CROWN THEORY FOR THE UPPER HALF PLANE

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Contents

1. **Vorwort**

This paper features no introduction; it has a table of contents.

The material for this text is scattered throughout my work, often only found in unpublished notes of mine. I focus on the upper half plane but want to mention that most matters hold true for arbitrary Riemannian symmetric spaces of the non-compact type. When I think it is useful, then remarks and references to the more general literature are made.

Over the years I had the opportunity to lecture on the crown topic at various institutions; these are:

- Research Institute of Mathematical Sciences (R.I.M.S.), Kyoto, various lectures in the fall semester of 2004
- Indian Statistical Institute, Bangalore, Lectures on the crown domain, March 2005
- University of Hokkaido at Sapporo, Center of excellence lecture series "Introduction to complex crowns", May 2005
- Morningside Center of Mathematics, Academica Sinica, Beijing, "Introduction to complex crowns", lectures for a summer school, July 2005
- Max-Planck-Institut für Mathematik, various presentations.

It is my special pleasure to thank my various hosts at this opportunity again.

2. Symbols

Throughout this text capital Latin letters, e.g. G, will be used for real algebraic groups; $\mathbb{C}$-subscripts will denote complexifications, e.g. $G_\mathbb{C}$. Lie algebras of groups will be denoted by the corresponding lower case altdeutsche Frakturschrift, e.g. $\mathfrak{g}$ is the Lie algebra of G.

In this paper our concern is with

$$G = \mathrm{Sl}(2,\mathbb{R}) \quad \text{and} \quad G_\mathbb{C} = \mathrm{Sl}(2,\mathbb{C})\,.$$

The following subgroups of G and their complexifications will be of relevance for us:

$$A = \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \mid t > 0 \right\},$$
$$A_\mathbb{C} = \left\{ a_z = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mid z \in \mathbb{C}^* \right\},$$

$$H = \mathrm{SO}(1,1;\mathbb{R}) \quad \text{and} \quad H_\mathbb{C} = \mathrm{SO}(1,1;\mathbb{C})\,,$$
$$K = \mathrm{SO}(2,\mathbb{R}) \quad \text{and} \quad K_\mathbb{C} = \mathrm{SO}(2,\mathbb{C})\,,$$

and

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$
$$N_\mathbb{C} = \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

3. The upper half plane, its affine complexification and the crown

Our concern is with the Riemannian symmetric space

$$X = G/K$$

of the non-compact type. We usually identify X with the upper half-plane $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ via the map

$$X \to \mathbf{H}, \quad gK \mapsto \frac{ai+b}{ci+d} \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

We use $x_0 = K$ for the base point $eK \in X$ and note that $x_0 = i$ within our identification.

We view $X = \mathbf{H}$ inside of the complex projective space $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and note that $\mathbb{P}^1(\mathbb{C})$ is homogeneous for $G_\mathbb{C}$ with respect to the usual fractional linear action:

$$g(z) = \frac{az+b}{cz+d} \qquad \left(z \in \mathbb{P}^1(\mathbb{C}), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\mathbb{C}\right).$$

Upon complexifiying G and K we obtain the *affine complexification*

$$X_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$$

of X. Observe that the map

$$X \hookrightarrow X_\mathbb{C}, \quad gK \mapsto gK_\mathbb{C} \tag{3.1}$$

constitutes a G-equivariant embedding which realizes X as a totally real submanifold of $X_\mathbb{C}$. We will use a more concrete model for $X_\mathbb{C}$: the mapping

$$X_\mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \backslash \mathrm{diag}, \quad gK_\mathbb{C} \mapsto (g(i), g(-i))$$

is a $G_\mathbb{C}$-equivariant diffeomorphism. With this identification of $X_\mathbb{C}$ the embedding of (3.1) becomes

$$X \hookrightarrow X_\mathbb{C}, \quad z \mapsto (z, \overline{z}). \tag{3.2}$$

We will denote by $\overline{X}$ the lower half plane and arrive at the object of our desire:

$$\Xi = X \times \overline{X}$$

the *crown domain* for $\mathrm{Sl}(2,\mathbb{R})$. Let us list some obvious properties of Ξ and emphasize that they hold for arbitrary crowns:

- Ξ is a G-invariant Stein domain in $X_\mathbb{C}$.
- G acts properly on Ξ.
- $\Xi = X \times \overline{X}$ is the *complex double* – this always holds if the underlying Riemannian space $X = G/K$ is already complex.

