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Low-Dimensional Topology and Number Theory

Organised by
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Abstract. The workshop brought together topologists and number theorists with the intent of exploring the many tantalizing connections between these areas.

Mathematics Subject Classification (2000): 11xx,57xx.

Introduction by the Organisers

The workshop *Low-Dimensional Topology and Number Theory*, organised by Paul E. Gunnels (Amherst), Walter Neumann (New York), Don Zagier (Bonn) and Adam S. Sikora (New York) was held August 26th – September 1st, 2012. This meeting was a part of a long-standing tradition of collaboration of researchers in these areas. The preceeding meeting under the same name took place in Oberwolfach two years ago. At the moment the topic of most active interaction between topologists and number theorists are quantum invariants of 3-manifolds and their asymptotics. This year's meeting showed significant progress in the field.

The workshop was attended by many researchers from around the world, at different stages of their careers – from graduate students to some of the most established scientific leaders in their areas. The participants represented diverse backgrounds. There were 22 talks ranging from 30 to 50 minutes intertwined with informal discussions.

Workshop: Low-Dimensional Topology and Number Theory

Table of Contents

Abstracts

Quantum hyperbolic invariants of cusped manifolds and their asymptotical behaviour

Stéphane Baseilhac
(joint work with Riccardo Benedetti, Charles Frohman)

Let M be a cusped hyperbolic 3-manifold; it is diffeomorphic to the interior of a compact 3-manifold V with torus boundary. Denote by $X(V)$ the variety of *augmented* $PSL(2,\mathbb{C})$-characters of V, and by $res : X(V) \to X(\partial V)$ the restriction map. In this talk we have presented the relations between:

- The $PSL(2,\mathbb{C})$-Chern-Simons theory of M, embodied in the Chern-Simons line bundle $\mathcal{L} \to X(\partial V)$ and the Chern-Simons section s_V of the pull-back bundle : $res^*\mathcal{L} \to X(V)$;
- The quantum hyperbolic invariants $\mathcal{H}_N(M)$, defined in [1] for each odd integer $N \geq 3$ as scalars associated to M equipped with its hyperbolic holonomy, and extended in [2] as regular functions on a tower of covering spaces of degree N^2 of the geometric component of $X(M)$.

Roughly, the functions $\mathcal{H}_N(M)$ are defined on a sequence of finite approximations of a subdomain of s_V. This leads us to formulate questions regarding the exponential growth rate of the sequence $(\mathcal{H}_N(M))_N$, like its finiteness, continuity, and relation with the volume and Chern-Simons invariants of $PSL(2,\mathbb{C})$-characters of M ("volume conjecture" type problem).

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Asymptotics of invariants of 3-manifolds and topological recursion

Gaëtan Borot
(joint work with Bertrand Eynard)

We push forward the general idea that a non-perturbative version of the topological recursion, applied to the A-polynomial of a 3-manifold with 1 cusp, should be identified to asymptotic series of knot invariants. In this text, I explain the notions involved in this statement, and give a precise conjecture for the asymptotics of the colored Jones polynomial. The presentation is based on [1]. I thank all the participants for questions and discussions that helped improving this abstract.

1. Topological recursions

We call *spectral curve*, the data of a compact Riemann surface Σ_g of genus g, a symplectic basis of cycles $(\mathcal{A}_j, \mathcal{B}_j)_j$, and a couple (x, y) of analytic functions on Σ_g. These functions may have singularities, and we require for simplicity that $\mathrm{d}x$ has only simple zeroes, denoted $a_i \in \Sigma_g$. The *topological recursion* (TR) is an algorithm which computes, for any spectral curve, a sequence of numbers $(F^h)_{h\geq 0}$ and for any $k \geq 1$ a sequence $(\omega_k^h(p_1, \ldots, p_k))_{h\geq 0}$, where p_i are points on Σ_g and ω_k^h is a 1-form with respect to each p_i, which is symmetric in all variables. It is natural to repackage them in formal generating series: we define the *perturbative partition function*

$$Z_\hbar^{\mathrm{pert}} = \exp\Big(\sum_{h\geq 0} \hbar^{2h-2}\, F^h\Big),$$

and for any $n \geq 1$, the *perturbative wave functions*:

$$\psi_{\hbar,n}^{\mathrm{pert}}(p_1, q_1; \ldots; p_n, q_n) = \exp\Big(\sum_{k\geq 1}\sum_{h\geq 0} \frac{\hbar^{2h-2+k}}{k!} \int_\bullet \cdots \int_\bullet \omega_k^h\Big),$$

which depend on $2n$ points $p_i, q_i \in \Sigma_g$, and where $\int_\bullet$ stands for $\sum_{i=1}^n \int_{q_i}^{p_i}$. The F^h and ω_n^h have been introduced in [2] so that $Z_\hbar^{\mathrm{pert}}$ is a power series solution to Virasoro-type constraints satisfying some analyticity requirements, and ω_k^h encode the k^{th}-order derivatives of F^h with respect to deformation parameters of the spectral curve. The full definition (not given here) is recursive, and involves only algebraic geometry on the curve Σ_g: ω_n^h can be written as a sum over residues at a_i, of a certain 1-form build out of $\omega_{k'}^{h'}$ for which $2 - 2h' - k' > 2 - 2h - k$. The initial values for the recursion are $\omega_1^0 = y\mathrm{d}x$, $\omega_2^0 =$ fundamental bidifferential of the 2^{nd} kind normalized on the $\mathcal{A}$-cycles.

The *non-perturbative topological recursion* (n.p.TR) is another algorithm which, to any spectral curve and an extra data $\mu, \nu \in \mathbb{C}^g$, associates a *non-perturbative partition function* $Z_\hbar$, and for any $n \geq 1$ a *non-perturbative wave function* $\psi_{n,\hbar}$. These are formal generating series in powers of $\hbar$ (as before), whose coefficients themselves depend on $\hbar$ but are either constant, or do not have an expansion in powers of $\hbar$. They are defined as follows:

$$Z_\hbar = Z_\hbar^{\mathrm{pert}}\Big\{\sum_{r\geq 1}\sum_{\substack{h_j\geq 0,\, k_j\geq 1\\ 2h_j-2+k_j>0}} \frac{\hbar^{\sum_j 2h_j-2+k_j}}{r!} \bigotimes_{j=1}^r \frac{\overbrace{\oint_{\mathcal{B}}\cdots\oint_{\mathcal{B}}}^{k_j} \omega_{k_j}^{h_j}}{(2\mathrm{i}\pi)^{k_j}\, k_j!} \cdot \vartheta^{(\sum_j k_j)}\Big\},$$

$$\psi_{\hbar,n} = \exp\Big(\hbar^{-1}\int_\bullet y\mathrm{d}x + \frac{1}{2}\int_\bullet\int_\bullet \omega_2^0\Big)\frac{\vartheta_\bullet}{\vartheta}$$

$$\times \Big\{\sum_{r\geq 0}\sum_{\substack{h_j,l_j\geq 0,\, k_j\geq 1\\ 2h_j-2+k_j+l_j>0}} \frac{\hbar^{\sum_j 2h_j-2+k_j+l_j}}{r!} \bigotimes_{j=1}^r \frac{\overbrace{\oint_{\mathcal{B}}\cdots\oint_{\mathcal{B}}}^{k_j}\overbrace{\int_\bullet\cdots\int_\bullet}^{l_j} \omega_{k_j+l_j}^{h_j}}{(2\mathrm{i}\pi)^{k_j}\, k_j!\, l_j!} \cdot \frac{\vartheta_\bullet^{(\sum_j k_j)}}{\vartheta_\bullet}\Big\}.$$

Some explanations are needed to read this formula: we denote $\mathrm{d}\mathbf{a}$, the vector of holomorphic 1-forms on Σ_g dual to the $\mathcal{A}$-cycles, and $\tau = \oint_{\mathcal{B}} \mathrm{d}\mathbf{a}$ the $g \times g$ matrix of periods ; we consider $\vartheta^{(k)}(\mathbf{w}|\tau)$, the tensor of k^{th}-order derivatives with respect to $\mathbf{w}$, of the theta function of characteristics (μ, ν) ; we denote $\zeta = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}-\tau\mathcal{A}} y \, \mathrm{d}x$; then, we use the notations $\vartheta^{(k)} = \vartheta^{(k)}(\hbar^{-1}\zeta|\tau)$ and $\vartheta_{\bullet}^{(k)} = \vartheta^{(k)}(\hbar^{-1}\zeta + \int_{\bullet} \mathrm{d}\mathbf{a}|\tau)$. $Z_\hbar$ is a special solution of Virasoro constraints introduced in [3]. It is an interesting object *per se*, because it is modular covariant under change of basis of cycles (it transforms like a theta function of characteristics (μ, ν), cf. [4]), and it is conjecturally the Tau function of an integrable system:

Conjecture 1. [5] *$Z_\hbar$ satisfies formally Hirota equations with respect to an infinite number of deformation parameters of the spectral curve.*

This was checked to first subleading order. The non-perturbative effects (the oscillations when $\hbar \to 0$ encoded in the theta functions) arise from multiple connectedness of the spectral curve. Such a phenomenon is indeed observed in large matrix integrals and solutions of integrable equations (like Korteweg-de Vries) in the small dispersion limit. The intuition behind the n.p. TR comes from these topics.

2. Spectral curves from A-polynomials

For any 3-manifold M with 1-cusp, the $\mathrm{SL}_2(\mathbb{C})$-character variety is essentially the zero locus $\mathcal{C}$ of a polynomial $A_M(m, l) \in \mathbb{Z}[m, l]$, where m and l denote longitude and meridian holonomies along the cusp [6]. In general, $\mathcal{C}$ has several irreducible components $\mathcal{C}_i$, and each of them is a singular curve. Besides, when M is a knot complement in a homology sphere, A_M is even in m and we want also to mod out this double covering. In this way, we obtain a smooth Riemann surface Σ_g of genus g, with two functions $x = \ln m$ and $y = \ln l$ defined on it, and we choose (arbitrarily) $(\mathcal{A}, \mathcal{B})$ cycles. This defines a spectral curve. We remark that it carries an involution $\iota : (m, l) \to (1/m, 1/l)$, since reversing meridian and longitude simultaneously for a given $\mathrm{SL}_2(\mathbb{C})$ representation lead to a conjugate representation. A-polynomials are very special from the K- theoretical viewpoint (they define torsion elements in the K_2 group of the curve), but we will not discuss it here, see [6, 9, 1]. To give an example, the geometric component of the figure 8-knot is isomorphic to the elliptic curve 15A8, which can be put in the form $Y^2 + XY + Y = X^3 + X^2$, and it admits 4 ramification points.

We have observed that for many knots with low number of crossings, the quotient $\mathcal{C}/\iota$ has genus $g_\iota = 0$. This happens for the figure 8-knot, and $\mathbf{8}_{21}$ is the simplest knot we found for which it is not the case. When this property holds true, the n.p. TR becomes much simpler: it yields power series in $\hbar$ involving only derivatives of Thetanullwerten with respect to their matrix of periods. The knot-theoretical interpretation of $g_\iota = 0$ thus becomes an interesting (open) question.

3. Asymptotics of the colored Jones polynomial

The A-polynomial has an irreducible component $\mathcal{C}^{\text{geom}}$, and there is a choice of branch $p_u \in \mathcal{C}^{\text{geom}}$, such that $\int_o^{p_u} \ln l \, \mathrm{d} \ln m$ is closely related to the complexified volume of M for uncomplete hyperbolic metrics on M parametrized by u [7]. When M is a hyperbolic knot complement in $\mathbb{S}_3$, according to the generalized volume conjecture [9, 12], the Jones polynomial of the knot behaves as

$$J_N(q) \sim \hbar^{\delta/2} \exp\Big(\frac{1}{\hbar}\int_o^{p_u} \ln l \, \mathrm{d}\ln m + \sum_{\chi \geq 0} \hbar^{\chi}\, \jmath_{\chi}(u)\Big),$$

with identifications $q = e^{2\hbar}$, $u = N\hbar \notin i\pi\mathbb{Q}\setminus\{1\}$ fixed and close enough to $\mathrm{i}\pi$ (this point correspond to the complete hyperbolic metrics on M), in the limit $N \to \infty$, $\hbar \to 0$. Dijkgraaf, Fuji and Manabe proposed that this series can be computed from TR, and their conjecture can be reformulated as:

Conjecture 2. [10] *If M is a hyperbolic 3-manifold, there exists a choice of basepoint o, a function $B(u)$ independent of $\hbar$, such that, within the assumption of the generalized volume conjecture:*

$$J_N(q) \sim B(u)\,\big[\psi_{\hbar,2}^{\text{pert}}(p_u, o; \iota(p_u), \iota(o))\big]^{1/2},$$

This conjecture was actually wrong, but computing the first orders for the 8-knot complement and the once-punctured torus bundle L^2R, they could match the left-hand side from TR by inserting to all orders renormalizations by certain rational numbers. We explain those discrepancies by proposing:

Conjecture 3. [1] *Keeping the previous notations, there exists a choice of characteristics (probably among even-half integer characteristics) such that*

$$J_N(q) \sim \tilde{B}(u)\,\big[\psi_{\hbar,2}(p_u, o; \iota(p_u), \iota(o))\big]^{1/2},$$

Notice that we have to exclude the case where $\hbar = \mathrm{i}\pi/k$ with k integer $\neq N$, because the behavior of the colored Jones polynomial is special at roots of unity. We checked that Conjecture 3 agrees with the results of [13] for the 8-knot complement up to $o(\hbar^3)$. We retrieve the subleading terms known in the expansion of the Kashaev invariant of the figure-eight knot by specializing at $u = \mathrm{i}\pi$:

$$J_N(q = e^{\frac{2\mathrm{i}\pi}{N}}) = 3^{-1/4}\, N^{3/2}\, e^{\frac{N}{2\pi}\,\mathrm{Vol}(\mathbf{4}_1)}\Big(1 + \frac{11}{12}\epsilon + \frac{697}{288}\epsilon^2 + \frac{724351}{51840}\epsilon^3 + o(\epsilon^3)\Big),$$

where $\epsilon = \frac{\mathrm{i}\pi}{3\sqrt{-3N}} \to 0$. Such an expansion has been proved with help of numerics in [13]. This non-trivial check supports the general idea that n.p. TR of the A-polynomial for any M should be compared to asymptotics of the corresponding manifold invariants.

When $g_\iota \neq 0$, $\psi_{\hbar,2}$ is no more a power series in $\hbar$, and if Conjecture 3 is trusted in general, it predicts that new asymptotic phenomena should be discovered for the colored Jones. Asymptotics for knots having $g_\iota \neq 0$ are numerically under investigation.

The relevance of Virasoro-type constraints in quantum topology is quite unexpected and mysterious up to now. The relationship between manifold invariants and integrable systems through Conjecture 1 might be related to the existence of integrable perturbations of the Wess-Zumino-Witten conformal field theory which underlies Chern-Simons theory. The generalization of our conjecture to asymptotics of Wilson loops for large representation in other gauge groups, and of asymptotics of refined and categorified invariants [11], still need to be explored.

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Non-Abelian Cohen-Lenstra Heuristics

Nigel Boston

(joint work with Michael R. Bush, Farshid Hajir)

Let p be an odd prime and consider how the p-class group $Cl_p(K)$ varies as K runs through imaginary quadratic fields of increasing absolute discriminant. In 1983, Cohen and Lenstra proposed the heuristic that a particular abelian p-group A should occur with frequency proportional to $1/|\mathrm{Aut}(A)|$. By Class Field Theory,

$Cl_p(K)$ is isomorphic to the Galois group of the maximal unramified abelian p-extension of K and we might ask how often a group G arises as the corresponding Galois group (the "p-class tower group") when the word "abelian" is removed.

