Pattern formation from projectively dynamical systems and iterations by families of maps

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Introduction

Pattern formation in dynamical systems is a phenomena that connects several different dynamics passing through some scaling limits. In some cases, dynamics which behaves very randomly is transformed into another which satisfies some rigid properties. This can be possible since procedure of scaling limits wastes detailed information and picks up rough movement of dynamics.

In this paper we introduce a fomulation of dynamical rescaling between two dynamics passing through another parametrized ones, and have explicit constructions. Iteration dymanics has been studied quite deeply and known to show various aspects of dynamical properties. Typically they behave quite chaostic manner, and so in general it would be impossible to trace their movements rigorously. On the other hand several integrable systems show some predictable dynamics and create some patterns. Of particular interest are solitons in KdV solutions. We will call a dynamical rescaling between such dynamics as a *dynamical pattern formation*. We construct dynamical pattern formations between iteration dynamics by families of maps and some partial differential equations. As an intermediate dynamics, we pass through parametrized complex dynamics which arise from tropical geometry. Our main theorem is a construction of dynamical pattern formations in our sense, from the iteration dynamics by piecewise linear maps to the KdV and Lotka Volterra solutions.

Let (Z, d') be a metric space and consider a family of dynamical spaces on it:

$$\sigma_t : Z \to Z$$

where $t \in [1, \infty)$ and σ_t are continuous maps.

Let us take another metric space (X, d, τ) equipped with a continuous map $\tau : X \to X$ on it.

A *contracting map* between these dynamical systems consistes of a parametrized maps:

$$\varphi_t : Z \to X, \quad t \in [1, \infty)$$

so that for any $m, \sigma_t(m) \in Z$:

$$d(\tau(\varphi_t(m)), \varphi_t(\sigma_t(m))) \to 0,$$
$$d(\varphi_t(m), \varphi_t(m')) \to 0$$

hold as $t \to \infty$. Thus as $t \to \infty$, φ_t looks as though equivariant, but the maps are collapsing neighbourhood of points and approaching constant. We denote it as $\varphi_t : (Z, \sigma_t) \to (X, \tau)$.

Similarly $\varphi_t : (Z, \sigma_t) \to (X, \tau)$ is an *expanding map*, if the following two conditions hold:

$$d(\tau(\varphi_t(m)), \varphi_t(\sigma_t(m))) \to 0,$$
$$d(\varphi_t(m), \varphi_t(m')) \to \infty$$

Let (X, d, τ) and (Y, d'', μ) be two metric spaces equipped with continuous maps. Here we introduce the following:

Definition 0.1 *A dynamical rescaling from (X, d, τ) to (Y, d'', μ) is the set $\{(Z, d', \sigma_t), \varphi_t, \phi_t\}$, where:*

$$\varphi_t : (Z, \sigma_t) \to (X, \tau)$$

is a contracting map, and:

$$\phi_t : (Z, \sigma_t) \to (Y, \mu)$$

is an expanding map.

In some cases the dynamics (X, d, τ) will show chaostic behaviour in its nature, on the other hand the dynamics (Y, d'', σ) may satisfy some rigidity. This will be possible by changing their scalings of dynamics as above. We construct dynamical pattern formations from dynamics of families of piecewise linear maps on $\mathbf{R}$ to some integrable partial differential equations including KdV and Lotka Volterra equations.

In general these spaces may be of infinite dimension. But in some cases these can be reduced to dynamics on finite dimensional spaces. These are the cases for us, where $X = \mathbf{R}^\infty$ and $Z = \mathbf{C}^\infty$. In some cases we have a reduction on $\mathbf{R}^\infty$ to finite automata, and on $\mathbf{C}^\infty$, we always have a reduction to a family of dynamics on a parametrized affine algebraic varieties.

Let us say that a contracting map from (X, d, τ) to (Z, d', σ_t) admits a *finite dimensional reduction*, if there are parametrized finite dimensional manifolds $V_t \subset \mathbf{C}^N$, families of functions:

$$P_n : Z \to \mathbf{C}, \quad n = 0, 1, \ldots,$$
$$Q : \mathbf{C}^\infty \to Y$$

and $a_1, a_2, b_1, b_2 \geq 0$ so that for each $n \geq a_i, b_j$ and $z \in Z$, the following conditions hold:

$$(P_{n-a_1}(z), \dots, P_{n+a_2}(z), P_{n-b_1}(\sigma_t(z)), \dots, P_{n+b_2}(\sigma_t(z))) \in V_t,$$
$$Q(P_0(z), P_1(z), \dots) = \varphi_t(z).$$

We denote such reduction as:

$$\varphi_t : (V_t, \sigma_t) \to (X, \tau).$$

Later when we construct dynamical rescalings, we will choose $a_1 = 0, \quad a_2 = 1, \quad b_1 = 1, \quad b_2 = 0$, since they are related to cell automata. In that case, V_t are all hypersurfaces in $\mathbf{C}^4$.

Let $f_1, \dots, f_a : [0,1] \to [0,1]$ be a family of continuous maps. In [K3], we have introduced a family of dynamics on the one sided fullshift $X_a = \{(k_0, k_1, \dots) : k_i \in \{1, \dots, a\}\}$ induced from the family:

$$\Phi(\{f_i\}_i) : [0,1] \times X_a \to [0,1] \times X_a$$

as $\Phi(x, (m_0, m_1, \dots)) = (x, \Phi(x)(m_0, m_1, \dots))$. We call it an *interaction map*.

Such families of maps are induced from another families of dynamical systems:

$$\bar{\Phi} : [0,1] \times [0,1]^\infty \to [0,1] \times [0,1]^\infty$$

$\bar{\Phi}(x, (y_0, y_1, \dots)) = (x, \bar{\Phi}(x)(y_0, y_1, \dots))$. In a sense the dynamical systems generalize the one dimensional iteration dynamics, and they will behave quite complicated manner.

