ALTERNATING EULER SUMS AND SPECIAL VALUES OF WITTEN MULTIPLE ZETA FUNCTION ATTACHED TO $\mathfrak{so}(5)$

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Abstract. In this note we shall study the Witten multiple zeta function associated to the Lie algebra $\mathfrak{so}(5)$ defined by Matsumoto. Our main result shows that its special values at nonnegative integers are always expressible by alternating Euler sums. More precisely, every such special value of weight $w \geq 3$ is a finite rational linear combination of alternating Euler sums of weight w and depth at most two, except when the only nonzero argument is one of the two last variables in which case $\zeta(w-1)$ is needed.

1. INTRODUCTION

The Witten multiple zeta function associated to the Lie algebra $\mathfrak{so}(5)$ is defined by Matsumoto as follows:

$$\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}}, \tag{1}$$

which converges whenever $\Re(s_1 + s_3 + s_4) > 1$, $\Re(s_2 + s_3 + s_4) > 1$ and $\Re(s_1 + s_2 + s_3 + s_4) > 2$. We call $s_1 + s_2 + s_3 + s_4$ the *weight*. Essouabri [2] and Matsumoto [9] have defined more general multiple zeta functions and studied their analytic continuations. However, the function in (1) itself already generalizes both the zeta function $\zeta_{\mathfrak{so}(5)}(s, s, s, s)$ suggested by Zagier [17, §7] after Witten [16] and the Mordell-Tornheim double zeta function [12, 13] (see §2.2). Zagier and Garoufalidis independently showed that for every positive integer m there is some $c(m) \in \mathbb{Q}$ such that

$$\zeta_{\mathfrak{so}(5)}(2m, 2m, 2m, 2m) = c(m) \cdot \pi^{8m}. \tag{2}$$

Special values like these are the main objects of study in this note.

In [15] Tsumura considered the special values of of (1) at nonnegative integers. In particular, when the weight is an odd number he showed that the special values of (1) are $\mathbb{Q}$-linear combinations of products of Riemann zeta values at positive integers, with slightly stronger restrictions on the arguments than just to guarantee convergence. Since the function $\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4)$ has depth two this type of results is commonly referred to as a "parity" relation. For example, the (Euler-Zagier) multiple zeta value (MZV for short) at positive integers

$$\zeta(s_1, \ldots, s_d) := \sum_{m_1 > \cdots > m_d \geq 1} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d} \tag{3}$$

has the well-known property that if the weight (> 2) and the depth have different parities then it can be written as a $\mathbb{Q}$-linear combination of products of MZVs of lower depths (see [4, 14]). In general it is expected that when the weight is even (and large enough) we do not always have such relations.

In this note we will investigate all the convergent special values of $\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4)$ at nonnegative integers without any parity restriction on the weight. It turns out that they are closely related to the *alternating Euler sums* (see §2.1). Our main result is

Theorem 1.1. *Let $s_1, \ldots, s_4$ be nonnegative integers such that $s_1 + s_3 + s_4 > 1$, $s_2 + s_3 + s_4 > 1$ and $w := s_1 + s_2 + s_3 + s_4 > 2$. Then $\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4)$ can be expressed as a finite $\mathbb{Q}$-linear combination of alternating Euler sums of weight w and depths at most two, except when $s_1 = s_2 = s_3 = 0$ or $s_1 = s_2 = s_4 = 0$ in which cases $\zeta(w-1)$ is needed.*

As a final remark we point out that the alternating Euler sums or more generally, special values of multiple polylogarithms at roots of unity [20], can be used to study many other types of multiple zeta functions such as those appearing in the recent work of Komori, Matsumoto and Tsumura [5, 6, 7]. This will be carried out in detail in another work.

2. SOME PRELIMINARIES

2.1. **Alternating Euler sum.** For positive integers $s_1, \ldots, s_d \in$ we define the *alternating Euler sum* by

$$\zeta(s_1, \ldots, s_d; x_1, \ldots, x_d) := \sum_{m_1 > \cdots > m_d \geq 1} \frac{x_1^{m_1} \cdots x_d^{m_d}}{m_1^{s_1} \cdots m_d^{s_d}}, \quad (s_1, x_1) \neq (1,1), \tag{4}$$

where $x_j = \pm 1$ for all $1 \leq j \leq l$. We call $s_1 + \cdots + s_d$ the *weight* and d the *depth*. To save space, if $x_j = -1$ then $\bar{s}_j$ will be used. For example, we have $\zeta(\bar{1}) = \zeta(1; -1) = -\ln 2$ and the striking identity [19] that for every positive integer n

$$\zeta(\{3\}^n) = 8^n \zeta(\{\bar{2}, 1\}^n),$$

where $\{S\}^n$ means the string S repeats n times. Identities like these which are derived by (regularized) double shuffle relations will be crucial to simplify our computations in the last section.

