

REMARKS ON MODULAR SYMBOLS FOR MAASS WAVE FORMS

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*To Professor F. Hirzebruch, with friendship and admiration,
for his anniversary.*

Abstract. In this paper I introduce modular symbols for Maass wave cusp forms. They appear in the guise of finitely additive functions on the Boolean algebra generated by intervals with non–positive rational ends, with values in analytic functions (pseudo–measures in the sense of [MaMar2]). After explaining the basic issues and analogies in the extended Introduction, I construct modular symbols in the sec. 1 and the related Lévy–Mellin transforms in the sec. 2. The whole paper is an extended footnote to the Lewis–Zagier fundamental study [LZ2].

§0. Introduction

0.1. Period polynomials and period functions. Let $u(\tau) \in S_{2k}(SL(2, \mathbf{Z}))$ be a cusp form of an integer *weight* $2k > 0$ for the full modular group. This means that it is holomorphic in the upper half plane, the tensor $u(\tau)(d\tau)^k$ is $SL(2, \mathbf{Z})$ –invariant, and $u(\tau)$ vanishes at cusps.

Its *period polynomial* is defined as the following integral:

$$\psi(z) = \psi_u(z) := \int_0^{i\infty} u(\tau)(z - \tau)^{2k-2} d\tau \quad (0.1)$$

Here z is, for the time being, an auxiliary formal variable.

One remarkable discovery in the theory of modular functions was a possibility to develop its versions for a certain set of *complex weights* $2s$ (replacing the former $2 - 2k$). This spectrum consists of the (doubled) zeroes of Selberg’s zeta function $Z(s)$ of $SL(2, \mathbf{Z})$ acting on the upper half–plane, or equivalently, those values of s for which the Mayer transfer operator \mathcal{L}_s^2 (cf. [May1], [May2]) has 1 as its eigenvalue: see [LZ1] for a short review and [LZ2] for a comprehensive exposition.

0.2. Classical modular symbols. The classical modular symbols of weight $2k$ for $SL(2, \mathbf{Z})$, in one of their guises, can be defined simply as integrals

$$\int_{\alpha}^{\beta} u(\tau)(z - \tau)^{2k-2} d\tau \quad (0.2)$$

where this time $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ are arbitrary cusps, and the integration is taken along, say, the hyperbolic geodesic connecting β to α .

More precisely, the modular symbol $\{\alpha, \beta\}_k$ (for the full modular group) is the integral (0.2) considered as a linear map

$$\{\alpha, \beta\}_k : S_{2k}(SL(2, \mathbf{Z})) \rightarrow \mathbf{C}[z]. \quad (0.3)$$

In the next subsections, we will briefly recall the number-theoretic motivations for considering (0.3). A geometric interpretation of (0.3), after a dualization, runs as follows: this integral expresses the pairing between the Hodge cohomology and the Betti homology of the moduli space $\overline{M}_{1, 2k-2}$ of elliptic curves with marked points (cf. [Sh1], [Sh2] for a version involving Kuga varieties rather than moduli spaces).

The modular symbols (0.3) satisfy the following simple functional equations:

$$\{\alpha, \beta\}_k + \{\beta, \gamma\}_k + \{\gamma, \alpha\}_k = 0, \quad \{\alpha, \beta\}_k + \{\beta, \alpha\}_k = \{\alpha, \alpha\}_k = 0, \quad (0.4)$$

Thus they can be extended to a $\mathbf{C}[z]$ -valued finitely additive function on the Boolean algebra generated by (positively oriented) segments with rational ends in $\mathbf{P}^1(\mathbf{R})$. We sometimes call such a function a *pseudo-measure*, as in [MaMar2]. The variable change formula applied to (0.2) leads to an additional property of this particular pseudo-measure, which we call its *modularity*:

$$\{g(\alpha), g(\beta)\}_k = g\{\alpha, \beta\}_k. \quad (0.5)$$

Here $g \in SL(2, \mathbf{Z})$ acts on $\mathbf{P}^1(\mathbf{Q})$ by fractional linear transformations, and on polynomials of degree $\leq 2k - 2$ by a natural twisted action.

A pseudo-measure can in principle take values in any abelian group, and modularity condition (0.5) makes sense if this group is a left $SL(2, \mathbf{Z})$ -module. If the group of values has no 2- and 3-torsion, the last two equations in (0.4) follow from the first one.

0.3. Modular symbols for Maass cusp forms. The first goal of this note is to extend the definition of $\{\alpha, \beta\}_k$ to complex weights for which there exist non-trivial Maass cusp forms. We take the formula (0.2) as our starting point and look for its analogs in the Lewis-Zagier theory. We are interested mostly in complex critical zeroes/weights for which $\operatorname{Re} s = \frac{1}{2}$.

Tracing parallels with the classical theory, one should keep in mind that certain classical objects have more (or less) than one parallel in the new setting.

For example, the most straightforward analogs of $u(\tau) \in S_{2k}(SL(2, \mathbf{Z}))$ apparently are the Maass wave cusp forms, introduced in [M], – smooth $SL(2, \mathbf{Z})$ -invariant functions on H satisfying the hyperbolic Laplace equation $\Delta u = s(1-s)u$

and certain growth/vanishing conditions. An appropriate version of the period polynomial (0.1) for such a form is its period function $\psi_u(z)$, this time a holomorphic function of our former auxiliary variable z .

However, the relationship between u and ψ_u as it is first explained in sec. 1 of Ch. I of [LZ2], does not look at all like (0.1) and passes through three intermediate steps $u \leftrightarrow L_\varepsilon \leftrightarrow f \leftrightarrow \psi$.

To the contrary, the structure of (0.1) is reproduced in the formula

$$\psi(z) = \int_{-\infty}^0 (z-t)^{-2s} U(t) dt. \quad (0.6)$$

(see [LZ2], page 221), in which $U(t)dt$ denotes a certain distribution on \mathbf{R} , called “the boundary value” of $u(\tau)$. Therefore, it is this distribution that in our context seems to be a more adequate analog of a classical cusp form, the more so that its $SL(2, \mathbf{Z})$ -invariance property involves an explicitly weighted action of the modular group,

$$U\left(\frac{at+b}{ct+d}\right) = |ct+d|^{2-2s} U(t). \quad (0.7)$$

whereas a Maass form is simply $SL(2, \mathbf{Z})$ -invariant.

The formula (0.6) seems to offer a straightforward way to do it – just consider the integrals

$$\int_{\alpha}^{\beta} (z-t)^{-2s} U(t) dt.$$

Formal manipulations with such integrals are simple and seemingly prove (0.4) and (0.5), and we reproduce them for their heuristic value. However, these calculations cannot be taken literally, because the characteristic functions of the intervals with rational ends do not belong to the space of test functions for the distribution $U(t)$.

Thus we have to find a way around this difficulty.

In fact, there are at least two different ways. One of them starts with the three-term functional equation for the period function $\psi(z)$, proceeds with pure algebra, and works also for Lewis–Zagier’s “period-like functions”.

Another method is applicable only to the period functions of Maass forms u and uses the Lewis–Zagier formula of the form

$$\psi(z) = \int_{-\infty}^0 \{u, R_z^s\}(\tau)$$

where the integrand is a closed 1-form depending on z as a parameter (its structure is described in the main text below). One can then integrate this form along a path

that may this time connect two arbitrary cusps, thus getting another analog of (0.2).