4. Geometric structure theory

4.1. Basic structure theory

4.1.1. *Ξ as a union of elliptic G-orbits.* We note that

$$\mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

and focus on a domain inside:

$$\Omega = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in (-\pi/4, \pi/4) \right\}.$$

We note that Ω is invariant under the Weyl group $\mathcal{W} = N_K(A)/Z_K(A) \simeq \mathbb{Z}_2$ and that that $\exp(i\Omega)$ consist of elliptic elements in $G_{\mathbb{C}}$.

The following proposition constitutes of what we call the *elliptic parameterization* of the crown domain.

Proposition 4.1. $\Xi = G\exp(i\Omega)\cdot x_0$.

Proof. (cf. [24], Th. 7.5 for the most general case). We first show that $G\exp(i\Omega)\cdot x_0 \subset \Xi$. By G-invariance of Ξ, this reduces to verify that

$$\exp(i\Omega).x_0 \in \Xi\,.$$

Explicitly this means

$$(e^{2i\phi}i, -e^{2i\phi}i) \in X \times \overline{X}$$

for $\phi \in (-\pi/4, \pi/4)$; evidently true.

Conversely, we want to see that every element in Ξ lies on a G-orbit through $\exp(i\Omega)$. Let $S = G\times G$ and $U = K\times K$ and observe, that $\Xi = S/U$ as homogeneous space. Now

$$S = \operatorname{diag}(G)\operatorname{antidiag}(H)U$$

and all what we have to see is that

$$\operatorname{antidiag}(H)\cdot x_0 \subset G\exp(i\Omega)\cdot x_0,$$

or, more concretely,

$$\{\left(\frac{i\cosh t+\sinh t}{i\sinh t+\cosh t}, -\frac{i\cosh t+\sinh t}{i\sinh t+\cosh t}\right) \mid t\in\mathbb{R}\} \subset G\exp(i\Omega)\cdot x_0\,. \tag{4.1}$$

Now we use that $A\exp(i\Omega)(i) = X$ and conclude that the LHS of (4.1) is contained in $A\exp(i\Omega)\cdot x_0$. $\square$

4.1.2. Ξ *as a union of unipotent G-orbits.* The following parameterization of Ξ is relevant for our discussion of automorphic cusp forms at the end of this article. It was discovered in [25].

We consider the Lie algebra of N:

$$\mathfrak{n} = \left\{\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix} \mid x\in\mathbb{R}\right\}$$

and focus on the subdomain

$$\Lambda = \left\{\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix} \mid x\in(-1,1)\right\}\,.$$

The following proposition constitutes of what we call the *unipotent parameterization* of the crown domain, see [25], Th. 3.4 for $G=\mathrm{Sl}(2,\mathbb{R})$ and [25], Th. 8.3 for G general.

Proposition 4.2. $\Xi = G\exp(i\Lambda)\cdot x_0$.

Proof. We wish to give the more conceptual proof. Let us first see that $G\exp(i\Lambda)\cdot x_0 \subset \Xi$, i.e.

$$\exp(i\Lambda)\cdot \subset \Xi\,.$$

Concretely this means that

$$(ix+i, -i+ix) \in X\times \overline{X}$$

for all $x\in(-1,1)$; evidently true.

For the reverse inclusion we will borrow in content and notation from Subsubsection 4.2.1 from below. It is a conceptional argument. Fix $Y\in\Omega$. Then, according to the complex convexity theorem 4.12 there exist a $k\in K$ such that

$$\mathrm{Im}\,\log a_{\mathbb{C}}(k\exp(iY)\cdot x_0) = 0\,.$$

In other words,

$$k\exp(iY)\cdot x_0 \in N_{\mathbb{C}}A\cdot x_0 = AN_{\mathbb{C}}\cdot x_0\,.$$

We conclude that $\exp(iY)\cdot G\exp(i\mathfrak{n})\cdot x_0$. From our discussion in (i) we deduce that $\exp(iY)\cdot x_0\in G\exp(i\Lambda)\cdot x_0$. $\square$

Another way to prove Prop. 4.2 is by means of matching elliptic and unipotent G-orbits. We cite [25], Lemma 3.3:

Lemma 4.3. *For all $\phi\in(-\pi/4,\pi/4)$ the following identity holds:*

$$G\begin{pmatrix}1 & i\sin 2\phi\\ 0 & 1\end{pmatrix}\cdot x_0 = G\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}\cdot x_0\,.$$

Proof. This is best seen in the hyperbolic model of the crown which we discuss in Appendix A; the proof of the lemma will be given there, too. $\square$

4.1.3. *Realization in the tangent bundle.* Let

$$\mathfrak{p} = \mathrm{Sym}(2,\mathbb{R})_{\mathrm{tr}=0}$$

and recall that:

- $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, the Cartan decomposition;
- $\mathfrak{p}$ is a linear K-module which naturally identifies with $T_{x_0}X$, the tangent space of X at x_0.