2.2. **Mordell-Tornheim zeta functions.** They are defined by (see [12, 13])

$$\zeta_{\mathrm{MT}}(s_1, \ldots, s_d; s) = \sum_{m_1, \ldots, m_d = 1}^{\infty} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d} (m_1 + \cdots + m_d)^{-s}. \tag{5}$$

Recently Zhou and Bradley have shown [22, Thm. 4] that (5) converges absolutely if $\Re(s) + \sum_{j=1}^{\ell} \Re(s_{i_j}) > \ell$ for each nonempty subset $\{i_1, \ldots, i_\ell\}$ of $\{1, 2, \ldots, d\}$. We can use integral test and the well-known formula

$$\sum_{m=1}^{n} m^t = \frac{1}{t+1} \Big(B_{t+1}(n+1) - B_{t+1}(0) \Big),$$

where $B_{t+1}(x)$ is the Bernoulli polynomial, to extend their proof to the following necessary and sufficient conditions for convergence when all arguments are integers.

Proposition 2.1. *Let $s_1, \ldots, s_d$ and s be arbitrary integers. Then the Mordell-Tornheim zeta function $\zeta_{\mathrm{MT}}(s_1, \ldots, s_d; s)$ converges if and only if*

$$s + \sum_{j=1}^{\ell} s_{i_j} > \ell$$

for each nonempty subset $\{i_1, \ldots, i_\ell\}$ of $\{1, 2, \ldots, d\}$.

The main result of [22] is the following

Proposition 2.2. *([22, Thm. 5]) Let $s_1, \ldots, s_d$ and s be nonnegative integers. If at most one of them is equal to 0 then the Mordell-Tornheim zeta value $\zeta_{\mathrm{MT}}(s_1, \ldots, s_d; s)$ can be expressed as a $\mathbb{Q}$-linear combination of MZVs of the same weight and depth.*

In this note we will only need this proposition when the depth is two.

2.3. **Convergence domain of $\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4)$.** In the following proposition we only consider integer arguments although it is not hard to extend it to the complex variable situation. The result can be derived from the concrete singularity set given in [6] but the following proof is more straight-forward.

Proposition 2.3. *Let $s_1, \ldots, s_4$ be nonnegative integers. Then*

$$\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}}$$

converges if and only if

$$s_1 + s_3 + s_4 > 1, s_2 + s_3 + s_4 > 1, \text{ and } s_1 + s_2 + s_3 + s_4 > 2. \tag{6}$$

Proof. First we observe that for all $m, n > 0$

$$m + n < m + 2n < 2(m+n).$$

Hence

$$\frac{1}{2^{s_4}} \zeta_{\mathrm{MT}}(s_1, s_2; s_3 + s_4) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (2m+2n)^{s_4}}$$

$$\leq \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}} \leq \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3+s_4}} = \zeta_{\mathrm{MT}}(s_1, s_2; s_3 + s_4).$$

The proposition now follows immediately from the the convergence criterion of the Mordell-Tornheim double zeta function in Prop. 2.1. $\square$

2.4. **A combinatorial lemma.** The following lemma will be used heavily throughout the proof of Theorem 1.1.

Lemma 2.4. ([22, Lemma 1]) *Let r and $n_1, \ldots, n_r$ be positive integers, and let $x_1, \ldots, x_r$ be non-zero real number such that $x_1 + \cdots + x_r \neq 0$. Then*

$$\prod_{j=1}^{r} \frac{1}{x_j^{n_j}} = \sum_{j=1}^{r} \left(\prod_{\substack{k=1 \\ k \neq j}}^{r} \sum_{a_k=0}^{n_k-1} \right) \frac{M_j}{x_j^{n_j + A_j}} \prod_{\substack{k=1 \\ k \neq j}}^{r} \frac{1}{x_k^{n_k - a_k}},$$

where the multi-nomial coefficient

$$M_j = \frac{(n_j + A_j - 1)!}{(n_j - 1)!} \prod_{\substack{k=1 \\ k \neq j}}^{r} \frac{1}{a_k!} \quad \text{and} \quad A_j = \sum_{\substack{k=1 \\ k \neq j}}^{r} a_k.$$

The notation $\prod_{\substack{k=1 \\ k \neq j}}^{r} \sum_{a_k=0}^{n_k-1}$ *means the multiple sum* $\sum_{a_1=0}^{n_1-1} \cdots \sum_{a_{j-1}=0}^{n_{j-1}-1} \sum_{a_{j+1}=0}^{n_{j+1}-1} \cdots \sum_{a_r=0}^{n_r-1}.$

3. Proof of Theorem 1.1

We now use a series of reductions to prove the theorem.