Let $\pi_a : [0,1] \to \{1, \dots, a\}$ be the projection. It gives a map $\pi : [0,1]^\infty \to X_a$ by $\pi(y_0, y_1, \dots) = (\pi(y_0), \pi(y_1), \dots)$. Then one has commutativity:

$$\pi \circ \bar{\Phi} = \Phi \circ \pi.$$

Let us divide the square $[0,1]^2$ into a^2 number of cells of length $1/a$. When the graphs of a family of continuous maps sit into each cell in a nice way, which we call the *cell type*, then the dynamics on X_a is reduced to a cell automaton A. Thus general interaction maps can be regarded as perturbations of cell automata. There are cell automata which do not come from families of continuous maps, on the other hand such class of automata include Lotka Volterra cell automaton, and other important integrable automata ([K3,4]).

On the above constrution, the number of the maps must be finite, since one has normalized domains and used projection $\pi_a : [0,1] \to \{1, \dots, a\}$. Thus when one has infinite number of maps, then it will be natural to use denormalized projection $\pi : \mathbf{R} \to \mathbf{Z}$. Thus for a family of maps $f_i : \mathbf{R} \to \mathbf{R}$, $i \in \mathbf{Z}$, one obtains another dymanics and the induced dymanics as:

$$\bar{\Phi} : \mathbf{R} \times \mathbf{R}^\infty \to \mathbf{R} \times \mathbf{R}^\infty,$$
$$\Phi : \mathbf{R} \times X_\infty \to \mathbf{R} \times X_\infty.$$

In the case of Lotka Volterra cell automaton, it is known that one can obtain subdynamics where the number of alphabets are bounded, and so it induces $\Phi : \mathbf{R} \times X \to \mathbf{R} \times X$, $X \subset X_a$ for some a.

When one has a map $F : \mathbf{R}^n \to \mathbf{R}$, then it induces a family of maps:

$$f_{i_1,\dots,i_{n-1}} : \mathbf{R} \to \mathbf{R},$$
$$f_{i_1,\dots,i_{n-1}}(x) = f(i_1, \dots, i_{n-1}, x) \in \mathbf{R}.$$

A family of maps $\{f_{i_1,\dots,i_{n-1}}\}$ is said to be *piecewise linear*, if the corresponding dynamics on X_∞ is induced from a piecewise linear maps $F : \mathbf{R}^n \to \mathbf{R}$.

When a family of maps is piecewise linear, then we will obtain contracting maps, which translate its dynamics into complex dynamical systems, where for the process, we use tropical geometry ([Mi],[V]).

Let $F : \mathbf{C}^n \to \mathbf{C}$ be a polynomial. Then it induces a dynamical systems:

$$\tilde{\Phi}(F)(z) : \mathbf{C}^\infty \to \mathbf{C}^\infty$$

by a similar way as above. The above dynamics admits a finite dimensional reduction over $V \subset \mathbf{C}^{n+1}$.

Let $(\mathbf{R}_+^\infty, \bar{\Phi})$ be its restriction, and let:

$$\mathrm{Log}_t : \mathbf{C} \to \mathbf{R}, \quad \mathrm{Log}_t(z) = \log_t |z|$$

Our main construction is the following:

Theorem 0.1 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a picewise linear map. Then there is a parametrized polynomials F_t, $t \in [1, \infty)$ so that there is a contracting map:*

$$Log_t : (\mathbf{R}_+^\infty, \tilde{\Phi}(F_t)) \to (\mathbf{R}^\infty, \bar{\Phi}(f))$$

which are reduced to parametrized affine algebraic varieties V_t.

In our notation, it is expressed as:

$$\mathrm{Log}_t : (V_t, \tilde{\Phi}(F_t)) \to (\mathbf{R}^\infty, \bar{\Phi}(f)).$$

When the latter dymanics $(\mathbf{R}^\infty, \bar{\Phi}(f))$ is reduced to an automaton A, then we will say that Log_t gives a contracting map from $(V_t, \tilde{\Phi})$ to an automaton A and denote as:

$$\mathrm{Log}_t : (V_t, \tilde{\Phi}) \to A.$$

Thus once one finds expanding maps, then they consiste of a dynamical rescaling.

Proposition 0.1 (H) *There are continuous deformations both from discrete KdV to KdV equation, and from discrete LV to LV equation.*

In our formulation these continuous procedures are interpreted as expanding maps. Combining this with the above contracting map, one obtains constructions of dynamical pattern formations:

Theorem 0.2 *(1) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a partially defined piecewise linear map given by:*

$$f(x_2, x_3, x_4) = x_2 - \max(0, x_2 + x_3), \quad x_4 \leq \max(0, x_2 + x_3) - x_2.$$

Then it gives a dynamical rescaling from a cell automaton:

$$A : V_1 + \max(0, V_2 + V_3) = V_2 + \max(0, V_1 + V_4)$$

to KdV flows:

$$\begin{aligned} &Log_t : (V, \sigma) = \{(z_1, z_2, z_3, z_4) : z_2 + z_1 z_2 z_4 = z_1 + z_1 z_2 z_3\} \subset \mathbf{C}^4 \to A, \\ &\phi_t : (V, \sigma) \to \{u(x.s) : u_s - \frac{1}{p^3} u u_x + \frac{1}{48p^2}(1 - \frac{1}{p^4}) u_{3x} = 0\}. \end{aligned}$$

(2) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a piecewise linear map given by:

$$f(x_2, x_3, x_4) = x_2 + \max(0, x_3) - \max(0, x_4).$$

Then it gives a dynamical rescaling from a cell automaton:

$$B : V_1 + \max(0, V_4) = V_2 + \max(0, V_3)$$

to LV flows:

$$\begin{aligned} &Log_t : (V, \sigma) = \{(z_1, z_2, z_3, z_4) : z_2 + z_2 z_3 = z_1 + z_1 z_4\} \subset \mathbf{C}^4 \to B, \\ &\phi_t : (V, \sigma) \to \{u(x, s) : u_n' = u_n(u_{n+1} - u_{n-1})\} \end{aligned}$$

Algebraic varieties admit various operations on themselves. Passing through them, one can induce operations on the dynamics on piecewise linear maps or on automata. The projective duality is an involution on the set of algebraic varieties, which comes from Legendre transformation ([GKZ]). Passing through our contracting maps, we will introduce *duality* on piecewise linear maps or on automata:

$$\{f_{i,j}\}_{i,j} \to \{f_{i,j}^{\vee}\}_{i,j}, \quad A \to A^{\vee}.$$

The projective duality uses global geometry of spaces, and so it seems hard to obtain dual cell automata directly. We have an example of such duality for the case of some curves.

