,n} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} .$$

And $SO_{n,n}$ is a Chevalley group scheme. Now [1, section 4.9 p.509] gives vanishing results for $SO_{n,n}$:

Lemma 3.19. *Let $\Bbbk$ be a commutative ring and let J be an injective in the category $\mathcal{P}$. Then $H^i_{\mathrm{rat}}(SO_{n,n}, J_n) = 0$ for $i > 0$.*

Proof. By lemma 2.4, it suffices to prove the statement for the injectives $I^d_k : V \mapsto S^d(V^{\oplus k})$, for $k, d \geq 0$. By lemma 3.3, it suffices to prove the vanishing over a field $\Bbbk$. In that case, [1, section 4.9 p.509] yields a good filtration on $S^*((\Bbbk^{2n\vee})^{\oplus k})$, whence the result. $\square$

But we want a vanishing result for the cohomology of $O_{n,n}$, not for $SO_{n,n}$. The Lyndon-Hochschild-Serre spectral sequence [12, Part I, Prop 6.6(3)] yields a graded isomorphism

$$H^*_{\mathrm{rat}}(\mathbb{Z}/2\mathbb{Z}, H^0_{\mathrm{rat}}(SO_{n,n}, J_n)) \simeq H^*_{\mathrm{rat}}(O_{n,n}, J_n) .$$

Here comes our restriction on the characteristic. If 2 is invertible in $\Bbbk$, then $\mathbb{Z}/2\mathbb{Z}$ is linearly reductive (Maschke's theorem) hence has no cohomology, so we get:

Lemma 3.20. *Assume 2 is invertible in $\Bbbk$. Then for all J injective in $\mathcal{P}$, and for all positive i, $H^i_{\mathrm{rat}}(O_{n,n}, J_n)$ equals zero. So $F \mapsto H^*_{\mathrm{rat}}(O_{n,n}, F_n)$ is a universal δ-functor.*

Remark 3.21. If 2 is not invertible in $\Bbbk$, then the finite group $\mathbb{Z}/2\mathbb{Z}$ may have non trivial cohomology, so the above argument does not work. In fact, not only the proof but also the statement of lemma 3.20 is false when 2 is not invertible in $\Bbbk$. So our restriction on the characteristic is necessary. Indeed, consider the constant functor $\Bbbk \in \mathcal{P}$. Then $\Bbbk$ is injective in $\mathcal{P}$, and $H^0_{\mathrm{rat}}(SO_{n,n}, \Bbbk) = \Bbbk$, so $H^*_{\mathrm{rat}}(O_{n,n}, \Bbbk) \simeq H^*_{\mathrm{rat}}(\mathbb{Z}/2\mathbb{Z}, \Bbbk)$. Take $\Bbbk$ a field of characteristic 2. Then $H^i_{\mathrm{rat}}(\mathbb{Z}/2\mathbb{Z}, \Bbbk) \simeq \Bbbk$ for all i, so $F \mapsto H^*_{\mathrm{rat}}(O_{n,n}, F_n)$ is not universal.

Lemma 3.22 (Step 3). *Let $F \in \mathcal{P}$ be a functor defined over a commutative ring $\Bbbk$. If $2n$ is greater than the total degree of F, then $\phi^0_{O_{n,n},F}$ is injective.*

Proof. We use lemma 3.14 to produce a suitable epimorphism $\widetilde{\theta} : P^{2d}_{(\Bbbk^{2n})^\vee} \twoheadrightarrow \Gamma^d(S^2)$, so that $\phi^0_{O_{n,n},F}$ is the composite of a Yoneda isomorphism and the injective map $\mathrm{Hom}_{\mathcal{P}}(\widetilde{\theta}, F)$. □

Lemma 3.23 (Step 4). *Let $F \in \mathcal{P}$ be a functor defined over a commutative ring $\Bbbk$. If $2n \geq \deg F$, then $\phi^0_{O_{n,n},F}$ is an isomorphism.*

Proof. As in the symplectic case, it suffices to prove surjectivity for the functors $I^d_k(V) = S^d(V^{\oplus k})$, $d \geq 0, k \geq 1$. Using compatibility with cup products, the proof reduces furthermore to proving that $\phi^0_{O_{n,n},I^*_k}$ hits the generators of the invariant ring $H^0_{\mathrm{rat}}(O_{n,n}, (I^*_k)_n) = H^0_{\mathrm{rat}}(O_{n,n}, S^*((\Bbbk^{2n\vee})^{\oplus k}))$, for all $k \geq 1$.

Let b_n be the bilinear form associated to q_n. By [6, Thm 5.6], a set of generators is given by the homogeneous polynomials $(i|j)_{1\leq i<j\leq k}$ of degree 2, which send $(v_1, \dots, v_k)$ to $b_n(v_i, v_j)$, and by the $(i|i)_{1\leq i\leq n}$ of degree 2, which send $(v_1, \dots, v_k)$ to $q(v_i)$. For $1 \leq i \leq k$, let $\iota_{i,i}$ be the inclusion of V into the i-th term of $V^{\oplus k}$. Then $\phi^0_{O_{n,n},I^2_k}$ sends $S^2(\iota_{i,i})$ to $(i|i)$. Assume now that $1 \leq i < j \leq k$. Denote by $\iota_{i,j}$ the inclusion of $V \oplus V$ in the i-th and the j-th terms of $V^{\oplus k}$. Let also ρ be the composite $S^2(V) \to V \otimes V \to S^2(V \oplus V)$, where the second map is induced by the exponential isomorphism for S^2. If we take $V = \Bbbk^{2n\vee}$, then ρ sends q_n to the sum $\sum_{i=1}^n (e^\vee_i, 0)(0, e^\vee_{i+n}) + \sum_{i=1}^n (e^\vee_{i+n}, 0)(0, e^\vee_i)$, which is nothing but the polynomial $\Bbbk^{2n} \oplus \Bbbk^{2n} \to \Bbbk$, $(v, w) \mapsto b_n(v, w)$. Thus, $\phi^0_{O_{n,n},I^2_k}$ sends the natural transformation $S^2(\iota_{i,j}) \circ \rho$ to $(i|j)$. This concludes the proof. □

Theorem 3.24 (The $O_{n,n}$ case). *Let $\Bbbk$ be a commutative ring, and let n be a positive integer. For all $F \in \mathcal{P}$ we have a $*$-graded map, natural in F:*

$$\phi_{O_{n,n},F} : \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(S^2), F) \to H^*_{\mathrm{rat}}(O_{n,n}, F_n)\ .$$

The map $\phi_{O_{n,n},F}$ is compatible with cup products:

$$\phi_{O_{n,n},F\otimes F'}(x \cup y) = \phi_{O_{n,n},F}(x) \cup \phi_{O_{n,n},F'}(y)\ .$$

Moreover, if $2n$ is greater or equal to the degree of F and if 2 is invertible in $\Bbbk$, then $\phi_{O_{n,n},F}$ is an isomorphism.

4 Products of classical groups and cohomological stabilization

In this section, we use Künneth formulas to extend the link between functor cohomology and rational cohomology to products of classical groups. We also prove the cohomological stabilization property for classical groups and their products.

4.1 External tensor products and Künneth isomorphisms

Let $\Bbbk$ be a commutative ring and for $i = 1, 2$, let $\mathcal{A}_i$ be a finite product of copies of $\mathcal{V}_\Bbbk$ or its opposite category. If $F_i \in \mathcal{P}_{\mathcal{A}_i}$, $i = 1, 2$, their external tensor product $F_1 \boxtimes F_2$ is the functor sending (X, Y) to $F(X) \otimes F(Y)$. This yields a biexact bifunctor

$$- \boxtimes - : \mathcal{P}_{\mathcal{A}_1} \times \mathcal{P}_{\mathcal{A}_2} \to \mathcal{P}_{\mathcal{A}_1 \times \mathcal{A}_2} \,.$$

Let us give some well-known [18, 9, 8] properties of external tensor products:

Lemma 4.1. *For all $X_i \in \mathcal{A}_i$, external tensor product of standard injectives satisfy the formula $I^*_{X_1} \boxtimes I^*_{X_2} \simeq I^*_{(X_1,X_2)}$, and we have a commutative diagram:*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}_1}}(F_1, I^*_{X_1}) \otimes \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}_2}}(F_2, I^*_{X_2}) & \xrightarrow{-\boxtimes-} & \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}_1\times\mathcal{A}_2}}(F_1 \boxtimes F_2, I^*_{(X_1,X_2)}) \\
\downarrow \simeq & & \downarrow \simeq \\
F_1(X_1)^\vee \otimes F_2(X_2)^\vee & = & F_1(X_1)^\vee \otimes F_2(X_2)^\vee \,,
\end{array}$$

where the vertical arrows are Yoneda isomorphisms. Moreover, if $\Bbbk$ is a field, then for all F_1, F_2, G_1, G_2, $-\boxtimes-$ induces an isomorphism:

$$\mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}_1}}(F_1, G_1) \otimes \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}_2}}(F_2, G_2) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}_1\times\mathcal{A}_2}}(F_1 \boxtimes F_2, G_1 \boxtimes G_2) \,.$$