Case (i). If $s_4 = 0$ then we just get a Mordell-Tornheim double zeta value so the theorem is mostly handled by Prop. 2.2 except for the case $s_1 = s_2 = 0$. Then assuming $s \geq 3$ we have

$$\zeta_{\mathfrak{so}(5)}(0, 0, s, 0) = \zeta(s, 0) = \sum_{m > n \geq 1} \frac{1}{m^s} = \sum_{m=1}^{\infty} \frac{m-1}{m^s} = \zeta(s-1) - \zeta(s). \tag{7}$$

Thus Theorem 1.1 is true in this case. This is the first one of the two exceptional cases in which we need the Riemann zeta value with the weight lowered by one.

We now assume $s_4 > 0$ in the rest of the proof. If $s_3 = 0$ and $s_2 > 0$ (resp. $s_2 = 0$ and $s_1 > 0$, resp. $s_1 = s_2 = 0$) then one can go directly to Case (iii.a) (resp. Case (iii.b), resp. Case (iii.c)) below. Otherwise we must have $s_2, s_3, s_4 > 0$ which is Case (ii) next.

Case (ii). Assume $s_2, s_3, s_4 > 0$. In Lemma 2.4 taking $x_1 = n$ and $x_2 = m + n$ we get:

$$\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4) = \sum_{a_2=0}^{s_2-1} \binom{s_3 + a_2 - 1}{a_2} \zeta_{\mathfrak{so}(5)}(s_1, s_2 - a_2, 0, s_4 + s_3 + a_2) \tag{8}$$

$$+ \sum_{a_3=0}^{s_3-1} \binom{s_2 + a_3 - 1}{a_3} \zeta_{\mathfrak{so}(5)}(s_1, 0, s_3 - a_3, s_4 + s_2 + a_3). \tag{9}$$

We recommend the interested reader to check the convergence of the above values by (6). The rule of thumb is as follows: if we apply Lemma 2.4 with each x_j a positive combination of indices then the convergence is automatically guaranteed. In each of the following steps we often omit this convergence checking since it is straight-forward in most cases. The only exception is (13) which in fact poses the most difficulty.

Note that the weight is kept unchanged in the above so we are led to the following three cases:

(iii.a). $s_3 = 0, s_2, s_4 > 0$ from (8) since $s_2 - a_2 > 0$,

(iii.b). $s_2 = 0, s_1, s_4 > 0$ from (9) if we started with $s_1 > 0$,

(iii.c). $s_1 = s_2 = 0, s_4 > 0$ from (9) if we started with $s_1 = 0$.

Case (iii.a). Suppose $s_3 = 0$, $s_2 > 0$ and $s_4 > 0$. With $x_1 = m$, $x_2 = m + 2n$ Lemma 2.4 yields

$$\zeta_{\mathfrak{so}(5)}(s_1, s_2, 0, s_4) = 2^{s_2} \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1}(2n)^{s_2}(m+2n)^{s_4}} = 2^{s_2-1} \sum_{m,n=1}^{\infty} \frac{1 + (-1)^n}{m^{s_1}n^{s_2}(m+n)^{s_4}}. \tag{10}$$

Breaking this into two parts and applying the Lemma with $x_1 = m$, $x_2 = n$ to the second part we have

$$(10) = 2^{s_2-1} \left\{ \zeta_{\mathrm{MT}}(s_1, s_2; s_4) + \sum_{a_1=0}^{s_1-1} \binom{s_2 + a_1 - 1}{a_1} \sum_{m,n=1}^{\infty} \frac{(-1)^n}{m^{s_1-a_1}(m+n)^{s_4+s_2+a_1}} \right.$$

$$\left. + \sum_{a_2=0}^{s_2-1} \binom{s_1 + a_2 - 1}{a_2} \sum_{m,n=1}^{\infty} \frac{(-1)^n}{n^{s_2-a_2}(m+n)^{s_4+s_1+a_2}} \right\}$$

$$= 2^{s_2-1} \left\{ \zeta_{\mathrm{MT}}(s_1, s_2; s_4) + \sum_{a_1=0}^{s_1-1} \binom{s_2 + a_1 - 1}{a_1} \zeta(\overline{s_4 + s_2 + a_1}, s_1 - a_1) \right.$$

$$\left. + \sum_{a_2=0}^{s_2-1} \binom{s_1 + a_2 - 1}{a_2} \zeta(s_4 + s_1 + a_2, \overline{s_2 - a_2}) \right\}.$$

Observe that the last component of every alternating Euler sum (or double zeta value) in the above sums is positive and its weight is unchanged. So Theorem 1.1 holds in this case.