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1 Dynamical rescaling

1.A Interacting maps: Let $f_i : [0,1] \to [0,1]$, $i = 1, \dots, a$ be a family of maps and

$$X_a = \{(k_0, k_1, \dots) : k_i \in \{1, \dots, a\}\}$$

be the one sided full shift.

For each element $\bar{k} = (k_0, k_1, \dots) \in X_a$, we will associate a family of maps:

$$\{h^m(x)\}_{k=0,1,\dots}, \quad h^m : [0,1] \to [0,1]$$

by:

$$h^m(x) \equiv f_{k_m} \circ f_{k_{m-1}} \circ \cdots \circ f_{k_0}(x).$$

We call the famliy as the *interaction maps*.

Let us put a subset $S(f_1, \dots, f_a; \bar{k}) = \{x \in [0,1] : h^m(x) \in \{\frac{i}{a}\}_{i=1}^{a-1}$ for some $m\}$ in $[0,1]$. We call it the *singular set*. The regular set with respect to $\bar{k}$ is given by $R(f_1, \dots, f_a; \bar{k}) \equiv [0,1] \backslash S(f_1, \dots, f_a; \bar{k})$.

The *regular set* of the family of maps $\{f_1, \dots, f_a\}$ is defined by:

$$R(f_1, \dots, f_a) \equiv \cap_{\bar{k} \in X_a} R(f_1, \dots, f_a; \bar{k}) \subset [0,1].$$

Let:

$$\pi : [0,1] \backslash \{\frac{1}{a}, \frac{2}{a}, \dots, \frac{a-1}{a}\} \to \{0, 1, \dots, a\}$$

be a measurable map given by $\pi((\frac{i-1}{a}, \frac{i}{a})) \equiv i$ for $i = 1, \dots, a$.

Let $\bar{k} \in X_a$ and $\{h^m\}_m$ be the corresponding famliy of maps. For each $x \in R(f_1, \dots, f_a)$, one can compose $\{h^m(x)\}_m$ with $\pi : [0,1]^\infty \to X_a$, $\pi(x_0, x_1, \dots) \equiv (\pi(x_0), \pi(x_1), \dots)$, and obtains another element:

$$\bar{k}' \equiv \pi((h^0(x), h^1(x), \dots)) \equiv (\pi \circ h^0(x), \pi \circ h^1(x), \dots) \in X_a.$$

Thus for each element $\bar{k} \in X_a$, one has assigned $\bar{k}' \in X_a$. We denote it as $\Phi(\{f_i\}_i)(x) : X_a \to X_a$ by $\Phi(\{f_i\}_i)(x)(\bar{k}) = \pi((h^0(x), h^1(x), \dots))$. It gives a family of symbolic dynamics:

$$\Phi(f_1, \dots, f_a) : [0,1] \times X_a \to [0,1] \times X_a,$$
$$\Phi(\{f_i\}_i)(x, \bar{k}) = (x, \Phi(\{f_i\}_i)(x)(\bar{k}))$$

with domain $R(f_1, \dots, f_a) \times X_a$. This is the most basic dynamics in this paper. We call it the *interaction map*.

Φ above is a reduction of the map $\bar{\Phi}(x) : [0,1]^\infty \to [0,1]^\infty$ defined below. For $(y_0, y_1, \dots) \in [0,1]^\infty$, let us denote:

$$\bar{k} = (\pi(y_0), \pi(y_1), \dots) \in X_a$$

and let $\{h^m\}_m$ be the family of maps on $[0,1]$ corresponding to $(\{f_i\}_i, \bar{k})$. Then we put:

$$\bar{\Phi}(x)(y_0, y_1, \dots) = (h^0(x), h^1(x), \dots) \in [0,1]^\infty.$$

Its reduction by the projection gives a commutative maps:

$$\pi \circ \bar{\Phi} = \Phi \circ \pi.$$

Suppose one has infinite number of family of maps. Then one cannot follow the above construction for $a = \infty$. In this case one can use denormalized projections. Let $f_i : \mathbf{R} \to \mathbf{R}$, $i \in \mathbf{Z}$ be infinite family of maps, and put $\pi : \mathbf{R} \to \mathbf{Z}$ be a measurable map given by $\pi((i-1,i)) \equiv i$. Then by use of this π, a parallel construction gives a family of dynamics:

$$\bar{\Phi}(\{f_i\}_i) : \mathbf{R} \times \mathbf{R}^\infty \to \mathbf{R} \times \mathbf{R}^\infty$$

which reduces to the family of symbolic dynamics:

$$\Phi : \mathbf{R} \times X_\infty \to \mathbf{R} \times X_\infty.$$

1.B Reduction to cell automata: Let $\{f_1, \dots, f_a\}$ be a family of maps. We say that the family is *of cell type*, if each map satisfies:

$$f_l((\frac{i}{a}, \frac{i+1}{a})) \subset (\frac{j}{a}, \frac{j+1}{a}), \quad l = 1, \dots, a$$

for all $i = 0, \dots, a-1$, where $j = 0, \dots, a-1$ depends on l and i.

Let $\Phi(x) : X_a \to X_a$ be the corresponding interacting map, and denote the flows as:

$$\Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots), \quad t = 0, 1, \dots$$

Let $\pi : [0,1] \to \{1, \dots, a\}$ be the projection.

Lemma 1.1 *Suppose the family of maps $\{f_1, \dots, f_a\}$ be of cell type. Then there is a map:*

$$\varphi : \{1, \dots, a\}^2 \to \{1, \dots, a\}$$

so that the flows above are determined by the finite automaton:

$$A : k_{i+1}^{t+1} = \varphi(k_{i+1}^t, k_i^{t+1})$$

for all $i, t = 0, 1, 2, \dots$.

Thus when a family of maps are of cell type, then the reduction to symbolic dynamics as in $1.A$ becomes in fact the one to a finite automaton:

$$\pi : (\bar{\Phi}, [0,1] \times X_a) \to (A, W)$$

where W is the set of strings of infinite length determined by A.

