

# ON TITS' CENTRE CONJECTURE FOR FIXED POINT SUBCOMPLEXES

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**ABSTRACT.** We give a short and uniform proof of a special case of Tits' Centre Conjecture using a theorem of J-P. Serre [8] and a result from [1]. We consider fixed point subcomplexes  $X^H$  of the building  $X = X(G)$  of a connected reductive algebraic group  $G$ , where  $H$  is a subgroup of  $G$ .

## 1. INTRODUCTION

Let  $G$  be a connected reductive linear algebraic group defined over an algebraically closed field  $k$ . Let  $X = X(G)$  be the spherical Tits building of  $G$ , cf. [10]. Recall that the simplices in  $X$  correspond to the parabolic subgroups of  $G$ , [8, §3.1]; for a parabolic subgroup  $P$  of  $G$ , we let  $x_P$  denote the corresponding simplex of  $X$ . The conjugation action of  $G$  on itself naturally induces an action of  $G$  on the building  $X$ , so the image of  $G$  is a subgroup of the automorphism group of  $X$ . Given a subcomplex  $Y$  of  $X$ , let  $N_G(Y)$  denote the subgroup of  $G$  consisting of elements which stabilize  $Y$  (in this induced action).

Recall the *geometric realization* of  $X$  as a bouquet of  $n$ -spheres. A subcomplex  $Y$  of  $X$  is called *convex* if whenever two points of  $Y$  (in the geometric realization) are not opposite in  $X$ , then  $Y$  contains the unique geodesic joining these points, [8, §2.1]. A convex subcomplex  $Y$  of  $X$  is *contractible* if it has the homotopy type of a point, [8, §2.2]. The following is a version due to J-P. Serre of the so-called "Centre Conjecture" by J. Tits, cf. [9, Lem. 1.2], [6, §4], [8, §2.4], [11]. This has been proved by B. Mühlherr and J. Tits for spherical buildings of classical type [5]. The simplex referred to in the conjecture is called a *centre* for  $Y$ .

**Conjecture 1.1.** *Let  $Y$  be a convex contractible subcomplex of  $X$ . Then there is a simplex in  $Y$  which is stabilized by all automorphisms of  $X$  which stabilize  $Y$ .*

For a subgroup  $H$  of  $G$  let  $X^H$  be the fixed point subcomplex of the action of  $H$ , i.e.,  $X^H$  consists of the simplices  $x_P \in X$  such that  $H \subseteq P$ . Thus, if  $H \subseteq K \subseteq G$  are subgroups of  $G$ , then we have  $X^K \subseteq X^H$ ; observe that  $X^H$  is always convex, cf. [8, Prop. 3.1]. Our main result, Theorem 3.1, gives a short, conceptual proof of a special case of Conjecture 1.1; namely, we consider subcomplexes of the form  $Y = X^H$  for  $H$  a subgroup of  $G$ , and we consider automorphisms only from  $N_G(Y)$ . The special case  $G = \mathrm{GL}(V)$  in Theorem 3.1 generalizes the classical construction of upper and lower Loewy series, see Remark 3.2(ii).

The initial motivation for Tits' Conjecture 1.1 was a question about the existence of a certain parabolic subgroup associated with a unipotent subgroup of a Borel subgroup of  $G$  (cf. [6, §4.1], [8, §2.4]). This existence theorem was ultimately proved by other means, [3, §3]. In Example 3.6 below we show that the existence of such a parabolic subgroup can be viewed as a special case of Theorem 3.1.

## 2. SERRE'S NOTION OF COMPLETE REDUCIBILITY

Following Serre [8, Def. 2.2.1], we say that a convex subcomplex  $Y$  of  $X$  is  $X$ -completely reducible ( $X$ -cr) if for every simplex  $y \in Y$  there exists a simplex  $y' \in Y$  opposite to  $y$  in  $X$ . The following is part of a theorem due to Serre, [6, Thm. 2]; see also [8, §2] and [11].

**Theorem 2.1.** *Let  $Y$  be a convex subcomplex of  $X$ . Then  $Y$  is  $X$ -completely reducible if and only if  $Y$  is not contractible.*

The notion of convexity for subcomplexes of  $X$  has the following nice characterization in terms of parabolic subgroups due to Serre, [8, Prop. 3.1].