These two constructions form the content of sec. 1 below.

0.4. Mellin transform and classical modular symbols. Now we will explain some of our motivations.

Briefly, we want to describe a construction presenting the Maass Dirichlet series as an integral over, say, $[0, 1/2]$, formally similar close to the Mazur–Mellin transform in the theory of p -adic interpolation. We call such a representation *the ∞ -adic Lévy–Mellin transform*, cf [MaMar2]. The integration measure in both cases is constructed out of modular symbols.

Here is a sketch of the classical p -adic constructions.

The classical theory of modular symbols as it was presented in [Ma1], [Ma2], started with the following observations. Suppose that we are interested in the calculation of some values (say, at integer points ρ) of a Dirichlet series

$$L_\kappa(\rho) = \sum_{n=1}^{\infty} a_n \kappa(n) n^{-\rho}. \quad (0.8)$$

where (a_n) is a certain “arithmetic” function, and κ is an additive character of \mathbf{Z} of finite order. In the standard approach one first introduces the Fourier series

$$u_\kappa(\tau) := \sum_{n=1}^{\infty} a_n \kappa(n) e^{2\pi i n \tau} \quad (0.9)$$

and then works with the Mellin transform

$$\Lambda_\kappa(\rho) := \int_0^{i\infty} u_\kappa(\tau) \left(\frac{\tau}{i}\right)^{\rho-1} d\tau. \quad (0.10)$$

which is related to (0.8) by the simple formula $\Lambda(\rho) = i(2\pi)^{-\rho} \Gamma(\rho) L(\rho)$.

Now, let $u(\tau) := u_{\kappa_0}(\tau)$ where κ_0 is identically 1. Clearly, $u_\kappa(\tau) = u(\tau + \alpha)$ for a rational number α such that $\kappa(n) = e^{2\pi i \alpha n}$, so that we can write, shifting the integration path,

$$\Lambda_\kappa(\rho) := \int_\alpha^{i\infty} u_\kappa(\tau) \left(\frac{\tau - \alpha}{i}\right)^{\rho-1} d\tau \quad (0.11)$$

Thus, if $\rho \geq 1$ is an integer, varying κ in (0.8) reduces to replacing $\tau^{\rho-1}$ in (0.10) by an arbitrary polynomial of degree $\leq \rho - 1$ and allowing the integration paths $(\alpha, i\infty)$ with an arbitrary rational α .

Furthermore, if $u \in S_{2k}(SL(2, \mathbf{Z}))$ as above, and $1 \leq \rho \leq 2k - 1$, applying to α the “continued fractions trick”, we can replace $(\alpha, i\infty)$ by a sum of geodesic paths in the upper half-plane, joining pairwise cusps of the form $g^{-1}(0)$ and $g^{-1}(i\infty)$, where g varies in $SL(2, \mathbf{Z})$, and then return to $(0, i\infty)$ by transforming the integrand via $\tau \mapsto g\tau$. Thus, in particular, all values of (0.8), corresponding to integer ρ ’s inside the critical strip and arbitrary characters κ , can be expressed as linear combinations of modular symbols *with rational coefficients*, and span a finite-dimensional space over \mathbf{Q} .

0.4. p -adic Mellin–Mazur transform. Such expressions were used in [Ma2], [Ma3] in order to produce a p -adic interpolation of values (0.8). This problem will make sense, if (after an appropriate normalization) these values will lie in a finitely generated \mathbf{Z} -module, so the basic problem is to control the denominators.

As we already said, the main tool for such an interpolation was a p -adic integral (the Mellin–Mazur transform) with respect to a p -adic *pseudo-measure* (see below) constructed using modular symbols. This transform integrates $\tau^{\rho-1}$ twisted by κ against this pseudo-measure, and for finite order κ produces the classical values $L_\kappa(\rho)$ more or less by definition. (In fact, one works usually with Dirichlet characters in place of κ , but the only difference consists in the appearance of auxiliary Gauss sums).

Here are some details.

(a) *The p -adic integration domain and a naive pseudo-measure.* The following tentative construction applies to any (absolutely convergent) series of the type (0.8) considered as a function of variable κ with fixed ρ .

At the first approximation, consider \mathbf{Z}_p with \mathbf{Z} densely embedded in it. The Boolean algebra of closed/open subsets of \mathbf{Z}_p is generated by the *primitive subsets* $a + p^m\mathbf{Z}_p$, $m = 0, 1, 2, \dots; a \bmod p^m$. Put

$$\mu_L(a + p^m\mathbf{Z}) := \sum_{n \equiv a \bmod p^m} a_n n^{-\rho}. \quad (0.12)$$

Any two primitive subsets either do not intersect, or one of them is contained in the other. If one primitive subset I is a disjoint union of a finite family of other primitive subsets I_j , then $\mu_L(I) = \sum_j \mu_L(I_j)$. Thus μ_L extends to a \mathbf{C} -valued finitely additive function on the Boolean algebra of closed/open subsets of \mathbf{Z}_p . We will call such objects *pseudo-measures* on \mathbf{Z}_p .

Generally, there is no chance that such a pseudo-measure will tend p -adically to zero when $m \rightarrow \infty$, even if its values lie in a finite-dimensional \mathbf{Q} -space. As is explained in [Ma2], a Mazur’s p -adic integral of a function against such a pseudo-measure typically converges not because the smaller primitive subsets have asymptotically vanishing pseudo-measure, but because in a typical Riemann sum, *many*

approximately equal terms of not very large p -adic size are involved, and the quantity of summands $\approx p^m$, tending to zero p -adically, produces an unconventional non-Archimedean convergence effect.

If the pseudo-measure of small subsets does not tend to zero, the best one may hope for is that it will be *bounded*, i.e. its values will lie in a \mathbf{Z} -module of finite type. Even this usually will not happen: for example, one can suspect that

$$\mu_L(p^m \mathbf{Z}) = \sum_{n \equiv 0 \pmod{p^m}} a_n n^{-\rho} = p^{-m\rho} \sum_n a_n p^m n^{-\rho}$$

will have denominator of order $p^{-m\rho}$.

A radical way to avoid this danger is to postulate that $a_n = 0$ if n is divisible by p . One can achieve this cheaply, if L admits an Euler product: simply discard the p -th Euler factor of L .

(Notice an interesting Archimedean analogy: the Mellin transform Λ in (0.10) produces L supplemented by the initially missing “Euler factor at arithmetical infinity”.)

Returning to $L_{(p)} := L$ divided by its p -factor, we may from now on look only at the group of p -adic units $\mathbf{Z}_p^* \subset \mathbf{Z}_p$ by which our pseudo-measure is now supported.

We repeat in conclusion, that the classical values (0.8) are tautologically integrals of the locally constant function κ against our pseudo-measure (0.12). (Of course, this is why chose it in the first place). Only when we start to interpolate and allow, say, continuous p -adically valued multiplicative characters in place of κ , we will need the basics of such p -adic integration.

(b) *Normalized p -adic pseudo-measure.* Let now L be the Mellin transform of an $SL(2, \mathbf{Z})$ -cusp form of weight $2k$ as above. Representing the characteristic function of the set $a + p^m \mathbf{Z}$ by a linear combination of the additive characters κ modulo p^m , and calculating $\Lambda_\kappa(\rho)$ as in (0.4), we see that $\mu_L(a + p^m \mathbf{Z}_p)$ is a linear combination of modular symbols $\{bp^{-m}, i_\infty\}$, $b \in \mathbf{Z}$.