In our first joint paper [1], we came up with a heuristic, which informally says that the relations in a (pro-p) presentation of G have a particular form and the frequency with which G arises should be proportional to the number of ways of picking such relations. Formally, G is a Schur σ-group, which means that its generator rank $d(G)$ equals its relation rank $r(G)$, its abelianization is finite, and it has an automorphism σ of order 2 acting as inversion on this abelianization. If F is the free pro-p group on $x_1, ..., x_g$, which has automorphism σ sending $x_i \mapsto x_i^{-1}$, then any Schur σ-group G with $d(G) = g$ is presented by picking g relations from $X := \{u^{-1}\sigma(u) \mid u \in \Phi(F)\}$.

In [1], we computed the Haar measure of the subset of X^g consisting of g-tuples of relations that present a given finite p-group G. The upshot is that the frequency with which G should arise as a p-class tower group is proportional to $1/|\mathrm{Aut}_\sigma(G)|$, where $\mathrm{Aut}_\sigma(G)$ is the centralizer of σ in $\mathrm{Aut}(G)$. We obtained much computational evidence and many consequences of this heuristic. In particular, it generalizes and implies the original Cohen-Lenstra heuristic. We also obtained a refinement concerning the maximal unramified p-extension of a given p-class.

We next considered the case of real quadratic fields in [2]. The main difference here is that, for the groups G that arise, $r(G)$ equals $d(G)$ or $d(G) + 1$. The heuristic now is to see how often $g + 1$ relations picked from X present G. Once again, we obtained a formula for the measure of the subset of X^{g+1} consisting of $(g + 1)$-tuples that present a given finite p-group G, this time giving a frequency proportional to $1/(|G||\mathrm{Aut}_\sigma(G)|)$, and gave a refinement for fixed p-class. One new phenomenon here is that a group can arise both as a p-class tower group and as a proper p-class quotient of a p-class tower group, and so the refinement is important in sorting this out.

A convenient way to express the original Cohen-Lenstra heuristics for imaginary quadratic fields is via their equivalent moments version. This says that if A is any abelian p-group, then the average number of unramified A-extensions of an imaginary quadratic field K, as K varies, should be 1. We can ask the same question if A is replaced by any finite p-group G and so deduce an equivalent moments version of our non-abelian Cohen-Lenstra heuristics.

In work with Daniel Ross and Melanie Matchett Wood, we have computed a formula for the average number of unramified G-extensions of imaginary quadratic fields. It turns out always to be an integer, namely the value of the $a(G)$th Rogers-Szëgo polynomial evaluated at $p^{d(G)}$, where $a(G)$ is an invariant of G (which equals 0 if G is abelian). This then is an equivalent form of the main heuristics of [1]. Jordan Ellenberg, Akshay Venkatesh, and Craig Westerland have made much progress in proving the moments version of Cohen-Lenstra in the function field case and Ross is working to extend this to our new heuristics. One interesting point is that the above integer should equal the number of components of a related Hurwitz space.

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L-spaces, left-orderability and foliations

Steven Boyer
(joint work with Cameron Gordon and Liam Watson; with Michel Boileau; and with Adam Clay)

In this talk we discussed relations between three measures of "largeness" for a closed, connected, orientable prime 3-manifold W:

- **CTF**: W admits a co-oriented, taut foliation (a topological condition).
- **NLS**: W is *not* a Heegaard-Floer L-space (an analytic condition).
- **LO**: $\pi_1(W)$ is a left-orderable group (a group theoretic condition).

We say that W is CTF, NLS, or LO when it possesses the corresponding property.

If W has positive first Betti number, it is CTF ([Ga, Theorem 55, page 477]) and LO ([BRW, Theorem 1.1(1)]). It is NLS by the definition of an L-space ([OSz3, Definition 1.1]). Thus we restrict our attention to the case that W is a rational homology sphere.

Ozsváth and Szabó have shown that if W is CTF, it is NLS ([OSz1, Theorem 1.4]) and have asked whether the converse holds. Calegari and Dunfield applied Thurston's universal circle construction to show that if W is an atoroidal CTF rational homology sphere, then the commutator subgroup of $\pi_1(W)$ is left-orderable ([CD, Corollary 7.6]). Hence W has an $|H_1(W)|$-fold abelian cover which is LO. Levine and Lewallen have shown that the fundamental groups of *strong* L-spaces are not left-orderable [LL].

Our first result deals with the case that W is either Seifert fibred or a Sol manifold. Equivalently, W is a non-hyperbolic geometric 3-manifold.

Theorem 1. ([BGW]) *Let W be a non-hyperbolic geometric 3-manifold. Then W is CTF if and only if it is NLS, and if and only if it is LO.*

Important components of the proof are contained in the work of Eisenbud, Hirsch and Neumann ([EHN]) on horizontal foliations in Seifert manifolds, in the work of Lisca and Stipsicz ([LS]) concerning L-spaces which are Seifert manifolds with base orbifold $S^2(a_1, \ldots, a_n)$, and in the work of Boyer, Rolfsen and Wiest ([BRW]) which characterised the Seifert and Sol manifolds which are LO. The new components found in [BGW] were the proofs that Seifert rational homology 3-spheres with base orbifold $P^2(a_1, \ldots, a_n)$ and Sol manifolds rational homology 3-spheres are L-spaces. Verification of the latter necessitated the use of bordered Heegaard-Floer theory.

Clay, Lidman and Watson proved that the fundamental groups of $\mathbb{Z}$-homology 3-sphere graph manifolds other than S^3 and the Poincaré homology sphere are left-orderabilty ([CLW]). Their result is also a consequence of the following theorem.

Theorem 2. ([BB]) *Let W be a $\mathbb{Z}$-homology 3-sphere graph manifold other than S^3 and the Poincaré homology 3-sphere. Then W is CTF. In fact, W admits a horizontal foliation. Hence W is NLS and LO.*

The left-orderability of the fundamental group of a graph manifold with a co-oriented horizontal foliation follows by combining [BRW, Theorem 1.1(1)] with Brittenham's result that such foliations are $\mathbb{R}$-covered ([Br, Proposition 7]).

Ozsváth and Szabó have conjectured that a prime $\mathbb{Z}$-homology 3-sphere is an L-space if and only if it is the 3-sphere or the Poincaré homology sphere. (See [Sz, Problem 11.4 and the remarks which follow it].) Hedden and Watson verified the conjecture for manifolds obtained by Dehn surgery on knots in the 3-sphere ([HW, Proposition 5]). Work of Rachel Roberts ([Ro1], [Ro2], [Ro3]) shows that $\mathbb{Z}$-homology 3-spheres obtained by surgery on many knots in the 3-sphere are CTF, and therefore LO by [CD]. See also [BGW, Proposition 1] and the discussion which follows it.

Work in progress of the speaker and Adam Clay ([BC]) indicates that the conditions CTF and LO are equivalent for graph manifolds and suggests that they are equivalent to NLS.

Infinite families of hyperbolic rational homology 3-spheres for which the conditions CTF, NLS and LO are equivalent are given by the next result.

Theorem 3. *Let L be a non-split alternating link and $\Sigma(L)$ its 2-fold branched cover.*
(1) ([OSz2, Proposition 3.3]) *$\Sigma(L)$ is an L-space. Hence it is not CTF.*
(2) ([BGW, Theorem 4]) *$\pi_1(\Sigma(L))$ is not left-orderable.*

Here are two corollaries of the second part of this theorem which are of independent interest.

Corollary 4. ([BGW, Corollary 2]) *Let K be an alternating knot and $\rho : \pi_1(S^3 \setminus K) \to Homeo_+(S^1)$ a homomorphism. If $\rho(\mu^2) = 1$ for some meridional class $\mu \in \pi_1(S^3 \setminus K)$, then the image of ρ is either trivial or isomorphic to $\mathbb{Z}/2$.*

Corollary 5. ([BGW, Corollary 3]) *Suppose that K is an alternating knot and let $\mathcal{O}_K(2)$ denote the orbifold with underlying set S^3 and singular set K with cone angle π. Suppose further that $\mathcal{O}_K(2)$ is hyperbolic. If the trace field of $\pi_1(\mathcal{O}_K(2))$ has a real embedding, then it must determine a $PSU(2)$-representation. In other words, the quaternion algebra associated to $\pi_1(\mathcal{O}_K(2))$ is ramified at that embedding.*

Many other examples of hyperbolic manifolds for which the conditions CTF, NLS and LO are equivalent are known ([Pe], [BGW, Proposition 2], [CW1], [CW2], [LW]). These examples and the results above suggest the following conjecture.

Conjecture 6. ([BGW]) *Let W be a closed, connected, orientable, prime 3-manifold. Then W is LO if and only if it is NLS.*

Nathan Dunfield has explored this conjecture through a computer-assisted search of over 10,000 hyperbolic rational homology 3-spheres W in the Hodgson-Weeks census. To date he has verified it in all cases for which it can be determined whether W is NLS or not NLS, and whether W is LO or not LO.

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Hyperbolic and Seifert volume of three-manifolds

Pierre Derbez
(joint work with Shicheng Wang)

1. Introduction

Let (G, X) be either $\mathrm{PSL}(2;\mathbb{C})$ with homogeneous space $X = \mathbf{H}^3$ or $\mathrm{Iso}_e\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with $X = \widetilde{\mathrm{SL}_2(\mathbb{R})}$. Denote by ω_X the corresponding G-invariant volume form. Let M be an oriented closed 3-manifold. To each representation $\rho\colon \pi_1 M \to G$ one can associate a developing map $D_\rho\colon \widetilde{M} \to X$ from the universal covering of M to X and a volume $\mathrm{vol}_{\mathrm{G}}(M, \rho)$ can be defined as the absolute value of $D_\rho^*\omega_X$ integrated over M. In both cases Reznikov [9] and Goldman-Brooks [2], [3] proved that the set $\mathrm{vol}\,(M, G)$ of volumes of all representations $\rho\colon \pi_1 M \to G$, where M is a closed oriented three-manifold, is finite. One can therefore define the hyperbolic, resp. Seifert, volume of M by $HV(M)$, resp. $SV(M)$, as the maximal value of $\mathrm{vol}\,(M, G)$.

Question 1. *For which M are these volume positive?*

2. the volumes of geometric manifolds

The answer to this question is known for geometric manifolds.

Theorem 2. *[[9], [2], [3]] Let M be a closed oriented and geometric 3-manifold.*

If M is hyperbolic then $HV(M) = \mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(M, \rho)$ iff ρ is a discrete and faithful representation and $HV(M) = \mathrm{vol}_{\mathbf{H}^3}(M)$.

If M supports an $\widetilde{\mathrm{SL}_2(\mathbb{R})}$-geometry the same statement is true and $SV(M) = \mathrm{vol}_{\widetilde{\mathrm{SL}_2(\mathbb{R})}}(M) = 4\pi^2\chi^2(O_M)/|e(M)|$, where O_M is the base 2-orbifold of M with rational Euler characteristic χ and where $e(M)$ denotes the rational Euler number of the Seifert fibration $M \to O_M$.

If M supports any of the six remaining geometries then $HV(M) = SV(M) = 0$.

Example 3. *Suppose M supports the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$-geometry and that its base 2-orbifold has a positive genus g. Using [6] and [2] one can compute*

$$\mathrm{vol}\left(M, \mathrm{Iso}\widetilde{\mathrm{SL}_2(\mathbb{R})}\right) = \left\{ \frac{4\pi^2}{|e(M)|} \left(\sum_{i=1}^{r} \left(\frac{n_i}{a_i} \right) - n \right)^2 \right\} \subset 4\pi^2\mathbf{Q}$$

where $n_1, ..., n_r, n$ are integers such that

$$\sum_{i=1}^{r} \llcorner n_i/a_i \lrcorner - n \leq 2g - 2, \quad \sum_{i=1}^{r} \ulcorner n_i/a_i \urcorner - n \geq 2 - 2g$$

and $a_1, ..., a_r$ are the indices of the singular points of the orbifold of M and where $\llcorner a \lrcorner$ and $\ulcorner a \urcorner$, for $a \in \mathbb{R}$, denote resp. the greatest integer $\leq a$ and the least integer $\geq a$. Moreover, choosing $n_i = k_i a_i + (a_i - 1)$ for $i = 1, ..., r$ we retrieve the maximal volume $SV(M)$.

3. The Seifert volume of graph manifolds

In [4] we answer Question 1 for graph manifolds with non-trivial geometric decomposition.

Theorem 4. *Any closed non-geometric graph manifold has a virtually positive Seifert volume.*

Remark 5. *Since we still don't know if there are non-geometric graph manifolds with zero Seifert volume it is unclear whether the condition "virtual" is necessary.*

Remark 6. *The geometric pieces of a non-geometric graph manifold are either Euclidean or* $\mathbf{H}^2 \times \mathbb{R}$ *and therefore they cannot contribute individually to the Seifert volume of* M*. It turns out that the volume of* M *is positive rather because the geometric pieces are glued along their boundary in such a way that their geometry do not extend. Accordingly it can be proved, see* [5]*, that there exists a finite covering* $\widetilde{M}$ *of* M *such that the set* $\mathrm{vol}(\widetilde{M}, \mathrm{Iso}_e\widetilde{\mathrm{SL}_2(\mathbb{R})})$ *contains the informations of the gluing involution when* M *is made of two Seifert pieces with connected boundary.*

4. The hyperbolic volume of three-manifolds

By a result of Reznikov stated in [9] $HV(M) \leq \mu_3 \|M\|$, where $\|.\|$ denotes the Gromov simplicial volume defined in [7]. Therefore the condition $\|M\| > 0$ is necessary in Question 1 for the hyperbolic volume. We conjecture that a closed 3-manifold has a virtually positive hyperbolic volume iff its Gromov simplicial volume is positive. In this statement the virtual condition cannot be dropped:

Proposition 7. *There are (infinitely many) 3-manifolds* M *with* $\|M\| > 0$ *but* $HV(M) = 0$*.*

On the other hand we found out that the manifolds constructed in Proposition 7 have all a virtually positive hyperbolic volume.

One can check the conjecture in particular when the dual graph of M is a tree (therefore when M is a rational homology sphere) and when M is a virtual surface bundle. Notice that this latter point can be related with [1, conjecture 9.1].

Besides one can construct a finite covering $\widetilde{M}$ of M such that the informations of the gluing involution can be estimated, using [8], by the elements of $\mathrm{vol}(\widetilde{M}, \mathrm{PSL}(2;\mathbb{C}))$ when M is made of two geometric pieces with connected boundary, one of them being hyperbolic.

The proofs of these results use the Chern Simons gauge theory with structural Lie groups $\mathrm{PSL}(2;\mathbb{C})$ and $\mathrm{Iso}_e(\widetilde{\mathrm{SL}_2(\mathbb{R})})$ as well as specific properties and results of Seifert and hyperbolic geometry developed resp. in [6] and [10].

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Galois action on knots

Hidekazu Furusho

In my talk, I discussed the following topics:

Absolute Galois action on profinite braids: I recalled the definitions of the profinite braid group $\widehat{B}_n$ $(n \geq 2)$, the absolute Galois group $G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of the rational number field $\mathbb{Q}$ and the profinite Grothendieck-Teichmüller group $\widehat{GT}$. The last one is the group suggested implicitly by Grothendieck [G] and constructed explicitly by Drinfel'd [Dr]. I reviewed a method to calculate explicitly the action of $G_{\mathbb{Q}}$ and $\widehat{GT}$ on $\widehat{B}_n$ (cf. Drinfel'd [Dr], Ihara-Matsumoto [IM]) and explained that $G_{\mathbb{Q}}$ is mapped to $\widehat{GT}$ (cf. Drinfel'd [Dr], Ihara [I]). I noted that the map is injective by Belyĭ's result [Be].