**Proposition 2.2.** *Let  $Y$  be a subcomplex of  $X$ . Then  $Y$  is convex if and only if whenever  $P, P'$ , and  $Q$  are parabolic subgroups in  $G$  with  $x_P, x_{P'} \in Y$  and  $Q \supseteq P \cap P'$ , then  $x_Q \in Y$ .*

Note that many subcomplexes which arise naturally in the building are fixed point subcomplexes. For example, the apartments of  $X$  are the subcomplexes  $X^T$  for maximal tori  $T$  of  $G$  and, more generally, the smallest convex subcomplex containing two simplices  $x_P$  and  $x_{P'}$  is  $X^{P \cap P'}$ .

Following Serre [8], we say that a (closed) subgroup  $H$  of  $G$  is  $G$ -completely reducible ( $G$ -cr) provided that whenever  $H$  is contained in a parabolic subgroup  $P$  of  $G$ , it is contained in a Levi subgroup of  $P$ ; for an overview of this concept see for instance [7] and [8]. In the case  $G = \mathrm{GL}(V)$  ( $V$  a finite-dimensional  $k$ -vector space) a subgroup  $H$  is  $G$ -cr exactly when  $V$  is a semisimple  $H$ -module, so this faithfully generalizes the notion of complete reducibility from representation theory. An important class of  $G$ -cr subgroups consists of those that are not contained in any proper parabolic subgroup of  $G$  at all (they are trivially  $G$ -cr). Following Serre, we call them  $G$ -irreducible ( $G$ -ir), [8]. As before, in the case  $G = \mathrm{GL}(V)$ , this concept coincides with the usual notion of irreducibility. If  $H$  is a  $G$ -completely reducible subgroup of  $G$ , then  $H^0$  is reductive, [7, Property 4].

Since  $X^H$  is a convex subcomplex of  $X = X(G)$  for any subgroup  $H$  of  $G$ , Theorem 2.1 applies in this case and we have the following result (see [7, p19], [8, §3]):

**Theorem 2.3.** *Let  $H$  be a subgroup of  $G$ . Then  $H$  is  $G$ -completely reducible if and only if the subcomplex  $X^H$  is not contractible.*

*Remark 2.4.* By convention, the empty subcomplex of  $X$  is not contractible. This is consistent with Theorem 2.1, because  $H$  is  $G$ -ir if and only if  $X^H = \emptyset$ , and a  $G$ -ir subgroup is  $G$ -cr.

Our next result [1, Thm. 3.10] gives an affirmative answer to a question by Serre, [7, p. 24]. The special case when  $G = \mathrm{GL}(V)$  is just a particular instance of Clifford Theory.

**Theorem 2.5.** *Let  $N \subseteq H \subseteq G$  be subgroups of  $G$  with  $N$  normal in  $H$ . If  $H$  is  $G$ -completely reducible, then so is  $N$ .*

## 3. TITS' CENTRE CONJECTURE FOR FIXED POINT SUBCOMPLEXES

Here is the main result of this note.

**Theorem 3.1.** *Let  $Y$  be a convex, contractible subcomplex of  $X$ . Suppose that  $Y$  is of the form  $Y = X^H$  for a subgroup  $H$  of  $G$ . Then there is a simplex in  $Y$  which is stabilized by all elements in  $N_G(Y)$ .*

*Proof.* Let  $M$  be the intersection of all parabolic subgroups of  $G$  corresponding to simplices in  $Y$ . Since  $H \subseteq M$ , we have  $X^M \subseteq X^H$ . But every parabolic subgroup containing  $H$  contains  $M$ , by definition of  $M$ . Hence  $X^M = X^H$ . Set  $K := N_G(Y)$ . It is clear that  $M$  is normal in  $K$ . Since  $X^K \subseteq X^M$ , it suffices to show that  $X^K \neq \emptyset$ . Now  $Y = X^M$  is contractible, so Theorem 2.3 implies that  $M$  is not  $G$ -cr. Thus, by Theorem 2.5, it follows that  $K$  is not  $G$ -cr and again by Theorem 2.3 that  $X^K$  is contractible. In particular,  $X^K$  is non-empty, by Remark 2.4. Thus  $K$  stabilizes a simplex in  $X^M$ , as claimed.  $\square$

*Remarks 3.2.* (i). Let  $H \subseteq K \subseteq G$  be subgroups of  $G$  with  $H$  normal in  $K$ . Suppose that  $X^H$  is contractible. Since  $H$  is normal in  $K$ , the latter permutes the simplices in  $X^H$ , and so  $K \subseteq N_G(X^H)$ . It thus follows from Theorem 3.1 that  $K$  fixes a simplex in  $X^H$ .