We write TX for the tangent bundle which is naturally isomorphic with $G\times_K\mathfrak{p}$ via the map

$$G\times_K\mathfrak{p}\to TX,\quad [g,Y]\mapsto \frac{d}{dt}\Big|_{t=0} g\exp(tY)\cdot x_0\,.$$

Inside $\mathfrak{p}$ we consider the disc

$$\hat{\Omega} = \{Y \in \mathfrak{p} \mid \operatorname{spec}(Y) \subset (-\pi/4, \pi/4)\}$$

and note that $\hat{\Omega}$ is K-invariant and

$$\hat{\Omega} \cap \mathfrak{a} = \Omega \, .$$

Therefore we can form the disc-bundle $G \times_K \hat{\Omega}$ inside of TX.

The following result was obtained in [1], in full generality.

Proposition 4.4. *The map*

$$G \times_K \hat{\Omega} \to \Xi, \quad [g, Y] \mapsto g \exp(iY) \cdot x_0$$

is a G-equivariant diffeomorphism.

Proof. Ontoness is clear. Injectivity can be obtained by direct computation. □

Remark 4.5. *The above proposition becomes more interesting when one considers more general groups G – the statement is literally the same. One deduces that G acts properly on Ξ (the action of G on TX is proper) and that Ξ is contractible: Ξ is a fiber bundle over $X = G/K \simeq \mathfrak{p}$ with convex fiber $\hat{\Omega}$.*

4.1.4. *The various boundaries of the crown.* In this part we discuss the various boundaries of Ξ. First and foremost there is the topological boundary $\partial\Xi$ of Ξ in $X_{\mathbb{C}}$. We will see that $\partial\Xi$ carries a natural structure of a cone bundle over the affine symmetric space $Y = G/H$. In particular $Y \subset \partial\Xi$ and Y and we will show that Y is some sort of Shilov boundary of Ξ (we will call it the *distinguished boundary* though).

We write $\mathfrak{q}$ for the tangent space of Y at the base point $y_0 = H \in Y$. Note that

$$\mathfrak{q} = \mathbb{R} \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}_{:=\mathbf{e}} \oplus \mathbb{R} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}_{:=\mathbf{f}}$$

is the decomposition of the H-module in eigenspaces. In particular,

$$C := \mathbb{R}_{\geq 0}\mathbf{e} \cup \mathbb{R}_{\geq 0}\mathbf{f}$$

is an H-invariant cone in $\mathfrak{q}$ and we can form the cone bundle

$$\mathcal{C} := G \times_H C$$

inside of TY.

We note that Y is naturally realized in $X_{\mathbb{C}}$ via the map

$$Y \to X_{\mathbb{C}}, \quad gH \mapsto g(1, -1)\, ,$$

i.e. y_0 identifies with $(1, -1)$.

Proposition 4.6.

$$\mathcal{C} = G \times_H C \to \partial\Xi, \quad [g, Z] \mapsto g \exp(iZ) \cdot y_0$$

is a G-equivariant homeomorphism.

Proof. Direct computation; see [25], Th. 3.1 for details. □

Corollary 4.7. $\pi_1(\partial\Xi) = \pi_1(G/H) = \mathbb{Z}$.

Henceforth we call write $\partial_d\Xi = G \cdot y_0 \simeq Y$ and call $\partial_d\Xi$ the distinguished boundary of Ξ. Its relevance is as follows. Write $\mathbb{P}(\Xi)$ for the cone of strictly plurisubharmonic functions on Ξ which extend continuously up to the boundary. A simple exercise in one complex variable then yields (cf. citeGKI, Th. 2.3).