Motivic Galois action on proalgebraic braids: I introduced the proalgebraic braid group $B_n(\mathbb{Q})$ $(n \geq 2)$ and recalled the definition of the proalgebraic Grothendieck-Teichmüller group $GT(\mathbb{Q})$ ([Dr]). I reviewed my result [F10] on defining equations of $GT(\mathbb{Q})$, which reduces two hexagon equations into one pentagon equation. Then it was explained that the motivic Galois group $Gal^{\mathcal{M}}(\mathbb{Z})(\mathbb{Q})$ (:the tannakian Galois group of the category $\mathcal{MTM}(\mathbb{Z})_{\mathbb{Q}}$ of unramified mixed Tate motives in Deligne-Goncharov [DeG]) is mapped to $GT(\mathbb{Q})$. I noted that the results of Brown [Ba] and of Zagier [Z] imply that the map is injective.

Motivic Galois action on proalgebraic knots: I introduced the space $\widehat{\mathbb{Q}\mathcal{K}}$ of proalgebraic knots by taking completion of the $\mathbb{Q}$-vector space $\mathbb{Q}\mathcal{K}$ generated by all oriented knots with respect to the singular knot filtration. Then I explained a method to construct $GT(\mathbb{Q})$-action there by following the ideas in Bar-Natan [Ba], Kassel-Turaev [KT] and Le-Murakami [LM]. By the embedding from $Gal^{\mathcal{M}}(\mathbb{Z})(\mathbb{Q})$ into $GT(\mathbb{Q})$, an action of the motivic Galois group $Gal^{\mathcal{M}}(\mathbb{Z})(\mathbb{Q})$ on $\widehat{\mathbb{Q}\mathcal{K}}$ is obtained. It implies that the space $\widehat{\mathbb{Q}\mathcal{K}}$ of proalgebraic knots carries a structure of unramified mixed Tate motive. I noted that a result of Le-Murakami [LM] leads that this action factors through $\mathbb{G}_m$, that means, a proalgebraic knot admits a non-trivial decomposition into Tate motives. One of my results in [F12], an explicit formula of the first term of this decomposition, was presented.

Absolute Galois action on profinite knots: My definition [F12] of profinite knots was introduced. It is defined to be finite 'consistent' products of 'annihilations', 'creations' and oriented profinite braids modulo a profinite analogue of the Turaev moves. They form a topological monoid $\widehat{\mathcal{K}}$. I explained my construction [F12] of an action of $\widehat{GT}$ on the group $G\widehat{\mathcal{K}}$ of profinite knots (which is defined to be the group of fraction of $\widehat{\mathcal{K}}$). It was noted that one of important consequences of my construction is that the absolute Galois group $G_{\mathbb{Q}}$ acts continuously on the group $G\widehat{\mathcal{K}}$ of profinite knots. Various properties of this Galois action and its related questions were discussed. Particularly, related to Belyĭ's result in the profinite braids setting and Brown's result in the proalgebraic braids setting, a question whether this action is faithful or not was emphasized.

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The 3D index of a cusped hyperbolic manifold

Stavros Garoufalidis

1. Introduction

1.1. **The 3D index of Dimofte-Gaiotto-Gukov.** The goal of the talk given in Oberwolfach August 28, 2012 is to discuss the index $I_{\mathcal{T}}$ of an ideal triangulation $\mathcal{T}$, a remarkable collection of Laurent series in $q^{1/2}$ with integer coefficients introduced by Dimofte-Gaiotto-Gukov [5, 6]. The talk reports on recent work of the author [9] and joint work in progress with Hodgson-Rubinstein-Segerman [10]. Explitily,

- In [9] we give necessary and sufficient conditions for the existence of the index $I_{\mathcal{T}}$ of an ideal triangulation in terms of the existence of an index structure of $\mathcal{T}$. The later is a weakened version of a strict-angle structure and can be checked efficiently given the gluing equation matrix of $\mathcal{T}$.
- In [9] we show that if $\mathcal{T}$ and $\mathcal{T}'$ are relatex by a 2-3 move and both admit an index structure, then $I_{\mathcal{T}} = I_{\mathcal{T}'}$.
- In [10] we show that $\mathcal{T}$ admits an index structure if and only if it is 1-efficient [13, 12]. Apart from the rather unexpected connection between the index of an ideal triangulation (a recent quantum object) and the classical theory of normal surfaces, Theorem 2 places restrictions in the topology of M; see Remark 3 below.
- In [10] we use triangulations of the canonical Epstein-Penner ideal cell decomposition of a cusped hyperbolic 3-manifold to show that the invariant of ideal triangulations can be promoted to an invariant of cusped hyperbolic 3-manifolds.

Let us point out that normal surfaces were also used in [8] in an attempt to construct topological invariants of 3-manifolds, in the style of a Turaev-Viro TQFT. Recently strict angle structures (a stronger form of an index structure) were used in [1] to prove convergence of state-integral invariants of ideal triangulations which are also expected to give analytic invariants of cusped hyperbolic 3-manifolds that generalize the Kashaev invariant [15]. The q-series of Theorem 6 below are q-holonomic, of Nahm-type and apart from a meromorphic singulatity at $q = 0$, admit analytic continuation in the punctured unit disk.

Before we get to the details, the reader should keep in mind that the origin of the 3D index is the exciting work of Dimofte-Gaiotto-Gukov [5, 6] in mathematical physics, where they studied 3-dimensional gauge theories with $N = 2$ supersymmetry that are associated to an ideal triangulation $\mathcal{T}$ of an oriented 3-manifold M with r cusps. The low-energy limit of these theories is a partially defined function (the so-called *3D index*)

$$I : \{\text{ideal triangulations}\} \longrightarrow \mathbb{Z}((q^{1/2}))^{\mathbb{Z}^r \times \mathbb{Z}^r}, \qquad \mathcal{T} \mapsto I_{\mathcal{T}}(m_1, \dots, m_r, e_1, \dots, e_r) \in \mathbb{Z}((q^{1/2})) \tag{1}$$

for integers m_i and e_i, which is invariant under some *partial* 2-3 moves. The above gauge theories are in a sense an analytic continuation of the colored Jones polynomial and play an important role on Chern-Simons perturbation theory and in categorification. Although the gauge theory depends on the ideal triangulation $\mathcal{T}$, and the 3D index in general may not converge, physics predicts that the gauge theory ought to be a topological invariant of the underlying 3-manifold M. Recall that every two ideal triangulations of a cusped 3-manifold are related by a sequence of 2-3 moves [16, 17, 18]. In [9] the following was shown. For a definition of an index structure.

Theorem 1. (a) $I_{\mathcal{T}}$ is well-defined if and only if $\mathcal{T}$ admits an *index structure.*
(b) If $\mathcal{T}$ and $\mathcal{T}'$ are related by a 2-3 move and both admit an index structure, then $I_{\mathcal{T}} = I_{\mathcal{T}'}$.

1.2. **Index structures and 1-efficiency.**

Theorem 2. *$\mathcal{T}$ admits an index structure if and only if $\mathcal{T}$ is 1-efficient.*

The above theorem has some consequences for our goal of constructing topological invariants.

Remark 3. *1-efficiency of $\mathcal{T}$ implies restrictions on the topology of M: it follows that M is irreducible and atoroidal. It follows by the Geometrization Theorem in dimension* 3 *that M is hyperbolic or small Seifert-fibered.*

Remark 4. *If K is the connected sum of 4_1 with the 5_2 knot, or K' is the Whitehead double of the 4_1 knot and $\mathcal{T}$ is any ideal triangulation of the complement of K or K', then $\mathcal{T}$ is not 1-efficient, thus $I_{\mathcal{T}}$ never exists. On the other hand, the (colored) Jones polynomial, the Kashaev invariant and the* $\mathrm{PSL}(2,\mathbb{C})$*-character variety of K and K' happily exist; see* [14, 15, 3].

Remark 5. *If $\mathcal{T}$ admits a semi-angle structure (in particular, a taut or a strict angle structure) and M is atoroidal then $\mathcal{T}$ is 1-efficient* [**?**, Thm.2.6]*, thus $I_{\mathcal{T}}$ exists.*

1.3. **Regular ideal triangulations.** In view of Remark 3, we will restrict our goal to construct the index of a hyperbolic 3-manifold M. All we need is a canonical set $\mathcal{X}_M$ of 1-efficient ideal triangulations of M such that every two triangulations are related by 2-3 moves within $\mathcal{X}_M$. Every cusped hyperbolic 3-manifold M has a canonical *cell decomposition* [7] where the cells are convex ideal polytopes in $\mathbb{H}^3$. The cells can be triangulated into ideal tetrahedra, with flat ones inserted when the triangulations of their faces do not match. Unfortunately, it is not known whether any two triangulations of a 3-dimensional polytope are related by 2-3 moves; the corresponding result trivially holds in dimension 2 and nontrivially fails in dimension 5; [4, 19]. Nontheless, it was shown by Gelfand-Kapranov-Zelevinsky that any two *regular* triangulations of a polytope in $\mathbb{R}^n$ are related by a sequence of geometric bistellar flips; [11]. Using the Klein model of $\mathbb{H}^3$, we define the notion of a regular ideal triangulation of an ideal polytope and observe that every two regular ideal triangulations are related by a sequence of geometric 2-2 and 2-3 moves. This allows us to define a finite set $\mathcal{X}_M^{\mathrm{EP}}$ of ideal triangulations of a cusped hyperbolic manifold M such that any two are related by a sequence of 2-3 moves within $\mathcal{X}_M^{\mathrm{EP}}$. Combining Theorems 1 and 2 we obtain a topological invariant of cusped hyperbolic 3-manifolds M.

Theorem 6. *If M is a cusped hyperbolic 3-manifold, and $\mathcal{T} \in \mathcal{X}_M^{\mathrm{EP}}$ we have $I_M := I_{\mathcal{T}}$ is well-defined.*

Remark 7. *If M has $r \geq 1$ cusps, then the Epstein-Penner cell decomposition is well-defined once we choose a scale vector $c_1 > c_2 \cdots > c_r > 0$ for the relative size of the cusps. The scale vector is well-defined up to multiplication by a positive real number.*

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Generalized Volume Conjecture: Categorified

Sergei Gukov
(joint work with Hiroyuki Fuji, Piotr Sułkowski)

The generalized volume conjecture states that "color dependence" of the colored Jones polynomial is governed by an algebraic variety, the zero locus of the A-polynomial (for knots) or, more generally, by character variety (for links or higher-rank quantum group invariants). This relation, based on $SL(2, C)$ Chern-Simons theory, explains known facts and predicts many new ones.

In particular, since the colored Jones polynomial can be categorified to a doubly-graded homology theory, one may wonder whether the generalized (or quantum) volume conjecture admits a natural categorification. In this talk, I argue that the answer to this question is "yes" and introduce a two-parameter deformation of the A-polynomial that describes the "color behavior" of the HOMFLY homology, much like the ordinary A-polynomial does it for the colored Jones polynomial. This deformation, called the super-A-polynomial, is strong enough to distinguish mutants, and its most interesting properties include relation to knot contact homology and knot Floer homology. This talk is based on a joint work with Hiroyuki Fuji and Piotr Sulkowski [1, 2].

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Fundamental groups of number fields

Farshid Hajir

In this mostly expository lecture aimed at low-dimensional topologists, I outlined some basic facts and problems of algebraic number theory. My focus was on one particular aspect of the rich set of analogies between number fields and 3-manifolds dubbed *Arithmetic Topology.* Namely, I discussed the role played in number theory by "fundamental groups" of number fields, and related some of the history of the subject over the past fifty years, since the unexpected discovery by Golod and Shafarevich of number fields with infinite fundamental group; see the monograph of Neukirch, Schmidt, Wingberg [10] for a comprehensive account. A conjecture of Fontaine and Mazur [3] has been influential in stimulating work on the structure of these infinite fundamental groups in recent years. I presented a formulation of this conjecture as it relates to the asymptotic growth of discriminants [6]. This discussion then served as motivation for a question about non-compact, finite-volume, 3-manifolds inspired by the following dictionary.

Topology	Arithmetic
M non-compact, finite-volume hyperbolic 3-manifold	K a number field or, more precisely, $X = \mathrm{Spec}\mathcal{O}_K$
universal cover $\widetilde{M}$	$\widetilde{K}$ = max. unramified extension of K
fundamental group $\pi_1(M)$	$\mathrm{Gal}(\widetilde{K}/K) \approx \pi_1^{\mathrm{et}}(X)$
Klein-bottle cusps of M Torus cusps of M $r_1 = \#$ Klein-bottle cusps of M $r_2 = \#$ Torus cusps of M $r = r_1 + r_2 = \#$ cusps of M $n = r_1 + 2r_2 =$ weighted $\#$ cusps	Real ("unoriented") places of K Complex ("oriented") places of K $r_1 = \#$ Real places of K $r_2 = \#$ Complex places of K $r = r_1 + r_2 = \#$ places of K at ∞ $n = r_1 + 2r_2 = [K : \mathbb{Q}]$
$\mathrm{vol}(M)$ = volume of M	$\log \lvert d_K \rvert$, d_K = discriminant of K

There are multiple accounts of the dictionary of arithmetic topology; these include Reznikov [12], Ramachandran [11], Deninger [2], Morin [9], and Morishita [8]. For the subtle distinction between $\mathrm{Gal}(\widetilde{K}/K)$ and the étale fundamental group of Spec $\mathcal{O}_K$ when K is not "orientable," i.e. $r_1(K) \neq 0$, see Ramachandran [11]. I hit upon the analogy between cusps and infinite places as well as between volumes and discriminants during several conversations with Champanekar and Dunfield at the 2010 Oberwolfach meeting on *Low-dimensional topology and number theory*, and wish to thank them both for their patient explanations to a non-specialist. For Ramachandran's justification of the cusps-places analogy, see section 2 of Deninger [2]. As justification for drawing a parallel between volumes for hyperbolic 3-manifolds (or more generally Gromov norms of 3-manifolds) with logarithmic discriminants for number fields, I limit myself here to appealing to the "Riemann-Hurwitz genus formula for number fields,"

$$\log |d_L| = [L : K] \log |d_K| + \log |\mathbb{N}_{K/\mathbb{Q}} d_{L/K}|$$

where $d_{L/K}$ is the relative discriminant of L/K. Thus, when L/K is a covering, i.e. is unramified, the volume scales up by a factor of $[L : K]$, just as with coverings of manifolds. The relative discrimiant $d_{L/K}$ is made up of a "wild" component corresponding to prime ideals of K that divide a prime divisor of $[L : K]$ and a "tame" component. While the latter is easy to compute, the former can be quite intricate.

The Riemann-Hurwitz formula relates the existence of coverings to the rate of growth of discriminants. It was this fact which led Minkowski to create his "geometry of numbers" for the purpose of proving the following conjecture of Kronecker: $\widetilde{\mathbb{Q}} = \mathbb{Q}$. Minkowski actually showed much more, namely that the discriminant grows exponentially with the degree. For this reason, we define a normalized discriminant for number fields $\nu(K) := \frac{\log |d_K|}{[K:\mathbb{Q}]}$, called the logarithmic root discriminant. This quantity remains constant in unramified extensions and remains bounded for extensions which are *tamely* ramified at a finite number of primes.

In his proof that discriminants grow exponentially with the degree, Minkowski found that real and complex places give different contributions. Namely, he found constants $A > B > 0$ such that $\log |d_K| \geq Ar_1 + Br_2 - \delta(n)$, where $\delta(n)$ is a small error term that is in $o(n)$ as $n = r_1 + 2r_2 \to \infty$. To reformulate this type of bound in the language of normalized discriminants, we introduce the parameter $t = r_1/n$. The best known values of A, B come from the study of Dedekind zeta functions of number fields. If we admit the Generalized Riemann Hypothesis for these zeta functions, we have

$$\nu(K) \geq \log(8\pi) + \gamma + t\pi/2 - \varepsilon(n) \tag{1}$$

with an explicit error term $\varepsilon(n)$ that tends to 0 with $n = [K : \mathbb{Q}]$.