Representations of algebraic groups have a similar external product. For $i = 1, 2$, let G_i be an algebraic group over $\Bbbk$ and let M_i be a G_i-module. The $\Bbbk$-module $M_1 \otimes M_2$ is naturally endowed with the structure of a $G_1 \times G_2$-module, which we denote by $M_1 \boxtimes M_2$. A computation on the Hochschild complex gives:

Lemma 4.2. *For $i = 1, 2$, let G_i be a flat algebraic group over $\Bbbk$ and let M_i be a $\Bbbk$-flat acyclic G_i-module. Assume furthermore that $H^0_{\mathrm{rat}}(G_1, M_1)$ is $\Bbbk$-flat. Then $M_1 \boxtimes M_2$ is an acyclic $G_1 \times G_2$-module and we have an isomorphism:*

$$H^0_{\mathrm{rat}}(G_1, M_1) \otimes H^0_{\mathrm{rat}}(G_2, M_2) \simeq H^0_{\mathrm{rat}}(G_1 \times G_2, M_1 \boxtimes M_2) \,.$$

Moreover, if $\Bbbk$ is a field, then for all M_1, M_2, $-\boxtimes-$ induces an isomorphism:

$$H^*_{\mathrm{rat}}(G_1, M_1) \otimes H^*_{\mathrm{rat}}(G_2, M_2) \simeq H^*_{\mathrm{rat}}(G_1 \times G_2, M_1 \boxtimes M_2) \,.$$

4.2 Application to products of classical groups

Let $\Bbbk$ be a commutative ring. We want to extend the results of section 3 to algebraic groups G_n over $\Bbbk$ which are finite products of classical groups.

To deal with products, we need some notations. Assume that $G_n = \prod_{i=1}^N G_n^i$, where $G_n^i = GL_n, Sp_n$ or $O_{n,n}$. To each factor G_n^i we associate a category $\mathcal{A}_i$, a 'characteristic functor' $F_{G^i} \in \mathcal{P}_{\mathcal{A}_i}$ of degree two, a representation $V_i^n \in \mathcal{A}_i$ and an invariant $e_i^n \in F_{G^i}(V_i^n)$ like in section 3:

G_n^i	GL_n	Sp_n	$O_{n,n}$
$\mathcal{A}_i$	$\mathcal{V}_\Bbbk^{\mathrm{op}} \times \mathcal{V}_\Bbbk$	$\mathcal{V}_\Bbbk$	$\mathcal{V}_\Bbbk$
F_{G^i}	gl	Λ^2	S^2
V_i^n	$(\Bbbk^n, \Bbbk^n)$	$\Bbbk^{2n\vee}$	$\Bbbk^{2n\vee}$
e_i^n	$\mathrm{Id}_{\Bbbk^n}$	ω_n	q_n

For all $d \geq 0$, let $\boxplus : \prod_{i=1}^N \mathcal{P}_{\mathcal{A}_i} \to \mathcal{P}_{\prod \mathcal{A}_i}$ be the functor induced by the direct sum. We define:

$$\mathcal{A} := \textstyle\prod_i \mathcal{A}_i\,, \quad V^n := (V_i^n)\,, \quad F_G := \boxplus_i F_{G^i}\,, \quad e^n := (e_i^n)\,.$$

Terminology 4.3. Let G_n be a finite product of the GL_n, Sp_n or $O_{n,n}$. We shall often denote by $\mathcal{P}_G$ the category of strict polynomial functors with source $\mathcal{A}$ as above. We refer to these functors as the functors 'adapted to G_n'. Indeed for all $n \geq 1$, since the V_i^n have a structure of G_i-module, evaluation on $V^n \in \mathcal{A}$ yields a functor

$$\mathcal{P}_G \to G_n\text{-mod}\,, \quad F \mapsto F_n := F(V^n)\ .$$

Example 4.4. If $G_n = GL_n \times Sp_n$, then $\mathcal{P}_G$ is the category of strict polynomial functors with source $\mathcal{V}_\Bbbk^{\mathrm{op}} \times \mathcal{V}_\Bbbk \times \mathcal{V}_\Bbbk$. For all $n \geq 1$ and any functor F adapted to G_n, the rational G_n-module F_n equals $F(\Bbbk^n, \Bbbk^n, (\Bbbk^{2n})^\vee)$ as a $\Bbbk$-module, and an element $(g,s) \in G_n$ acts by the formula $v \mapsto F(g^{-1}, g, s)(v)$.

Theorem 4.5. *Let $\Bbbk$ be a commutative ring, let n be a positive integer and let G_n be a finite product of the algebraic groups (over $\Bbbk$) GL_n, Sp_n and $O_{n,n}$. For all $F \in \mathcal{P}_G$ we have a $*$-graded map, natural in F:*

$$\phi_{G_n,F} : \mathrm{Ext}^*_{\mathcal{P}_G}(\Gamma^\star(F_G), F) \to H^*_{\mathrm{rat}}(G_n, F_n)\ .$$

The map $\phi_{G_n,-}$ is compatible with cup products:

$$\phi_{G_n,F\otimes F'}(x \cup y) = \phi_{G_n,F}(x) \cup \phi_{G_n,F'}(y)\ .$$

Assume that $2n$ is greater or equal to the degree of F. If one of the factors of G_n equals $O_{n,n}$, assume furthermore that 2 is invertible in $\Bbbk$. Then $\phi_{G_n,F}$ is an isomorphism.

Proof. Once again we use a δ-functor argument. **Step 1.** We build $\phi_{G_n,F}$. First for all d, $\Gamma^d(F_G)$ is homogeneous of degree $2d$, so by homogeneity it suffices to do the construction for a degree $2d$ homogeneous functor F. The element $e^n \in (F_G)_n = F_G(V_n)$ is G_n-invariant, so we have a G_n-equivariant map $\iota^d : \Bbbk \to \Gamma^d((F_G)_n)$, $\lambda \mapsto \lambda(e^n)^{\otimes d}$.

Now a class in $x \in \mathrm{Ext}^i_{\mathcal{P}_{\mathcal{A}}}(\Gamma^\star(F_G), F)$ is represented by an extension $F \hookrightarrow \cdots \twoheadrightarrow \Gamma^{2d}(F_G)$. We define $\phi_{G_n,F}(x)$ as the class of the extension obtained by evaluation on V^n and pullback by ι^d. Following the proof of lemma 3.2, we check that $\phi_{G_n,F}(x)$ is a map of δ-functors, compatible with cup products.

Step 2. Using the exponential isomorphism for S^* and lemma 2.4, we see that the injectives of $\mathcal{P}_{\mathcal{A}}$ are (direct summands in) finite direct sums of injectives of the form $\boxtimes_{i=1}^N I^{d_i}$, where I^{d_i} is either an injective of the form $I^{d_i}_{k,\ell}$ or $I^{d_i}_k$, according to the fact that $\mathcal{A}_i = \mathcal{V}_{\Bbbk}^{\mathrm{op}} \times \mathcal{V}_{\Bbbk}$ or $\mathcal{V}_{\Bbbk}$.

Using this and lemma 4.2, we obtain that $F \mapsto H^*_{\mathrm{rat}}(G_n, F_n)$ is a universal δ-functor. By definition, $F \mapsto \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}}}(\Gamma^\star(F_G), F)$ is also a universal δ-functor.

Step 3. So to finish the proof, it suffices to prove that $\phi^0_{G_n,F}$ is an isomorphism if $2n \geq d$, where d is the degree of F. By left exactness of $F \mapsto H^0_{\mathrm{rat}}(G_n, F_n)$ and $F \mapsto \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(\Gamma^\star(F_G), F)$, it suffices to prove the isomorphism for $F = \boxtimes_{i=1}^N I^{d_i}$, with $\sum d_i \leq d$. But in that case we have a commutative diagram:

$$\begin{tikzcd}
\bigotimes_i \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}_i}}(\Gamma^\star(F_{G^i}), I^{d_i}) \arrow[rr, "\otimes \phi^0_{G^i_n, I^{d_i}}"] \arrow[d, "\simeq"] & & \bigotimes_i H^0_{\mathrm{rat}}(G^i_n, (I^{d_i})_n) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(\boxtimes_{i=1}^N \Gamma^\star(F_{G^i}), \boxtimes_{i=1}^N I^{d_i}) \arrow[d, "\simeq"] & & H^0_{\mathrm{rat}}(G_n, (\boxtimes_{i=1}^N I^{d_i})_n) \\
\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(\Gamma^\star(F_G), \boxtimes_{i=1}^N I^{d_i}) \arrow[urr, "\phi^0_{G_n, \boxtimes_{i=1}^N I^{d_i}}"'] & &
\end{tikzcd}$$

Since for all i, I^{d_i} is a functor of degree $d_i \leq d \leq 2n$, we deduce that the horizontal map of the diagram, hence $\phi^0_{G_n, \boxtimes_{i=1}^N I^{d_i}}$, is an isomorphism. This concludes the proof. $\square$

4.3 Cohomological stabilization

We keep the notations of paragraph 4.2. In particular, $\Bbbk$ is a commutative ring and $G_n = \prod_i G^i_n$, where the G^i_n are copies of the algebraic groups GL_n, Sp_n or $O_{n,n}$ over $\Bbbk$, and V^n denotes the tuple (V^n_i) where the V^n_i are G^i_n-modules (or pairs of G^i_n-modules in the general linear case).