Case (iii.b). Suppose $s_2 = 0$, $s_1 > 0$ and $s_4 > 0$. Applying Lemma 2.4 with $x_1 = m$, $x_2 = m + 2n$ to

$$\zeta_{\mathfrak{so}(5)}(s_1, 0, s_3, s_4) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1}(m+n)^{s_3}(m+2n)^{s_4}},$$

we get

$$\zeta_{\mathfrak{so}(5)}(s_1, 0, s_3, s_4) = \sum_{a_1=0}^{s_1-1} \binom{s_4 + a_1 - 1}{a_1} \frac{1}{2^{s_4+a_1}} \zeta(s_3 + s_4 + a_1, s_1 - a_1) \tag{11}$$

$$+ \sum_{a_4=0}^{s_4-1} \binom{s_1 + a_4 - 1}{a_4} \frac{1}{2^{s_1+a_4}} \zeta_{\mathfrak{so}(5)}(0, 0, s_3 + s_1 + a_4, s_4 - a_4). \tag{12}$$

Note that all the double zeta values in (11) have the same weight as the one we start with. We remind the reader that to determine the weight of a MZV it's not enough just to add all the components to see that weight does not change. We also need to check that every component is positive. In particular the last component $s_1 - a_1 > 0$ in (11). So we are reduced to the case (iii.c): $s_1 = s_2 = 0$, and because of the convergence restriction that the sum of all components is at least 3 we may also assume that $s_3 + s_4 \geq 3$ holds in case (iii.c). Further, since we assume $s_1 > 0$ and $s_4 - a_4 > 0$ in (12) we are in fact reduced to the subcase (iii.c.2) of (iii.c) where we can assume $s_3, s_4 > 0$.

Case (iii.c). Suppose $s_1 = s_2 = 0$ and $s_4 > 0$. We divide the case further into two subcases: (iii.c.1) $s_1 = s_2 = s_3 = 0$ and $s_4 > 0$, and (iii.c.2) $s_1 = s_2 = 0, s_3, s_4 > 0$.

Case (iii.c.1). In this case setting $s_4 = s \geq 3$ (by convergence restraint) we get

$$\zeta_{\mathfrak{so}(5)}(0,0,0,s) = \sum_{m,n=1}^{\infty} \frac{1}{(m+2n)^s} = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{1+(-1)^n}{(m+n)^s} = \frac{1}{2}\left\{ \zeta(s,0) + \sum_{k>m\geq 1} \frac{(-1)^{m+k}}{k^s} \right\}.$$

Now the sum over m is 0 unless k is even so by (7) we have

$$\zeta_{\mathfrak{so}(5)}(0,0,0,s) = \frac{1}{2}\left\{ \zeta(s-1) - \zeta(s) + \sum_{2k\geq 1} \frac{-1}{(2k)^s} \right\} = \frac{1}{2}\left\{ \zeta(s-1) - \zeta(s) - 2^{-s}\zeta(s) \right\}.$$

Thus Theorem 1.1 holds in this case. This is the second exceptional case when we need the Riemann zeta value with the weight decreased by one.

Case (iii.c.2). Suppose $s_3 = r > 0, s_4 = t > 0$ and $r + t \geq 3$ (by convergence restraint). Taking $x_1 = -m - n$, $x_2 = m + 2n$ in Lemma 2.4 we get

$$\zeta_{\mathfrak{so}(5)}(0,0,r,t) = \sum_{m,n=1}^{\infty} \frac{(-1)^r}{(-m-n)^r(m+2n)^t}$$

$$= \sum_{a=0}^{r-2} \binom{t+a-1}{a} \sum_{m,n=1}^{\infty} \frac{(-1)^a}{n^{t+a}(m+n)^{r-a}} + \sum_{a=0}^{t-2} \binom{r+a-1}{a} \sum_{m,n=1}^{\infty} \frac{(-1)^r}{n^{r+a}(m+2n)^{t-a}}$$

$$+ (-1)^r \binom{t+r-2}{r-1} \sum_{m,n=1}^{\infty} \left(\frac{1}{n^{r+t-1}(m+2n)} - \frac{1}{n^{r+t-1}(m+n)} \right) \qquad (13)$$

The inner infinite sum of the first sum is exactly $(-1)^a \zeta(r-a,t+a)$ while the second sum can be dealt with by the method similar to (10). For any positive integer u,v with $v > 1$ we have

$$\sum_{m,n=1}^{\infty} \frac{1}{n^u(m+2n)^v} = 2^{u-1} \sum_{m,n=1}^{\infty} \frac{1+(-1)^n}{n^u(m+n)^v} = 2^{u-1}\left\{ \zeta(v,u) + \zeta(v,\bar{u}) \right\}.$$