If we follow the analogy introduced in the table above, we are led to the question: does the volume of an r-cusped hyperbolic 3-manifold grow linearly with r? The answer is yes. Indeed, we have the following theorem of Adams [1]: *If M is an r-cusped hyperbolic 3-manifold, then* $\mathrm{vol}(M) \geq v_3 r$ *where v_3 is the volume of the regular ideal tetrahedron.*

We note that Adams' proof relies on Minkowski's geometry of numbers. Even without this fact as a provocation, it is natural for a number-theorist to wonder whether Adams' theorem can similarly be refined for contributions from torus cusps and Klein-bottle cusps. A somewhat vague form of the question is: What are the optimal values of positive constants v_1 and v_2 such that every hyperbolic 3-manifold having r_1 Klein-bottle and r_2 torus cusps satisfies $\mathrm{vol}(M) \geq r_1 v_1 + r_2 v_2$? To make the question more precise, let us define, for an r-cusped 3-manifold M with r_1 Klein bottle cusps and $r - r_1 = r_2$ torus cusps, the *orientation type* t of M to be $t = r_1/r$ and its *normalized volume* to be $\nu(M) := \mathrm{vol}(M)/r$. It is clear that we intend $\nu(M)$ to be a reasonable analogue of the logarithmic root discriminant for number fields.

In number theory, the estimate (1) is of great importance; in particular, an interesting problem to determine whether the constants in the linear function bounding the normalized discriminant from below are optimal; this is measured by a function defined by Martinet (see [7] and also [5]). As an analogue of the Martinet function, we define a function $\mathscr{A}(t)$ as follows: For a rational number $t \in [0, 1]$, define

$$\mathscr{A}(t) = \inf_{M \text{ of type } t} \nu(M),$$

the infimum being taken over all hyperbolic 3-manifolds of orientation type t.

The question then is to determine (upper and lower bounds for) $\mathscr{A}(t)$. If for no other reason than for the analogy with asymptotic problems of this type in number theory and many other contexts (graph theory, coding theory, curves over finite fields etc., see [4]), it would be very interesting if it can be established that $\mathscr{A}(t)$ is a linear function, or that it meets a fixed linear lower bound for many values of t.

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Decomposition of Elliptic Genera in terms of Superconformal Characters

Kazuhiro Hikami
(joint work with Tohru Eguchi)

Study of the elliptic genus by use of representation theory of the superconformal algebras was initiated in [9]. For instance, the elliptic genus of the $K3$ surface is a weak Jacobi form with weight 0 and index 1, and we have

$$2\,\phi_{0,1}(z;\tau) = 24\,\mathrm{ch}_{h=\frac{1}{4},\ell=0}(z;\tau) + \sum_{n=0}^{\infty} A(n)\,\mathrm{ch}_{h=n+\frac{1}{4},\ell=\frac{1}{2}}(z;\tau)$$

Here we use the $\mathcal{N}=4$ superconformal character $\mathrm{ch}_{h,\ell}(z;\tau)$ with central charge $c=6$, conformal weight h, and isospin ℓ. It is known [3, 4] that we have

$$-2 + \sum_{n=1}^{\infty} A(n)\,q^n = 8 \sum_{w=\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}} \mu(z;\tau),$$

where $\mu(z;\tau)$ is a mock modular form studied in detail in [11]

$$\mu(z;\tau) = \frac{i\, e^{\pi i z}}{\theta_{11}(z;\tau)} \sum_{n\in\mathbb{Z}} (-1)^n \, \frac{q^{\frac{1}{2}n(n+1)}\, e^{2\pi i n z}}{1 - q^n e^{2\pi i z}}.$$

It was observed [8] that the integral Fourier coefficients $A(n)$ are related to the dimensions of the irreducible representations of the largest Mathieu group M_{24}. This “Mathieu moonshine” still remains mysterious, though known is that the non-abelian automorphism group on $K3$ is a subgroup of M_{24} [10].

A generalization of Mathieu moonshine is proposed in [1]. Therein a piece of elliptic genus of $2k$-dimensional hyper-Kähler manifold is studied by use of $\mathcal{N}=4$ superconformal algebra with central charge $c = 6k$ following a method developed in [5]. In case of 4-dimension, we have

$$\begin{aligned}
&\frac{1}{12}\,[\phi_{0,1}(z;\tau)]^2 - \frac{1}{12}E_4(\tau)\,[\phi_{-2,1}(z;\tau)]^2 = 12\mathrm{c}h^{\mathcal{N}=4}_{k=2,h=\frac{2}{4},\ell=0}(z;\tau) \\
&\quad + q^{-\frac{1}{12}}\left(-2 + 32\,q + 110\,q^2 + 288\,q^3 + 660\,q^4 + 1408\,q^5 + 2794\,q^6 + \cdots\right) B^{\mathcal{N}=4}_{2,1}(z;\tau) \\
&\quad - q^{-\frac{1}{3}}\left(20\,q + 88\,q^2 + 220\,q^3 + 560\,q^4 + 1144\,q^5 + 2400\,q^6 + \cdots\right) B^{\mathcal{N}=4}_{2,2}(z;\tau).
\end{aligned}$$

Here the BPS characters $\mathrm{c}h^{\mathcal{N}=4}_{k=2,h=\frac{2}{4},\ell=0}(z;\tau)$ is a mock modular form, and bases of the non-BPS characters $B^{\mathcal{N}=4}_{2,a}(z;\tau) = \frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\mathrm{c}hi_{1,\frac{a-1}{2}}(z;\tau)$ are modular. Observed is another moonshine that the Fourier coefficients of the non-BPS characters are related to the dimensions of irreducible representations of $2.M_{12}$. See [1], where other moonshine phenomena are suggested for higher dimensional case.

We propose another possibility [7]. Ordinally the $\mathcal{N}=4$ superconformal algebra describes the geometry of hyper-Kähler manifolds, while the $\mathcal{N}=2$ superconformal algebra is for Calabi–Yau manifolds. We employ the $\mathcal{N}=2$ superconformal characters $\mathrm{c}h^{\mathcal{N}=2}_{D,h,Q}(z;\tau)$ for central charge $c = 3D$, conformal weight h, and $U(1)$ charge Q, to decompose a weak Jacobi form as a piece of elliptic genus of Calabi–Yau manifolds [6]. In the case of 4-dimension, the above weak Jacobi form is decomposed as

$$\begin{aligned}
&\frac{1}{12}\,[\phi_{0,1}(z;\tau)]^2 - \frac{1}{12}E_4(\tau)\,[\phi_{-2,1}(z;\tau)]^2 = 12\mathrm{c}h^{\mathcal{N}=2}_{D=4,h=\frac{4}{8},Q=0}(z;\tau) \\
&\quad + q^{-\frac{1}{24}}\left(-2 + 10\,q + 20\,q^2 + 42\,q^3 + 62\,q^4 + 118\,q^5 + 170\,q^6 + \cdots\right) B^{\mathcal{N}=2}_{4,1}(z;\tau) \\
&\quad + q^{-\frac{3}{8}}\left(12\,q + 36\,q^2 + 60\,q^3 + 120\,q^4 + 180\,q^5 + 312\,q^6 + \cdots\right) B^{\mathcal{N}=2}_{4,2}(z;\tau),
\end{aligned}$$

where the BPS character $\mathrm{c}h^{\mathcal{N}=2}_{D=4,h=\frac{4}{8},Q=0}(z;\tau)$ is mock modular, while bases of the non-BPS characters $B^{\mathcal{N}=2}_{D,Q}(z;\tau)$ are modular. Mathematically the decomposition in terms of $\mathcal{N}=4$ (resp. $\mathcal{N}=2$) superconformal characters such as () (resp. ()) corresponds to a theta expansion of weak Jacobi forms of integral weight (resp. half-odd integral weight). Remarkable is that the integral Fourier coefficients are related to the dimensions of the irreducible representations of $SL_2(11) \cong 2.L_2(11)$. It is well known that the group $L_2(11)$ is closely related to M_{12}, and that it plays a crucial role in the ternary Golay code [2]. It is expected [7] that there might exist similar moonshine phenomena for higher D.

Currently the real origin of these moonshine phenomena remains mysterious. We hope that geometrical meaning of the character decompositions of elliptic genus will be clarified in near future.

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Small dilatation pseudo-Anosov mapping classes

Eriko Hironaka

This report outlines some recent work and open questions surrounding the minimum dilatation problem for pseudo-Anosov mapping classes on oriented surfaces of finite type. Using the geometry of 3-manifolds and results of Thurston [13, 14], Fried [4] and McMullen [11], we study deformations of mapping classes within the space $\mathcal{P}$ of all pseudo-Anosov mapping classes. We also give two general constructions of convergent sequences of dilatation mapping classes, which conjecturally can be used to describe all pseudo-Anosov mapping classes with bounded normalized dilatation.

1. Minimum dilatation problem

Let $\phi : S \to S$ be a pseudo-Anosov mapping class on an oriented surface $S = S_{g,n}$ of genus g and n punctures. The *dilatation* $\lambda(\phi)$ is the expansion factor of ϕ along the stable transverse measured singular foliation associated to ϕ, and is a Perron algebraic unit greater than one. The set of dilatations for a fixed S is discrete [14].

Let $\mathcal{P}(S)$ be the set of all pseudo-Anosov mapping classes on S, and let $\delta(S)$ be the minimum dilatation for $\phi \in \mathcal{P}(S)$.

The minimum dilatation problem (cf. [12, 11, 2]) can be stated as follows.

Problem 1 (Minimum Dilatation Problem I). *What is the behavior of $\delta(S_{g,n})$ as a function of g and n?*

The exact value of $\delta(S_{g,n})$ is not known except for very small cases (for example, for closed surfaces, the answer is only known for $g = 2$ [5]). However, more is known about the asymptotic behavior of $\delta(S_{g,n})$ as a function of g and n, and the topological Euler characteristic $\chi(S_{g,n})$.

Let $\mathcal{P} = \bigcup_S \mathcal{P}(S)$. For $(S, \phi) \in \mathcal{P}$, the *normalized dilatation* is defined by

$$L(S, \phi) = \lambda(\phi)^{|\chi(S)|}.$$

For $\ell > 1$, we say ϕ is *ℓ-small* if $L(\phi) \leq \ell$. Let $\mathcal{P}(\ell)$ be the set of ℓ-small pseudo-Anosov maps.

The current smallest known accumulation point of the image of L is

$$\ell_0 = \left(\frac{3+\sqrt{5}}{2}\right)^2. \tag{1}$$

(See [7, 1, 10].)

Problem 2 (Assymptotic Minimum Dilatation Problem). *Is there an accumulation point for the image of L that is smaller than ℓ_0?*

One can also formulate the minimum dilatation problem from a geometric rather than numerical standpoint.

Problem 3 (Minimum Dilatation Problem II). *What do small dilatation mapping classes look like?*

We approach these three problems from two fronts. One is to study deformations of pseudo-Anosov mapping classes using Thurston's theory of fibered faces. The other is to explictly construct mapping classes with small dilatations.

2. Deformations of pseudo-Anosov mapping classes.

By a result of Thurston[14], a mapping class is hyperbolic if and only if the mapping class is pseuod-Anosov. Thus, $\mathcal{P}$ partitions into sets of monodromies $\Phi(M)$ of hyperbolic 3-manifolds M. The $\Phi(M)$ partition further into subsets $\Phi(M, F)$ that are in one-to-one correspondence with rational points on fibered face F in such a way that the topological Euler characteristic of S is the denominator of the corresponding rational point for $(S, \phi) \in \Phi(M)$. Each fibered face F is a top dimensional face of the Thurston norm ball in $H^1(M; \mathbb{Z})$, a convex polyhedron that is the convex hull of integral points. Thus $\mathcal{P}$ can be identified with the set of rational points on a disjoint union of open Euclidean polyhedra $\sqcup_\alpha F_\alpha$.

The following is a consequence of results of Fried [4] and McMullen [11].

Theorem 4. *The normalized dilatation function L is continuous on $\mathcal{P}$ and extends to a locally convex function on $\sqcup_\alpha F_\alpha$.*

Corollary 5. *The normalized dilatation function L is bounded on any compact subset of $\sqcup_\alpha F_\alpha$.*

A partial converse to this statement also holds. Consider the subcollection $\mathcal{P}^0 \subset \mathcal{P}$ consisting of elements (S, ϕ) whose stable and unstable foliations have no interior singularities. Let $\mathcal{P}^0(\ell)$ be the set of pseudo-Anosov mapping classes with normalized dilatation less than or equal to ℓ.

Theorem 6 (Farb-Leininger-Margalit [3]). *Given $\ell > 1$, there is a finite set of 3-manifolds $M_1, \dots, M_r$ so that*

$$\mathcal{P}^0(\ell) \subset \bigcup_{i=1}^{r} \Phi(M_i).$$

It follows from Theorem 6 that to understand the shape of all ℓ-small dilatation mapping it suffices to understand how mapping classes vary on small open neighborhoods in $\mathcal{P}$.

3. Nearly periodic mapping classes with small dilatation.

It is reasonable to guess that small dilatation mapping classes should be "nearly" periodic. We consider two descriptions of sequences of mapping classes that are of this form.

Penner-type sequences. Let (S, ϕ, τ) be such that $(S, \phi) \in \mathcal{P}$ and $(\tau, \partial\tau) \subset (S, \partial S)$ is a simple closed multi-curve relative to the boundary. Assume that $\phi = \delta \circ \eta$, where η leaves τ point-wise fixed, and δ is a Dehn twist on a simple-closed curve $\gamma \subset S$ whose algebraic intersection with τ is zero.

Theorem 7 ([9]). *There is a sequence $(S_k, \phi_k) \in \mathcal{P}$ such that*

1. *the Euler characteristic of S_k is mk for some $m < 0$,*
2. *$\phi_k = r_k\widehat{\phi}$, where $\widehat{\phi}$ has support on a subsurface of S_k whose homeomorphism type is independent of k and r_k is periodic of period k, and*
3. *(S_k, ϕ_k) converge to (S, ϕ) in $\mathcal{P}$.*

The sequence (S_k, ϕ_k) generalize Penner-sequence, and by continuity of L the sequence of normalized dilatations $L(S_k, \phi_k)$ converges to $L(S, \phi)$,

Twisted mapping classes. Let P_m be a closed $2m$-gon with alternate sides removed. Let (S_1, ϕ_1) and (S_2, ϕ_2) be two mapping classes with proper embeddings $P_m \subset S_i$, for $i = 1, 2$. Then the *Murasugi sum* of (S_1, ϕ_1) and (S_2, ϕ_2) equals (S, ϕ), where S is the result of gluing S_1 and S_2 along the corresponding mages of P_m and ϕ is the composition of the extensions of ϕ_1 and ϕ_2 by the identity on S.

In [8], we show the following.

Lemma 8. *For each m, there is a family of mapping classes (Σ_k, σ_k) so that*

1. *σ_k^{mk} is a composition of Dehn twists centered at boundary components of Σ_k,*

(2) *there exist mk disjoint embedded copies of P_m in Σ_k, and*
(3) *the mapping tori of (Σ_k, σ_k) are independent of k.*

The surfaces Σ_k constructed in [8] come with a distinguished proper embedding of P_m. Let (S_0, ϕ_0) be any mapping class with a proper embedding of P_m in S_0. Let (S_k, ϕ_k) be the mapping classes obtained by Murasugi sum of $(S_0.\phi_0)$ with (Σ_k, σ_k) along P_m.