Let $n \leq m$ be two positive integers. For all i we have a standard embedding $\iota_i : G^i_n \hookrightarrow G^i_m$ and a standard G^i_n-equivariant map $\pi_i : V^m_i \twoheadrightarrow V^n_i$.

Let $\iota = \prod \iota_i$ and $\pi = \prod \pi_i$. The pair (ι, π) induces a morphism in rational cohomology:

$$\phi_{n,m} := H^*_{\mathrm{rat}}(G_m, F(V^m)) \xrightarrow{\iota^*} H^*_{\mathrm{rat}}(G_n, F(V^m)) \xrightarrow{F(\pi)_*} H^*_{\mathrm{rat}}(G_n, F(V^n)) .$$

Now theorem 4.5 implies:

Corollary 4.6. *Let $\Bbbk$ be a commutative ring, let n be a positive integer and let G_n be a finite product of copies of GL_n, Sp_n or $O_{n,n}$. Let $F \in \mathcal{P}_G$ be a degree d functor adapted to G_n. Let n, m be two positive integers such that $2m \geq 2n \geq d$. If the orthogonal group appears as one of the factors of G_n, assume furthermore that 2 is invertible in $\Bbbk$. Then the morphism*

$$\phi_{n,m} : H^*_{\mathrm{rat}}(G_m, F_m) \xrightarrow{\simeq} H^*_{\mathrm{rat}}(G_n, F_n)$$

is an isomorphism.

Proof. We check that $(F_G)_m \xrightarrow{F_G(\pi)} (F_G)_n$ sends e^m to e^n. Thus $\phi_{G_n,F} = \phi_{n,m} \circ \phi_{G_m,F}$, and we apply theorem 4.5. □

Remark 4.7. Corollary 4.6 is a good illustration of the differences between our methods for classical algebraic groups and the methods of [9, 7] where classical groups over finite fields are considered as finite groups. Indeed, in our case the cohomological stabilization is a byproduct of the proof, whereas in the finite group case it is needed as an input for the proof.

Notation 4.8. If G_n is a product of copies of GL_n, Sp_n or $O_{n,n}$, and if F is a strict polynomial functor adapted to G_n, we denote by $H^*_{\mathrm{rat}}(G_\infty, F_\infty)$ the stable value of the $H^*_{\mathrm{rat}}(G_n, F_n)$.

5 Products and coproducts on functor cohomology

In this section, $\Bbbk$ is a field (we need this condition because we use in many places the Künneth isomorphism of lemma 4.1). We study product and coproduct structures which arise on functor cohomology $\mathrm{Ext}^*_{\mathcal{P}_\mathcal{A}}(E^*, -)$. Our purpose is to generalize and clarify the tools of [9, Lemma 1.10 and 1.11].

Sections 5.1, 5.2 and 5.3 are introductory. We recall the definition of 'Hopf algebra functor', we introduce the notion of 'Hopf monoidal functor' (which is useful to describe structures on strict polynomial functors E^*, as well as the structures on functor cohomology $\mathrm{Ext}^*(E^*, -)$). Then we recall a classical tool of functor categories [9, 8], namely, the sum-diagonal adjunction. This tool is the key for the existence of coproducts and more generally of Hopf monoidal structures on functor cohomology.

With these tools at our disposal, we make an attempt to classify the Hopf monoidal structure which may arise on extension groups of the form $\mathrm{Ext}^*(E^*, -)$. To be more specific, we give in section 5.4 bijections between:

(1) Hopf algebra structures on E^* (denoted by $(m_E, 1_E, \Delta_E, \epsilon_E)$),

(2) Hopf monoidal structures on E^* (denoted by $(\mu, \eta, \lambda, \epsilon)$),

(3) Hopf monoidal structures on $\mathrm{Hom}(E^*, -)$.

Taking injective resolutions, these structures yield Hopf monoidal structures on $\mathrm{Ext}^*(E^*, -)$.

In fact, we don't need the classification of Hopf monoidal structures on $\mathrm{Ext}^*(E^*, -)$ for our applications. We only need theorem 5.16 which states that a Hopf algebra structure on E^* induces a Hopf monoidal structure on $\mathrm{Ext}^*(E^*, -)$, and gives two equivalent descriptions of the external cup product. But [9, Lemma 1.10 and 1.11] use a superflous hypothesis (the functors need not be exponential), and also has a sign problem, so we thought it was worth clarifying the situation.

Convention on gradings

If $n \geq 0$ is an integer an n-graded object is a family of objects indexed by n-tuples of nonnegative integers (Thus, a 0-graded object is a family indexed by the empty tuple '$()$', in other words a 0-graded object is just a non-graded object). We denote n-gradations by a single '$*$' sign. If $* = (i_1, \dots, i_n)$ and $\star = (j_1, \dots, j_n)$, then $* + \star$ is the tuple $(i_1 + j_1, \dots, i_n + j_n)$, $*\star = (i_1 j_1, \dots, i_n j_n)$ and $|*|$ is the integer $\sum i_k$ (in particular $|()| = 0$).

We often drop the gradings and write X for a multigraded object instead of X^* when no confusion is possible.

5.1 Hopf algebra functors

In this section and in the remainder of the paper, we define Hopf algebras as in [14], that is without requiring an antipode.

Thus if F^* is a n-graded functor from a category $\mathcal{C}$ to the category of $\Bbbk$-vector spaces, a 'n-graded Hopf algebra structure on F^*' is a tuple $(m_F, 1_F, \Delta_F, \epsilon_F)$ of n-graded natural maps

$$F^*(X)^{\otimes 2} \xrightarrow{m_F} F^*(X)\,,\ \Bbbk \xrightarrow{1_F} F^*(X)\,,\ F^*(X) \xrightarrow{\Delta_F} F^*(X)^{\otimes 2}\,,\ F^*(X) \xrightarrow{\epsilon_F} \Bbbk\,,$$

such that for all $X \in \mathcal{C}$, $F^*(X)$ is an n-graded Hopf algebra.

5.2 Hopf monoidal functors

Let $\Bbbk$ be a field, and let $(\mathcal{C}, \Box, e)$ be a symmetric monoidal category [15, VII.7]. We consider the category $\Bbbk$-vect of $\Bbbk$-vector spaces as a symmetric monoidal category, with monoidal product the usual tensor product over $\Bbbk$. We fix an n-graded functor $F^* : \mathcal{C} \to \Bbbk\text{-vect}$. We regard $\Bbbk$ as an n-graded constant functor concentrated in degree $(0,\dots,0)$. A *n-graded monoidal structure* on F^* is a pair (μ,η) of n-graded maps:

$$\mu : F^*(X)\otimes F^\star(Y) \to F^{*+\star}(X\Box Y)\ , \qquad\qquad \eta : \Bbbk \to F^*(e)\ ,$$

which satisfy an associativity and a unit condition [15, XI.2]. By reversing the arrows, one obtains the notion of an n-graded comonoidal structure (λ,ϵ) on F^*. A monoid in $\mathcal{C}$ is an object equipped with a multiplication $M\Box M \to M$ and a unit $e \to M$ satisfying an associativity and a unit condition [15, VII.3]. By reversing the arrows one gets the definition of a comonoid in $\mathcal{C}$. The following lemma is straightforward from the axioms:

Lemma 5.1. *Let $F^* : \mathcal{C} \to \Bbbk$-vect be an n-graded monoidal functor and let M be a monoid in $\mathcal{C}$. The maps:*

$$F^*(M)\otimes F^\star(M) \xrightarrow{\mu} F^{*+\star}(M\Box M) \to F^{*+\star}(M)\ , \quad \Bbbk \xrightarrow{\eta} F^*(e) \to F^*(M)\ ,$$

make $F^*(M)$ into an n-graded algebra. In particular, $F^*(e)$ is an n-graded algebra. Similarly, an n-graded comonoidal functor sends a comonoid to an n-graded coalgebra, and $F^*(e)$ is an n-graded coalgebra.

Let τ be the isomorphism $X\otimes Y \xrightarrow{\simeq} Y\otimes X$, and let τ^* be its n-graded version, which sends the tensor product $x\otimes y$ of an element x of n-degree $*$ and an element y of n-degree $\star$ to $(-1)^{|*\star|}y\otimes x$.

Definition 5.2. A *n-graded Hopf monoidal structure* on F^* is a tuple $(\mu,\lambda,\eta,\epsilon)$ such that:

(0) (μ,η) is an n-graded monoidal structure on F^* and (λ,ϵ) is an n-graded comonoidal structure on F^*.

(1) $\eta : \Bbbk \to F^*(e)$ is a morphism of n-graded coalgebras.

(2) $\epsilon : F^*(e) \to \Bbbk$ is a morphism of n-graded algebras.