Notice that all the alternating Euler sums have the same weight (and the second components are all positive since $r,t > 0$). So now we need to consider the sum in (13). Assume $s = r + t - 1 > 1$ and let

$$S^{(N)} := \sum_{m,n=1}^{N} \left(\frac{1}{n^s(m+2n)} - \frac{1}{n^s(m+n)} \right) = \sum_{n=1}^{N} \frac{1}{n^s} \left(\sum_{m=1+2n}^{N+2n} - \sum_{m=1+n}^{N+n} \right) \frac{1}{m}$$

$$= \sum_{n=1}^{N} \frac{1}{n^s} \left(\sum_{m=N+n+1}^{N+2n} + \sum_{m=1}^{n} - \sum_{m=1}^{2n} \right) \frac{1}{m}.$$

Noticing that $s > 1$ and therefore

$$\sum_{n=1}^{N} \frac{1}{n^s} \sum_{m=N+n+1}^{N+2n} \frac{1}{m} < \sum_{n=1}^{N} \frac{1}{n^{s-1}N} \ll \log N/N \to 0 \quad \text{as } N \to \infty,$$

we quickly see that

$$\lim_{N \to \infty} S^{(N)} = \sum_{n \geq m \geq 1} \frac{1}{n^s m} - \sum_{2n \geq m \geq 1} \frac{1}{n^s m}$$

$$= \zeta(s+1) + \zeta(s,1) - 2^{s-1} \sum_{n \geq m \geq 1} \frac{1 + (-1)^n}{n^s m}$$

$$= (1 - 2^{s-1})\Big(\zeta(s+1) + \zeta(s,1)\Big) - 2^{s-1}\Big(\zeta(\bar{s},1) + \zeta(\overline{s+1})\Big),$$

where all the Euler sums have the same weight. This concludes the proof of Theorem 1.1.   □.

Remark 3.1. Notice that Theorem 1.1 does not imply Tsumura's result concerning odd weight values since the parity relations do not hold in general for alternating Euler sums. For example, we know the $\mathbb{Q}$-linear space generated by the Riemann zeta values of weight three has dimension one since $\zeta(3) = \zeta(2,1)$. Broadhurst conjectures that the dimension F_n of the space generated by weight n alternating Euler sums is a Fibonacci number: $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, and so on. It is easy to verify [21] that the weight three space is spanned by $\zeta(3)$, $\zeta(\bar{1}, 2)$ and $\zeta(\bar{1}, 1, 1)$. Also note that the depth one subspace is generated by $\zeta(3)$ since $\zeta(\bar{3}) = -\frac{3}{4}\zeta(3)$. Therefore the alternating Euler sum $\zeta(\bar{1}, 2)$ can not be reduced to depth one according to Broadhurst conjecture despite the parity difference between its weight and its depth.

4. Some examples and a conjecture

In this last section we present some numerical examples and put forward a conjecture on the space generated by the special values of $\zeta_{\mathfrak{so}(5)}$. In [15] Tsumura provided some evaluations of $\zeta_{\mathfrak{so}(5)}(s_1, \ldots, s_4)$ when the weight is odd. By our general approach we can now compute all the convergent values and in particular we are able to confirm all the odd weight values of $\zeta_{\mathfrak{so}(5)}$ found in [14]. In practice one may first convert our formulas to computer programs and then compute with Maple. As a safeguard we have checked numerically all the equations in this section with EZface [1]. In what follows we will only consider the regular cases, i.e., not the two exceptional cases in each weight.

We first list all the regular weight three values below:

$$\zeta_{\mathfrak{so}(5)}(1,0,0,2) = \tfrac{3}{2}\zeta(\bar{1},2) + \tfrac{1}{16}\zeta(3), \quad \zeta_{\mathfrak{so}(5)}(0,0,1,2) = -\tfrac{1}{2}\zeta(3) + 3\zeta(\bar{1},2),$$
$$\zeta_{\mathfrak{so}(5)}(0,1,0,2) = \tfrac{5}{4}\zeta(3) - 3\zeta(\bar{1},2), \quad \zeta_{\mathfrak{so}(5)}(1,0,2,0) = \zeta_{\mathfrak{so}(5)}(0,1,2,0) = \zeta(3),$$
$$\zeta_{\mathfrak{so}(5)}(0,1,1,1) = \tfrac{3}{4}\zeta(3), \quad \zeta_{\mathfrak{so}(5)}(1,1,1,0) = 2\zeta(3), \quad \zeta_{\mathfrak{so}(5)}(1,0,1,1) = \tfrac{5}{8}\zeta(3),$$
$$\zeta_{\mathfrak{so}(5)}(0,0,2,1) = \tfrac{1}{4}\zeta(3), \quad \zeta_{\mathfrak{so}(5)}(1,1,0,1) = \tfrac{11}{8}\zeta(3).$$

Note that the first three values do not satisfy the conditions of Tsumura's Theorem in [14] so there is no contradiction even if they can not be rational multiples of $\zeta(3)$ by Broadhurst conjecture. On the other hand, $\zeta_{\mathfrak{so}(5)}(0,1,1,1) = \tfrac{3}{4}\zeta(3)$ does not satisfy the conditions either but it is reduced to depth one. Therefore it would be interesting to find out the exact conditions on the reducibility of special values of $\zeta_{\mathfrak{so}(5)}$ to Riemann zeta values.