Lemma 4.8. *For all $f \in \mathbb{P}(\Xi)$:*

$$\sup_{z \in \Xi} |f(z)| = \sup_{z \in \partial_d\Xi} |f(z)|\,.$$

The complement of the distinguished boundary of Ξ we denote $\partial_u\Xi$, and refer to it as the unipotent boundary. A straightforward computation explains the terminology:

$$\partial_u\Xi = G \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \cdot x_0 \amalg G \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \cdot x_0\,. \tag{4.2}$$

4.2. **Fine structure theory**

4.2.1. *The complex convexity theorem.* We begin the standard horospherical coordinates for X: the map

$$N \times A \to X, \quad (n_x, a_{\sqrt{y}}) \mapsto n_x a_{\sqrt{y}} \cdot i = x + iy$$

is an analytic diffeomorphism. Accordingly we obtain a map $a : X \to A$, the so-called A-projection. Upon complexifying $X = NA \cdot x_0$ we obtain a Zariski-open subset

$$N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0 \subsetneq X_{\mathbb{C}}\,.$$

Upon extending the map a holomorphically we have to be more careful as the groups $A_{\mathbb{C}}$ and $K_{\mathbb{C}}$ intersect in the finite two-group

$$M = A_{\mathbb{C}} \cap K_{\mathbb{C}} = \{\pm \mathbf{1}\}\,.$$

Accordingly the extension $a_{\mathbb{C}}$ is only valued mod M:

$$a_{\mathbb{C}} : N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0 \to A_{\mathbb{C}}/M\,.$$

The second part of the following proposition is of fundamental importance.

Proposition 4.9. *The following assertions hold:*

(i) $N_{\mathbb{C}}A_{\mathbb{C}}\cdot x_0 = \mathbb{C}\times\mathbb{C}\setminus \mathrm{diag}$, *in other words* $N_{\mathbb{C}}A_{\mathbb{C}}\cdot x_0$ *is the affine open piece of* $X_{\mathbb{C}}$.

(ii) $\Xi \subset N_{\mathbb{C}}A_{\mathbb{C}} \cdot x_0$.

(iii) *The map* $a_{\mathbb{C}}$, *restricted to* Ξ, *admits a holomorphic logarithm* $\log a_{\mathbb{C}} : \Xi \to \mathfrak{a}_{\mathbb{C}}$ *such that* $\log a_{\mathbb{C}}(x_0) = 0$.

Proof. (i) We observe that

$$\begin{aligned} N_{\mathbb{C}}A_{\mathbb{C}} \cdot x_0 &= \{(iz + w, -iz + w) \mid z \in \mathbb{C}^*, w \in \mathbb{C}\} \\ &= \{(z + w, -z + w) \mid z \in \mathbb{C}^*, w \in \mathbb{C}\} \\ &= \mathbb{C} \times \mathbb{C}\setminus \mathrm{diag}\ . \end{aligned}$$

(ii) is immediate from (i).
(iii) follows from (ii) and the fact that Ξ is simply connected. □

Remark 4.10. *We wish to make a few remarks about the inclusion (ii) for more general groups. For classical groups (ii) was obtained in* [23] *and* [14] *by somewhat explicit, although efficient, matrix computations. For general simple groups a good argument based on complex analysis was given in* [17] *and* [18]. *The method of* [17] *was later simplified and slightly generalized in* [27].

From Proposition 4.9(i) we obtain the following

Corollary 4.11. $\left[\bigcap_{g\in G} gN_{\mathbb{C}}A_{\mathbb{C}} \cdot x_0\right]_0 = \Xi$, *where* $[\cdot]_0$ *denotes the connected component of* $[\cdot]$ *containing* x_0.

Proof. Let $D := \left[\bigcap_{g\in G} gN_{\mathbb{C}}A_{\mathbb{C}} \cdot x_0\right]_0$. Write D_1, D_2 for the projection of D to the first, resp. second, factor in $[\mathbb{C} \times \mathbb{C}]\setminus \mathrm{diag}$. Then $D_1 \subset \mathbb{C}$ is G-invariant. Hence $D_1 = X$, $D_1 = \overline{X}$ or $D_1 = X \cup \overline{X}$. The last case is excluded, as D is connexted. The second case is excluded as $x_0 \in D$ implies $i \in D_1$. Hence $D_1 = X$. By the same reasoning one gets $D_2 = \overline{X}$. As $\Xi \subset D$ we thus get $D = \Xi$. □