(3) The following diagram commutes :

$$\begin{tikzcd}
F(X\Box Y)\otimes F(Z\Box T) \arrow[r, "\lambda\otimes\lambda"'] \arrow[d, "\mu"] & F(X)\otimes F(Y)\otimes F(Z)\otimes F(T) \arrow[d, "F(X)\Box\tau^*\Box F(T)"'] \\
F(X\Box Y\Box Z\Box T) \arrow[d, "F(X\Box\tau\Box T)"] & F(X)\otimes F(Z)\otimes F(Y)\otimes F(T) \arrow[d, "\mu\otimes\mu"'] \\
F(X\Box Z\Box Y\Box T) \arrow[r, "\lambda"] & F(X\Box Z)\otimes F(Y\Box T)\ .
\end{tikzcd}$$

A Hopf monoid in $\mathcal{C}$ is an object M which is both a monoid and a comonoid, and such that (1) the unit $e \to M$ is a map of comonoids, (2) the counit $M \to e$ is a map of monoids, and (3) the comultiplication $M \to M\Box M$ is a map of monoids ($M\Box M$ can be made into a monoid in the obvious way because $\mathcal{C}$ is *symmetric* monoidal). For example, a Hopf monoid in $\Bbbk$-vect is nothing but a Hopf algebra. With this definition we immediately obtain the Hopf analogue of lemma 5.1:

Lemma 5.3. *Let $F^* : \mathcal{C} \to \Bbbk$-vect be an n-graded Hopf monoidal functor and let M be a Hopf monoid in $\mathcal{C}$. The monoid and the comonoid structures on $F^*(M)$ given by lemma 5.1 make $F^*(M)$ into an n-graded Hopf algebra. In particular, $F^*(e)$ is an n-graded Hopf algebra.*

We finish the presentation by giving examples.

Lemma 5.4. *Let $(\mathcal{C}, \Box, e)$ be a symmetric monoidal category and let (F^*, μ, η) be an n-graded symmetric monoidal functor from $\mathcal{C}$ to $\Bbbk$-vect, such that F^* has finite dimensional values, and for all X, Y, $\mu_{X,Y} : F^*(X) \otimes F^*(Y) \to F^*(X\Box Y)$ is an isomorphism. We have:*

(a) the unit η induces an isomorphism $\Bbbk \xrightarrow{\simeq} F^{(0,\dots,0)}(e)$.

(b) Let ϵ denote the composite $F^*(e) \twoheadrightarrow F^{(0,\dots,0)}(e) \simeq \Bbbk$. Then $(\mu, \eta, \mu^{-1}, \epsilon)$ is an n-graded Hopf monoidal structure on F^* if and only if for all Y, Z, the following diagram commutes:

$$\begin{tikzcd} F^*(Y) \otimes F^*(Z) \arrow[r, "\mu_{Y,Z}"] \arrow[d, "\tau^*"] & F^*(Y\Box Z) \arrow[d, "F^*(\tau)"] \\ F^*(Z) \otimes F^*(Y) \arrow[r, "\mu_{Z,Y}"] & F^*(Z\Box Y) . \end{tikzcd}$$

Proof. (a) Because of the unit axiom for (F^*, μ, η), we know that η is injective. Since the $\lambda_{X,Y}$ are isomorphisms, we have $F^{(0,\dots,0)}(e) \simeq F^{(0,\dots,0)}(e\Box e) \simeq F^{(0,\dots,0)}(e)^{\otimes 2}$. Using finite dimension of these vector spaces, we deduce that $F^{(0,\dots,0)}(e)$ is one dimensional, whence the result.

(b) A trivial verification shows that $(\mu, \eta, \mu^{-1}, \epsilon)$ satisfies axioms (0-2) of definition 5.2 (without assuming that the diagram commutes). Now we check that axiom (3) is satisfied if and only if the diagram commutes. To prove the 'only if' part, evaluate axiom (3) on $X = T = e$. To prove the 'if' part, tensor the commutative diagram on the left by $F^*(X)$, on the right by $F^*(Y)$ and use the associativity of μ. □

Example 5.5. Let $(\mathcal{C}, \Box, e)$ be the category $(\mathcal{V}_\Bbbk, \oplus, 0)$ of finite dimensional vector spaces. For all $V \in \mathcal{V}_\Bbbk$ we consider the divided powers $\Gamma^*(V)$ with $\Gamma^d(V)$ in degree $2d$. Then the exponential isomorphism (cf §2.1) $\Gamma^*(V) \otimes \Gamma^*(W) \xrightarrow{\simeq} \Gamma^*(V \oplus W)$ and the unit $\Bbbk = \Gamma^0(0) = \Gamma^*(0)$ satisfy the hypothesis of lemma 5.4. Thus, they induce a graded Hopf monoidal structure on Γ^*. Similarly, $S^*(V)$ (with $S^d(V)$ placed in degree $2d$) and $\Lambda^*(V)$ (with

$\Lambda^d(V)$ placed in degree d) have a Hopf monoidal structure defined by the exponential isomorphism.

Remark 5.6. We warn the reader that for $\Gamma^*(V)$ with $\Gamma^d(V)$ in degree d, the above structure is *not* a Hopf monoidal structure (axiom (3) fails). For an analogous reason, $\Gamma^*(V)$ with $\Gamma^d(V)$ in degree d is not a graded Hopf algebra. In particular, [9, Lemma 1.10] is false as stated, and our lemma 5.4(b) indicates the missing hypothesis. To be more specific, the only 'Hopf exponential functors' which satisfy the conclusion of [9, Lemma 1.10] are the 'skew commutative' ones.

In general, axiom (3) of definition 5.2 is a constraint for the gradings. For example if F^* is a graded Hopf monoidal functor, by 'forgeting' the grading, one does not obtain a non graded Hopf monoidal functor (except if $\Bbbk$ has characteristic two or if F^* is concentrated in even degrees). The same defect arises for multigraded Hopf algebras and lemma 5.3 explains the link.

5.3 The sum-diagonal adjunction

General statements about adjunction isomorphisms in functor categories are given for example in [16]. We sketch here the arguments in our specific case and give explicit formulas.

As usual, $\mathcal{A}$ is a finite product of copies of $\mathcal{V}_\Bbbk$ and $\mathcal{V}_\Bbbk^{\mathrm{op}}$. In particular, $\mathcal{A}$ is an additive category. The diagonal functor $D : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$, $X \mapsto (X, X)$ is left adjoint to the sum functor $\bigoplus : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$, $(X, Y) \mapsto X\oplus Y$. To be more specific, if δ_2 is the diagonal $V \to V \oplus V$, $v \mapsto (v, v)$ and $\mathrm{pr}_i : V_1 \oplus V_2 \to V_i$, $i = 1, 2$, is the projection onto the i-th factor, we easily check that the unit, resp. the counit, of this adjunction equals:

$$\delta_2 : \mathrm{Id}_{\mathcal{A}} \to \bigoplus \circ D\ , \text{ resp. } \quad (\mathrm{pr}_1, \mathrm{pr}_2) : D \circ \bigoplus \to \mathrm{Id}_{\mathcal{A}\times\mathcal{A}}\ .$$

Precomposition by D and $\bigoplus$ yields adjoint functors $-\circ \bigoplus : \mathcal{P}_{\mathcal{A}} \to \mathcal{P}_{\mathcal{A}\times\mathcal{A}}$ and $-\circ D : \mathcal{P}_{\mathcal{A}\times\mathcal{A}} \to \mathcal{P}_{\mathcal{A}}$. Let's be more explicit. We denote by $F(\bigoplus)$ and $G(D)$ the functors F and G precomposed by $\bigoplus$ and D. Then the adjunction isomorphism is given by:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(F(\bigoplus), G) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(F, G(D)) \\ f & \mapsto & f(D) \circ F(\delta_2) \end{array} ,$$

with inverse $g \mapsto G(\mathrm{pr}_1, \mathrm{pr}_2) \circ g(\bigoplus)$. For all $X, Y \in \mathcal{A}$ we have $I^*_{(X,Y)}(D) \simeq I^*_{X\oplus Y}$. Hence $-\circ D$ preserves the injectives. One easily computes:

Lemma 5.7. *Let $X, Y \in \mathcal{A}$. Denote by $\mathrm{pr}_X, \mathrm{pr}_Y$ the projections of $X \oplus Y$ onto X, Y. Then we have a commutative diagram:*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(K, I^*_{(X,Y)}) \arrow[r, "-\circ D"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(K(D), I^*_{X\oplus Y}) \arrow[d, "\simeq"] \\ K(X,Y)^\vee \arrow[r, "K(\mathrm{pr}_X, \mathrm{pr}_Y)^\vee"] & K(X\oplus Y, X\oplus Y)^\vee \end{tikzcd} ,$$

in which the vertical arrows are Yoneda isomorphisms. As a consequence, the adjunction fits into a commutative diagram, in which the vertical arrows are Yoneda isomorphisms:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(F(\bigoplus), I^*_{(X,Y)}) \arrow[r, "\alpha"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(F, I^*_{X\oplus Y}) \arrow[d, "\simeq"] \\
F(X\oplus Y)^\vee \arrow[r, equal] & F(X\oplus Y)^\vee \ .
\end{tikzcd}$$

Since $-\circ D$ preserves the injectives, we may take injective resolutions to obtain:

Lemma 5.8. *For all $F \in \mathcal{P}_{\mathcal{A}}$ and all $G \in \mathcal{P}_{\mathcal{A}\times\mathcal{A}}$, there is an isomorphism, natural in F, G:*

$$\alpha : \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(F(\bigoplus), G) \xrightarrow{\simeq} \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}}}(F, G(D)) \ .$$

Remark 5.9. The functors D and $\bigoplus$ are adjoint on both sides. Using that D is right adjoint of $\bigoplus$ one can get another adjunction isomorphism: $\beta : \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}}}(G(D), F) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(G, F(\bigoplus))$. We don't use this latter isomorphism in this section.