Remark 4.1. The smallest weight of $\zeta_{\mathfrak{so}(5)}(s_1, s_2, s_3, s_4)$ is three because of the convergence restraint (6). We have listed all the possible regular weight three values in the above and find that they do not span the whole weight three space of alternating Euler sums because the whole space has dimension three according to Broadhurst conjecture (see Remark 3.1). In fact, this was expected because $\zeta_{\mathfrak{so}(5)}(s_1, s_2, s_3, s_4)$ only has depth two while all the depth one and two alternating Euler sums only generate a proper subspace of dimension two over $\mathbb{Q}$, and in fact, this subspace can be generated by $\zeta(3)$ and $\zeta(\bar{1}, 2)$. We believe this phenomenon happens in general.

Conjecture 4.2. *Let w be a positive integer ≥ 3. Let V_w be the $\mathbb{Q}$-vector space spanned by all the weight w special values of $\zeta_{\mathfrak{so}(5)}(s_1, s_2, s_3, s_4)$ where s_1, s_2, s_3 and s_4 are nonnegative integers such that*

(6) *are satisfied,* $s_1 + s_2 + s_3 > 0$ *and* $s_1 + s_2 + s_4 > 0$. *Then* V_w *coincides with the* $\mathbb{Q}$-*vector space spanned by all the weight* w *alternating Euler sums of depth at most two.*

We have verified this conjecture for all the weights up to weight five.

We now list all the 25 regular weight four values below:

$$
\begin{aligned}
&\zeta_{\mathfrak{so}(5)}(0,0,2,2) = \tfrac{3}{8}\zeta(4) - 4\zeta(\bar{3},1), \quad &&\zeta_{\mathfrak{so}(5)}(2,1,0,1) = \tfrac{7}{8}\zeta(4) - \zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(0,1,2,1) = \tfrac{1}{2}\zeta(4) - 4\zeta(\bar{3},1), \quad &&\zeta_{\mathfrak{so}(5)}(0,2,0,2) = \tfrac{1}{8}\zeta(4) + 4\zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(0,2,1,1) = \tfrac{1}{4}\zeta(4) + 4\zeta(\bar{3},1), \quad &&\zeta_{\mathfrak{so}(5)}(1,2,0,1) = \tfrac{3}{4}\zeta(4) + 2\zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(1,0,1,2) = \tfrac{3}{16}\zeta(4) - \zeta(\bar{3},1), \quad &&\zeta_{\mathfrak{so}(5)}(2,0,1,1) = \tfrac{3}{8}\zeta(4) + \zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(1,1,0,2) = \tfrac{5}{16}\zeta(4) - \zeta(\bar{3},1), \quad &&\zeta_{\mathfrak{so}(5)}(1,1,1,1) = \tfrac{1}{2}\zeta(4) - 2\zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(0,1,0,3) = \tfrac{17}{24}\zeta(4) - \tfrac{7}{3}\zeta(\bar{1},3), \quad &&\zeta_{\mathfrak{so}(5)}(0,0,3,1) = -\tfrac{1}{4}\zeta(4) + 4\zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(0,0,1,3) = -\tfrac{7}{12}\zeta(4) + \tfrac{7}{3}\zeta(\bar{1},3), \quad &&\zeta_{\mathfrak{so}(5)}(1,0,2,1) = 2\zeta(\bar{3},1), \\
&\zeta_{\mathfrak{so}(5)}(1,0,3,0) = \zeta_{\mathfrak{so}(5)}(0,1,3,0) = \tfrac{1}{4}\zeta(4), \quad &&\zeta_{\mathfrak{so}(5)}(0,1,1,2) = \tfrac{1}{8}\zeta(4), \\
&\zeta_{\mathfrak{so}(5)}(2,0,2,0) = \zeta_{\mathfrak{so}(5)}(0,2,2,0) = \tfrac{3}{4}\zeta(4), \quad &&\zeta_{\mathfrak{so}(5)}(2,2,0,0) = \tfrac{5}{2}\zeta(4), \\
&\zeta_{\mathfrak{so}(5)}(2,1,1,0) = \zeta_{\mathfrak{so}(5)}(1,2,1,0) = \tfrac{5}{4}\zeta(4), \quad &&\zeta_{\mathfrak{so}(5)}(2,0,0,2) = \tfrac{9}{32}\zeta(4), \\
&\zeta_{\mathfrak{so}(5)}(1,0,0,3) = -\tfrac{19}{96}\zeta(4) + \tfrac{7}{6}\zeta(\bar{1},3) - \tfrac{1}{2}\zeta(\bar{3},1), \quad &&\zeta_{\mathfrak{so}(5)}(1,1,2,0) = \tfrac{1}{2}\zeta(4).
\end{aligned}
$$