Conversely, we may take an appropriate linear combination of such measures and obtain the one that was used in [Ma2], [Ma3], namely

$$\mu_p(a + p^m \mathbf{Z}_p) := \varepsilon^{-m} \{ap^{-m}, i_\infty\}_k - p^{2k-2} \varepsilon^{-m+1} \{ap^{-m+1}, i_\infty\}_k. \quad (0.13)$$

Here ε is a root of the (inverted) p -factor of L : $\varepsilon^2 - a_p \varepsilon + p^{2k-1} = 0$. If one of the two roots is a p -adic unit, we get a bounded measure. In any case, its growth can be controlled. The appearance of two summands and ε in (0.13) is a slightly more sophisticated solution than the total discarding of the p -th Euler factor.

0.6. ∞ -adic Lévy–Mellin transform. As it was suggested in [MaMar2], let us make the following replacements in the picture sketched above.

Replace p by the arithmetic infinity (archimedean valuation of \mathbf{Q}). Replace \mathbf{Z}_p^* by the semi-interval $(0, 1]$.

Call the classical Farey intervals with ends $(g^{-1}(i\infty), g^{-1}(0))$, $g \in SL(2, \mathbf{Z})$, *primitive segments*. They will be our replacement for the residue classes $a + p^m \mathbf{Z}_p$. Exactly as residue classes, two open primitive segments either do not intersect, or one of them is contained in another. For an abelian group W , call a pseudo-measure a W -valued finitely additive function on segments with rational ends (see additional details below).

A typical pseudo-measure in this sense is the modular symbol itself:

$$\mu(\alpha, \beta) = \{\alpha, \beta\}_k,$$

in particular, $\mu(\alpha, \infty) = \{\alpha, \infty\}_k$ which may be compared to (0.13).

As in the p -adic case, the pseudo-measure of a small segment is not small in the archimedean sense. However, now we cannot hope to compensate this by the non-Archimedean effect referred to above.

Instead, we suggest to use the following general feature of our constructions:

(*) *The Mellin transform of a cusp form, after a suitable normalization, can be naturally written as the sum over rational numbers in $(0, 1]$ of values of a certain arithmetic function a :*

$$A := \sum_{\beta \in (0, 1] \cap \mathbf{Q}} a(\beta). \quad (0.14)$$

The values $a(p/q)$ involved here are essentially modular symbols divided by a power of the denominator q . For details, see sec. 2 below.

Generally, a convergent series of the form (0.14) gives rise to an archimedean integral in two related ways:

(i) *The first construction.* We can define a pseudo-measure $\mu = \mu_a$ on the Boolean algebra generated by segments with *irrational* ends in $[0, 1]$ putting

$$\mu(\alpha, \beta) := \sum_{\gamma \in (\alpha, \beta) \cap \mathbf{Q}} a(\gamma). \quad (0.15)$$

so that

$$A = \int_0^1 d\mu. \quad (0.16)$$

One can also treat (0.14) as a distribution on an appropriate space of test functions.

This is a direct analog of (0.12), however, it is not the version that we will use in this paper.

(ii) *The second construction.* Let r be a function defined on pairs of positive coprime integers (p, q) , $p < q$ and decreasing sufficiently fast. For a real number ξ , denote by $q_i(\xi)$ the denominator of the i -th convergent to ξ , $i \geq 0$. We can introduce the Lévy 1-form $l(\xi)d\xi$, associated to r , and defined on $(0, \frac{1}{2}]$ by the following prescription:

$$l(\xi) = l_r(\xi) = \sum_{i=0}^{\infty} r(q_i(\xi), q_{i+1}(\xi)). \quad (0.17)$$

According to a lemma by P. Lévy, for any pair (p, q) as above, the set of all $\xi \in (0, \frac{1}{2}]$ for which there exists i with $(p, q) = (q_i(\xi), q_{i+1}(\xi))$, fills a primitive semi-interval of length $\frac{1}{(p+q)q}$. Moreover, this i is uniquely defined. Therefore, when $r(p, q)$ decreases sufficiently rapidly to assure convergence, we get

$$\int_0^{1/2} l_r(\xi)d\xi = \sum_{\alpha=p/q \in (0,1]} \frac{r(p, q)}{(p+q)q}. \quad (0.18)$$

In particular, we get A from (0.14) if we choose

$$r(p, q) := a(p/q)(p+q)q. \quad (0.19)$$

When A comes from a modular form (classical or Maass), so that the summands $a(\beta)$ are concocted of (classical or Maass) modular symbols, we will call the integral in (0.12) *the ∞ -adic Lévy-Mellin transform*.

The Lévy functions and their generalizations appear also in a different context: that of linearizations of the germs of analytic diffeomorphisms of one complex variable z with an indifferent fixed point. For example, a germ with linear part $e^{2\pi i\xi}z$ is linearizable iff the Brjuno number of ξ

$$b(\xi) := \sum_{i=0}^{\infty} \frac{\log q_{n+1}}{q_n}$$

is finite. In fact, an interesting theory is developed/reviewed in the papers [MarMouYo1] and [MarMouYo2] for another Brjuno function $B(\xi)$, which differs from

$b(\xi)$ by $O(1)$, but satisfies a functional equation and has a complex version closely resembling some constructions in the theory of modular forms. In our context, it can be used for calculation of the derivative of some classical L -series at certain points. This looks like an interesting variation on the subject of Lévy–Mellin transform.

0.7. A summary of p -adic/ ∞ -adic analogies. For clarity, we summarize the suggested analogies in the following lines:

$$\begin{array}{lll}
 \mathbf{Z}_p^* & \iff & (0, 1] \\
 \cup & & \cup \\
 \mathbf{Z} & \iff & \mathbf{Q} \cap (0, 1] \\
 a + p^m \mathbf{Z}_p & \iff & \text{primitive (Farey) segments} \\
 \sum_{m=1}^{\infty} \frac{a_m}{m^\rho} & \iff & \sum_{0 < p/q \leq 1} \frac{a(p/q)}{q^\rho} \\
 \text{Mazur–Mellin transform} & \iff & \text{Lévy–Mellin transform}
 \end{array}$$

§1. Pseudo–measures

associated with period–like functions

1.1. An heuristic construction. For the moment, we adopt the viewpoint of [LZ2], Ch. II, sec. 5. Fix a complex number s such that $s(1-s)$ is an eigenvalue of the standard hyperbolic Laplace operator on \mathbf{C} producing a $PSL(2, \mathbf{Z})$ -invariant Maass wave form $u(z) = u_s(z)$, $z \in H$. Define complex powers by the usual formula $t^s := e^{s \log t}$ where the branch of the logarithm is determined by the normalization $-\pi < \arg t \leq \pi$. As is shown in [LZ1], there exists a distribution $U(t) = U_s(t)$ on \mathbf{R} , whose values on the following test functions of t (z enters as a parameter)

$$(\operatorname{Im} z)^s |z - t|^{-2s}, \quad (z - t)^{-2s}, \quad \chi_{(-\infty, 0)}(t)(z - t)^{-2s}$$

are respectively $u(z)$ (the initial Maass form), a function $f(z)$ holomorphic in $\mathbf{C} \setminus \mathbf{R}$, and a *period function* $\psi(z)$ defined and holomorphic in $\mathbf{C}' := \mathbf{C} \setminus (-\infty, 0]$. Here χ is the characteristic function of \mathbf{R}_- , in other words

$$\psi(z) = \int_{-\infty}^0 (z - t)^{-2s} U(t) dt. \tag{1.1}$$

The distribution U is automorphic in the following sense: for all

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$$

we have

$$U\left(\frac{at+b}{ct+d}\right) = |ct+d|^{2-2s}U(t). \quad (1.2)$$

Thus, (1.1) has the same structure as (0.1).