5.4 Hopf monoidal structures on functor cohomology

In this paragraph, $\Bbbk$ is a field and we fix an n-graded functor $E^* \in \mathcal{P}_{\mathcal{A}}$. To avoid cumbersome notations, we drop the '$\mathcal{P}_{\mathcal{A}}$' index on Hom or Ext-groups, as well as the grading on E when no confusion is possible.

We first examine structures which may equip E^*. First, E^* may be endowed with a n-graded Hopf algebra structure on E^*' is a tuple $(m_E, 1_E, \Delta_E, \epsilon_E)$ of n-graded natural maps

$$E(V)^{\otimes 2} \xrightarrow{m_E} E(V)\,,\ \Bbbk \xrightarrow{1_E} E(V)\,,\ E(V) \xrightarrow{\Delta_E} E(V)^{\otimes 2}\,,\ E(V) \xrightarrow{\epsilon_E} \Bbbk\,,$$

such that for all $V \in \mathcal{A}$, $E^*(V)$ is an n-graded Hopf algebra.

On the other hand, the direct sum endows $\mathcal{A}$ with the structure of a symmetric monoidal category. So we may also consider n-graded Hopf monoidal structures on E^*, that is tuples $(\mu, \eta, \lambda, \epsilon)$ with $\mu : E(V)\otimes E(W) \to E(V\oplus W)$, etc. These two kinds of structure are equivalent:

Lemma 5.10. *To any n-graded Hopf monoidal structure $(\mu, \eta, \lambda, \epsilon)$ on E^*, we associate an n-graded Hopf algebra structure on E^* defined as follows:*

$$m_E : E(V)^{\otimes 2} \xrightarrow{\mu_{V,V}} E(V\oplus V) \xrightarrow{E(\Sigma_2)} E(V), \quad 1_E : \Bbbk \xrightarrow{\eta} E(0) \xrightarrow{E(0)} E(V),$$

$$\Delta_E : E(V) \xrightarrow{E(\delta_2)} E(V\oplus V) \xrightarrow{\lambda_{V,V}} E(V)^{\otimes 2}, \quad \epsilon_E : E(V) \xrightarrow{E(0)} E(0) \xrightarrow{\epsilon} \Bbbk.$$

This yields a bijection between the set of n-graded Hopf monoidal structures $(\mu, \eta, \lambda, \epsilon)$ on E^* and the set of n-graded Hopf algebra structures $(m_E, 1_E, \Delta_E, \epsilon_E)$ on E^*.

Proof. For all $V \in \mathcal{A}$, the sum $\Sigma_2 : V \oplus V \to V$ and the diagonal $\delta_2 : V \to V \oplus V$ turn V into a Hopf monoidal object in $(\mathcal{A}, \oplus, 0)$. Hence, by lemma 5.3, $(m_E, 1_E, \Delta_E, \epsilon_E)$ is actually a Hopf algebra structure. To prove the bijection, we give its inverse. If $V_i \in \mathcal{A}$, $i = 1, 2$, we denote by in_i the inclusion of V_i into $V_1 \oplus V_2$ and by pr_i the projection of $V_1 \oplus V_2$ onto its i-th factor. Now from a Hopf algebra structure $(m_E, 1_E, \Delta_E, \epsilon_E)$ we define:

$$\bigotimes E(V_i) \xrightarrow{\otimes E(\mathrm{in}_i)} E(\bigoplus V_i)^{\otimes 2} \xrightarrow{m_E} E(\bigoplus V_i)\,, \quad \Bbbk \xrightarrow{1_E} E(V) \xrightarrow{E(0)} E(0)\,,$$

$$E(\bigoplus V_i) \xrightarrow{\Delta_E} E(\bigoplus V_i)^{\otimes 2} \xrightarrow{\otimes E(\mathrm{pr}_i)} \bigotimes E(V_i)\,, \quad E(0) \xrightarrow{E(0)} E(V) \xrightarrow{\epsilon_E} \Bbbk\,.$$

A straightforward verification shows that this actually gives an n-graded Hopf monoidal structure on E^*, and that this yields the inverse of the map of the lemma. □

Lemma 5.11. *To any n-graded Hopf monoidal structure $(\mu, \eta, \lambda, \epsilon)$ on E^*, we associate an n-graded Hopf monoidal structure on $\mathrm{Hom}(E, -)$ defined as follows:*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}_\mathcal{A}}(E, F) \otimes \mathrm{Hom}_{\mathcal{P}_\mathcal{A}}(E, G) &\xrightarrow[\simeq]{\kappa} \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(E^{\boxtimes 2}, F \boxtimes G)\\ &\xrightarrow{\lambda^*} \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(E(\textstyle\bigoplus), F \boxtimes G) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{P}_\mathcal{A}}(E, F \otimes G)\,,\\ \Bbbk = \mathrm{Hom}(\Bbbk, \Bbbk) &\xrightarrow{\epsilon^*} \mathrm{Hom}(E(0), \Bbbk) \to \mathrm{Hom}(E, \Bbbk)\,,\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}_\mathcal{A}}(E, F \otimes G) &\xrightarrow[\simeq]{\alpha^{-1}} \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(E(\textstyle\bigoplus), F \boxtimes G)\\ &\xrightarrow{\mu^*} \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}\times\mathcal{A}}}(E^{\boxtimes 2}, F \boxtimes G) \xrightarrow[\simeq]{\kappa^{-1}} \mathrm{Hom}_{\mathcal{P}_\mathcal{A}}(E, F) \otimes \mathrm{Hom}_{\mathcal{P}_\mathcal{A}}(E, G)\,,\\ \mathrm{Hom}(E, \Bbbk) &\to \mathrm{Hom}(E(0), \Bbbk) \xrightarrow{\eta^*} \mathrm{Hom}(\Bbbk, \Bbbk) = \Bbbk\ .\end{aligned}$$

This yields a bijection between the n-graded Hopf monoidal structures on E^* and the n-graded Hopf monoidal structures on $\mathrm{Hom}(E^*, -)$.

Proof. By left exactness of the functor $\mathrm{Hom}(E, -)$ and its tensor products, it suffices to check the axioms when F and G are injective. Since the injectives of $\mathcal{P}_\mathcal{A}$ are direct summands of (sums of) standard injectives, we can furthermore assume that $F = I_X^*$ and $G = I_Y^*$, for $X, Y \in \mathcal{A}$. But in that case, by lemmas 4.1 and 5.7, we have a commutative diagram (in which the vertical arrows are Yoneda isomorphisms):

$$\begin{tikzcd} \mathrm{Hom}(E, I_X^*) \otimes \mathrm{Hom}(E, I_Y^*) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(E, I_{X\oplus Y}^*) \arrow[d, "\simeq"] \\ E(X)^\vee \otimes E(Y)^\vee \arrow[r, "\lambda^\vee"] & E(X \oplus Y)^\vee\ , \end{tikzcd}$$

and also a similar diagram involving $\mu^\vee$. Using these two diagrams, we easily check the Hopf monoidal axioms for $\mathrm{Hom}(E,-)$ from the axioms satisfied by $(\mu,\eta,\lambda,\epsilon)$.

Now it remains to show the bijection. Let us give the inverse. If we have an n-graded Hopf monoidal structure on $\mathrm{Hom}(E^*,-)$, we may restrict to the standard injectives I_X^*, $X \in \mathcal{A}$. By the Yoneda isomorphisms, we obtain a Hopf monoidal structure on E^*. The diagrams mentioned above make it clear that this yields the inverse. $\square$

We have proved:

Theorem 5.12. *Let $\Bbbk$ be a field, and let $E^* \in \mathcal{P}_{\mathcal{A}}$ be an n-graded functor. There are bijections between:*

(1) The set of n-graded Hopf algebra structures on E^*.

(2) The set of n-graded Hopf monoidal structures on E^*.

(3) The set of n-graded Hopf monoidal structures on $\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}}}(E^*,-)$.

Explicit formulas for the bijection between (2) and (1), and between (2) and (3) are given in lemmas 5.10 and 5.11. For further use, we also need an explicit link between the product $\mathrm{Hom}(E,F)\otimes\mathrm{Hom}(E,G) \to \mathrm{Hom}(E,F\otimes G)$ and the n-graded Hopf algebra structure of E^*.