(ii). Observe that Theorem 3.1 can be viewed as a generalization of the classical construction of upper and lower Loewy series in representation theory (for definitions, see e.g., [4]). Let  $V$  be a finite-dimensional  $k$ -vector space. Let  $H \subseteq K \subseteq \mathrm{GL}(V)$  be subgroups of  $\mathrm{GL}(V)$  with  $H$  normal in  $K$  and suppose that  $V$  is not  $H$ -semisimple. Then the upper and lower Loewy series of the  $H$ -module  $V$  are proper  $K$ -stable flags in  $V$ , and so they provide “natural centres” for the action of  $K$  on the complex  $X(V)^H$ , where  $X(V)$  is the flag complex of  $V$ .

(iii). In [8, Prop. 2.11], J-P. Serre showed that Theorem 2.5 is a consequence of Tits’ Centre Conjecture 1.1. So, Theorem 3.1 is just the reverse implication of Serre’s result [8, Prop. 2.11] in the special case when Theorem 2.5 applies.

(iv). Let  $k_0$  be any field and let  $k$  be the algebraic closure of  $k_0$ . Suppose that  $G$  is defined over  $k_0$ . One can define what it means for a subgroup  $H$  defined over  $k_0$  to be  $G$ -completely reducible over  $k_0$ , cf. [1, Sec. 5], [8, Sec. 3]. In [1, Thm. 5.8], it is proved that if  $k_0$  is perfect, then a subgroup  $H$  is  $G$ -cr over  $k_0$  if and only if it is  $G$ -cr. Using this, one can show that the proof of Theorem 3.1 goes through for buildings of the form  $X = X(G(k_0))$ . In particular, this includes many finite spherical buildings attached to finite groups of Lie type.

(v). In the Centre Conjecture 1.1, one considers all automorphisms of the building. If  $X = X(G)$ , then in many cases,  $\mathrm{Aut} X$  is generated by inner and graph automorphisms of  $G$ , together with field automorphisms (cf. [10, Intro.]). We will consider graph and field automorphisms in the setting of Theorem 3.1 in future work (see [2, Sec. 6]).

Our final result gives a characterization of subcomplexes of  $X$  of the form  $X^H$  for a subgroup  $H$  of  $G$ .

**Proposition 3.3.** *Let  $Y \subseteq X$  be a subset of simplices of  $X$ . Then  $Y$  is a subcomplex of  $X$  of the form  $Y = X^H$  for some subgroup  $H$  of  $G$  if and only if for every  $n \in \mathbb{N}$ , the following condition holds:*

$$(3.4) \quad \text{if } P_1, \dots, P_n, Q \text{ are parabolic subgroups with } x_{P_i} \in Y \text{ and } Q \supseteq \bigcap_{i=1}^n P_i, \text{ then } x_Q \in Y.$$

*Proof.* First suppose that  $Y = X^H$  for some subgroup  $H$  of  $G$ . Let  $n \in \mathbb{N}$  and let  $x_{P_1}, \dots, x_{P_n} \in Y$ . If  $Q$  is a parabolic subgroup of  $G$  containing  $\bigcap_{i=1}^n P_i$ , then  $Q$  contains  $H$ , because each  $P_i$  does, so  $x_Q \in Y$ .