There are 46 regular weight five values and 74 regular weight six values. The following are some interesting weight six ones:

$$
\begin{aligned}
&\zeta_{\mathfrak{so}(5)}(0,2,2,2) = \tfrac{1}{105}\zeta(2)^3 = \tfrac{1}{22680}\pi^6 = .04238929428\ldots \\
&\zeta_{\mathfrak{so}(5)}(2,0,2,2) = \tfrac{1}{210}\zeta(2)^3 + \tfrac{3}{8}\zeta(3,3) - \tfrac{2}{3}\zeta(\bar{3},3) = .03772580207\ldots \\
&\zeta_{\mathfrak{so}(5)}(2,2,0,2) = \tfrac{4}{105}\zeta(2)^3 - \tfrac{3}{8}\zeta(3,3) + \tfrac{2}{3}\zeta(\bar{3},3) = .15302602205\ldots \\
&\zeta_{\mathfrak{so}(5)}(2,2,2,0) = \zeta_{\mathrm{MT}}(2,2;2) = \tfrac{8}{105}\zeta(2)^3 = \tfrac{1}{2835}\pi^6 = .3391143543\ldots \\
&\zeta_{\mathfrak{so}(5)}(1,2,2,1) = \tfrac{1}{30}\zeta(2)^3 - \tfrac{3}{4}\zeta(3,3) + \tfrac{4}{3}\zeta(\bar{3},3) = .1153002199792\ldots \\
&\zeta_{\mathfrak{so}(5)}(2,1,1,2) = \tfrac{1}{60}\zeta(2)^3 = \tfrac{1}{12960}\pi^6 = .07418126500\ldots \\
&\zeta_{\mathfrak{so}(5)}(1,1,2,2) = \tfrac{1}{84}\zeta(2)^3 - \tfrac{3}{8}\zeta(3,3) + \tfrac{2}{3}\zeta(\bar{3},3) = .0364554628649\ldots \\
&\zeta_{\mathfrak{so}(5)}(1,2,1,2) = \tfrac{3}{140}\zeta(2)^3 - \tfrac{3}{8}\zeta(3,3) + \tfrac{2}{3}\zeta(\bar{3},3) = .0788447571142\ldots \\
&\zeta_{\mathfrak{so}(5)}(2,1,2,1) = \tfrac{3}{140}\zeta(2)^3 + \tfrac{3}{8}\zeta(3,3) - \tfrac{2}{3}\zeta(\bar{3},3) = .1119070670077\ldots \\
&\zeta_{\mathfrak{so}(5)}(2,2,1,1) = \tfrac{23}{420}\zeta(2)^3 - \tfrac{3}{8}\zeta(3,3) + \tfrac{2}{3}\zeta(\bar{3},3) = .2272072869870\ldots
\end{aligned}
$$

Finally, we return to the question related to Zagier's original version of Witten's zeta function attached to $\mathfrak{so}(5)$. We would like to know the rational coefficients in (2) so the following four values can shed some light on this:

$$
\begin{aligned}
&\zeta_{\mathfrak{so}(5)}(2,2,2,2) = \tfrac{3}{700}\zeta(2)^4 = \tfrac{2\cdot 3}{5\cdot 9!}\pi^8 = .031377417381\ldots \\
&\zeta_{\mathfrak{so}(5)}(4,4,4,4) = \tfrac{4311}{297797500}\zeta(2)^8 = \tfrac{2^5\cdot 479}{5\cdot 17!}\pi^{16} = .0007759700\ldots \\
&\zeta_{\mathfrak{so}(5)}(6,6,6,6) = \tfrac{2490861}{45593675752625}\zeta(2)^{12} = \tfrac{2^7\cdot 5\cdot 43\cdot 19309}{3^2\cdot 7\cdot 13\cdot 23!}\pi^{24} = .00002144010\ldots \\
&\zeta_{\mathfrak{so}(5)}(8,8,8,8) = \tfrac{138835874547}{670007833199392187500}\zeta(2)^{16} = \tfrac{2^8\cdot 13\cdot 241\cdot 64009163}{5\cdot 17\cdot 31!}\pi^{32} = .000000595384\ldots.
\end{aligned}
$$