Lemma 5.13 (Key formula)**.** *Let $(\mu,\eta,\lambda,\epsilon)$ be an n-graded Hopf monoidal structure on E, and let $(m_E, 1_E, \Delta_E, \epsilon_E)$ be the associated Hopf algebra structure (cf. lemma 5.10). For any functors F_i, $i=1,2$, the following two composites are equal:*

$$\begin{aligned}\mathrm{Hom}(E,F_1)\otimes\mathrm{Hom}(E,F_2) &\xrightarrow[\simeq]{\kappa} \mathrm{Hom}(E^{\boxtimes 2}, F_1\boxtimes F_2)\\ &\xrightarrow{\lambda^*} \mathrm{Hom}(E(\textstyle\bigoplus), F_1\boxtimes F_2) \xrightarrow{\alpha} \mathrm{Hom}(E, F_1\otimes F_2) \qquad (1)\end{aligned}$$

$$\mathrm{Hom}(E,F_1)\otimes\mathrm{Hom}(E,F_2) \xrightarrow{\otimes} \mathrm{Hom}(E^{\otimes 2}, \textstyle\bigotimes_i F_i) \xrightarrow{\Delta_E^*} \mathrm{Hom}(E, \textstyle\bigotimes_i F_i) \quad (2)$$

Proof. We proceed in the same way as in the proof of lemma 5.11. By left exactness of $\mathrm{Hom}(E,-)$ it suffices to prove the formula for standard injectives $F_1 = I_X^*$ and $F_2 = I_Y^*$. In that case, by lemmas 4.1 and 5.7, the first map identifies, through Yoneda isomorphisms, with the map $\lambda^\vee$: $E(X)^\vee \otimes E(Y)^\vee \to E(X\oplus Y)^\vee$. On the other hand, by lemmas 4.1 and 5.7, the second map identifies with the composite:

$$E(X)^\vee \otimes E(Y)^\vee \xrightarrow{E(\mathrm{pr}_X)^\vee \otimes E(\mathrm{pr}_Y)^\vee} E(X\oplus Y)^{\otimes 2} \xrightarrow{\Delta_E^\vee} E(X\oplus Y)^\vee \quad (*)$$

Now by definition (lemma 5.10) $\Delta_E = \lambda_{X\oplus Y, X\oplus Y}\circ E(\delta_2)$. By naturality of λ, $(E(\mathrm{pr}_X)\otimes E(\mathrm{pr}_Y))\circ\lambda_{X\oplus Y,X\oplus Y}\circ E(\delta_2)$ equals $\lambda_{X,Y}\circ E(\mathrm{pr}_X\oplus\mathrm{pr}_Y)\circ E(\delta_2)$ which in turn equals $\lambda_{X,Y}$. Thus $(*)$ equals $\lambda^\vee$, and this concludes the proof. $\square$

Now we turn to Hopf monoidal structures on Ext-groups. Let E^* be an n-graded functor in $\mathcal{P}_{\mathcal{A}}$ and suppose that $\mathrm{Hom}(E,-)$ has an n-graded monoidal structure $(\mu,\eta,\lambda,\epsilon)$. By taking injective resolutions, we obtain $(1+n)$-graded maps $\mu : \bigotimes_i \mathrm{Ext}^*(E,F_i) \to \mathrm{Ext}^*(E,\bigotimes_i F_i)$, $\lambda : \mathrm{Ext}^*(E,\bigotimes_i F_i) \to \bigotimes_i \mathrm{Ext}^*(E,F_i)$, and we also define $\eta : \Bbbk \to \mathrm{Hom}(E,\Bbbk) = \mathrm{Ext}^*(E,\Bbbk)$ and $\epsilon : \mathrm{Ext}^*(E,\Bbbk) = \mathrm{Hom}(E,\Bbbk) \to \Bbbk$. One easily sees that this defines a $(1+n)$-graded Hopf monoidal structure on $\mathrm{Ext}^*(E^*,-)$ which coincides with the Hopf monoidal structure of $\mathrm{Hom}(E^*,-)$ in degree $(0,*)$. Moreover, the resulting structure is a 'δ-Hopf monoidal structure' on $\mathrm{Ext}^*(E,-)$, that is, if we fix one of the two functors F_i, then μ and λ become maps of δ-functors. To sum up we have:

Lemma 5.14. *Let $\Bbbk$ be a field, and let $E^* \in \mathcal{P}_{\mathcal{A}}$ be an n-graded functor. Derivation induces an injection:*

$$\left\{ \begin{array}{c} n\text{-graded Hopf monoidal} \\ \text{structures on } \mathrm{Hom}(E^*,-) \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} (1+n)\text{-graded } \delta\text{-Hopf monoidal} \\ \text{structures on } \mathrm{Ext}^*(E^*,-) \end{array} \right\} .$$

Remark 5.15. The map of lemma 5.14 is not a bijection in general. Indeed, the condition of being a δ-Hopf monoidal structure does not guaranty that the structure is of derived type, i.e. obtained by applying a Hopf monoidal structure on $\mathrm{Hom}(E,-)$ to injective resolutions. To be more specific, the δ condition guaranties that the product μ is of derived type (cf. [14, XII, Thm 10.4]) but in general it is not sufficient to guaranty that the coproduct λ is of derived type (see also [14, Notes of XII.9]).

Now lemmas 5.14, 5.13 and theorem 5.12 yield:

Theorem 5.16. *Let $\Bbbk$ be a field, and let $E^* \in \mathcal{P}_{\mathcal{A}}$ be an n-graded functor, endowed with a Hopf monoidal structure $(\mu,\eta,\lambda,\epsilon)$. The functor cohomology cup product associated (cf. §2.3) to the comultiplication $\Delta_E : E(V) \xrightarrow{E(\delta_2)} E(V\oplus V) \xrightarrow{\lambda} E(V)^{\otimes 2}$ equals the composite:*

$$\mathrm{Ext}^*(E,F)\otimes \mathrm{Ext}^*(E,G) \underset{\simeq}{\xrightarrow{\kappa}} \mathrm{Ext}^*(E^{\boxtimes 2}, F\boxtimes G) \xrightarrow{\lambda^*} \mathrm{Ext}^*(E(\textstyle\bigoplus), F\boxtimes G) \underset{\simeq}{\xrightarrow{\alpha}} \mathrm{Ext}^*(E, F\otimes G) .$$

Together with the following unit, counit and coproduct, they make $\mathrm{Ext}^*(E^*,-)$ into a $(1+n)$-graded Hopf monoidal functor:

$$\Bbbk = \mathrm{Ext}^*(\Bbbk,\Bbbk) \xrightarrow{\epsilon^*} \mathrm{Ext}^*(E,\Bbbk) , \quad \mathrm{Ext}^*(E,\Bbbk) \xrightarrow{\eta^*} \mathrm{Ext}^*(\Bbbk,\Bbbk) = \Bbbk ,$$

$$\mathrm{Ext}^*(E,F\otimes G) \underset{\simeq}{\xrightarrow{\alpha^{-1}}} \mathrm{Ext}^*(E(\textstyle\bigoplus), F\boxtimes G) \xrightarrow{\mu^*} \mathrm{Ext}^*(E^{\boxtimes 2}, F\boxtimes G) \underset{\simeq}{\xrightarrow{\kappa^{-1}}} \mathrm{Ext}^*(E,F)\otimes \mathrm{Ext}^*(E,G) .$$

Corollary 5.17. *Let $\Bbbk$ be a field, and for $i = 1, 2$, let $A_i^* \in \mathcal{P}_{\mathcal{A}}$ be a n_i-graded Hopf algebra functor. The maps:*

$$\begin{aligned}
&\mathrm{Ext}^*(A_1, A_2) \otimes \mathrm{Ext}^*(A_1, A_2) \to \mathrm{Ext}^*(A_1, A_2 \otimes A_2) \to \mathrm{Ext}^*(A_1, A_2)\\
&\Bbbk \to \mathrm{Ext}^*(A_1, \Bbbk) \to \mathrm{Ext}^*(A_1, A_2)\\
&\mathrm{Ext}^*(A_1, A_2) \to \mathrm{Ext}^*(A_1, A_2 \otimes A_2) \to \mathrm{Ext}^*(A_1, A_2) \otimes \mathrm{Ext}^*(A_1, A_2)\\
&\mathrm{Ext}^*(A_1, A_2) \to \mathrm{Ext}^*(A_1, \Bbbk) \to \Bbbk
\end{aligned}$$

make $\mathrm{Ext}^*_{\mathcal{P}_{\mathcal{A}}}(A_1^*, A_2^*)$ *into a* $(1 + n_1 + n_2)$*-graded Hopf algebra.*

Proof. To get corollary 5.17 from theorem 5.16, we just apply a graded version of lemma 5.3 (which holds for additive functors). □

Remark 5.18. Corollary 5.17 is a generalization of [9, Lemma 1.11]. Indeed, we don't require our functors A_i^* to be exponential functors. In section 6, we apply this corollary to Hopf algebra functors which are *not* exponential functors.

Remark 5.19. In this section, the proofs rely on (1) Yoneda isomorphisms for standard injectives, (2) adjunction between the sum and the diagonal functors, (3) Künneth formulas. Properties (1) and (2) hold in the category $\mathcal{F}_{\mathcal{A}}$ of ordinary functors with source an additive category $\mathcal{A}$ and target $\Bbbk$-vect. The Künneth formula also holds if one assumes furthermore some finiteness conditions on the functors (either if F_1, F_2 have finite dimensional values and G_1, G_2 have injective resolutions by *finite* sums of standard injectives, or if F_1, F_2 have projective resolutions by *finite* sums of projectives). Up to these slight finiteness conditions, the results of this section holds in the category $\mathcal{F}_{\mathcal{A}}$. This gives interesting applications for the stable cohomology of the finite classical groups $O_{n,n}(\mathbb{F}_q)$ and $Sp_n(\mathbb{F}_q)$ with twisted coefficients [7].