In order to relate the interaction maps with cell automata, let us generalize the composition way as follows. Let $\{f_{i,j}\}_{1 \leq i,j \leq a}$ be a family of continuous maps on $[0,1]$ and choose $\bar{k} = (k_0, k_1, \dots) \in X_a$. Then we inductively define another family of maps $\{h^m : [0,1] \to [0,1]\}_{m \geq 0}$ as:

$$h^m(x) = f_{k_m, k_{m+1}} \circ h^{m-1}(x), \quad h^0(x) = x.$$

By the same way as above, one defines the interaction map:

$$\Phi(x) : X_a \to X_a, \quad \Phi(x)(\bar{k}) = (\pi \circ h^0(x), \pi \circ h^1(x), \dots).$$

Let us denote its iterations $\Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots)$ for $t = 0, 1, \dots$

Definition 1.1 *The interaction map Φ given by a family of maps $\{f_{i,j}\}_{1 \leq i,j \leq a}$ is called cell automaton type, if for each $x \in R(\{f_{i,j}\}_{i,j})$, there are two maps:*

$$\varphi_1, \varphi_2 : \{1, \dots, a\}^4 \to \{1, \dots, a\}$$

so that its reduction to symbolic dynamics is detemined by the equality:

$$\varphi_1(k_m^t, k_{m-1}^t, k_m^{t-1}, k_{m+1}^{t-1}) = \varphi_2(k_m^t, k_{m-1}^t, k_m^{t-1}, k_{m+1}^{t-1})$$

hold for all m and t, where $k_0^t = \pi(x)$.

This is exactly the case when the family of maps map $\{f_{i,j}\}_{i,j}$ is of cell type defined above.

Let $f_{i_1, \dots, i_{n-1}} : \mathbf{R} \to \mathbf{R}$, $i_1, \dots, i_{n-1} \in \mathbf{Z}$ be a family of maps., and let $\phi : \mathbf{R} \times X_\infty \to \mathbf{R} \times X_\infty$ be the corresponding interaction map. Similarly we say that Φ is called *automaton type*, if there are two maps:

$$\varphi_1, \varphi_2 : \mathbf{Z}^{n+2} \to \mathbf{Z}$$

so that

$$\varphi_1(k_m^t, k_{m-1}^t, k_m^{t-1}, \dots, k_{m+n-1}^{t-1}) = \varphi_2(k_m^t, k_{m-1}^t, k_m^{t-1}, \dots, k_{m+n-1}^{t-1})$$

hold for all m and t.

1.B.2 Lotka Volterra cell automaton: Lotka Volterra equation is an ordinary differential equation, known as describing the growth rate of competiting lives. The equation is obtained as a continuous limit of discrete Lotka Volterra equation. On the other hand by taking another way of limit, one obtains a cell automaton called the *Lotka Volterra cellular automaton* ([TTMS]):

$$CA(LV) : V_n^{t+1} - V_n^t = \max(L_0, V_{n+1}^t) - \max(L_0, V_{n-1}^{t+1})$$

for $n, t = 0, 1, \dots$ and $L_0 \geq 1$ is a fixed integer. One of the important behaviour of the dynamics is existence of *soliton*. It is known that solitary property is preserved under the continuous limits.

Proposition 1.1 (K3) *There is a family of maps $\{f_{i,j}\}_{1 \leq i,j \leq a}$ of cell type so that the Lotka Volterra cell automaton is described as flows $\{\Phi(x)^t(\bar{k})\}_{t=0,1,\dots}$ of the corresponding interaction map.*

Thus one obtains a reduction to CA(LV) from interacting dynamics by a family of maps:

$$\pi : (\bar{\Phi}, [0,1] \times [0,1]^\infty) \to CA(LV).$$

The dynamics of the former will be quite complicated, and passing through the projection, one can reduce it to the cell automaton which contains solitons.

1.B.3 Piecewise linear maps: Let $F : \mathbf{R}^n \to \mathbf{R}$ be a piecewise linear map.

Lemma 1.2 *A piecewise linear map F above is expressed by a* $(\max, \pm)$*-type equation as:*

$$y = F(\bar{x}) = \Sigma_{l=1}^s \pm \max(\alpha_1^l + \bar{z}_1^l \bar{x}, \dots, \alpha_m^l + \bar{z}_m^l \bar{x})$$

where $\bar{x} \in \mathbf{R}^n$, some families of constants $\{\bar{z}_i^l \in \mathbf{R}^n\}_{i=1,l=1}^{i=m,l=s}$, $\bar{z}^l \bar{x}$ is the inner product, and $\alpha_i^l \in \mathbf{R}$.

It can be written as:

$$F_1(\bar{x}) = F_2(\bar{x})$$

by two $(\max, +)$-type functions $F_1, F_2 : \mathbf{R}^{n+1} \to \mathbf{R}$ of the form:

$$F_1(\bar{x}) = \Sigma_{l=1}^{s_i} \max(\alpha_1^l + \bar{z}_1^l \bar{x}, \dots, \alpha_m^l + \bar{z}_m^l \bar{x}), \quad \bar{x} = (y, x_1, x_2, \dots, x_n)$$

and F_2 is similar.

A family of maps $\{f_{i,j}\}_{1 \leq i,j \leq a}$ is said to be *piecewise linear*, if there is a piecewise linear map $F : \mathbf{R}^3 \to \mathbf{R}$ so that the equalities hold:

$$\pi(F(a^{-1}i, a^{-1}j, z)) = \pi(f_{i,j}(z)) \in \{1, \dots, a\}$$

for all $i, j = 1, \dots, a$ and $z \in [0,1]$.

Conversely a piecewise linear map $F : \mathbf{R}^3 \to \mathbf{R}$ determines a piecewise linear family $\{f_{i,j}\}_{1 \leq i,j \leq a}$ by $f_{i,j}(z) = F(a^{-1}i, a^{-1}j, z)$.

Example: For the Lotka Volterra cell automaton, one can choose:

$$F(i, j, z) = i + \max(\frac{L_0}{a}, j) - \max(\frac{L_0}{a}, z).$$

1.C Tropical geometry: Let us consider a $(\max, \pm)$-function F:

$$F(\bar{x}) = \Sigma_{i=1}^s \pm \max(\alpha_1^i + \bar{m}_1^i \bar{x}, \dots, \alpha_l^i + \bar{m}_l^i \bar{x})$$

where $\bar{m}_j^i \in \mathbf{Z}^N$, $\alpha_j^i \in \mathbf{R}$ and $\bar{x} = (x_1, \dots, x_N) \in \mathbf{R}^N$. When all $\bar{m}_j^i \in \mathbf{Z}_+^N$ have potive components, then we say that F is a positive $(\max, \pm)$-type function.