Consider now only $g \in SL(2, \mathbf{Z})$ with non-negative entries. Then for any $z \in \mathbf{C}'$ we have also $gz := \frac{az+b}{cz+d} \in \mathbf{C}'$. From (1.2) we find formally

$$\psi(gz) = \int_{-\infty}^0 (gz-t)^{-2s}U(t) dt = \int_{g^{-1}(-\infty)}^{g^{-1}(0)} (gz-g\tau)^{-2s}U(g\tau) d(g\tau). \quad (1.3)$$

The direct calculation using (1.2) reduces the integrand to the form

$$\left[\frac{z-\tau}{(cz+d)(c\tau+d)}\right]^{-2s} |c\tau+d|^{-2s+2}U(\tau) \frac{d\tau}{|c\tau+d|^2}. \quad (1.4)$$

Since $a \neq 0$, we have

$$g^{-1}(-\infty) = -\frac{d}{c} < -\frac{b}{a} = g^{-1}(0),$$

and hence for $\tau \in (g^{-1}(-\infty), g^{-1}(0))$ we have $c\tau+d > 0$. This shows that all terms involving $c\tau+d$ in (1.4) cancel, so that finally we find formally

$$\psi(gz) = (cz+d)^{2s} \int_{-d/c}^{-b/a} (z-\tau)^{-2s}U(\tau) d\tau. \quad (1.5)$$

Thus if $(\alpha, \beta) = (g^{-1}(-\infty), g^{-1}(0))$ with g as above, and if we put

$$\mu(\alpha, \beta)(z) := (cz+d)^{-2s}\psi(gz) = \int_{\alpha}^{\beta} (z-t)^{-2s}U(t) dt, \quad (1.6)$$

then for three intervals of this type (α, β) , (β, γ) , (α, γ) we would have from (1.6)

$$\mu(\alpha, \beta)(z) + \mu(\beta, \gamma)(z) = \mu(\alpha, \gamma)(z). \quad (1.7)$$

As we will see, all primitive intervals in \mathbf{R}_- are of this form, so that we have formally constructed a pre-measure (see below) on the (left half of) $\mathbf{P}^1(\mathbf{R})$, extendable to a pseudo-measure on this half with values in the space of holomorphic functions on \mathbf{C}' , in view of [MaMar2], Theorem 1.8.

The weak point of this reasoning, about which the word “formally” is supposed to warn the reader, is this: the functions $\chi_{(\alpha,\beta)}(t)(z-t)^{-2s}$ generally do not belong to the space of test functions as it is defined in [LZ1], p. 225. Therefore the integrals in the r.h.s. of (1.5), (1.6) *a priori* make no sense.

Our heuristic reasoning is in fact a simple extension of the formal argument on p. 222 of [LZ2], “proving” the three-term functional equation for $\psi(z)$.

In the next subsections, we will provide a precise construction of the pseudo-measures, whose values on the intervals considered above are given by

$$\mu(g^{-1}(-\infty), g^{-1}(0))(z) := (cz + d)^{-2s} \psi(gz) \quad (1.8)$$

without appealing to the integral representation (1.6), but making use of the theory developed in [LZ1].

1.2. Preliminaries I: left primitive segments. As in [MaMar2], we consider $\mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$ as points of an affine line with a fixed coordinate z . Completing this line by one point $\infty = -\infty = i\infty$, we get points of the projective line $\mathbf{P}^1(\mathbf{Q}) \subset \mathbf{P}^1(\mathbf{R}) \subset \mathbf{P}^1(\mathbf{C})$ (Riemannian sphere). $GL(2, \mathbf{C})$ acts on $\mathbf{P}^1(\mathbf{C})$ by fractional linear transformations. *Segments* are defined as non-empty connected subsets of $\mathbf{P}^1(\mathbf{R})$. A segment is called infinite if ∞ is in its closure, otherwise it is called finite. The boundary of each segment generally consists of an unordered pair of points (α, β) in $\mathbf{P}^1(\mathbf{R})$. We will identify a segment with an *ordered* pair of its ends: the additional element of structure is its orientation from α to β . For our purposes, it is usually inessential whether one or two boundary points belong to the segment. In this section we will consider mostly *left* segments, that is, ones for which $-\infty \leq \alpha, \beta \leq 0$. One-point segments are sometimes called improper ones.

A segment is called *rational* if its ends are in $\mathbf{P}^1(\mathbf{Q})$, and *primitive*, or Farey, if it is of the form $(g(\infty), g(0))$ for some $g \in GL(2, \mathbf{Z})$.

In [MaMar2] we called a *pseudo-measure with values in an abelian group* W a finitely additive W -valued function on the Boolean algebra of rational segments, vanishing on improper segments. We extended it to oriented segments by the condition that $\mu(\alpha, \beta) = -\mu(\beta, \alpha)$.

In this section, we will construct pseudo-measures supported by left segments. As it follows from [MaMar2], each such pseudo-measure is defined by its restriction to the set P of positively oriented left primitive segments. We will use the following enumeration of the latter.

Denote by $S \subset SL(2, \mathbf{Z})$ the sub-semigroup of matrices with non-negative entries a, b, c, d . For any $g \in S$, $(g^{-1}(-\infty), g^{-1}(0))$ is in P . In fact, if $c \neq 0$,

$$g^{-1}(\infty) = \frac{d}{-c} < g^{-1}(0) = \frac{-b}{a},$$

because $ad - bc = 1$. If $c = 0$, then $a = d = 1$, and again

$$g^{-1}(\infty) = -\infty < g^{-1}(0) = -b.$$

Finally, the case $a = 0$ does not occur in S .

One easily sees that this map $S \rightarrow P: g \mapsto (g^{-1}(-\infty), g^{-1}(0))$ is in fact a bijection.

1.3. Preliminaries II: the slash operators of complex weight. Here we summarize the considerations of [LZ2], p. 240, and [HiMaMo], sec 3. They determine a partial map

$$(\varphi, g) \mapsto \varphi|_s g \tag{1.9}$$

allowing us to make sense of and correctly calculate expressions as those appearing in (1.4), (1.6).

For proofs, see [HiMaMo].

(i) *Definition domain.* The argument $\varphi = \varphi(z)$ in (1.9) can be an arbitrary function holomorphic in some domain of the form $\mathbf{C} \setminus (-\infty, r]$, $r \in \mathbf{R}$. Such functions form a \mathbf{C} -algebra which we will denote \mathcal{F} . Period functions $\psi = \psi_s$ belong to \mathcal{F} .