6 Applications

6.1 Stable products and coproducts for classical groups

Theorem 6.1. *Let $\Bbbk$ be a field. Let G_n be a product of copies of the groups GL_n, Sp_n or $O_{n,n}$, and let F_1, F_2 be strict polynomial functors adapted to G_n (cf. terminology 4.3). If $O_{n,n}$ is a factor in G_n, assume that $\Bbbk$ has odd characteristic. The stable rational cohomology of G_n is equipped with a coproduct:*

$$H^*_{\mathrm{rat}}(G_\infty, (F_1 \otimes F_2)_\infty) \to H^*_{\mathrm{rat}}(G_\infty, F_{1\,\infty}) \otimes H^*_{\mathrm{rat}}(G_\infty, F_{2\,\infty})\ .$$

Together with the usual cup product (cf. §2.4), they endow $H^*_{\mathrm{rat}}(G_\infty, -)$ with the structure of a graded Hopf monoidal functor.

Moreover, the cup product is a section of the coproduct.

Proof. We consider the usual graded Hopf algebra structure $\Gamma^*(V)$, with $\Gamma^d(V)$ in degree $2d$, cf. paragraph 2.1. Let $F_G \in \mathcal{P}_G$ be the characteristic functor associated to G_n. If we set $V = F_G$, the divided powers of F_G are a graded Hopf algebra, or equivalently a graded Hopf monoidal functor. To be more specific, the product μ and the coproduct λ are given by the formulas:

$$\begin{aligned}\mu : \Gamma^*(F_G(V)) \otimes \Gamma^*(F_G(W)) &\to \Gamma^*(F_G(V \oplus W))^{\otimes 2}\\ &\simeq \Gamma^*(F_G(V \oplus W)^{\oplus 2}) \to \Gamma^*(F_G(V \oplus W))\ ,\\ \lambda : \Gamma^*(F_G(V \oplus W)) &\to \Gamma^*(F_G(V \oplus W)^{\oplus 2})\\ &\simeq \Gamma^*(F_G(V \oplus W))^{\otimes 2} \to \Gamma^*(F_G(V)) \otimes \Gamma^*(F_G(W))\ .\end{aligned}$$

In particular, one checks that $\lambda \circ \mu = \mathrm{Id}$. Thus, by theorem 5.16, $\mathrm{Ext}^*(\Gamma^\star(F_G), -)$ is a bigraded Hopf monoidal functor, and $\lambda \circ \mu = \mathrm{Id}$ implies that the external cup product is a section of the coproduct.

Since the divided powers of F_G are concentrated in even degree, we may forget the grading arising from the divided powers (cf. remark 5.6) and $\mathrm{Ext}^*(\Gamma^\star(F_G), -)$ is a $*$-graded Hopf monoidal functor. Then it suffices to apply theorem 4.5 to conclude the proof. □

Corollary 6.2. *Let $\Bbbk$ be a field. Let G_n be a product of copies of the groups GL_n, Sp_n or $O_{n,n}$, and let F_1, F_2 be two functors of degree d_1, d_2 adapted to G_n. If $O_{n,n}$ is a factor in G_n, assume that $\Bbbk$ has odd characteristic. For all n such that $2n \geq d_1 + d_2$, the cup product induces a injection:*

$$H^*_{\mathrm{rat}}(G_n, (F_1)_n) \otimes H^*_{\mathrm{rat}}(G_n, (F_2)_n) \hookrightarrow H^*_{\mathrm{rat}}(G_n, (F_1)_n \otimes (F_2)_n)\ .$$

Remark 6.3. The injectivity in odd degree cohomological degree does not contradict the usual commutativity formula $x \cup y = (-1)^{\deg(x)\deg(y)} y \cup x$. Indeed, this latter formula holds only for *internal* cup products. If τ denotes the isomorphism $(F_1)_n \otimes (F_2)_n \simeq (F_2)_n \otimes (F_1)_n$, the commutativity relation for external cup products is $x \cup y = (-1)^{\deg(x)\deg(y)} H^*_{\mathrm{rat}}(G_n, \tau)(y \cup x)$.

Corollary 6.4. *Let $\Bbbk$ be a field. Let G_n be a product of copies of the groups GL_n, Sp_n or $O_{n,n}$, and let A^* be an n-graded strict polynomial functor adapted to G_n, endowed with the structure of a Hopf algebra. If $O_{n,n}$ is a factor in G_n, assume that $\Bbbk$ has odd characteristic. The usual cup product $H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)^{\otimes 2} \to H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)$ may be supplemented with a coproduct $H^*_{\mathrm{rat}}(G_\infty, A^*_\infty) \to H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)^{\otimes 2}$ which endow $H^*_{\mathrm{rat}}(G_\infty, A^*_\infty)$ with the structure of a $(1 + n)$-graded Hopf algebra.*

6.2 A new statement for the universal classes

As an application of theorem 6.1, we give a nicer formulation of the existence of the universal cohomology classes [20, Thm 4.1]. Consider the divided powers $\Gamma^*(H^*_{\mathrm{rat}}(GL_\infty, gl^{(1)}_\infty))$, with the usual Hopf algebra structure but regraded in the following way: the bidegree of an element in $\Gamma^i(H^j_{\mathrm{rat}}(GL_\infty, gl_\infty^{(1)}))$ is $(2ij, 2i)$. We easily get:

Corollary 6.5. *Let $\Bbbk$ be a field of positive characteristic p. The existence of the universal cohomology classes is equivalent to the following statement.*

There is a bigraded Hopf algebra morphism

$$\psi : \Gamma^*(H^*_{\mathrm{rat}}(GL_\infty, gl_\infty^{(1)})) \to H^*_{\mathrm{rat}}(GL_\infty, \Gamma^*(gl_\infty^{(1)}))\ ,$$

such that for all $n \geq p$ the following composite is a monomorphism:

$$\Gamma^*(H^*_{\mathrm{rat}}(GL_\infty, gl_\infty^{(1)})) \xrightarrow{\psi} H^*_{\mathrm{rat}}(GL_\infty, \Gamma^*(gl_\infty^{(1)})) \xrightarrow{\phi_{n,\infty}} H^*_{\mathrm{rat}}(GL_n, \Gamma^*(gl_n^{(1)}))\ .$$

6.3 Cohomology computations for the orthogonal and symplectic groups

In [7], Djament and Vespa showed how to obtain cohomological computations for the orthogonal and symplectic groups from the computations of [9]. Their method adapts easily to the strict polynomial functor setting. For all $r \geq 0$, we denote by $I^{(r)} \in \mathcal{P}$ the r-th Frobenius twist [10, (v) p.224]. We consider the Hopf algebra $S^*(I^{(r)})$ (resp. $\Lambda^*(I^{(r)})$) with $S^d(I^{(r)})$ in degree $2d$ (resp. with $\Lambda^d(I^{(r)})$ in degree d). We have:

Theorem 6.6. *Let $\Bbbk$ be a field of odd characteristic. Let r be a nonnegative integer.*

(i) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(O_{\infty,\infty}, S^\star(I^{(r)})_\infty)$ is a symmetric Hopf algebra on generators e_m of bidegree $(2m, 4)$ for $0 \leq m < p^r$.

(ii) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(Sp_\infty, S^\star(I^{(r)})_\infty)$ is trivial.

(iii) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(O_{\infty,\infty}, \Lambda^\star(I^{(r)})_\infty)$ is trivial.

(iv) The bigraded Hopf algebra $H^*_{\mathrm{rat}}(Sp_\infty, \Lambda^\star(I^{(r)})_\infty)$ is a divided power Hopf algebra on generators e_m of bidegree $(2m, 2)$ for $0 \leq m < p^r$.

We get a proof by following closely [7, Section 4]. For sake of completeness, we give some details in the remainder of this paragraph. Let $\Bbbk$ be a field of characteristic $p > 2$. As in section 3, we denote by $\mathcal{P}(1,1)$ the category of strict polynomial functors with source $\mathcal{V}_\Bbbk^{\mathrm{op}} \times \mathcal{V}_\Bbbk$ and by $\mathcal{P}$ the category of strict polynomial functors with source $\mathcal{V}_\Bbbk$. We also denote by $\mathcal{P}(2)$ the category strict polynomial functors with source $\mathcal{V}_\Bbbk \times \mathcal{V}_\Bbbk$.

Let $E^* = S^*(I^{(r)})$ (with $S^d(I^{(r)})$ in degree $2d$) or $E^* = \Lambda^*(I^{(r)})$ (with $\Lambda^d(I^{(r)})$ in degree d), or more generally let E^* be a 'skew-commutative Hopf exponential functor' (cf. [9, p. 675 and Def. 1.9]. Equivalently, E^* is a graded functor in $\mathcal{P}$ satisfying all the hypotheses of lemma 5.4). We wish to compute the bigraded Hopf algebra $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(F_G), E^*)$, with $F_G = S^2$

or $F_G = \Lambda^2$ (it is a trigraded Hopf algebra by corollary 5.17, and we may drop the gradation on $\Gamma^\star(F)$, since this gradation is concentrated in even degrees, cf. remark 5.6). Indeed, by theorem 4.5, this bigraded Hopf algebra is isomorphic to the bigraded Hopf algebra $H^*_{\mathrm{rat}}(G_\infty, E^*_\infty)$ with $G_n = O_{n,n}$ for $F_G = S^2$, and $G_n = Sp_n$ for $F_G = \Lambda^2$.