Tropical geometry associates with parametrized rational maps f_t to $(\max,+)$-type functions for $t \in [1,\infty)$ ([Mi]). If F is a positive $(\max,+)$-type function, then f_t are parametrized polynomials.

Let F be a $(\max,+)$-type functions as above.

Definition 1.2 *The associated family of rational maps is given by:*

$$f_t(\bar{z}) = \Pi_{i=1}^s (t^{\alpha_1^i} \bar{z}^{\bar{m}_1^i} + \cdots + t^{\alpha_l^i} \bar{z}^{\bar{m}_l^i})$$

where $\bar{z}^{\bar{m}} = \Pi_{i=1}^N z_i^{m_i}$.

Conversely F can be recovered from the family f_t. Thus the correspondence $F \leftrightarrow f_t$ is one to one.

Let (F_1, F_2) be a pair of $(\max,+)$-type functions, and (f_t^1, f_t^2) be the corresponding rational families.

The *associated affine algebraic variety* is defined by:

$$V_t(F_1, F_2) = \{z \in \mathbf{C}^N : f_1^t(z) = f_2^t(z)\}.$$

We denote its Zariski closure $\bar{V}_t(F_1, F_2) \subset \mathbf{C}P^N$.

1.C.2 From pl maps to polynomials: Let $\{f_{i,j}\}_{i,j}$ be a piecewise linear familiy, and denote the corresponding interacting map by $\Phi(x)^s(\bar{k}) = (k_0^s, k_1^s, \dots)$, $s = 0, 1, \dots$

Let $F : \mathbf{R}^3 \to \mathbf{R}$ be the piecewise linear map corresponding to the family $\{f_{i,j}\}_{i,j}$. It is given by a pair of $(\max,+)$-type functions (F_1, F_2) by $F_1(\bar{x}) = F_2(\bar{x})$ for $\bar{x} = (x_1, x_2, x_3, x_4)$. From dynamical view point, we fix the correspondence of the variables as:

$$x_1 \leftrightarrow k_n^{s+1}, x_2 \leftrightarrow k_n^s, x_3 \leftrightarrow k_{n+1}^s, x_4 \leftrightarrow k_{n-1}^{s+1}.$$

Now we denote the associated varieties as:

$$V_t(\{f_{i,j}\}_{i,j}) \equiv V_t(F_1, F_2) \subset \mathbf{C}^4.$$

Example: The Lotka Volterra cell automaton has the associated variety:

$$V_t(z_1, z_2, z_3, z_4) = \{(z_1, z_2, z_3, z_4) : f_t(z_1, z_4) = f_t(z_2, z_3)\}$$

where $f_t(z, w) = t^L z + zw$.

1.C.3 The associated complex dynamics: Let $\{f_{i,j}\}_{i,j}$ be a piecewise linear familiy. It gives the corresponding pair of $(\max,+)$-type functions (F_1, F_2), and the pair of the rational families (f_t^1, f_t^2).

The pair of the rational families gives flows:

$$\tilde{\Phi}(z)_t : \mathbf{C}^\infty \to \mathbf{C}^\infty, \quad \tilde{\Phi}(z)_t(z_0, z_1, \dots) = (z_0', z_1', \dots)$$

determined by the equality:

$$f_t^1(z'_m, z_m, z_{m+1}, z'_{m-1}) = f_t^2(z'_m, z_m, z_{m+1}, z'_{m-1}).$$

We say that it is the *associated complex dynamics*, and denote their iterations as $\tilde{\Phi}(z)_t^s(z_0, z_1, \dots) = (z_0^s, z_1^s, \dots)$, $s = 0, 1, \dots$

Lemma 1.3 *For each n and s, the sets lie on the varieties:*

$$p_{n,s} = (z_n^s, z_n^{s-1}, z_{n+1}^{s-1}, z_{n-1}^s) \in V_t(\{f_{i,j}\}_{i,j}) \subset \mathbf{C}^4$$

Thus for $n \geq 1$, the families:

$$\{p_{n,s}\}_{s=0}^{\infty}$$

give flows on $V_t(\{f_{i,j}\}_{i,j})$.

Example: For the LV case, the complex dynamics is determined successively by the equalities:

$$z_n^{s+1} = (t^L + z_{n-1}^{s+1})^{-1}(t^L z_n^s + z_n^s z_{n+1}^s).$$

When domains of piecewise linear maps are $\mathbf{R}^n$, then by the same way one can generalize the associated complex dynamics determined by the equalities of the type:

$$f_t^1(z_{n-a_1}, \dots, z_{n+a_2}, z'_{n-b}, \dots, z'_n) = f_t^2(z_{n-a_1}, \dots, z_{n+a_2}, z'_{n-b}, \dots, z'_n)$$

for some a_1, a_2 and b.

In $1.C.4$, we consider the relationships of dynamics between automata and the corresponding complex dynamics as above.

1.C.3.2 Periodic complex dynamics The domain of assoiciated complex dynamics is $\mathbf{C}^{\infty}$. When one puts some periodicity conditions, then it becomes dynamics on finite dimensional complex planes. It turns out that the dynamics is given by a correspondince between varieties in the case of cell automata.

Let $f : \mathbf{C}^N \to \mathbf{C}$ be a polynomial, and $a_1, a_2, b \geq 0$ be non negative integers with $N = a_1 + a_2 + 1 + b + 1$. Suppose $\tilde{\Phi} : \mathbf{C}^{\infty} \to \mathbf{C}^{\infty}$ is given by the equation:

$$f(z_{n-a_1}, \dots, z_{n+a_2}, z_{n-b}^1, \dots, z_n^1) = 0$$

where $\tilde{\Phi}(z_0, z_1, \dots) = (z_0^1, z_1^1, \dots)$. Put $c = \max\{a_1 + a_2 + 1, b + 1\}$. Then the associated periodic complex dynamics is given by a multivalued map:

$$\tilde{\Phi}_p : \mathbf{C}^{\infty} \to \mathbf{C}^{\infty}$$

defined by $f(z_{n-a_1}, \dots, z_{n+a_2}, z_{n-b}^1, \dots, z_n^1) = 0$ for all n mod $c + 1$. It is determined by the first c components, and so can be expressed as $\tilde{\Phi}_p : \mathbf{C} \times \mathbf{Z}_c \to \mathbf{C} \times \mathbf{Z}_c$. These are multi-valued, since here we do not impose any initial conditions on z_0.