Conversely, suppose that condition (3.4) holds for all  $n \in \mathbb{N}$ . Let  $H$  be the intersection of all  $P$  such that  $x_P \in Y$ . By the descending chain condition, we have  $H = \bigcap_{i=1}^m P_i$  for some  $m \in \mathbb{N}$  and some  $P_i$  with  $x_{P_i} \in Y$ . It follows from condition (3.4) for  $n = m$  that for any parabolic subgroup  $P$  containing  $H$ ,  $x_P \in Y$ , so  $X^H \subseteq Y$ . It is clear from the definition of  $H$  that  $Y \subseteq X^H$ .  $\square$

*Remark 3.5.* Note that  $Y$  is a subcomplex of  $X$  precisely when condition (3.4) holds for  $n = 1$ . Further, by Proposition 2.2,  $Y$  is convex if and only if condition (3.4) holds for  $n = 2$ .

As indicated in the Introduction, a fundamental theorem of Borel and Tits on unipotent subgroups of Borel subgroups of  $G$  [3, §3] yields a key example for Theorem 3.1.

**Example 3.6.** Let  $U$  be a non-trivial unipotent subgroup of  $G$  contained in a Borel subgroup  $B$  of  $G$ . Let  $Y = X^U$ . Note that  $U$  is not  $G$ -cr; for if  $U$  is contained in a Borel subgroup  $B^-$  opposite to  $B$ , then  $U$  is contained in the maximal torus  $B^- \cap B$  of  $G$ , which is absurd. So  $Y$  is contractible, by Theorem 2.3. Thus, by Theorem 3.1,  $N_G(U)$  stabilizes a simplex in  $Y$ , i.e., there is a parabolic subgroup  $P$  of  $G$  containing  $N_G(U)$ . Now, the construction of Borel and Tits in [3] yields such a parabolic subgroup  $P$  which enjoys additional properties; for example, it is stabilized by automorphisms of  $G$  which stabilize  $U$ . The framework for  $G$ -complete reducibility developed in [1] and subsequent papers allows one to associate such *canonical* parabolic subgroups to all non- $G$ -cr subgroups of  $G$ , see [2, Sec. 5].

**Acknowledgements:** The authors acknowledge the financial support of EPSRC Grant EP/C542150/1 and Marsden Grant UOC0501. Part of the paper was written during a visit of the third author to the Max-Planck-Institute for Mathematics in Bonn.

## REFERENCES

- [1] M. Bate, B. Martin, G. Röhrle, *A geometric approach to complete reducibility*, Inv. math. **161**, no. 1 (2005), 177–218.
- [2] M. Bate, B. Martin, G. Röhrle, R. Tange, *Closed orbits and uniform  $S$ -instability in Invariant Theory*, preprint (2009).
- [3] A. Borel, J. Tits, *Éléments unipotents et sous-groupes paraboliques des groupes réductifs, I*, Invent. Math. **12** (1971), 95–104.
- [4] J. Dauns, *Chains of modules with completely reducible quotients*, Pacific J. Math. **17** (1966), 235–242.
- [5] B. Mühlherr, J. Tits, *The Centre Conjecture for non-exceptional buildings*, J. Algebra **300** (2006), no. 2, 687–706.
- [6] J-P. Serre, *La notion de complète réductibilité dans les immeubles sphériques et les groupes réductifs*, Séminaire au Collège de France, résumé dans [12, pp. 93–98], (1997).
- [7] ———, *The notion of complete reducibility in group theory*, Moursund Lectures, Part II, University of Oregon, 1998, [arXiv:math/0305257v1](https://arxiv.org/abs/math/0305257v1) [math.GR].
- [8] ———, *Complète Réductibilité*, Séminaire Bourbaki, 56ème année, 2003–2004, n° 932.
- [9] J. Tits, *Groupes semi-simples isotropes*, Colloq. Théorie des Groupes Algébriques, Bruxelles, 1962.
- [10] ———, *Buildings of spherical type and finite  $BN$ -pairs*, Lecture Notes in Math. **386**, Springer-Verlag (1974).
- [11] ———, *Quelques cas d’existence d’un centre pour des ensembles de chambres qui sont convexes, non vides et ne contiennent pas de paires de chambres opposées*, Séminaire au Collège de France, résumé dans [12, pp. 98–101], (1997).
- [12] ———, *Théorie des groupes*, Résumé des Cours et Travaux, Annuaire du Collège de France, 97<sup>e</sup> année, (1996–1997), 89–102.

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