In [HiHaMo], any point r such that $\varphi \in \mathcal{F}$ is holomorphic in $\mathbf{C} \setminus (-\infty, r]$, is called a *branching point* of φ .

The argument g in (1.9) can be an arbitrary $(2, 2)$ -matrix g with integer entries (a, b, c, d) and non-zero determinant such that either $c > 0$, or $c = 0; a, d > 0$. Denote by \mathcal{G} the set of such matrices. The set S describing left primitive segments in 1.2 is a subset of \mathcal{G} . When $g \in \mathcal{G}$ and $s \in \mathbf{C}$, the function $(cz + d)^s$ belongs to \mathcal{F} .

A pair $(\varphi, g) \in \mathcal{F} \times \mathcal{G}$ belongs to the definition domain \mathcal{DS} of the slash operator (1.9) if φ admits a branching point r such that either $a - cr > 0$, or $a - cr = 0$ and $dr - b < 0$. For a period function $\phi = \psi$, we can take $r = 0$, and g will do if $a > 0$ or else $a = 0, b > 0$.

Let \mathcal{G}^+ be the set of matrices in \mathcal{G} such that $b, d \geq 0$ and either $a > 0$, or $a = 0, b > 0$. Again, $S \subset \mathcal{G}^+$. Denote by \mathcal{F}_0 the subspace of \mathcal{F} admitting 0 as a branch point. Then $\mathcal{F}_0 \times \mathcal{G}^+ \subset \mathcal{DS}$.

(ii) *Slash operator of weight s .* It is the map $\mathcal{DS} \rightarrow \mathcal{F}$ defined by

$$(\varphi(z), g) \mapsto (\varphi|_s g)(z) := |\det g|^s (cz + d)^{-2s} \varphi(gz). \tag{1.10}$$

It is well defined. Moreover, it sends $\mathcal{F}_0 \times \mathcal{G}^+$ to \mathcal{F}_0 .

(iii) *Properties of the slash operator.* The basic property is that slash operator is a honest action: if $g_1, g_2 \in \mathcal{G}$ and $(\varphi, g_1), (\varphi|_s g_1, g_2), (\varphi, g_1 g_2) \in \mathcal{DS}$, then

$$\varphi|_s(g_1 g_2) = (\varphi|_s g_1)|_s g_2.$$

(Formally, it is the associativity of the triple product of φ, g_1, g_2 .) Applying this to $\mathcal{F}_0 \times \mathcal{G}^+$, one can check that $|_s$ defines a right action of the multiplicative semigroup \mathcal{G}^+ on \mathcal{F}_0 ([HiMaMo], Remark 3.4).

From 1.2 one sees that if $(g^{-1}(-\infty), g^{-1}(0))$ is a left primitive segment, then $g \in \mathcal{G}^+$. Since $\psi \in \mathcal{F}_0$ in the first equality of (1.6), this expression for $\mu(\alpha, \beta)(z)$ (disregarding the second equality and the poorly defined integral) makes sense, and the slash action can be further iterated.

1.4. The pre-measures related to period-like functions. Choose a complex number s and a function $\psi(z) \in \mathcal{F}_0$ satisfying the three term functional equation

$$\psi(z) = \psi(z+1) + (z+1)^{-2s} \psi\left(\frac{z}{z+1}\right) \quad (1.11)$$

Thus, ψ is a period-like function in the sense of [LZ], Ch. III.

For a left primitive segment $(\alpha, \beta) = (g^{-1}(-\infty), g^{-1}(0))$, put

$$\tilde{\mu}(\alpha, \beta)(z) = (cz+d)^{-2s} \psi(gz) = \psi|_s(z). \quad (1.12)$$

Consider now three left primitive segments $(\alpha, \beta) = (g_1^{-1}(-\infty), g_1^{-1}(0))$, $(\beta, \gamma) = (g_2^{-1}(-\infty), g_2^{-1}(0))$, $(\alpha, \gamma) = (g_3^{-1}(-\infty), g_3^{-1}(0))$. In plain words, the third segment is broken into two others by a point β in the middle.

1.4.1. Lemma. *We have*

$$\tilde{\mu}(\alpha, \beta)(z) + \tilde{\mu}(\beta, \gamma)(z) = \tilde{\mu}(\alpha, \gamma)(z). \quad (1.13)$$

Proof. *Case 1:* $(\alpha, \beta, \gamma) = (-\infty, -1, 0)$. In this case

$$g_1 = T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 = T' := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_3 = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the equation (1.13) coincides with (1.11) which can be written as

$$\psi|_s I = \psi|_s T + \psi|_s T'. \quad (1.14)$$

Case 2: $g_1 = Tg, g_2 = T'g, g_3 = g$, where $g \in SL(2, \mathbf{Z})$ is a matrix with non-negative entries.

In this case, (1.13) reads

$$\psi|_s g = \psi|_s Tg + \psi|_s T'g$$

which obviously holds in view of (1.14) and the associativity of the slash operator restricted to $\mathcal{F}_0 \times \mathcal{G}^+$.

General case. In fact, the previous case *is* general: we necessarily have $g_1 = Tg_3$ and $g_2 = T'g_3$.

Let us check this for the case when $\alpha \neq -\infty$ leaving the remaining case to the reader. Put $\alpha = \frac{d}{-c}, \gamma = \frac{-b}{a}$ as in 1.2 where (a, b, c, d) are the entries of g_3 . Then the only possible value of β is $\beta = \frac{-(b+d)}{a+c} = \frac{b+d}{-(a+c)}$ as is well known from the classical theory of Farey series. This fact directly translates into $g_1 = Tg_3, g_2 = T'g_3$.

This completes the proof of the lemma.

Remark. Notice that if $\psi(z)$ is an actual period function for a Maass wave form, the lemma becomes obvious in view of the integral representation of $\psi(z)$ proven in [LZ], Ch. II, sec. 1. The relevant formula on the p. 212 of [LZ] reads (we have replaced the notation $\psi_1(\zeta)$ by $\psi(\zeta)$ and changed the sign):

$$(c\zeta + d)^{-2s} \psi(g\zeta) = \int_{g^{-1}(\infty)}^{g^{-1}(0)} \{u, R_\zeta^s\}(z). \quad (1.15)$$

In this formula, we integrate a closed form along an arbitrary path leaving ζ and $\bar{\zeta}$ to the right of it. Additivity (1.13) becomes evident.

We will use this integral representation in the next section.

1.4.2. The pre-measure on left segments. To define a pre-measure in the sense of [MaMar], *supported by the subset of left primitive segments*, it remains to complete the definition (1.12) of the function $\tilde{\mu}$ by putting for $\alpha < \beta \leq 0$

$$\tilde{\mu}(\beta, \alpha) := -\tilde{\mu}(\alpha, \beta), \quad \tilde{\mu}(\alpha, \alpha) = 0.$$

One easily checks that (1.13) continues to hold on this extended domain.

1.5. The pseudo-measure related to a period-like function. Now we can state the main result of this section.

1.5.1. Theorem. *There exists a unique finitely additive function μ with values in \mathcal{F}_0 coinciding with $\tilde{\mu}$ on left primitive segments and vanishing on all rational segments in $(0, \infty)$.*

Sketch of proof. We simply recall the plan of proof of the Theorem 1.8 of [MaMar2]. It consists of the following steps.