Let us consider the case of cell automata we have considred so far, where $a_1 = 0, a_2 = 1, b = 1$. Thus $c = 2$ and $N = 4$. let $\mathbf{C}^4 = \{(z_1, z_2, w_1, w_2)\}$ be a coordinate. Let $M_1, M_2 \subset \mathbf{C}^4$ be hypersurfaces, and $S = M_1 \cap M_2 \subset \mathbf{C}^4$ be the surface of the intersection.

Let f be a polynomials of 4 variables so that

$$M_1 = \{(z_1, z_2, w_1, w_2) : f(z_1, z_2, w_1, w_2) = 0\},$$
$$M_2 = \{(z_1, z_2, w_1, w_2) : f(z_2, z_1, w_2, w_1) = 0\}$$

hold respectively. Thus:

$$S = \{(z_1, z_2, w_1, w_2) : f(z_1, z_2, w_1, w_2) = 0, f(z_2, z_1, w_2, w_1) = 0\}.$$

So given $(z_1, z_2) \in \mathbf{C}^2$, there are at most finitely many points (w_1, w_2) so that (z_1, z_2, w_1, w_2) are points on S. This determines a correspondence:

$$\varphi : \mathbf{C}^2 \to \mathbf{C}^2$$

which is a multivalued, and its iteration gives two dimensional interaction dynamics determined by f.

Example: For the LV case, $f = f_{LV}$ and $f_{LV}(z_1, z_2, w_1, w_2) = w_2(t^L + w_1) - (t^L z_1 + z_1 z_2)$. Thus S is defined by the equations:

$$\begin{aligned} S = \{(z_1, z_2, w_1, w_2) :& w_2(t^L + w_1) = (t^L z_1 + z_1 z_2), \\ & w_1(t^L + w_2) = (t^L z_2 + z_1 z_2)\}. \end{aligned}$$

1.C.4 Scaling limit from pl to complex: Maslov introduced *dequantization* of the real line $\mathbf{R}$ ([LM], [V]). It is given by a family of semirings R_t for $t > 1$, which are all the real number $\mathbf{R}$ as sets. The multiplication and the addition are respectively given by:

$$x \oplus_t y = \log_t(t^x + t^y), \quad x \otimes_t y = x + y.$$

The important feature for us is the behaviour as $t \to \infty$, and in fact one has the equality:

$$x \oplus_\infty y = \max\{x, y\}.$$

Corresponding to polynomials in the usual real numbers, one has R_t-*polynomials* whose limit $t \to \infty$ satisfies a max plus equation:

$$\varphi_t(x) = \oplus_t(\alpha_j + m_j x), \quad x \in \mathbf{R}^n, \quad m_j \in \mathbf{Z}^n,$$
$$\varphi_\infty(x) = \max(\alpha_1 + m_1 x, \ldots, \alpha_k + m_k x).$$

Let F be a positive $(\max, +)$-type function of the form:

$$F(\bar{x}) = \Sigma_{i=1}^s \max(\alpha_1^i + \bar{m}_1^i \bar{x}, \ldots, \alpha_l^i + \bar{m}_l^i \bar{x})$$

on $\mathbf{R}^N$, and f_t be the associated polynomials with respect to F.

We define the corresponding *R_t-polynomials* F_t by:

$$F_t(\bar{x}) = \Sigma_{i=1}^s (\alpha_1^i + \bar{m}_1^i \bar{x}) \oplus_t \cdots \oplus_t (\alpha_l^i + \bar{z}_l^i \bar{x}).$$

Then we have the equality

$$\lim_{t \to \infty} F_t = F.$$

Let us denote:

$$\mathrm{Log}_t : \mathbf{C}^N \to \mathbf{R}_+$$

by $\mathrm{Log}_t(z_1, \dots, z_N) = (\log_t |z_1|, \dots, \log_t |z_N|)$.

Proposition 1.2 (LM,V) *The equality holds:*

$$\log_t^{-1} \circ F_t \circ Log_t = f_t$$

on $\mathbf{R}_+^N$.

Let $F : \mathbf{R}^n \to \mathbf{R}$ be a positive piecewise lieanr map (positive $(\max, \pm)$-type function), and $\{f_{i_1, \dots, i_{n-1}}\}$ be the corresponding piecewise linear family. As in $1.A$, it gives a dynamical systems:

$$\bar{\Phi}(F) : \mathbf{R} \times \mathbf{R}^\infty \to \mathbf{R} \times \mathbf{R}^\infty.$$

Let (F_1, F_2) be the pair of positive $(\max, +)$-type functions corresponding to F, and (f_t^1, f_t^2) be the pairs of the parametrized polynomials. It gives a complex dymanics:

$$\tilde{\Phi}(f_t^1, f_t^2) : \mathbf{C} \times \mathbf{C}^\infty \to \mathbf{C} \times \mathbf{C}^\infty$$

and the associated algebraic varieties:

$$V_t(F) = V_t(f_t^1, f_t^2) \subset \mathbf{C}^{n+1}.$$

By this way, F above produces two dymanics $\bar{\Phi}(F)$ and $\tilde{\Phi}(f_t^1, f_t^2)$ with a parametrized algebraic varieties $V_t(F)$.

Let $(\tilde{\Phi}(f_t^1, f_t^2), \mathbf{R}_+^\infty)$ be the restriction of $\tilde{\Phi}$ on $\mathbf{R}^+$.

Theorem 1.1 *There is a contracting map:*

$$Log_t : (\tilde{\Phi}(f_t^1, f_t^2), \mathbf{R}_+^\infty) \to (\bar{\Phi}(F), \mathbf{R}^\infty)$$

which admits a reduction to parametrized affine algebraic varieties $V_t(F)$.