For an element $Y \in \mathfrak{a}$ we note that the convex hull of the Weyl-group orbit of Y, in symbols $\mathrm{conv}(\mathcal{W} \cdot Y)$, is just the line segment $[-Y, Y]$. With that we turn to a deep geometric fact for crown domains, the *complex convexity theorem*:

Theorem 4.12. *For* $Y \in \Omega$:

$$\mathrm{Im}\, \log a_{\mathbb{C}}(K \exp(iY) \cdot x_0) = [-Y, Y]\, .$$

Proof. Direct computation. For $G = \mathrm{Sl}(2,\mathbb{R})$ there is an explicit formula for $a_\mathbb{C}$: with $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K$ one has

$$a_\mathbb{C}\left(k_\theta \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}\right) = a_z$$

with

$$z = \sqrt{e^{2i\phi} + \sin^2\theta(e^{-2i\phi} + e^{2i\phi})}\,,$$

see [23], Prop. A.1 (i). From that the assertion follows. For the general case we refer to [10] for the inclusion "$\subset$" and to [26] for actual equality. $\square$

4.2.2. *Realization in the complexified Cartan decomposition.* The Cartan or polar decomposition of X says that the map

$$K/M \times A \to X, \quad (kM, a) \mapsto ka \cdot x_0$$

is onto with faithful restriction to $K/M \times A^+$. Here, as usual

$$A^+ = \{a_t \mid t > 1\}\,.$$

Thus

$$X = KA \cdot x_0$$

and we wish to complexify this equality. We have to be a little more careful here, as $K_\mathbb{C}A_\mathbb{C} \cdot x_0$ is no longer a domain (it fails to be open at the base point x_0). The remedy comes from a little bit of invariant theory. We note that $X_\mathbb{C}$ is an affine variety and write $\mathbb{C}[X_\mathbb{C}]$ for its ring of regular function. We denote by $\mathbb{C}[X_\mathbb{C}]^{K_\mathbb{C}}$ for the subring of regular function. According to Hilbert, the invariant ring is finitely generated, i.e.

$$\mathbb{C}[X_\mathbb{C}]^{K_\mathbb{C}} = \mathbb{C}[p]\,.$$

In order to describe p we use a different realization of $X_\mathbb{C}$, namely

$$X_\mathbb{C} = \mathrm{Sym}(2,\mathbb{C})_{\det=1}\,.$$

In this model the generator p is given by

$$p : X_\mathbb{C} \to \mathbb{C}, \quad z \mapsto \operatorname{tr} z\,.$$

For a symmetric, i.e. $\mathcal{W}$-invariant, open segment $\omega \subset \Omega$ we define a $K_\mathbb{C}$-invariant domain $X_\mathbb{C}(\omega) \subset X_\mathbb{C}$ by

$$X_\mathbb{C}(\omega) = p^{-1}(p(A\exp(i\omega)\cdot x_0))\,.$$

We note that

- $K_\mathbb{C}A\exp(i\omega)\cdot x_0 \subset X_\mathbb{C}(\omega)$
- $\exp(i\omega')\cdot x_0 \not\subset X_\mathbb{C}(\omega)$ if $\omega \subsetneq \omega'$.

Hence we may view $X_{\mathbb{C}}(\omega)$ as the $K_{\mathbb{C}}$-invariant open envelope of $K_{\mathbb{C}}A\exp(i\omega)\cdot x_0$ in $X_{\mathbb{C}}$. The main result here is as follows:

Theorem 4.13. *For all open symmetric segments $\omega \subset \Omega$ one has*

$$G\exp(i\omega)\cdot x_0 \subset X_{\mathbb{C}}(\omega)\,.$$

In particular

$$\Xi \subset X_{\mathbb{C}}(\Omega)\,.$$

Proof. For $G = \mathrm{Sl}(2,\mathbb{R})$ this was established in [?]; in general in [20]. $\square$

5. Holomorphic extension of representations

I want to explain a few things on representations first. For the beginning G might be any connected unimodular Lie group, for simplicity even contained in its universal complexification $G_{\mathbb{C}}$. By a *unitary representation* of G we understand a group homomorphism

$$\pi : G \to U(\mathcal{H})$$

from G into the unitary group of some complex Hilbert space $\mathcal{H}$ such that for all $v \in \mathcal{H}$ the orbit maps

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

are continuous. We call a vector $v \in \mathcal{H}$ *analytic* if f_v is a real analytic $\mathfrak{h}$-valued map. The entity of all analytic vectors of π is denoted by $\mathcal{H}^\omega$ and we observe that $\mathcal{H}^\omega$ is a G-invariant vector space. The following result was obtained by Nelson; the idea is already found in the approximation theorem of Weierstraß.