Lemma 9. *For any choice of (S_0, ϕ_0), the mapping classes (S_k, ϕ_k) correspond to a convergent sequence on a fibered face (possibly converging to the boundary).*

Theorem 10 ([8])**.** *There exists (S_0, ϕ_0) so that (S_k, ϕ_k) are orientable pseudo-Anosov mapping classes with unbounded genus that converge to a point in the interior of a fibered facem and whose normalized dilatations converge to ℓ_0.*

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Small covolume and growth of hyperbolic Coxeter groups

Ruth Kellerhals

Consider a hyperbolic n-orbifold, that is, a quotient of $\mathbb{H}^n$ by a discrete group of isometries of $\mathbb{H}^n$. Simplest examples are orbit spaces of *hyperbolic Coxeter groups* which are groups generated by finitely many reflections with respect to hyperplanes in $\mathbb{H}^n$. In the case of few generators, such groups as well as their fundamental polytopes are most conveniently represented by their Coxeter graph ([5], for example). We are interested in describing hyperbolic n-orbifolds of finite volume, compact or non compact, arithmetically defined or not, by means of characteristic invariants such as volume, Euler characteristic, growth rate of the fundamental group, length of a shortest closed simple geodesic, small eigenvalues of the Laplacian, and so on.

For small n, minimal volume hyperbolic n-orbifolds are identified. For example, by a well-known result of C. L. Siegel, the quotient of $\mathbb{H}^2$ by the triangle group $(2,3,7)$ has minimal volume among *all* hyperbolic 2-orbifolds. The 1-cusped quotient space $\mathbb{H}^3/G_\infty$, where G_∞ is the tetrahedral Coxeter group $(3,3,6)$ with Coxeter graph

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\overset{6}{-\!\!\!-\!\!\!-}\!\!\bullet \quad ,$$

has minimal volume among *all non-compact* hyperbolic 3-orbifolds [7]. For corresponding results about minimal volume cusped hyperbolic n-orbifolds with $4 \leq n \leq 9$, see [2] and [3]. Recently, Gehring, Marshall and Martin [6] (see also [1]) completed their work proving that the oriented double cover of the quotient of $\mathbb{H}^3$ by the $\mathbb{Z}_2$-extension of the Coxeter group $(3,5,3)$ with graph

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\overset{5}{-\!\!\!-\!\!\!-}\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

has minimal volume among *all oriented* hyperbolic 3-orbifolds which was known before in the arithmetic case, only.

Due to the apparent importance of hyperbolic Coxeter groups with few generators and finite covolume, we study these groups with respect to some of their relevant algebraic features. More specifically, consider a cofinite hyperbolic Coxeter group $G = (G, S)$ generated by a finite set S of reflections. Its *growth series* is given by

$$f_S(x) = 1 + |S|\,x + \sum_{k\geq 2} a_k x^k \quad ,$$

where a_k is the number of words in G of S-length k, and which is the series expansion of a rational function $p(x)/q(x)$ with coprime polynomials p, q defined over $\mathbb{Z}$. Notice that the value $1/f_S(1)$ is proportional to the Euler characteristic $\chi(G)$, and proportional to the covolume of $G \subset \mathrm{Isom}(\mathbb{H}^n)$ if n is even. The *growth rate* τ_G is given by the reciprocal of the radius of convergence R of $f_S(x)$. It is known that $\tau_G > 1$ is a root of maximal absolute value of $q(x)$ and an algebraic integer. As such τ_G is an interesting object and closely related to Salem numbers, Pisot numbers and Perron numbers. Recently, in [5], we proved the following result.

Theorem. *Among all hyperbolic Coxeter groups with non-compact fundamental polyhedron of finite volume in* $\mathbb{H}^3$*, the tetrahedral group* $G_\infty = (3,3,6)$ *has minimal growth rate, and as such the group is unique.*

The above result completes the picture of growth rate minimality for cofinite hyperbolic Coxeter groups in three dimensions. Indeed, in collaboration with A.

Kolpakov [4], we showed that the growth rate of the Coxeter group $(3, 5, 3)$ is minimal among all growth rates (being Salem numbers) of Coxeter groups acting *cocompactly* on $\mathbb{H}^3$.

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On homology growth of finite covering

Thang Lê

1. Torsion Growth and volume

1.1. **Sequence of subgroups and limits.** Suppose π is a finitely presented group. Any finite set $S = \{s_1, \dots, s_n\}$ of generators of π defines a metric on π. For a subgroup G of π, let

$$d_S(G) := \min\{\ell_S(x), x \in G \setminus \{1\}\},$$

where ℓ_S is the word length of x in the metric defined by S.

Suppose that f is a function defined on a set D of subgroups of π. We say that

$$\lim_{G \to \infty, G \in D} f(G) = L$$

if for some finite set of generators S one has

$$\lim_{d_S(G) \to \infty, G \in D} f(G) = L. \tag{1}$$

It is easy to see that (1) holds if and only if it holds when S is replaced by any other finite set of generators. One define $\limsup_{G \to \infty, G \in D} f(G)$ similarly.

Note that if $\lim_{k \to \infty} d_S G_k = \infty$ then $\cap G_k = \{1\}$, i.e. $\{G_k\}$ is co-final. The converse is not true (true if G_k is nested).

1.2. **A volume conjecture.** Let X be an irreducible, orientable 3-manifold with boundary either empty or union of tori. By the JSJ decomposition and Thurston-Perelman geometrization, one can cut X along some embedded tori such that the result consists of several pieces, each is either Seifert fibered or hyperbolic. Defined $\mathrm{Vol}(K)$ as the sum of the hyperbolic volumes of the hyperbolic pieces. Another way to define $\mathrm{Vol}(K)$ is to use the Gromov norm.

Let $\pi = \pi_1(X)$, the fundamental group. For a finite-index normal subgroups of G of π let X_G be the of X corresponding to G. We are interested in the asymptotics of the homology of X_G when $G \to \infty$. It follows from a result of Kazhdan and Lück [Lu] that

$$\lim_{G\to\infty, |\pi:G|<\infty} \frac{b_1(X_G)}{[\pi : G]} = 0.$$

Here b_1 is the rank of $H_1(X_G, \mathbb{Z})$. Hence we will look at the torsion:

$$t(X, G) := |\mathrm{Tor} H_1(X_G, \mathbb{Z})|.$$

Theorem 1. *One has*

$$\limsup_{G\to\infty, |\pi:G|<\infty} t(X, G)^{1/[\pi:G]} \leq \exp(\frac{1}{6\pi}\mathrm{Vol}(X)).$$

We suggest the following conjecture (circa 2007).

Conjecture 2 (See also [Le2])**.** *One has*

$$\limsup_{G\to\infty, |\pi:G|<\infty} t(X, G)^{1/[\pi:G]} = \exp(\frac{1}{6\pi}\mathrm{Vol}(X)).$$

Theorem 1 says that the left hand side is less than or equal to the right hand side. It follows that Conjecture 2 holds true if $\mathrm{Vol}(X) = 0$, i.e. when X is a graph manifold.

Remark 3. *Similar, slightly different, conjectures were also formulated by Lück and Begeron-Venkatesh.*

2. Abelian case

Let $\mathcal{C}$ be a finite free complex over the ring $\Lambda = \Lambda_n := \mathbb{Z}[\mathbb{Z}^n] \equiv \mathbb{Z}[t^{\pm 1}, \dots, t^{\pm n}]$. For a finite index subgroup $G < \mathbb{Z}^n$ let

$$t_j(G) = |\mathrm{Tor}_{\mathbb{Z}} (H_j(\mathcal{C} \otimes_\Lambda \mathbb{Z}[\mathbb{Z}^n/G]))|.$$

Suppose $f(t_1^{\pm 1}, \dots, t_n^{\pm 1}) \in \mathbb{C}[\mathbb{Z}^n] \equiv \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Assume $f \neq 0$. The Mahler measure of f is defined by

$$\mathrm{Mah}(f) := \int_{\mathcal{T}^n} \log |f| d\sigma$$

where $\mathcal{T}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| = 1\}$, the n-torus, and $d\sigma$ is the invariant measure normalized so that $\int_{\mathcal{T}^n} d\sigma = 1$.

For a finitely generated Λ-module M with a presentation

$$\Lambda^k \xrightarrow{A} \Lambda^l \twoheadrightarrow M \to 0,$$

where A is an $k \times l$ matrix with entry in Λ, let $\Delta_j(M)$ be the greatest common divisor of all $(l-j) \times (l-j)$ minor of A. Then $\Delta_{j+1} | \Delta_j$, and

$$\Delta_{j+r}(M) = \Delta_j(\mathrm{Tor}_\Lambda(M)),$$

where $r = \mathrm{rk}(M)$, the dimension of $M \otimes_\Lambda F(\Lambda)$ over the fractional field $F(\Lambda)$ of Λ. Moreover, $\Delta_0(M) \neq 0$ if and only if M is a torsion module, and $\Delta_j(M) = 0$ for $j < r$. Let $\Delta(M) := \Delta_r(M) = \Delta_0(\mathrm{Tor}_\Lambda(M))$, which is known as the first non-trivial Alexander polynomial of M.

Since $H_j(\mathcal{C})$, for each $j \geq 0$, is a Λ-module, one can define $\Delta(H_j(\mathcal{C}))$.

Theorem 4. *One has*

$$\limsup_{G \to \infty, G < \mathbb{Z}^n, |\mathbb{Z}^n : G| < \infty} \frac{\ln t_j(G)}{|\mathbb{Z}^n : G|} = \mathrm{Mah}(\Delta(H_j(\mathcal{C}))).$$

If $n = 1$, then one can replace $\limsup$ *by the ordinary* $\lim$.

For the case when X is the complement of a link in S^3 this proved a conjecture of Silver and Williams [SW], who proved a similar result for the case when the first Alexander polynomial of the link is non-zero.

The special case $j = 0$ of Theorem 4 can be reformulated as follows.

Theorem 5. *Suppose M is a finitely generated Λ-module. Then*

$$\limsup_{G \to \infty, G < \mathbb{Z}^n, |\mathbb{Z}^n : G| < \infty} \frac{\ln |\mathrm{Tor}_{\mathbb{Z}}(M \otimes_\Lambda \mathbb{Z}[\mathbb{Z}^n / G])}{|\mathbb{Z}^n : G|} = \mathrm{Mah}(\Delta(M)).$$

Theorem 5 was formulated as a conjecture by K. Schmidt [Sch], in another language. Schmidt proved Theorem 5 in the case when M is a Λ-torsion module, using tools from symbolic dynamical system.

There is no known direct proof of Theorem 5, even in the case when M is a torsion module. In our proof, we use Bourbaki's pseudo-isomorphism theory and a Manin-Mumford principle for sets of torsion points on algebraic sets (a result of Laurent), to reduce the case of general M to the case of Λ-torsion modules.

For $n = 1$, one can replace $\limsup$ by the ordinary $\lim$. One can replace $\limsup$ by the ordinary $\lim$ in Theorems 4 and 5 for every n if one can prove the following conjecture.

Conjecture 6. *Let $f \in \mathbb{Z}[t_1, \dots, t_n]$. There exists a positive constant E such that for every roots of unity $z_1, \dots, z_n$ of order $\leq d$, either $f(z_1, \dots, z_n) = 0$, or*

$$f(z_1, \dots, z_n) > \frac{1}{d^E}.$$

When $n = 1$, the conjecture holds true, due to Gelfond-Baker theory.

The conjecture can be easily reduced to the case when f is a linear polynomial (by increasing the number of variables), and is eventually equivalent to the following conjecture. Suppose $C_d = \{\exp(2pik/d) \mid k \in \mathbb{Z}\}$ be the set of all roots of unity of orders dividing d. Let $(C_d)^{\#m}$ be the Minkowski sum of m copies of C_d, i.e. $(C_d)^{\#m} = \{x_1 + \cdots + x_m \mid x_j \in C_d\}$. For a finite subset $A \subset \mathbb{C}$ let $\min_0(A) = \min_{x \in A \setminus \{0\}} |x|$.

Conjecture 7. *For a fixed positive integer m, there is a constant $E = E(m)$ such that as $d \to \infty$,*

$$\min_0((C_d)^{\#m}) > \frac{1}{d^E}.$$

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Homological growth and L^2-invariants

Wolfgang Lück

Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $[G : G_i]$ is finite for all $i \in I$ and $\bigcap_{i \in I} G_i = \{1\}$. Let K be a field. We denote by d the minimal number of generators, by $\rho^{\mathbb{Z}}$ the integral torsion, by $b_n^{(2)}$ the n-th L^2-Betti number, and by $\rho^{(2)}$ the L^2-torsion. The starting point of this talk is the following result by Lück [11]).

Theorem: Let X be a finite connected CW-complex and let $\overline{X} \to X$ be a G-covering. Then

$$b^{(2)}(\overline{X}) = \lim_{i \to \infty} \frac{b_n(G_i \backslash \overline{X}; \mathbb{Q})}{[G : G_i]}.$$

The analogous result for signatures and η-invariants has been proved by Lück-Schick [15].

Meanwhile the question has occurred whether a result like this is true also for characteristic p. Then the theory of von Neumann algebras is not available anymore. A partial result is given by Linnell-Lück-Sauer [10].

Theorem: Let X be a finite connected CW-complex and let $\overline{X} \to X$ be a

G-covering. Suppose that G is torsionfree and elementary amenable. Then one can assign to a $\mathbb{F}_pG$-module M its Ore dimension $\dim_{\mathrm{Ore}}$ and one has

$$\dim_{\mathrm{Ore}}\bigl(H_n(\overline{X};\mathbb{F}_p)\bigr) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\overline{X};\mathbb{F}_p)}{[G:G_i]};$$

The following result is taken from Bergeron-Lück-Sauer [2] which also follows from the methods in Calegari-Emerton [4].

Theorem: Let X be a finite connected CW-complex and let $\overline{X}\to X$ be a G-covering. Let p be a prime, let n be a positive integer, and let $\phi\colon G\to GL_n(\mathbb{Z}_p)$ be an injective homomorphism. The closure of the image of ϕ, which is denoted by Γ, is a p-adic analytic group admitting an exhausting filtration by open normal subgroups $\Gamma_i = \ker\bigl(\Gamma\to GL_n(\mathbb{Z}/p^i\mathbb{Z})\bigr)$. Let $d=\dim(\Gamma)$. Set $G_i=\phi^{-1}(\Gamma_i)$.

Then, for any integer n we have

$$b_n(\overline{X}/G_i) = \beta_n(\overline{X},\overline{\Gamma})[\Gamma:\Gamma_i] + O\left([\Gamma:\Gamma_i]^{1-1/d}\right),$$

where $\beta_n(\overline{X},\overline{\Gamma})$ is a certain Betti number defined in terms of the Iwasawa algebra $K[[\Gamma]]$.

The case $n=1$ is of special interest for group theory. For instance the following conjecture is open.

Conjecture: Let G be finitely presented. Then the limit $\lim_{i\in I}\frac{b_1(G_i;K)}{[G:G_i]}$ exist for all systems $(G_i)_{i\in I}$ with $\bigcap_{i\in I} G_i=\{1\}$ and fields K and is independent of the choice of $(G_i)_{i\in I}$ and K.

Abért-Nikolov [1, Theorem 3] have shown for a finitely presented residually finite group G which contains a normal infinite amenable subgroup that the conjecture above is true in these cases.

The conjecture above is not true if we drop the condition that the system $\{G_i \mid i\in I\}$ has non-trivial intersection, as an example by Lück [13] shows. It also fails if we weaken the condition "finitely presented" to "finitely generated", see Lück-Osin [14] and Ershof-Lück [5]

The questions above is related to questions of Gaboriau (see [6, 7, 8]), whether every essentially free measure preserving Borel action of a group has the same cost, and whether the difference of the cost and the first L^2-Betti number of a measurable equivalence relation is always equal to 1.

The following two conjectures are motivated by [3, Conjecture 1.3] and [12, Conjecture 11.3 on page 418 and Question 13.52 on page 478].

Conjecture: (*Approximation Conjecture for L^2-torsion*)

Let X be a finite connected CW-complex and let $\overline{X}\to X$ be a G-covering.