1) Using the “continued fractions trick”, we show that for any non-positive rational (or infinite) α, β one can find a sequence of rational non-positive numbers $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_n = \beta$ such that (α_i, α_{i+1}) is a left primitive segment for all $i = 0, \dots, n - 1$. Such sequence is called a primitive chain connecting α to β .

2) Having chosen such a primitive chain, we put

$$\mu(\alpha, \beta) := \sum_{i=0}^{n-1} \tilde{\mu}(\alpha_i, \alpha_{i+1}). \quad (1.16)$$

3) The fact that (1.16) does not depend on the choice of the connecting primitive chain is checked by proving that any two chains can be transformed one to another by using “elementary moves” compatible with relations that hold for $\tilde{\mu}$. An elementary move essentially replaces a Farey interval $\left(\frac{a}{c}, \frac{b}{d}\right)$ by the chain $\left(\frac{a}{c}, \frac{a+b}{c+d}\right)$, $\left(\frac{a+b}{c+d}, \frac{b}{d}\right)$, or vice versa.

4) Finally, we have to check that (1.16) implies finite additivity and the sign change after the change of orientation. This is straightforward.

1.6. Modularity. Let Γ be a subgroup of $SL(2, \mathbf{Z})$, W a left Γ -module.

In [MaMar2], a pseudo-measure μ with values in W is called Γ -modular, if for all $g \in \Gamma$ and $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ we have

$$\mu(g\alpha, g\beta) = g\mu(\alpha, \beta).$$

It was checked that such pseudo-measures correspond to parabolic 1-cocycles.

In our context, this is replaced by the following property: for all g with $g^{-1} \in S$ and any left segment (α, β) ,

$$\mu(g^{-1}(\alpha), g^{-1}(\beta)) = \mu(\alpha, \beta)|_s g. \quad (1.17)$$

In fact, it suffices to check this for left primitive segments $(\alpha, \beta) = (h^{-1}(-\infty), h^{-1}(0))$, in which case we have

$$\mu(g^{-1}(\alpha), g^{-1}(\beta)) = \mu((hg)^{-1}(-\infty), (hg)^{-1}(0)) =$$

$$\psi|_s(hg) = (\psi|_s h)|_s g = \mu(\alpha, \beta)|_s g.$$

Since the right slash action of g can be considered as the left action of g^{-1} , we can say that (1.17) expresses the modularity of μ with respect to the multiplicative semigroup $S^{-1} \subset SL(2, \mathbf{Z})$.

§2. Maass L -functions and their Mellin–Lévy transforms

2.1. Maass L -series as sums over rational numbers. Let $u = u_s$ be a Maass cusp form, which is an eigenfunction with respect to all Hecke operators

$$T_m := \sum_{\substack{ad=m \\ 0 < b \leq d}} \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} \quad (2.1)$$

acting via slash operator of weight 0: $u \mapsto u|_0 T_m = \lambda_m u$.

Put

$$L_u(\rho) := \sum_{m=1}^{\infty} \frac{\lambda_m}{m^\rho}. \quad (2.2)$$

The action of the Hecke operators on u induces an action on the period functions, which can be explicitly described by a nice formula, for example, as in [Müh]. However, we will need a different expression, involving the pseudo-measure μ_u , and we will start with an heuristic derivation of it, as in 1.1.

Let us formally apply the slash operator $|_{-s}$ (see (1.10)) to the boundary measure $U(t)dt$ and denote the resulting action upon the respective period function ψ by T_m^* . In this heuristic calculation we “define” ψ by (1.1). The choice of weight $-s$ is motivated by the invariance property (1.2). We get:

$$\begin{aligned} (\psi|T_m^*)(\zeta) &:= \int_{-\infty}^0 (\zeta - t)^{-2s} (U(t) dt |_{-s} T_m) = \\ &\sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{d}{a}\right)^s \int_{-\infty}^0 (\zeta - t)^{-2s} U\left(\frac{at-b}{d}\right) d\left(\frac{at-b}{d}\right). \end{aligned}$$

Make the change of variable $\tau = \frac{at-b}{d}$. The last integral takes form

$$\sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{d}{a}\right)^s \int_{-\infty}^{-b/d} \left(\zeta - \frac{d\tau + b}{a}\right)^{-2s} U(\tau) d\tau =$$

$$\sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{d}{a}\right)^s \int_{-\infty}^{-b/d} \left(\frac{dz+b}{a} - \frac{d\tau+b}{a}\right)^{-2s} U(\tau) d\tau,$$

where $z = \frac{a\zeta - b}{d}$. The integral in the last sum can be rewritten as

$$\left(\frac{a}{d}\right)^{2s} \int_{-\infty}^{-b/d} (\zeta - \tau)^{-2s} U(\tau) d\tau.$$

Thus heuristically

$$\begin{aligned} (\psi|T_m^*)(\zeta) &= (\mu(-\infty, 0)|T_m^*)(\zeta) = \sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{a}{d}\right)^s \mu\left(-\infty, -\frac{b}{d}\right) \left(\frac{a\zeta - b}{d}\right) = \\ &= \sum_{\substack{ad=m \\ 0 < b \leq d}} \mu\left(-\infty, -\frac{b}{d}\right) \Big|_s \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} (\zeta). \end{aligned} \quad (2.3)$$

This expression is useful for our purposes because it allows us to represent (the somewhat normalized) Dirichlet series $L_u(s)$ as a natural sum over rational numbers. We will state now the respective theorem:

2.2. Theorem. *We have*

$$\psi(z) \sum_{m=1}^{\infty} \frac{\lambda_m}{m^\rho} = \zeta(\rho - s)\zeta(\rho + s) \sum_{q=1}^{\infty} \frac{1}{q^\rho} \sum_{\substack{0 \leq p < q \\ (p,q)=1}} \mu(-\infty, -p/q) \Big|_s \begin{pmatrix} 1 & -p \\ 0 & q \end{pmatrix} (z). \quad (2.4)$$

Proof. *Step 1.* First, we have to supply an honest proof of (2.3). In [LZ2], Ch. II, sec. 2, the authors construct a differential 1-form $\{u, R_\zeta^s\}(z)$ which we invoked at the end of 1.4.1. It has the following properties:

(i) $\{u, R_\zeta^s\}(z)$ is a closed smooth form of z varying in the complex upper half-plane H . It depends on the parameter $\zeta \in \mathbf{C}$ holomorphically when $z \neq \zeta, \bar{\zeta}$. Generally, it is multivalued, but a well defined branch can be chosen on the complement in H of a path joining ζ to $\bar{\zeta}$.

(ii) The period function $\psi(\zeta), \zeta \in H$ for u (up to a constant proportionality factor) can be then written as an integral

$$\psi(\zeta) = \int_{-\infty}^0 \{u, R_\zeta^s\}(z) \quad (2.5)$$

taken along any path in H leaving ζ to the left of it.