Proof: Let us put $\tilde{\Phi}(z)(z_0, z_1, \dots) = (z_0', z_1', \dots)$, and:

$$\bar{x} = (z_{n-a_1}, \dots, z_{n+a_2}, z_{n-b}', \dots, z_n') \in \mathbf{R}^{n+1}.$$

These sets satisfy the equality:

$$f_t^1(\bar{x}) = f_t^2(\bar{x}).$$

We check that $F_1(\mathrm{Log}_t(\bar{x})) - F_2(\mathrm{Log}_t(\bar{x}))$ approaches to zero as $t \to \infty$.

Let (F_t^1, F_t^2) be the pair of the R_t-polynomials corresponding to F. By the proposition 1.2, one has the equality:

$$F_t^1 \circ \mathrm{Log}_t(\bar{x}) = F_t^2 \circ \mathrm{Log}_t(\bar{x})$$

on positive and real points $\bar{x} \in \mathbf{R}_+^{n+1}$. Since their limits satisfy the equalities:

$$\lim_{t \to \infty} F_t^i = F_i, \quad i = 1, 2$$

Log_t gives a contracting map as desired. We put:

$$P_i : \mathbf{C}^\infty \to \mathbf{C}, \quad P_i((z_0, z_1, \dots)) = z_i,$$
$$Q(z_0, z_1, \dots) = (\mathrm{Log}_t(z_0), \mathrm{Log}_t(z_1), \dots).$$

By lemma 1.3, this gives a reduction of $\tilde{\Phi}|\mathbf{R}_+^\infty$ to a parametrized affine algebraic varieties $V_t(F)$. This completes the proof.

In particular, if the dymanics $\bar{\Phi}$ is reduced to a call automaton A, then one obtains a contracting map:

$$\mathrm{Log}_t : (V_t(F), \tilde{\Phi}) \to A.$$

1.C.5 Duality on cell automata: By invertibility of our contracting maps, one can obtain various operations on automata arising from geometric operations on varieties.

[GKZ] introduced the *projective duality* for projective varieties:

$$X \subset P^N(V) \to X^\vee \subset P^N(V^*)$$

for a $\mathbf{C}$-vector space V and its projectivization $P(V)$. V^* is the dual vector space.

Suppose both are hypersurfaces. Then they have the defining polynomials unique up to constant multiplications. This implies that one obtains an assignment from polynomials to themselves by the above duality, unique up to constant multiplications.

Let $\{f_{i,j}\}_{i,j}$ be a piecewise linear family, (f_1^t, f_2^t) be the associated polynomials and $V_t(\{f_{i,j}\}_{i,j})$ be the associated hypersurfaces.

Definition 1.3 (K4) *Let $\{f_{i,j}\}_{i,j}$ be a picewise linear family. Another picewise linear family $\{g_{i,j}\}_{i,j}$ is called the dual picewise linear family, if the parametrized hypersurfaces $V_t(\{g_{i,j}\}_{i,j})$ satisfy the equality:*

$$V_t(\{f_{i,j}\}_{i,j})^\vee = V_t(\{g_{i,j}\}_{i,j})$$

for all $t \in [1, \infty)$.

Similarly let A be an cell automaton. Another cell automaton $A^\vee$ is called the *dual cell automaton*, if the corresponding hypersurfaces satisfy the equality:

$$V_t(A)^\vee = V_t(A^\vee)$$

for all t.

In general it is not so easy to find the defining polynomials of the projective dual varieties. We have calculated dual automata in the case of some curves:

Lemma 1.4 (K4)

$$[\max\{au_n, \alpha + au_{n+1}\} = c]^\vee =$$
$$\max\{\frac{a}{a-1}(c - \frac{\alpha}{a}) + \frac{a}{a-1}u_{n+1}, \frac{ac}{a-1} + \frac{a}{a-1}u_n\} = c.$$

1.D Scaling limits: Let (f_1, f_2) be two polynomials, and correspondingly $\{z_i^t\}_{i,t}$ be the iterated complex dynamics.

A smooth function $\alpha : \mathbf{R}_+ \times \mathbf{R}_+ \times (0, 1] \to \mathbf{R}_+$ is called a *scaling function*.

Let us take two scaling functions α and β, and have the change of variables as:

$$n = \alpha(x, s, \epsilon), \quad t = \beta(x, s, \epsilon).$$

Let us fix a constant p and a small $\epsilon > 0$. A *scaled function* with respect to $\{z_i^t\}_{i,t}$ is given by a function u with variables (x, s) so that it satisfies the equation:

$$z_n^t = p + \epsilon u(x, s).$$

As $\epsilon \to 0$, the values of u may go to infinity, and so this is an expanding change of dynamics.

Suppose there are polynomials F and $\{f_i, g_i\}_{i=1}^m$ with $f_i(x, 0) = x$, $g_i(s, 0) = s$ so that the equality:

$$F(\epsilon, p, u(f_1(x, \epsilon), g_1(s, \epsilon)), \dots, u(f_m(x, \epsilon), g_m(s, \epsilon))) = 0$$

holds, which is induced from the iterated complex dynamics.

The *formal Taylor expansion of the scaled equation* is given by the one of the above equation:

$$\begin{aligned} F(\epsilon, p, &u(f_1(x, \epsilon), g_1(s, \epsilon)), \dots, u(f_m(x, \epsilon), g_m(s, \epsilon))) \\ &= \epsilon^l D(u) + \epsilon^{l+1} D_1(u) + \dots \end{aligned}$$

for some $l \geq 0$, where D is a partial differential operator on u.

The *PDE at infinity* induced from the complex dynamics $\{z_n^t\}_{n,t}$, is given by the above partial differential equation $D(u) = 0$. As $\epsilon \to 0$, u approaches to solutions of a PDE at infinity. Thus one has:

Lemma 1.5 *Suppose a complex dymanics arising from* (f_1, f_2) *admits a scaling change as above so that a formal Taylor expansion* $\epsilon^l D(u) + \epsilon^{l+1} D_1(u) + \ldots$ *is obtained.*

Then this gives an expanding map from $\tilde{\Phi}(f_1, f_2)$ *to solutions of PDE* $D(u) = 0$.