(1) If the G-CW-structure on $\overline{X}$ and for each $i\in I$ the CW-structure on $G_i\backslash\overline{X}$ come from a given CW-structure on X, then

$$\rho^{(2)}(\overline{X}) = \lim_{i\to\infty}\frac{\rho(G_i\backslash\overline{X})}{[G:G_i]};$$

(2) If X is a closed Riemannian manifold and we equip $G_i\backslash\overline{X}$ and $\overline{X}$ with the induced Riemannian metrics, one can replace the torsion in the equality appearing in (1) by the analytic versions;

(3) If $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n \geq 0$, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}(G_i\backslash\overline{X})}{[G:G_i]}.$$

Conjecture (*Homological growth and L^2-torsion for aspherical closed manifolds*) Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Then

(1) For any natural number n with $2n \neq d$ we have

$$b_n^{(2)}(M;\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = 0.$$

If $d = 2n$ is even, we get

$$b_n^{(2)}(M;\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = (-1)^n \cdot \chi(M) \geq 0;$$

(2) For any natural number n with $2n+1 \neq d$ we have

$$\lim_{i\in I} \frac{\ln\left(\left|\operatorname{tors}\bigl(H_n(G_i\backslash M)\bigr)\right|\right)}{[G:G_i]} = 0.$$

If $d = 2n+1$, we have

$$\lim_{i\in I} \frac{\ln\left(\left|\operatorname{tors}\bigl(H_p(G_i\backslash M)\bigr)\right|\right)}{[G:G_i]} = (-1)^n \cdot \rho^{(2)}\bigl(M;\mathcal{N}(G)\bigr) \geq 0.$$

Some evidence for the two conjectures above comes from results of Koch-Lück [9] for graphs and the the following result of Lück [13].

Theorem Let M be an aspherical closed manifold with fundamental group $G = \pi_1(X)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$ that the sequences $\frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]}$, $\frac{\operatorname{mg}\left(H_n(G_i\backslash M)\right)}{[G:G_i]}$, $\frac{\ln\left(\left|\operatorname{tors}\left(H_n(G_i\backslash M)\right)\right|\right)}{[G:G_i]}$, $\frac{\rho^{(2)}\left(G_i\backslash\overline{X};\mathcal{N}(\{1\})\right)}{[G:G_i]}$, and $\frac{\rho^{\mathbb{Z}}\left(G_i\backslash\overline{X}\right)}{[G:G_i]}$ converge to zero, and we have $b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) = \rho^{(2)}(\widetilde{M};\mathcal{N}(G)) = 0$.

In particular the two conjectures above are true.

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All finite groups are involved in the Mapping Class Group

Gregor Masbaum
(joint work with Alan W. Reid)

Let Γ_g denote the orientation-preserving Mapping Class Group of the genus g closed orientable surface.

A group H is *involved* in a group G if there exists a finite index subgroup $K < G$ and an epimorphism $K \twoheadrightarrow H$. The question as to whether every finite group is involved in Γ_g was raised by U. Hamenstädt in her talk at the 2009 Georgia Topology Conference. This was known to be the case in genus $g = 1$ and $g = 2$, but in genus $g \geq 3$ Hamenstädt's question was open. The main result of our joint work [9] is the following.

Theorem 1 ([9])**.** *For all $g \geq 1$, every finite group is involved in Γ_g.*

When $g = 1$, $\Gamma_1 \cong \mathrm{SL}(2, \mathbf{Z})$ and in this case the result follows since $\mathrm{SL}(2, \mathbf{Z})$ contains free subgroups of finite index (of arbitrarily large rank). For the case of $g = 2$, it is known that Γ_2 is *large* [6]; that is to say, Γ_2 contains a finite index subgroup that surjects a free non-abelian group, and again the result follows. In genus $g \geq 3$, one cannot argue in this way, as it is not known whether Γ_g is large. In fact, if $g \geq 3$, it is not even known whether Γ_g contains a finite index subgroup that surjects $\mathbf{Z}$.

Let us assume $g \geq 3$ from now on. Although Γ_g is well-known to be residually finite [5], and therefore has a rich supply of finite quotients, apart from those finite quotients obtained from

$$\Gamma_g \twoheadrightarrow \mathrm{Sp}(2g, \mathbf{Z}) \twoheadrightarrow \mathrm{Sp}(2g, \mathbf{Z}/N\mathbf{Z})$$

very little seems known explicitly about what finite groups can arise as quotients of Γ_g (or of subgroups of finite index). Note that one cannot expect to prove Theorem 1 simply using the subgroup structure of the groups $\mathrm{Sp}(2g, \mathbf{Z}/N\mathbf{Z})$. The reason for this is that since $\mathrm{Sp}(2g, \mathbf{Z})$ has the Congruence Subgroup Property [1], it is well-known that not all finite groups are involved in $\mathrm{Sp}(2g, \mathbf{Z})$ (see [8] Chapter 4.0 for example).

Our main new idea to prove Theorem 1 and thus to answer Hamenstädt's question, was to exploit the unitary representations of mapping class groups arising in Topological Quantum Field Theory (TQFT) first constructed by Reshetikhin and Turaev [11]. We actually use the so-called SO(3)-TQFT following the skein-theoretical approach of [2] and its Integral TQFT refinement [4].

Using these TQFT representations, we prove the following result which gives many new finite simple groups of Lie type as quotients of Γ_g. Let $\mathbf{F}_q$ denote a finite field of order q.

Theorem 2 ([9])**.** *For each $g \geq 3$, there exist infinitely many N such that for each such N, there exist infinitely many primes q such that Γ_g surjects* $\mathrm{PSL}(N, \mathbf{F}_q)$.

Theorem 1 follows easily from Theorem 2 (see [9]).

In addition we show that Theorem 2 also holds for the Torelli group (with $g \geq 2$).

A proof of these results was also given by Funar [3].

We briefly indicate the strategy of the proof of Theorem 2. The unitary representations that we consider are indexed by primes p congruent to 3 modulo 4. For each such p we use Integral SO(3)-TQFT [4] to exhibit a group Δ_g which is the image of a certain central extension $\widetilde{\Gamma}_g$ of Γ_g and satisfies

$$\Delta_g \ \subset \mathrm{SL}(N_p, \mathbf{Z}[\zeta_p]) \ ,$$

where ζ_p is a primitive p-th root of unity, and $\mathbf{Z}[\zeta_p]$ is the ring of integers in $\mathbf{Q}(\zeta_p)$. Moreover, the dimension $N_p \to \infty$ as we vary p. In fact, N_p is the dimension of the SO(3)-TQFT vector space (with quantum parameter $q = \zeta_p$) associated to the genus g surface.

The key part of the proof is the following. We use strong approximation in the form proved by Weisfeiler [12] (see also Nori [10]) and a density result for the SO(3)-TQFT-representations due to Larsen and Wang [7] to exhibit infinitely many rational primes q, and prime ideals $\tilde{q} \subset \mathbf{Z}[\zeta_p]$ satisfying

$$\mathbf{Z}[\zeta_p]/\tilde{q} \simeq \mathbf{F}_q \ ,$$

for which the reduction homomorphism

$$\mathrm{SL}(N_p, \mathbf{Z}[\zeta_p]) \twoheadrightarrow \mathrm{SL}(N_p, \mathbf{F}_q)$$

(induced by the isomorphism $\mathbf{Z}[\zeta_p]/\tilde{q} \simeq \mathbf{F}_q$) restricts to a surjection

$$\Delta_g \twoheadrightarrow \mathrm{SL}(N_p, \mathbf{F}_q) \ .$$

From this, it is then easy to get surjections

$$\Gamma_g \twoheadrightarrow \mathrm{PSL}(N_p, \mathbf{F}_q) \ ,$$

which will complete the proof of Theorem 2.

For more details about how all this is achieved, see [9].

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Johnson maps in non-Abelian Iwasawa theory

Masanori Morishita

1. Introduction. This is the joint work with Yuji Terashima. We propose an approach to non-Abelian Iwasawa theory, following the idea of Johnson maps in low dimensional topology.

We fix an odd prime number p throughout this report. Let $k_\infty := \mathbb{Q}(\sqrt[p^\infty]{1})$ be the field obtained by adjoining all p-power roots of unity to the rationals $\mathbb{Q}$ and let $\tilde{k}$ be the maximal pro-p extension of k_∞ which is unramified outside p. A basic problem of non-Abelian Iwasawa theory is then to study the conjugate action of $\Gamma_p := \mathrm{Gal}(k_\infty/\mathbb{Q})$ on $F_p := \mathrm{Gal}(\tilde{k}/k_\infty)$, while the classical Iwasawa theory deals with the action of Γ_p on the Abelianization $F_p^{\mathrm{ab}} = H_1(F_p, \mathbb{Z}_p)$. In terms of the standard algebraic geometry, one has the tower of étale pro-finite covers

$$\tilde{X}_p := \mathrm{Spec}(\mathcal{O}_{\tilde{k}}[1/p]) \to X_p^\infty := \mathrm{Spec}(\mathbb{Z}[\sqrt[p^\infty]{1}, 1/p]) \to X_p := \mathrm{Spec}(\mathbb{Z}[1/p]) \tag{1.1}$$

with Galois groups

$$\Gamma_p = \mathrm{Gal}(X_p^\infty/X_p), \quad F_p = \mathrm{Gal}(\tilde{X}_p/X_p^\infty). \tag{1.2}$$

Now, based on the analogy between a knot and a prime ([Ms])

$$\begin{array}{ccc} \text{knot} & \longleftrightarrow & \text{prime} \\ K : S^1 = K(\mathbb{Z}, 1) \hookrightarrow \mathbb{R}^3 & & \mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}), \end{array}$$

the topological counterpart of (1.1) and (1.2) may be the tower of covers

$$\tilde{X}_K \to X_K^\infty \to X_K := \mathbb{R}^3 \setminus K \tag{1.3}$$

and Galois groups

$$\Gamma_K := \mathrm{Gal}(X_K^\infty/X_K), \quad F_K := \mathrm{Gal}(\tilde{X}_K/X_K^\infty), \tag{1.4}$$

where X_K^∞ is the infinite cyclic cover of X_K and $\tilde{X}_K$ is the universal cover of X_K. To push our idea further, suppose K is fibered so that X_K is the mapping torus of the monodromy $\phi : S \to S$, S being the Seifert surface of genus g. The mapping class ϕ, a generator of Γ_K, induces the automorphism ϕ_* of $F_K = \pi_1(S)$. The theory of Johnson maps provides a framework to describe this action ([J], [Ka], [Ki], [Mt]).

In the following, we shall introduce arithmetic analogues of the Johnson maps and use them for non-Abelian Iwasawa theory.

2. Pro-p Johnson maps. Let F be a free pro-p group on $x_1, \ldots, x_r$, and let $H = F^{\mathrm{ab}} = \mathbb{Z}_p^r$ be the Abelianization of F. We let $[f] := f \bmod [F, F]$. Let $T = T(H)$ be the complete tensor algebra on H, $T = \prod_{m \geq 0} H^{\otimes m}$, which is identified with the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p\langle\langle X_1, \ldots, X_r\rangle\rangle$ of non-commutative power series, where $X_j = [x_j]$ $(1 \leq j \leq r)$. Let $T_n := \prod_{m \geq n} H^{\otimes m}$ be the two-sided ideal of T made up by power series of degree $\geq n$. A $\mathbb{Z}_p$-algebra automorphism φ of T is called filtration-preserving if $\varphi(T_n) = T_n$ for all $n \geq 0$ and we denote by $\mathrm{Aut}^{\mathrm{fil}}(T)$ the group of filtration-preserving $\mathbb{Z}_p$-algebra automorphisms of T. Each $\varphi \in \mathrm{Aut}^{\mathrm{fil}}(T)$ induces a $\mathbb{Z}_p$-module automorphism of $H = T_1/T_2$, by which we denote $[\varphi]$. Note that the homomorphism $\mathrm{Aut}^{\mathrm{fil}}(T)\ ni \varphi \mapsto [\varphi] \in \mathrm{GL}(H)$ splits. (The splitting $\iota : \mathrm{GL}(H) \to \mathrm{Aut}^{\mathrm{fil}}(T)$ is given by $\iota([\varphi])(t_m) = ([\varphi]^{\otimes m}(t_m))$ $(t_m \in H^{\otimes m})$.) Set $\mathrm{IA}(T) := \mathrm{Ker}(\mathrm{Aut}^{\mathrm{fil}}(T) \to \mathrm{GL}(H))$.

Lemma 2.1. (1) *One has an isomorphism* $\mathrm{Aut}^{\mathrm{fil}}(T) \simeq \mathrm{IA}(T) \rtimes \mathrm{GL}(H)$ *given by* $\varphi \mapsto (\varphi \circ [\varphi]^{-1}, [\varphi])$.
(2) *One has a bijection* $\mathrm{IA}(T) \simeq \mathrm{Hom}(H, T_2)$ *given by* $\varphi \mapsto \varphi|_H - \mathrm{id}_H$.

Let $\mathbb{Z}_p[[F]]$ be the complete group algebra of F over $\mathbb{Z}_p$ with augmentation ideal I. The Magnus expansion $\theta : F \hookrightarrow T^\times$ defined by $\theta(x_j) = 1 + X_j$ is extended to a $\mathbb{Z}_p$-algebra isomorphism $\hat{\theta} : \mathbb{Z}_p[[F]] \xrightarrow{\sim} T$, which satisfies $\hat{\theta}(I^n) = T_n$ for all n.

Now, let $\phi \in \mathrm{Aut}(F)$. Then ϕ induces a $\mathbb{Z}_p$-algebra automorphism $\hat{\phi}$ of $\mathbb{Z}_p[[F]])$ satisfying $\hat{\phi}(I^n) = I^n$. We then define the *extended pro-p Johnson homomorphism* by

$$\hat{\tau} : \mathrm{Aut}(F) \longrightarrow \mathrm{Aut}^{\mathrm{fil}}(T); \;\; \hat{\tau}(\phi) := \hat{\theta} \circ \hat{\phi} \circ \hat{\theta}^{-1}. \tag{2.2}$$

Noting $[\hat{\theta} \circ \hat{\phi} \circ \hat{\theta}^{-1}] = [\phi]$, we let $(\tau(\phi), [\phi])$ be the pair in $\mathrm{IA}(T) \rtimes \mathrm{GL}(H)$ which corresponds to $\hat{\tau}(\phi)$ under the isomorphism of Lemma 2.1 (1). Thus we have a map

$$\tau : \mathrm{Aut}(F) \longrightarrow \mathrm{IA}(T) \tag{2.3}$$

which we call the *pro-p Johnson map*. Composing τ with $\mathrm{IA}(T) \stackrel{\sim}{\to} \mathrm{Hom}(H, T_2) \to \mathrm{Hom}(H, H^{\otimes})$ $(m \geq 2)$, where the 1st map is the bijection of Lemma 2.1 (2) and the second is the map induced by the projection $T_2 \to H^{\otimes m}$, we have the *m-th pro-p Johnson map*

$$\tau_m : \mathrm{Aut}(F) \longrightarrow \mathrm{Hom}(H, H^{\otimes m}) \;\; (m \geq 2). \tag{2.4}$$

Let $F = F_1 \supset \cdots \supset F_m := [F_{m-1}, F] \supset \cdots$ be the lower central series of F, and let $\mathrm{Aut}_m(F) := \mathrm{Ker}(\mathrm{Aut}(F) \to \mathrm{Aut}(F/F_m))$ for $m \geq 2$.