Let us now assume that $u|_0T_m = \lambda_m u$ for T_m from (2.1) and a constant λ_m . Then we have from (2.5) and (2.1)

$$\lambda_m \psi(\zeta) = \int_{-\infty}^0 \left\{ \sum_{\substack{ad=m \\ 0 < b \leq d}} u \left(\frac{az-b}{d} \right), R_\zeta^s \right\} (z). \quad (2.6)$$

For each a, b, d fixed, we first want to make the implicit argument z of R_ζ^s the same as that of u , that is, $\frac{az-b}{d}$. We have (see [LZ2], p. 211):

$$R_\zeta(z) = \frac{i}{2} ((z-\zeta)^{-1} - (\bar{z}-\zeta)^{-1}) = \frac{a}{d} \cdot \frac{i}{2} \left(\left(\frac{az-b}{d} - \frac{a\zeta-b}{d} \right)^{-1} - \left(\frac{a\bar{z}-b}{d} - \frac{a\zeta-b}{d} \right)^{-1} \right) = \frac{a}{d} R_\xi \left(\frac{az-b}{d} \right),$$

where $\xi := \frac{a\zeta-b}{d}$.

Substituting this into (2.6), we obtain:

$$\lambda_m \psi(\zeta) = \sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{a}{d} \right)^s \int_{-\infty}^0 \left\{ u \left(\frac{az-b}{d} \right), R_{\frac{a\zeta-b}{d}}^s \left(\frac{az-b}{d} \right) \right\}. \quad (2.7)$$

Considering now $z \mapsto \frac{az-b}{d}$ as a holomorphic change of variables, we infer from the Lemma on p. 210 of [LZ2] that the integrand in the respective term of (2.7) can be rewritten as

$$\left\{ u, R_{\frac{a\zeta-b}{d}}^s \right\} \left(\frac{az-b}{d} \right).$$

Hence finally

$$\begin{aligned} \lambda_m \psi(\zeta) &= \sum_{\substack{ad=m \\ 0 < b < d}} \left(\frac{a}{d} \right)^s \int_{-\infty}^{-b/d} \left\{ u, R_{\frac{a\zeta-b}{d}}^s \right\} (z) = \\ &= \sum_{\substack{ad=m \\ 0 < b \leq d}} \mu(-\infty, -b/d)|_s \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} (\zeta). \end{aligned} \quad (2.8)$$

This is formula (2.3), written for u which is an eigenfunction of T_m , and its respective period function.

Step 2. Multiply now the identity (2.8) by $m^{-\rho}$ and sum over all $m = 1, 2, \dots$. Again replacing the notation of the free variable ζ by z (in order not to confound it with Riemann's zeta in (2.11) below), we obtain

$$\psi(z) \sum_{m=1}^{\infty} \frac{\lambda_m}{m^\rho} = \sum_{m=1}^{\infty} \frac{1}{m^\rho} \sum_{\substack{ad=m \\ 0 < b \leq d}} \mu(-\infty, -b/d)|_s \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} (z). \quad (2.9)$$

Each matrix in (2.9) can be uniquely written in the following way:

$$\begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} = \begin{pmatrix} d_2 & -pd_1 \\ 0 & qd_1 \end{pmatrix} = \begin{pmatrix} 1 & -p \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} d_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.10)$$

where $m = d_1 d_2 q$, $d_i \geq 1$, $0 < p \leq q$, $(p, q) = 1$. Moreover, arbitrary quadruple (d_1, d_2, p, q) satisfying these conditions produces one term in (2.9).

From (2.10) and the associativity of the slash operator (1.10) it follows that

$$|_s \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} = |_s \begin{pmatrix} 1 & -p \\ 0 & q \end{pmatrix} \cdot d_1^{-s} d_2^s.$$

Hence we can rewrite (2.9) as follows:

$$\begin{aligned} \psi(z) \sum_{m=1}^{\infty} \frac{\lambda_m}{m^\rho} &= \sum_{q, d_1, d_2=1}^{\infty} \frac{1}{q^\rho d_1^{\rho-s} d_2^{\rho+s}} \sum_{\substack{0 < p \leq q \\ (p, q)=1}} \mu(-\infty, -p/q)|_s \begin{pmatrix} 1 & -p \\ 0 & q \end{pmatrix} (z) = \\ &\zeta(\rho - s) \zeta(\rho + s) \sum_{q=1}^{\infty} \frac{1}{q^\rho} \sum_{\substack{0 < p \leq q \\ (p, q)=1}} \mu(-\infty, -p/q)|_s \begin{pmatrix} 1 & -p \\ 0 & q \end{pmatrix} (z). \end{aligned} \quad (2.11)$$

2.3. Lévy–Mellin transform. Put now

$$r_u(p, q) := (p + q)q^{1-\rho} \mu(-\infty, -p/q)|_s \begin{pmatrix} 1 & -p \\ 0 & q \end{pmatrix} (z) \cdot \psi(z)^{-1}.$$

and

$$l_u(\xi) := \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} r(q_i(\xi), q_{i+1}(\xi)).$$

From (2.11) and (0.18) we get the following

2.3.1. Corollary. *Let u be a Maass cusp form, $\Delta u = s(1-s)u$, $u|T_m = \lambda_m u$ for all $m \geq 1$. Put*

$$\Lambda_u(\rho) := \zeta(\rho-s)^{-1} \zeta(\rho+s)^{-1} \sum_{m=1}^{\infty} \frac{\lambda_m}{m^\rho}.$$

Then

$$\Lambda_u(\rho) = \int_0^{1/2} l_u(\xi) d\xi.$$

2.3.2. Remark. The class of series of the form (0.18) involving modular symbols includes also D. Goldfield's Eisenstein series, cf. [Go2]. They certainly deserve further study.

2.4. Hecke operators on period functions via continued fractions. Consider the sequence of normalized convergents b/d as in [MaMar2], (1.5). When $0 < b/d < 1$, it starts with

$$-\infty = \frac{1}{0} =: \frac{b_{-1}}{d_{-1}}, \quad 0 = \frac{0}{1} =: \frac{b_0}{d_0}, \quad \dots, \quad b/d = \frac{b_n}{d_n},$$

where $n = n(b/d)$ is the length of the continued fraction expansion.

The following sequence of left primitive segments $I_k = I_k(b/d)$ connects $-\infty$ to $-b/d$. We order their ends from the left one to the right one, and put minus before those that should be run in the opposite direction in our chain:

$$I_0 = (-\infty, 0) = \left(-\frac{b_{-1}}{d_{-1}}, -\frac{b_0}{d_0} \right), \quad I_1 = - \left(-\frac{b_1}{d_1}, -\frac{b_0}{d_0} \right),$$

$$I_2 = \left(-\frac{b_1}{d_1}, -\frac{b_2}{d_2} \right), \quad I_3 = - \left(-\frac{b_3}{d_3}, -\frac{b_2}{d_2} \right),$$

and generally

$$I_k = (-1)^k \left(-\frac{b_{k-\varepsilon_k}}{d_{k-\varepsilon_k}}, -\frac{b_{k-\varepsilon_{k+1}}}{d_{k-\varepsilon_{k+1}}} \right)$$

where $\varepsilon_k = 1$ for even k and 0 for odd k .