To do the computation, it suffices to compute the bigraded Hopf algebra $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), E^*)$, together with the involution θ of bigraded Hopf algebras, induced by the permutation $V\otimes V \simeq V\otimes V$ which exchanges the factors of $\otimes^2$. Indeed, since $\Bbbk$ has characteristic $p\neq 2$, F_G is a direct summand in $\otimes^2$. As a result, $H^*_{\mathrm{rat}}(G_\infty, E^*_\infty)$ equals the image of $(1+\theta)/2$ in the orthogonal case and of $(1-\theta)/2$ in the symplectic case. So we now concentrate on $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), E^*)$.

Let $I\in\mathcal{P}$ denote the identity functor of $\mathcal{V}_\Bbbk$. Using the sum diagonal adjunction of remark 5.9, and the exponential isomorphism for E^*, we obtain an isomorphism of multigraded vector spaces (on the right, we take the total degree of $E^*\boxtimes E^*$):

$$\mathrm{Ext}^*_{P}(\Gamma^\star(\otimes^2), E^*) \simeq \mathrm{Ext}^*_{P(2)}(\Gamma^\star(I\boxtimes I), E^*\boxtimes E^*)\ .$$

Now if $B\in\mathcal{P}(2)$ is a strict polynomial functor with 2 covariant variables, one may precompose the first variable of B by the duality functor $-^\vee : \mathcal{V}_\Bbbk \to \mathcal{V}_\Bbbk^{\mathrm{op}}, V\mapsto V^\vee$. One obtains a strict polynomial functor $B(-^\vee,-)\in\mathcal{P}(1,1)$. This yields an equivalence of categories between $\mathcal{P}(2)$ and $\mathcal{P}(1,1)$. Let $(E^i)^\sharp$ denote the strict polynomial functor $V\mapsto E^i(V^\vee)^\vee$. Using [8, Thm 1.5 (1.11)] we obtain isomorphisms of bigraded vector spaces (recall that we don't take the gradation of $\Gamma^\star(\otimes^2)$ into account):

$$\begin{aligned}\mathrm{Ext}^*_{P(2)}(\Gamma^\star(I\boxtimes I), E^*\boxtimes E^*) &\simeq \mathrm{Ext}^*_{P(1,1)}(\Gamma^\star(gl), E^*(-^\vee)\boxtimes E^*)\\ &\simeq \mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*)\ .\end{aligned}$$

To sum up, we have an isomorphism of bigraded vector spaces (on the left we don't take the gradation of $\Gamma^*(\otimes^2)$ into account, on the right we take the total gradation associated to the gradations of $E^{*\,\sharp}$ and E^*):

$$\mathrm{Ext}^*_{P}(\Gamma^\star(\otimes^2), E^*) \simeq \mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*)\quad (*)$$

Both objects have a bigraded Hopf algebra structure by corollary 5.17. We need the Hopf algebra structure of $\Gamma^\star(\otimes^2)$ to define the bigraded Hopf algebra structure on the left but not to define the one on the right. Nonetheless:

Lemma 6.7 ([7, Prop 4.10])**.** *For all 'skew commutative exponential functor' $E^*\in\mathcal{P}$, the isomorphism $(*)$ is compatible with the bigraded Hopf algebra structures.*

For $E^* = S^*(I^{(r)})$ or $E^* = \Lambda^*(I^{(r)})$, the Hopf algebra $\mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*)$ is computed in [9, Thm 5.8]. So it remains to describe how the involution θ

acts on these extension groups. For all F, G, we have an isomorphism (see for example [9, lemma 1.12]) $\mathrm{Ext}^*_{\mathcal{P}}(F^\sharp, G) \simeq \mathrm{Ext}^*_{\mathcal{P}}(G^\sharp, F)$. With $F = G = E^*$, we obtain an involution:

$$\widetilde{\theta} : \mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*) \xrightarrow{\simeq} \mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*) \;.$$

Lemma 6.8 ([7, Lemme 4.12])**.** *For all 'skew commutative Hopf exponential functor' $E^* \in \mathcal{P}$, we denote by $\widetilde{\theta}^*$ the involution of $\mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*)$ whose restriction to $\mathrm{Ext}^*_{\mathcal{P}}(E^{i\,\sharp}, E^j)$ equals $(-1)^{ij}\widetilde{\theta}$. We have a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), E^*) \arrow[r, "\simeq", "(*)"'] \arrow[d, "\theta"] & \mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*) \arrow[d, "\widetilde{\theta}^*"] \\
\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), E^*) \arrow[r, "\simeq", "(*)"'] & \mathrm{Ext}^*_{\mathcal{P}}(E^{*\,\sharp}, E^*) \;.
\end{tikzcd}$$

We are now ready to use the computations of [9]. We first recall the results we need. If V^* is a graded vector space concentrated in even degrees, we consider the vector spaces $S^*(V^*)$ bigraded in the following way: the bidegree of an element $S^i(V^j)$ is $(ij, 4i)$. With this grading, the usual Hopf algebra structure on the symmetric powers makes $S^*(V^*)$ into a bigraded Hopf algebra. We have (recall that an element in $\mathrm{Ext}^k_{\mathcal{P}}(\Gamma^\ell(I^{(r)}), S^m(I^{(r)}))$ has bidegree $(k, 2\ell + 2m)$):

Lemma 6.9 ([9, Thm 4.5 and Thm 5.8])**.** *For all $n \geq 0$, the bigraded Hopf algebra multiplication*

$$\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^1(I^{(r)}), S^1(I^{(r)})))^{\otimes n} \to \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^n(I^{(r)}), S^n(I^{(r)})))$$

is surjective. It induces an isomorphism of Hopf algebras:

$$S^*(\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})) \simeq \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^*(I^{(r)}), S^*(I^{(r)}))) \;.$$

Since the involution θ is compatible with the Hopf algebra structure, the first part of lemma 6.9 shows that knowing the involution $\widetilde{\theta}$ on $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$is sufficient to determine θ. The involution $\widetilde{\theta}$ is already computed by Djament and Vespa:

Lemma 6.10 ([7, Lemme 4.13])**.** *The involution $\widetilde{\theta}$ equals the identity map.*

Thus, by lemmas 6.8, 6.9 and 6.10, the map $(1 + \theta)/2$: $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), S^*(I^{(r)})) \to \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), S^*(I^{(r)}))$ equals the identity map, so that we have:

$$\begin{aligned}
&\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(S^2), S^*(I^{(r)})) \simeq \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\otimes^2), S^*(I^{(r)})) \;, \\
&\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\star(\Lambda^2), S^*(I^{(r)})) \simeq 0 \;.
\end{aligned}$$

Now we may apply lemmas 6.7 and 6.9 to conclude the proof of 6.6(i) and (ii). The computation for the exterior powers $\Lambda^*(I^{(r)})$ is similar.

References

[1] H.H. Andersen, J.C. Jantzen, Cohomology of induced representations for algebraic groups. Math. Ann. 269 (1984), no. 4, 487–525.

[2] T. Bühler, Exact Categories, Expositiones Mathematicae 28 (2010) 1–69, doi: 10.1016/j.exmath.2009.04.004

[3] M. Chałupnik, Extensions of strict polynomial functors. Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 5, 773–792.

[4] E. Chaput, M. Romagny, On the adjoint quotient of Chevalley groups over arbitrary base schemes, `ArXiv:0805.2140`

[5] E. Cline, B. Parshall, L. Scott, W. van der Kallen, Rational and generic cohomology, Invent. Math. 39 (1977), 143-163.

[6] C. de Concini, C. Procesi, A characteristic free approach to invariant theory. Advances in Math. 21 (1976), no. 3, 330–354.

[7] A. Djament, C. Vespa, Sur l'homologie des groupes orthogonaux et symplectiques à coefficients tordus, Ann. Sci. École Norm. Sup. (4) 43 (2010), no. 3.

[8] V. Franjou, E. M. Friedlander, Cohomology of bifunctors, Proceedings of the London Mathematical Society (2008), doi:10.1112/plms/pdn005

[9] V. Franjou, E. Friedlander, A. Scorichenko, A. Suslin, General linear and functor cohomology over finite fields, Ann. of Math. (2) 150 (1999), no. 2, 663–728.

[10] E. Friedlander, A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (1997), 209–270.

[11] A. Grothendieck, Sur quelques points d'algèbre homologique. (French) Tôhoku Math. J. (2) 9 1957 119–221.

[12] J.-C. Jantzen, Representations of Algebraic Groups, Mathematical Surveys and Monographs vol. 107, Amer. Math. Soc., Providence, 2003.

[13] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, The book of involutions. American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI, 1998. xxii+593 pp. ISBN: 0-8218-0904-0.

[14] S. Mac Lane, Homology. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. x+422 pp. ISBN: 3-540-58662-8.

[15] S. Mac Lane, Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp. ISBN: 0-387-98403-8.

[16] T. Pirashvili, Introduction to functor homology. Rational representations, the Steenrod algebra and functor homology, 1–26, Panor. Synthèses, 16, Soc. Math. France, Paris, 2003.

[17] D. Quillen, Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[18] A. Suslin, E. Friedlander, C. Bendel, Infinitesimal 1-parameter subgroups and cohomology. J. Amer. Math. Soc. 10 (1997), no. 3, 693–728.

[19] A. Touzé, Universal classes for algebraic groups, Duke Math. J. 151 (2010), no. 2, 219–249.

[20] A. Touzé, W. van der Kallen, Bifunctor cohomology and cohomological finite generation for reductive groups, Duke Math. J. 151 (2010), no. 2, 251–278.