Proposition 2.5. *The restriction of τ_m to $\mathrm{Aut}_m(F)$ is a homomorphism given by $\tau_m(\phi)([f]) = \theta(\phi(f)f^{-1})$ mod T_{m+1} for $f \in F$.*

3. Non-Abelian Iwasawa theory. Let k be a number field of finite degree over $\mathbb{Q}$. Let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k with $\Gamma := \mathrm{Gal}(k_\infty/k) = \langle \gamma \rangle \simeq \mathbb{Z}_p$. Let M/k_∞ be a subextension of the maximal, unramified outside p, pro-p extension $\tilde{k}$ of k_∞ such that M/k is a Galois extension. Set $F := \mathrm{Gal}(M/k_\infty)$ and $G := \mathrm{Gal}(M/k)$. Take a section (lift) $\Gamma \to G$ and then Γ acts on F via conjugation, $\Gamma \to \mathrm{Aut}(F); \gamma \mapsto \phi_\gamma$. Now we suppose that F is a free pro-p group on $x_1, \ldots, x_r$ in order to apply the tools in Section 2. This assumption is satisfied in the following cases:

- k is totally real and $M = \tilde{k}$ with the Iwasawa μ-invariant $\mu(F^{\mathrm{ab}}) = 0$ ([W1]).
- $k \ni \sqrt[p]{1}$ and M is the maximal, unramified outside S, positively ramified over S_p, pro-p extension of k, where S is a finite set of primes of k containing properly the set S_p of primes over p. The Iwasawa μ-invariant of k_∞ is assumed to be 0 ([W2], [S]).

Now, let $\hat{\tau} : \mathrm{Aut}(F) \to \mathrm{Aut}^{\mathrm{fil}}(T)$ be the extended pro-p Johnson map and let $\tau_m : \mathrm{Aut}(F) \to \mathrm{Hom}(H, H^{\otimes m})$ be the m-th pro-p Johnson map $(m \geq 2)$. We propose the following arithmetic invariants derived from Johnson maps.

(3.1) Let $[\phi_\gamma]_n$ be the $\mathbb{Z}_p$-module automorphism of T_n/T_{n+1} induced by $\hat{\tau}(\phi_\gamma)$ for $n \geq 1$. We then define the *n-th Iwasawa polynomial* by

$$L_n(T) := \det(1 + T - [\phi_\gamma]_n \,|\, (T_n/T_{n+1}) \otimes \mathbb{Q}_p).$$

Note that $L_1(T)$ is nothing but the classical Iwasawa polynomial (p-adic L-function).

(3.2) For $f \in F$ and $m \geq 2$, we write

$$\tau_m(\phi_\gamma)([f]) = \sum_{1 \leq i_1, \ldots, i_m \leq r} \tau(i_1 \cdots i_m; [f]) X_{i_1} \cdots X_{i_m}.$$

Similarly, denoting by $\hat{\tau}_m(\phi_\gamma)([f])$ the degree m-part of $\hat{\tau}(\phi_\gamma)([f])$, we write

$$\hat{\tau}_m(\phi_\gamma)([f]) = \sum_{1\le i_1,\ldots,i_m\le r} \hat{\tau}(i_1\cdots i_m;[f])X_{i_1}\cdots X_{i_m}.$$

These coefficients $\tau(i_1\cdots i_m;[f]), \hat{\tau}(i_1\cdots i_m;[f]) \in \mathbb{Z}_p$ are numerical datum encoded in the Johnson maps.

We may write $G = \langle x_1,\ldots,x_r,y | R_j := [x_j,y]\phi_\gamma(x_j)x_j^{-1}\,(1\le j\le r)\rangle$ where the word y corresponds to (a lift of) γ. Let η_j be a homology class in $H_2(G,\mathbb{Z}_p)$ corresponding to the relator R_j and let x_j^*'s are cohomology class in $H^1(G,\mathbb{Z}_p)$ dual to x_j's.

Theorem 3.3. *One has*

$$\begin{aligned}&\hat{\tau}(i_1\cdots i_m;[x_j]) = \tau(i_1\cdots i_m;[x_j]) + \tau(i_1\cdots i_m;[\phi_\gamma(x_j)x_j^{-1}])\\ &\equiv \langle x_{i_1}^*,\ldots,x_{i_m}^*\rangle(\eta_j) \mod \Delta(i_1\cdots i_{m-1}j),\end{aligned}$$

where $\langle x_{i_1}^*,\ldots,x_{i_m}^*\rangle$ stands for the Massey product and $\Delta(i_1\cdots i_{m-1}j)$ is the ideal of $\mathbb{Z}$ generated by the Magnus coefficient of $\phi_\gamma(x_j)x_j^{-1}$ at $X_{i_1}\cdots X_{i_{m-1}}$.

This theorem may be regarded as a generalization of Kitano's result [Ki, Theorem 4.1] in the context of non-Abelian Iwasawa theory.

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On the growth of the first Betti number of arithmetic hyperbolic 3-manifolds

Joachim Schwermer

1. Arithmetically defined hyperbolic 3-manifolds

Every orientable hyperbolic 3–manifold is isometric to the quotient H^3/Γ of hyperbolic 3–space H^3 by a discrete torsion free subgroup Γ of the group $Iso(H^3)^0$ of orientation – preserving isometries of H^3. The latter group is isomorphic to the (connected) group $PGL_2(\mathbb{C})$, the real Lie group $SL_2(\mathbb{C})$ modulo its center $\{\pm Id\}$. Hyperbolic 3-space can be realized in various models. In the given framework H^3 is best described as the symmetric space attached to the real Lie group $G = SL_2(\mathbb{C})$, that is, $H^3 = K\backslash G$ where K denotes a maximal compact subgroup in $SL_2(\mathbb{C})$. By definition, a Kleinian group Γ is a discrete subgroup of the group $Iso(H^3)^0$ of orientation – preserving isometries of H^3. The group Γ is said to have finite covolume if H^3/Γ has finite volume, and is said to be cocompact if H^3/Γ is compact. If the Kleinian group Γ has torsion, then H^3/Γ is an orbifold (that is, it locally looks like the quotient of a Euclidean space by a finite group), otherwise it is a manifold.

Among hyperbolic 3–manifolds, the ones originating with arithmetically defined Kleinian groups form a class of special interest. These arithmetic Kleinian groups fall naturally into two classes, according to whether H^3/Γ is compact or not. However, this quotient always has finite volume with respect to the hyperbolic metric.

Let Γ be a discrete subgroup of $PGL_2(\mathbb{C})$. Then Γ is said to be arithmetically defined if there exist an algebraic number field $k/\mathbb{Q}$ with exactly one complex place w (that is, $t = 1$ in the usual enumeration of the places of an algebraic number field), an arbitrary (but possibly empty) set T of real places, a k–form G of the algebraic group PGL_2/k such that $G(k_v)$ is compact for $v \in T$ and an isomorphism

$$PGL_2(\mathbb{C}) \tilde{\to} G(k_w),\ w \text{ the complex place}$$

which maps Γ onto an arithmetic subgroup of $G(k)$ naturally embedded into $G(k_w)$.[1]

[1]We briefly describe all k–forms of the algebraic group PGL_2 (or SL_2) over an algebraic number field k. By definition, a linear algebraic group G defined over k is a k–form of the k–group PGL_2 (or SL_2) if there exists a field extension k'/k such that G is isomorphic as a k'–group to PGL_2/k' (or SL_2/k').

The k-forms in question can be described in the following way. Let A be a quaternion algebra over the field k, that is, A is a central simple algebra over k of degree 2. Let $GL(A)$ be the algebraic group defined over k whose rational points over an extension k'/k equal the group of invertible elements in the k'–algebra $A \otimes_k k'$. The reduced norm defines a surjective homomorphism $Nrd : GL(A) \to G_m$ of $GL(A)$ into the multiplicative group G_m over k. The kernel of the morphism Nrd is a semisimple, simply connected algebraic group over k, to be denoted $SL_1(A)$. The k–group $GL(A)$ has a one–dimensional center, and its derived group is $SL_1(A)$. Then the quotient G of $GL(A)$ by its center is a k–form of PGL_2/k. This construction exhausts all possible k–forms of PGL_2/k.

First, the case of Bianchi groups is subsumed under this construction. Given an imaginary quadratic extension k of $\mathbb{Q}$, that is, k is of the form $\mathbb{Q}(\sqrt{d})$, $d < 0$, d a square free integer, G is the split form PGL_2/k itself, that is, T is the empty set and the choice of $A = M_2(k)$ is equivalent to the specification that the ramification set $\mathrm{Ram}(A) = \emptyset$. Then a subgroup Γ of the group $G(k)$ is arithmetically defined (or an arithmetic group) if it is commensurable with the group $\Gamma_d := PGL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ denotes the ring of integers of k. As early as 1892 L. Bianchi studied this class of groups, today named after him. These groups and all their subgroups of finite index have finite covolume but are not cocompact.

Second, there are groups originating with orders in division algebras. Given an algebraic number field k with exactly one complex place and an arbitrary non-empty set T of real places we consider a k–form G of PGL_2/k which is of the form $SL_1(D)$ where D is a division quaternion algebra over k which ramifies (at least) at all real places $v \in T$. Then an arithmetically defined subgroup Γ originates with an order Λ in D. By definition, an order Λ in D is a subring of D containing the unit element 1_D which is a finitely generated $\mathcal{O}_k$-module with $k\Lambda = D$. The latter condition characterizes a full $\mathcal{O}_k$-lattice in D. Then any subgroup Γ of $G(k)$ which is commensurable with G_Λ gives rise to a compact hyperbolic 3-manifold H^3/Γ. This latter construction exhausts all possible types of arithmetically defined subgroups of $PGL_2(\mathbb{C})$ that give rise to a compact hyperbolic 3-manifold H^3/Γ.

Examples. We discuss some families of examples. Suppose that the defining field k (which has exactly one complex place) contains a subfield k' such that the degree $[k : k']$ of the extension k/k' is 2. Due to the assumption on k, k' is a totally real extension field of $\mathbb{Q}$. Let $Gal(k/k') = \{Id_k, c\}$ denote its Galois group.

Let D be a quaternion division algebra over k underlying a given inner form G'/k of $G/k = PGL_2/k$ so that the finite set S of places $v \in V$ where $G'(k_v)$ is not isomorphic to $G(k_v)$ contains T. As a quaternion division algebra D is isomorphic to its opposite algebra, the class of D is of order 2 in the Brauer group $Br(k)$ of k. In our situation at hand, given a central simple k-algebra A of degree $\deg(A)$ there is the associated central simple k'-algebra $N_{k/k'}(A)$ of degree $\deg(A)^2$, to be called the norm of the k-algebra A. This construction induces a group homomorphism

$$N_{k/k'} : Br(k) \longrightarrow Br(k'), \quad [A] \mapsto [N_{k/k'}(A)],$$

of the respective Brauer groups In our context we have to distinguish the two cases

(I) The class $[N_{k/k'}(D)]$ has order 1 in $Br(k')$
(II) The class $[N_{k/k'}(D)]$ has order 2 in $Br(k')$.

In case (I), the class of the k'-algebra $N_{k/k'}(D)$ of degree 4 is the unit element in $Br(k')$. As a consequence, $N_{k/k'}(D)$ is isomorphic to the matrix algebra $M_4(k')$, that is, the algebra splits over k'. In such a case, by using results of Albert, the quaternion algebra D posseses an involution τ of the second kind of a particular type. There exists a unique quaternion k'-subalgebra $D_0 \subset D$ such that $D = D_0 \otimes_{k'} k$ and τ is of the form $\tau = \gamma_0 \otimes c$ where γ_0 is the quaternionic conjugation.

In case (II), the k'-algebra $N_{k/k'}(D)$ of degree 4 is (up to isomorphism) of the form $M_2(Q)$ where Q is a quaternion division algebra over k'.

2. Construction of (co)-homology classes

In this subsection we discuss various approaches to construct non-trivial classes in the (co)-homology of an arithmetically defined hyperbolic 3-manifold.

Bianchi Groups. From the geometric point of view, the arithmetically defined non-compact hyperbolic 3 - manifolds of Bianchi type admit totally geodesic submanifolds. In particular, totally geodesic hypersurfaces arise as 2-dimensional components $F(\gamma)$ of the set of fixed points under the involution induced by the non-trivial Galois automorphism of the underlying imaginary quadratic extension $k/\mathbb{Q}$. Their existence made possible the construction of non-bounding cycles and eventually lead to non-vanishing results for the cohomology of Bianchi groups (see e.g. [2] [6])

Betti numbers in the compact case. A fundamental conjecture in 3-manifold theory, stated by Waldhausen in 1968, says: Given an irreducible 3-manifold M with infinite fundamental group there exists a finite cover M' of M which is Haken, that is, it is irreducible and contains an embedded incompressible surface. One knows that 3-manifolds which are virtually Haken are geometrizable. This so called virtual Haken conjecture is the source for the (even stronger) virtual positive Betti number conjecture which states within the class of hyperbolic 3-manifolds $M = H^3/\Gamma$ that there exists a finite cover M' with non-vanishing first Betti number $b_1(M')$. The following result confirms this conjecture in a specific case.

Theorem *Let $H^3/\Gamma = M$ be a compact arithmetically defined hyperbolic 3-manifold. Suppose that the defining field k contains a subfield k' so that the field extension k/k' has degree two. Then there exists a finite covering N of M with non-vanishing first Betti number $b_1(N)$.*

We refer to [9] and [10] for an overview over the various approaches (which are substantially different in nature) which lead to a proof of this result in spedific cases. However, within the realm of the theory of automorphic forms, there is a unified approach to the non-vanishing result ([5, Section 6]).

3. On the Growth of the first Betti number

Investigating the first Betti number, it is quite natural to consider its growth rate in a nested sequence $(\Gamma_i)_{i\in\mathbb{N}}$ of finite index (normal) subgroups $\Gamma_i \subset \Gamma$ (whose intersection is the identity) for a given arithmetically defined Kleinian group Γ. One defines the first Betti number gradient which is the limit of the ratio of the first Betti number $b_1(\Gamma_i)$ by the index $[\Gamma : \Gamma_i]$. This is a special case of a general concept: Let Γ be a lattice in a semi-simple real Lie group G. If $(\Gamma_i)_{i\in\mathbb{N}}$ is a nested sequence of finite index normal subgroups $\Gamma_i \subset \Gamma$ (whose intersection is the identity) one can form the quotients

$$\beta_j(\Gamma_i) = \frac{\dim H_j(\Gamma_i, \mathbb{C})}{[\Gamma : \Gamma_i]}.$$

It is known by a result of Lück [7] that the $\beta_j(\Gamma_i)$ converge to the j-th L^2-Betti number of Γ, that is, the limit $\lim_i \beta_j(\Gamma_i)$ exists for each j. The limit is non-zero if and only if the rank G of G equals the rank K of a maximal compact subgroup $K \subset G$ and $j = \frac{1}{2}\dim(G/K)$.

However, in the situation of arithmetically defined hyperbolic 3-manifolds, that is, G is the group $PGL_2(\mathbb{C})$ one has rank $G \neq$ rank K, thus, $\lim_i \beta_j(\Gamma_i) = 0$. In particular, this assertion is valid for $j = 1$. As a consequence, the sequence of first Betti numbers $b_1(\Gamma_i)$ grows sub-linearly as a function of the index $[\Gamma : \Gamma_i]$ whenever $(\Gamma_i)_{i\in\mathbb{N}}$ is a decreasing sequence of finite index normal subgroups in an arithmetically defined group $\Gamma \subset PGL_2(\mathbb{C})$. Recently there has been some progress on improved upper bounds for the growth of Betti numbers, e.g. in [1]. Our objective in joint work with Steffen Kionke is to obtain *lower bounds* for the growth of the first Betti number.

The main new result concerns a specific class of compact arithmetically defined hyperbolic 3-manifolds which originate with orders in suitable division quaternion algebras D defined over some number field E. Given an arithmetic subgroup in the algebraic group $SL_1(D)$ we show that there are a positive real number κ and a nested sequence $(\Gamma_i)_{i\in\mathbb{N}}$ of finite index subgroups $\Gamma_i \subset \Gamma$ (whose intersection is the identity) such that the first Betti number of the compact hyperbolic 3-manifold H/Γ_i corresponding to Γ_i satisfies the inequality $b_1(\Gamma_i) \geq \kappa[\Gamma : \Gamma_i]^{1/2}$ for all indices $i \in \mathbb{N}$. One obtains a similar result in the case of Bianchi groups, that is, the corresponding manifold is non-compact. In this case one can construct nested sequences such that the first Betti number grows at least as fast as $[\Gamma : \Gamma_i]^{2/3}$ up to a factor.