This means that

$$(-1)^k I_k = (g_k^{-1}(-\infty), g_k^{-1}(0)) \tag{2.12}$$

where

$$g_k = g_{k,b/d} = \begin{pmatrix} d_{k-\varepsilon_{k+1}} & b_{k-\varepsilon_{k+1}} \\ d_{k-\varepsilon_k} & b_{k-\varepsilon_k} \end{pmatrix} \in S. \quad (2.13)$$

Therefore, (2.8) can be rewritten as

$$\lambda_m \psi(\zeta) = \sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{a}{d}\right)^s \sum_{k=0}^{n(b/d)} (-1)^k \int_{-\infty}^0 \{u(g_{k,b/d}(z)), R_{\frac{a\zeta-b}{d}}^s(g_{k,b/d}(z))\}. \quad (2.14)$$

We have $u(g_{k,b/d}(z)) = u(z)$ and

$$R_{\frac{a\zeta-b}{d}}^s(g_{k,b/d}(z)) = (d_{k-\varepsilon_k} g_{k,b/d}^{-1} \left(\frac{a\zeta-b}{d}\right) + b_{k-\varepsilon_k})^{2s} R_{g_{k,b/d}^{-1}(\frac{a\zeta-b}{d})}^s(z) \quad (2.15)$$

This follows from the formula (2.6) on p. 211 of [LZ] and (2.13). To shorten notation, denote

$$j_k(b/d, \zeta)^{2s} := (d_{k-\varepsilon_k} g_{k,b/d}^{-1} \left(\frac{a\zeta-b}{d}\right) + b_{k-\varepsilon_k})^{2s}. \quad (2.16)$$

Then we get

$$\begin{aligned} \lambda_m \psi(\zeta) &= \sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{a}{d}\right)^s \sum_{k=0}^{n(b/d)} (-1)^k j_k(b/d, \zeta)^{2s} \int_{-\infty}^0 \{u(z), R_{g_{k,b/d}^{-1}(\frac{a\zeta-b}{d})}^s(z)\} = \\ &= \sum_{\substack{ad=m \\ 0 < b \leq d}} \left(\frac{a}{d}\right)^s \sum_{k=0}^{n(b/d)} (-1)^k j_k(b/d, \zeta)^{2s} \psi \left(g_{k,b/d}^{-1} \left(\frac{a\zeta-b}{d}\right) \right). \end{aligned} \quad (2.17)$$

In order to deduce from (2.17) a nice ‘‘explicit’’ formula for λ_m , as it was done in [Ma2] for the coefficients of the classical cusp forms, one could use an appropriate linear functional on functions of ζ . In the classical case, it was the highest coefficient (or the constant term) of the period polynomial.

In the Maass case, one could try to use asymptotic behaviours at 0 or ∞ . Other forms of Hecke operators, as (2.18) below, might be useful.

3.1. Hecke operators and transfer operator. T. Mühlenbruch, using the Choie–Zagier method ([ChZ]), shows in [Müh] that the Hecke operators acting on period functions for the full modular group can be written in the nice form

$$T_m^+ = \sum_{\substack{a > c \geq 0 \\ d > b \geq 0 \\ ad - bc = m}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.18)$$

Of course, they act on $\psi(z)$ via $|_s$ in our notation (Mhlenbruch denotes this slash operator $|_{2s}$.)

In particular, for $m = 1$ we have $T_m^+ = I$.

However, if we slightly change the summation domain replacing $a > c \geq 0$ by $a \geq c > 0$, then then the equations for case $m = 1$ will admit the following solutions. From $ad = 1 + bc \leq 1 + (d - 1)a$ it follows that $a = c = 1$ and $d = b + 1 \geq 1$ so that we will get the operator

$$T_1^* := \sum_{b=0}^{\infty} \begin{pmatrix} 1 & b \\ 1 & b+1 \end{pmatrix} \quad (2.19)$$

This completely *ad hoc* correction in fact makes sense, and moreover, T_1^* imitates the Hecke operator corresponding to the “improper prime $p = 1$ ”, with eigenvalue 1 on ψ :

2.5.1. Claim. *If $\psi(z)$ is a period function for a Maass cusp form of weight s with $\operatorname{Re} s > 0$, $s \neq \frac{1}{2}$, then*

$$\psi|_s T_1^*(z) = \psi(z). \quad (2.20)$$

Proof. Assume moreover that

$$\psi^\tau(z) := \psi|_s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (z) = \varepsilon \psi(z), \quad \varepsilon = \pm 1, \quad (2.21)$$

so that ψ is even or odd. This is not a restriction because any ψ is the sum of an even and odd period functions.

According to [LZ2], p. 255, the function

$$h(z) := \psi(z + 1) = \psi|_s \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (z) \quad (2.22)$$

satisfies the equation

$$\varepsilon h|_s \left(\sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \right) (z) = h(z). \quad (2.23)$$

Substituting first (2.22) into (2.23), and then (2.21) into resulting identity, we get:

$$\psi|_s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |_s \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} |_s \left(\sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \right) (z) = \psi|_s \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (z) \quad (2.24)$$

The associativity of the slash operator and the identity

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n-1 \\ 0 & n \end{pmatrix}$$

establish (2.20).

2.6. The Brjuno function and derivatives of the classical L -functions.

The Brjuno function which we referred to in sec. 0.6 is defined as a generalized Lévy sum

$$B(\xi) := \sum_{j=0}^{\infty} |p_j(\xi) - q_j(\xi)\xi| \log \frac{p_{j-1}(\xi) - q_{j-1}(\xi)\xi}{q_j(\xi)\xi - p_j(\xi)}. \quad (2.25)$$

This series diverges on a set of measure 0. Outside it converges to a measurable function, continuous at irrational points, with period 1. (cf. [MarMouYo2]).

The values of derivatives of Mellin transforms of classical forms were studied by D. Goldfield ([Go1]) and N. Diamantis ([Di]). Goldfield's idea consisted in replacing the $\log y$ initially appearing at the Mellin expression for the first derivative by the logarithm of the η -function, or a combination of such, to enhance the modular properties of the integrand. The same game can be played with the Brjuno function in place of η -function.

Consider a classical cusp form $u(z)$ for $SL(2, Z)$ of integral weight $2k = w + 2$ as in 0.1. Let $L_u(s)$ be its Mellin transform.

2.6.1. Proposition. *We have*

$$L'_u(w/2 + 2) = C \left[- \int_0^1 u(iy)y^{w/2} B(y) dy + \int_1^{\infty} u(iy)y^{w/2-1} B(y) dy \right], \quad (2.26)$$

where

$$C = \frac{(2\pi)^{(w+4)/2}}{\Gamma((w+2)/2)} (1 + i^{w+2}).$$

Proof. First of all, $B(\xi)$ satisfies the functional equation

$$B(\xi) = -\log \xi + \xi B(\xi^{-1}), \quad \xi \in (0, 1) \quad (2.27)$$

This is shown by an easy calculation.

Therefore, we have

$$\int_0^{\infty} u(iy)y^{w/2} \log y dy =$$

$$\int_0^1 u(iy)y^{w/2} [-B(y) + yB(y^{-1})] dy + \int_1^\infty u(iv)v^{w/2} [v^{-1}B(v) - B(v^{-1})] dv.$$

In the second summand of the second integrand, make the change of variable $v = y^{-1}$, and combine it with the first summand of the first integrand. Similarly, in the second summand of the first integrand, make the change of variable $y = v^{-1}$, and combine it with the first summand of the second integrand. This will result in

$$(1 + i^{w+2}) \left[- \int_0^1 u(iy)y^{w/2} B(y) dy + \int_1^\infty u(iy)y^{w/2-1} B(y) dy \right].$$

The remaining factor in C comes from the Mellin transform.

Acknowledgement. I am grateful to Don Zagier who read a preliminary draft of this paper and suggested a number of corrections and complements.

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