1.D.2 Dynamical pattern formations: Here we construct two dynamical rescalings from iteration dynamics by families of piecewise linear maps to some PDEs, KdV and LV. They are obtained by combination of our construction of contracting maps with expanding maps obtained by Hirota. Now we have our main theorem:

Theorem 1.2 *(1) Let* $f : \mathbf{R}^3 \to \mathbf{R}$ *be a partially defined piecewise linear map given by:*

$$f(x_2, x_3, x_4) = x_2 - \max(0, x_2 + x_3), \quad x_4 \leq \max(0, x_2 + x_3) - x_2.$$

Then it gives a dynamical rescaling from a cell automaton:

$$A : V_1 + \max(0, V_2 + V_3) = V_2 + \max(0, V_1 + V_4)$$

to KdV flows:

$$\begin{aligned} &Log_t : (V, \sigma) = \{(z_1, z_2, z_3, z_4) : z_2 + z_1 z_2 z_4 = z_1 + z_1 z_2 z_3\} \subset \mathbf{C}^4 \to A, \\ &\phi_t : (V, \sigma) \to \{u(x.s) : u_s - \frac{1}{p^3} u u_x + \frac{1}{48 p^2}(1 - \frac{1}{p^4}) u_{3x} = 0\}. \end{aligned}$$

(2) Let $f : \mathbf{R}^3 \to \mathbf{R}$ *be a piecewise linear map given by:*

$$f(x_2, x_3, x_4) = x_2 + \max(0, x_3) - \max(0, x_4).$$

Then it gives a dynamical rescaling from a cell automaton:

$$B : V_1 + \max(0, V_4) = V_2 + \max(0, V_3)$$

to LV flows:

$$\begin{aligned} &Log_t : (V, \sigma) = \{(z_1, z_2, z_3, z_4) : z_2 + z_2 z_3 = z_1 + z_1 z_4\} \subset \mathbf{C}^4 \to B, \\ &\phi_t : (V, \sigma) \to \{u(x, s) : u'_n = u_n(u_{n+1} - u_{n-1})\} \end{aligned}$$

Proof: (1) Let us consider the equation $x_1 = f(x_2, x_3, x_4) = x_2 - \max(0, x_2 + x_4)$, $x_3 \leq \max(0, x_2 + x_4) - x_2$. It is easy to see that it induces the automaton $x_1 + \max(0, x_2 + x_4) = x_2 + \max(0, x_1 + x_3)$. Then the associated polynomials are independent of the time t:

$$\begin{aligned} f_1(z_1, z_2, z_3, z_4) &= z_1 + z_1 z_2 z_3, \\ f_2(z_1, z_2, z_3, z_4) &= z_2 + z_1 z_2 z_4. \end{aligned}$$

The associated complex dynamics satisfy the equalities:

$$z_n^t + z_n^{t+1} z_{n-1}^{t+1} z_n^t = z_n^{t+1} + z_{n+1}^t z_n^{t+1} z_n^t.$$

which is the same as $z_{n+1}^t - z_{n-1}^{t+1} = \frac{1}{z_n^{t+1}} - \frac{1}{z_n^t}$.

Now we rewrite it as:

$$z_{n+1}^{t-\frac{1}{2}} - z_{n-1}^{t+\frac{1}{2}} = \frac{1}{z_n^{t+\frac{1}{2}}} - \frac{1}{z_n^{t-\frac{1}{2}}}$$

and put scaling parameters as:

$$\begin{aligned} n &= \frac{s}{\epsilon^2}, \quad t = \frac{x}{\epsilon} - \frac{cs}{\epsilon^3}, \\ z_n^t &= p + \epsilon^2 u(x,s) \end{aligned}$$

where c and p are constants satisfying $1 - 2c = 1/p^2$. The change of variables gives a parametrized dynamics:

$$\{z_n^t\}_{n,t} \to \{u_\epsilon(x,s)\}_\epsilon.$$

By applying these change of variables into the defining equation above, one obtains the equation:

$$\begin{aligned} &\epsilon^2 u(x - \frac{\epsilon}{2} + c\epsilon, s + \epsilon^3) - \epsilon^2 u(x + \frac{\epsilon}{2} - c\epsilon, s - \epsilon^3) \\ &= \frac{1}{p + \epsilon^2 u(x + \frac{\epsilon}{2}, s)} - \frac{1}{p + \epsilon^2 u(x - \frac{\epsilon}{2}, s)}. \end{aligned}$$

The Taylor expansion of the above equation at $\epsilon = 0$ gives a formula ([H]):

$$\epsilon^5 (u_s - \frac{1}{p^3} u u_x + \frac{1}{48p^2}(1 - \frac{1}{p^4}) u_{3x}) + o(\epsilon^7) = 0.$$

Now let us put $t = \epsilon^{-1}$, and denote the parametrized dynamics $\{u_\epsilon(x,s)\}_\epsilon$ as (σ_t, V). By the above estimate, it gives an expanding map from (σ_t, V) to KdV flows. Thus combining with our construction of contracting maps, one has obtained a dynamical rescaling from the above cell automaton to the KdV flows. This completes the proof of (1).

(2) The piecewise linear map f determines the cell automaton:

$$x_n^{t+1} + \max(0, x_{n-1}^{t+1}) = x_n^t + \max(0, x_{n+1}^t).$$

The associated polynomials are:

$$\begin{aligned} f_1(z_1, z_2, z_3, z_4) &= z_1 + z_1 z_4, \\ f_2(z_1, z_2, z_3, z_4) &= z_2 + z_2 z_3. \end{aligned}$$

Thus the associated complex dynamics satisfy the equalities:

$$z_n^{t+1} - z_n^t = z_n^t z_{n+1}^t - z_n^{t+1} z_{n-1}^{t+1}.$$

Let us put rescaling parameters $t = \frac{s}{\epsilon}, \quad z_n^t = \epsilon u_n(s)$. Then one obtains the the equality:

$$\epsilon(u_n(s+\epsilon) - u_n(s)) - \epsilon^2(u_n(s)u_{n+1}(s) - u_n(s+\epsilon)u_{n-1}(s+\epsilon)) = 0.$$

Thus the formal Taylor expansion of the above equation at $\epsilon = 0$ gives a formula ([H]):

$$(\frac{d}{ds}u_n - u_n(u_{n+1} - u_{n-1})) + o(\epsilon) = 0.$$

By this way we have an expanding map from the above cell automaton to the Lotka Volterra flows at infinity. Thus one has obtained a dynamical rescaling from the above cell automaton to the Lotka Volterra flows.

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