Theorem (joint with S. Kionke, [3]) *Let F be a totally real algebraic number field, and let E be a quadratic extension field of F so that E has exactly one complex place. Let Γ be an arithmetic subgroup in the algebraic group $SL_1(D)$ where D is a quaternion division algebra over E which belongs to case (I). Then there are a positive number $\kappa > 0$ and a nested sequence $(\Gamma_i)_{i\in\mathbb{N}}$ of torsion-free, finite index subgroups $\Gamma_i \subset \Gamma$ (whose intersection is the identity) such that the first Betti number of the compact hyperbolic 3-manifold H/Γ_i corresponding to Γ_i satisfies the inequality*

$$b_1(\Gamma_i) \geq \kappa[\Gamma : \Gamma_i]^{1/2}$$

for all indices $i \in \mathbb{N}$. Further, Γ_i is normal in Γ_1 for all $i \in \mathbb{N}$.

The proof of this result relies on the following methodological approach which goes back to the work of Rohlfs [8]: The non-trivial Galois automorphism σ of the extension E/F induces an orientation-reversing involution on the hyperbolic 3-manifold H/Γ, whenever Γ is σ-stable. In the case the extension E/F is unramified over 2 one can determine the Lefschetz number $L(\sigma, \Gamma)$ of the induced homomorphism in the cohomology of H/Γ where Γ is a suitable congruence subgroup in $SL_1(D)$. In the general case, one gets the analogous value as a lower bound for $L(\sigma, \Gamma)$. This bound is given up to sign and some power of two as

$$\pi^{-2d}\zeta_F(2)|d_F|^{3/2}\Delta(D_0) \times [K_0 : K_0(\mathfrak{a})],$$

where $\zeta_F(2)$ denotes the value of the zeta-function of F at 2, $|d_F|$ denotes the absolute value of the discriminant of F, $[K_0 : K_0(\mathfrak{a})]$ denotes a global index attached to the congruence subgroup of level $\mathfrak{a} \subseteq \mathcal{O}_F$, and $\Delta(D_0) = \prod_{\mathfrak{p}_0 \in Ram_f(D_0)} (\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{p}_0) - 1)$ depends on the set of finite places of F in which the quaternion division algebra D_0 ramifies. In turn, this bound can be used to give a lower bound for the first Betti number of the hyperbolic 3-manifold in question. This result implies that the first Betti number becomes arbitrarily large when we vary over the congruence condition since the term $[K_0 : K_0(\mathfrak{a})]$ is unbounded.

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On the cusp shape of hyperbolic knots

Yoshiyuki Yokota

Let K be a hyperbolic knot in S^3. Then, we can suppose that the holonomies of the meridian and longitude of K are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

respectively. The *cusp shape* of K is nothing but this c. A nice table for c^{-1} can be found in [1]. In the previous meeting, the author reported the following result, which is related to the leading term of the asymptotic expansion of the Kashaev invariant of knots.

Theorem 1([2]). Let K be a hyperbolic knot in S^3. Then, we can construct a potential function $V(x_1, \ldots, x_n)$ to an appropriate diagram D of K, such that the hyperbolicity equations of M are given by

$$x_\nu \frac{\partial V}{\partial x_\nu} = 2\pi\sqrt{-1}\, r_\nu, \quad r_\nu \in \mathbb{Z}.$$

Furthermore, if $x_\nu = z_\nu$ is the geometric solution, the complex volume of M is

$$V(z_1, \dots, z_n) - 2\pi\sqrt{-1}\sum_{\nu=1}^{n} r_\nu \log z_\nu \mod \pi^2.$$

This result continued to the following new result, which should be related to the sub-leading term of the asymptotic expansion of the Kashaev invariant of knots.

Theorem 2. Under the same assumption as in Theorem 1, there exists a natural deformation $V(x_1, \dots, x_n; m)$ of the potential function such that the cusp shape of K is given by

$$-2\left(\begin{vmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nn}\end{vmatrix}\right)^{-1} \begin{vmatrix} V_{00} & V_{01} & \cdots & V_{0n} \\ V_{10} & V_{11} & \cdots & V_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n0} & V_{n1} & \cdots & V_{nn}\end{vmatrix},$$

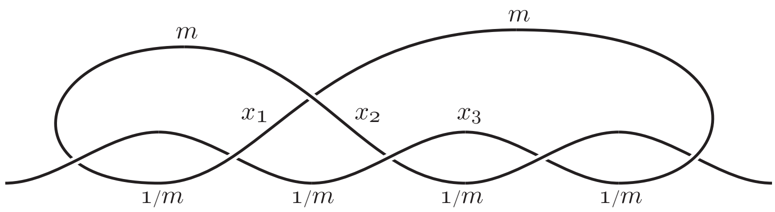
where we put $x_0 = m^2$ and

$$V_{ij} = \left(x_j x_i \frac{\partial^2 V}{\partial x_j \partial x_i}\right)(z_1, \dots, z_n; 1).$$

Remark. $V(x_1, \dots, x_n; m)$ is related to an incomplete hyperbolic structure of M, where the holonomies of the meridian and longitude become

$$\begin{pmatrix} m & 1 \\ 0 & m^{-1}\end{pmatrix}, \quad \begin{pmatrix} \ell & (\ell - \ell^{-1})/(m - m^{-1}) \\ 0 & \ell^{-1}\end{pmatrix}.$$

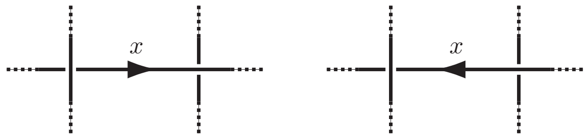
Example. Suppose K is represented by the following diagram.



Then, the potential function $V(x_1, x_2, x_3; m)$ is given by

$$\begin{aligned} &- \mathrm{Li}_2(1/mx_1) + \mathrm{Li}_2(x_1/m) - \mathrm{Li}_2(x_1/x_2) + \mathrm{Li}_2(m/x_2) \\ &- \mathrm{Li}_2(mx_2) + \mathrm{Li}_2(x_2/x_3) - \mathrm{Li}_2(1/mx_3) - \mathrm{Li}_2(mx_3) + \pi^2/3 \\ &+ 2\log m\,(\log 1/m - \log x_3 + \log 1/m - \log m + \log x_1 \\ &\qquad\qquad - \log 1/m + \log m - \log x_2 + \log 1/m), \end{aligned}$$

where the dilogarithm part is defined as in [2]. The logarithm part consists of the following terms which correspond to the edges of the diagram.



$2\log x \log m$ $\quad$ $-2\log x \log m$

The partial derivatives of $V(x_1, x_2, x_3; 1)$ with respect to x_1, x_2, x_3 are

$$
\begin{aligned}
x_1\frac{\partial V}{\partial x_1} &= -\log\left(1-\frac{1}{x_1}\right) - \log(1-x_1) + \log\left(1-\frac{x_1}{x_2}\right),\\
x_2\frac{\partial V}{\partial x_2} &= \log\left(1-\frac{1}{x_2}\right) - \log\left(1-\frac{x_1}{x_2}\right) + \log(1-x_2) - \log\left(1-\frac{x_2}{x_3}\right),\\
x_3\frac{\partial V}{\partial x_3} &= -\log\left(1-\frac{1}{x_3}\right) + \log\left(1-\frac{x_2}{x_3}\right) - \log(1-x_3),
\end{aligned}
$$

and the hyperbolicity equations for an ideal triangulation of M are given by

$$\frac{1-x_1/x_2}{(1-1/x_1)(1-x_1)} = \frac{(1-1/x_2)(1-x_2)}{1-x_1/x_2} = \frac{1-x_2/x_3}{(1-1/x_3)(1-x_3)} = 1$$

due to Theorem 1, where the moduli of the tetrahedra in the triangulation are

$$x_1,\ \frac{x_2}{x_1},\ \frac{1}{x_2},\ \frac{x_2}{x_3},\ x_3,\ \frac{1}{x_3}.$$

The solutions to the equations above are given by

$$\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix} 0.629714\\ 0.517119\\ -0.482881\end{pmatrix}, \begin{pmatrix} 0.87122 \mp 1.107662\sqrt{-1}\\ 2.20635 \pm 0.340852\sqrt{-1}\\ 1.20635 \pm 0.340852\sqrt{-1}\end{pmatrix}, \begin{pmatrix} -0.186078 \mp 0.874646\sqrt{-1}\\ 0.0350866 \pm 0.621896\sqrt{-1}\\ -0.964913 \pm 0.621896\sqrt{-1}\end{pmatrix},$$

each of which satisfies that

$$x_1,\ \frac{x_2}{x_1},\ \frac{1}{x_2},\ \frac{x_2}{x_3},\ x_3,\ \frac{1}{x_3} \notin \{0, 1, \infty\},$$

and the values of $V(x_1, x_2, x_3; 1)$ at these solutions are

$$-0.888787,\ -2.96077 \mp 1.53058\sqrt{-1},\ 2.58269 \mp 4.40083\sqrt{-1}$$

respectively. Therefore, the geometric solution (z_1, z_2, z_3) is the fifth one, and the complex volume of M is given by

$$2.58269 + 4.40083\sqrt{-1} \mod \pi^2$$

due to Theorem 1, see [2] for detail. Furthermore, by Theorem 2, the cusp shape of K is given by

$$-2 \cdot \frac{\begin{vmatrix} \frac{1}{4}\cdot\frac{1+z_1}{1-z_1} - \frac{1}{4}\cdot\frac{1+z_2}{1-z_2} - \frac{7}{4} & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1+z_1}{1-z_1} + \frac{1}{1-\frac{z_2}{z_1}} & -\frac{1}{1-\frac{z_2}{z_1}} & 0 \\ -\frac{1}{2} & -\frac{1}{1-\frac{z_2}{z_1}} & \frac{1}{1-\frac{z_2}{z_1}} - \frac{1+z_2}{1-z_2} - \frac{1}{1-\frac{z_3}{z_2}} & \frac{1}{1-\frac{z_3}{z_2}} \\ -\frac{1}{2} & 0 & \frac{1}{1-\frac{z_3}{z_2}} & \frac{1}{1-\frac{z_2}{z_3}} \end{vmatrix}}{\begin{vmatrix} \frac{1+z_1}{1-z_1} + \frac{1}{1-\frac{z_2}{z_1}} & -\frac{1}{1-\frac{z_2}{z_1}} & 0 \\ -\frac{1}{1-\frac{z_2}{z_1}} & \frac{1}{1-\frac{z_2}{z_1}} - \frac{1+z_2}{1-z_2} - \frac{1}{1-\frac{z_3}{z_2}} & \frac{1}{1-\frac{z_3}{z_2}} \\ 0 & \frac{1}{1-\frac{z_3}{z_2}} & \frac{1}{1-\frac{z_2}{z_3}} \end{vmatrix}}$$

which is numerically equal to

$$-6.74431 + 3.49859\sqrt{-1}.$$

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Gluing equations for PGL(n, $\mathbb{C}$)-representations

Christian Zickert

(joint work with Stavros Garoufalidis, Matthias Goerner, Dylan Thurston)

Thurston's gluing equations were developed to explicitly compute a hyperbolic structure on a compact 3-manifold M with a topological ideal triangulation $\mathcal{T}$. The gluing equations have the form

$$\prod_j z_j^{A_{ij}} \prod_j (1-z_j)^{B_{ij}} = 1, \tag{1}$$

where A and B are matrices whose columns are parametrized by the simplices of $\mathcal{T}$. Each variable z_j may be thought of as an assignment of an ideal simplex shape to a simplex of $\mathcal{T}$. The gluing equations have many interesting properties including

(a) The symplectic property of the exponent matrix $(A|B)$ of the gluing equations due to Neumann and Zagier [4].

(b) The link to PGL(2, $\mathbb{C}$) representations via a *developing map*

$$V_2(\mathcal{T}) \to \{\rho\colon \pi_1(M) \to \mathrm{PGL}(n,\mathbb{C})\}/\mathrm{Conj}$$

where $V_2(\mathcal{T})$ denotes the affine variety of solutions in $\mathbb{C} \setminus \{0,1\}$ to the gluing equations.

In [2] we define shape coordinates for $\mathrm{PGL}(n,\mathbb{C})$-representations that satisfy gluing equations of a form similar to (1). Both properties above still hold. Among the interesting new features of the higher gluing equations are

- They give rise to new quantum invariants.
- There is remarkable duality between the shape coordinates and the Ptolemy coordinates of Garoufalidis, D. Thurston and Zickert [3].

The shape, and Ptolemy coordinates are inspired by the $\mathcal{X}$ and $\mathcal{A}$ coordinates on higher Teichmüller spaces due to Fock and Goncharov [1]. Their coordinates parametrize representations of surfaces, whereas ours parametrize representations of 3-manifold groups. The duality property above may be a 3-diamensional aspect of a Langlands duality discussed by Fock and Goncharov.

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Identities related to Nahm's conjecture

Sander Zwegers

(joint work with Masha Vlasenko (partially))

Let $r \geq 1$ be a positive integer, A a real positive definite symmetric $r \times r$-matrix, B a vector of length r, and C a scalar. We are interested in the q-series

$$F_{A,B,C}(q) := \sum_{n=(n_1,\dots,n_r)\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^T An + n^T B + C}}{(q)_{n_1}\cdots(q)_{n_r}}.$$

which converges for $|q| < 1$. Here we use the notation $(q)_n = \prod_{k=1}^n (1-q^k)$ for $n \in \mathbb{Z}_{\geq 0}$. We are concerned with the following problem due to Werner Nahm (see [2]): describe all such A, B and C with rational entries for which $F_{A,B,C}$ is a modular form ($q = e^{2\pi i\tau}$). The first (non-trivial) example is for $A = 2$, where modularity is obtained from the Rogers-Ramanujan equation (slightly rewritten)

$$F_{2,0,-\frac{1}{60}}(q) = \frac{\sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{5}{2}(k+\frac{1}{10})^2}}{\eta(\tau)}.$$

Nahm's conjecture states that for given A: there is a B and C such that $F_{A,B,C}$ is modular if and only if all solutions of Nahm's equation $1 - x = x^A$ give torsion elements in the Bloch group.

Nahm's conjecture is known to hold for $r = 1$ (see [4]). In this talk we present several counterexamples to Nahm's conjecture for $r \geq 2$ (see [3]), like

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

In these counterexamples, not all solutions of Nahm's equation give torsion elements in the Bloch group, but there do exist B and C such that $F_{A,B,C}$ is modular. The modularity is obtained from explicit identities also presented in the talk.

Further we consider the family of matrices of the form $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where $\mathcal{C}(X)$ and $\mathcal{C}(X')$ are on of A, D, E, T Cartan matrices. It has been shown by Lee (see [1]) that for these matrices, all solutions of Nahm's equation give torsion elements in the Bloch group, so we expect the corresponding q-series to be modular (since Nahm's conjecture still seems to hold in this direction). In this talk we discuss several examples of identities for q-series for matrices belonging to this family and show that

$$F_{\mathcal{C}(E_8)^{-1},0,-\frac{1}{33}}(q) = \frac{\sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{11}{2}(k+\frac{1}{22})^2}}{\eta(\tau)},$$

by making repeated use of

$$\frac{1}{(q)_m (q)_n} = \sum_{\substack{r,s,t \\ r+t=m \\ s+t=n}} \frac{q^{rs}}{(q)_r (q)_s (q)_t}.$$

In general, this identity can be used to relate the q-series for a given matrix, to that for a matrix in rank one higher. In terms of the Bloch group this corresponds to the five term relation.

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