For $\zeta_{\mathfrak{so}(5)}(2,2,2,2)$ we can reduce our linear combination of alternating Euler sums to the correct multiple of $\zeta(2)^4$. But we can not perform the same for the other three because there isn't any table of relations available for alternating Euler sums of weight greater than 15 although we can check numerically that our formulas are all correct (with the errors bounded by 10^{-100}). However, it turns out that by analytical methods Komori et al. [8] have found a closed formula of $\zeta_{\mathfrak{so}(5)}(2m,2m,2m,2m)$ which implies that

$$
\begin{aligned}
c(m) = \frac{2^{8m-3}}{(8m)!} \sum_{\nu=0}^{m} B_{2\nu} B_{8m-2\nu} \binom{8m}{2\nu} \Biggl\{ &\sum_{\mu=0}^{2m-1} \left(\frac{2^{2\nu-1}}{2^{2m+\mu}} - (-1)^\mu \right) \binom{4m-\mu-2}{2m-1} \binom{2m-2\nu+\mu}{2m-2\nu} \\
&+ \sum_{\mu=0}^{2m-2\nu} \left(\frac{1}{2^{2m+\mu}} + (-1)^\mu \right) \binom{4m-2\nu-\mu-1}{2m-1} \binom{2m-1+\mu}{2m-1} \Biggr\},
\end{aligned}
$$

where $B_{2\nu}$ are Bernoulli numbers. So we have come around a full circle and back to the original values we began with, but this time the rational coefficients of the π-powers are determined precisely.

References

[1] J. Borwein, P. Lisonek, and P. Irvine, *An interface for evaluation of Euler sums.* Available online at `http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi`

[2] D. Essouabri, Singularités des séries de Dirichlet associes à des polynômes de plusieurs variables et applications en théorie analytique des nombres. Ann. Inst. Fourier **47** (1997), 429–483.

[3] M. E. Hoffman, *Multiple harmonic series.* Pacific J. Math. **152** (1992), 275–290.

[4] K. Ihara, M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values.* Compositio Math. **142** (2006), 307–338.

[5] Y. Komori, K. Matsumoto and H. Tsumura, Zeta and *L*-functions and Bernoulli polynomials of root systems, Proc. Japan Acad. Ser. A **84** (2008), 57–62.

[6] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras II. Preprint.

[7] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras III. Preprint.

[8] Y. Komori, K. Matsumoto and H. Tsumura, Functional equations for zeta-functions of root systems. In preparation.

[9] K. Matsumoto, *On Mordell-Tornheim and other multiple zeta-functions.* In: Proc. of the Session in analytic number theory and Diophantine equations, ed. D.R. Heath-Brown and B.Z. Moroz, Bonner Mathematische Schriften, **25** (2003), 17 pages.

[10] K. Matsumoto, *Analytic properties of multiple zeta-functions in several variables.* (English summary) Number theory, 153–173, Dev. Math. **15**, Springer, New York, 2006.

[11] K. Matsumoto and H. Tsumura, *On Witten multiple zeta-functions associated with semisimple Lie algebras I*, Ann. Inst. Fourier **56** (2006), 1457–1504.

[12] L.J. Mordell, *On the evaluation of some multiple series.* J. London Math. Soc. **33** (1958), 368–271.

[13] L. Tornheim, *Harmonic double series.* Amer. J. Math. **72** (1950), 303–314.

[14] H. Tsumura, *Combinatorial relations for Euler-Zagier sums.* Acta Arith. **111** (2004), 27–42.

[15] H. Tsumura, On Wittens type of zeta values attached to $SO(5)$, Arch. Math. **84** (2004), 147–152.

[16] E. Witten, *On quantum gauge theories in two-dimensions.* Commun. Math. Phys. **141**(1) (1991), 153–209.

[17] D. Zagier, *Values of zeta function and their applications.* Proc. of the First European Congress of Math. **2** (1994), 497–512.

[18] J. Zhao, *Analytic continuation of multiple zeta functions.* Proc. of Amer. Math. Soc. **128** (1999), 1275–1283.

[19] J. Zhao, *On a conjecture of Borwein, Bradley and Broadhurst.* In press: J. reine angew. Math.

[20] J. Zhao, *Multiple polylogarithm values at roots of unity.* C. R. Acad. Sci. Paris, Ser. I. **346** (2008), 1029–1032.

[21] J. Zhao, Integral structures of multi-polylogs at roots of unity. In preparation.

[22] X. Zhou and D.M. Bradley, On Mordell-Tornheim sums and multiple zeta values. Preprint, 2008.