Lemma 5.1. *$\mathcal{H}^\omega$ is dense in $\mathcal{H}$.*

Proof. (Sketch) We first recall that with π comes a Banach-$*$-representation Π of the group algebra $L^1(G)$ given by

$$\Pi(f)v = \int_G f(g)\pi(g)v \, dg \qquad (f \in L^1(G), v \in \mathcal{H})$$

with dg a Haar-measure. For a Dirac-sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(G)$ one immediately verifies that

$$\Pi(f_n)v \to v \tag{5.1}$$

for all $v \in \mathcal{H}$. We choose a good Dirac sequence: Fix a left invariant Laplace operator on G and write ρ_t for the corresponding heat kernel. We use the theory of parabolic PDE's as black box and just state:

- $\rho_t \in L^1(G)$ for all $t > 0$,
- ρ_t is analytic and of Gaußian decay,
- $(\rho_{1/n})_{n\in\mathbb{N}}$ is a Dirac-sequence.

As a result $\Pi(\rho_t)v \in \mathcal{H}^\omega$ and

$$\lim_{t\to 0^+} \Pi(\rho_t)v = v \qquad (v \in \mathcal{H})$$

by (5.1). □

Let us now sharpen the assumptions on G and π. In the next step we request:

- G is semisimple.
- π is irreducible.

Harish-Chandra observed that screening the representation π under a maximal compact subgroup $K < G$ is meaningful. He introduced the space of K-finite vectors:

$$\mathcal{H}_K = \{v \in \mathcal{H} \mid \operatorname{span}_{\mathbb{C}}\{\pi(K)v\} \text{ is finite dim.}\}$$

Observe that $\mathcal{H}_K$ is dense in $\mathcal{H}$ by the theorem of Peter and Weyl. Harish-Chandra made a key-observation:

Lemma 5.2. *$\mathcal{H}_K \subset \mathcal{H}^\omega$.*

Proof. The following sketch of proof is non-standard. We will use a little bit of functional analysis. It is known that $\mathcal{H}^\omega$ is a locally convex vector space of compact type. As such it is sequentially complete. This makes the Peter-Weyl-Theorem for the representation of K on $\mathcal{H}^\omega$ applicable. In particular the K-finite vectors in $\mathcal{H}^\omega_K$ in $\mathcal{H}^\omega$ are dense in $\mathcal{H}^\omega$. Apply the previous Lemma combined with the density of $\mathcal{H}_K$ in $\mathcal{H}$. □

The upshot of our discussion is that $\mathcal{H}_K$ is the vector space consisting of the best possible analytic vectors. It is a module of countable dimension for the Lie algebra $\mathfrak{g}$ and as such irreducible.

Given $v \in \mathcal{H}_K$ we consider the real analytic orbit map

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

and ask the following :

Question: What is the natural domain $D_v \subset G_{\mathbb{C}}$ to which f_v extends holomorphically?

It turns out that D_v does only depend on the type of the representation π but not on the specific vector $v \neq 0$ (this is reasonable as v generates $\mathcal{H}_K$ as a $\mathfrak{g}$-module). We will give this classification in the subsection below. At this point we only remark that the domain D_v is naturally left G-invariant and right $K_{\mathbb{C}}$-invariant, in symbols:

$$D_v = G D_v K_{\mathbb{C}} \,.$$

A little bit more terminology is good for the purpose of the discussion. We write

$$q : G_{\mathbb{C}} \to X_{\mathbb{C}}, \quad g \mapsto gK_{\mathbb{C}}$$

for the canonical projection and for a domain $D \subset X_{\mathbb{C}}$ we write

$$DK_{\mathbb{C}} = q^{-1}(D)$$

for the pre-image of D in $G_{\mathbb{C}}$.

To get a feeling for that I want to discuss one class of examples first.

5.1. **The spherical principal series**

For the rest of this section we return to our basic setup: $G = \mathrm{Sl}(2,\mathbb{R})$.

We fix a parameter $\lambda \in \mathbb{R}$, let $\mathcal{H} = L^2(\mathbb{R})$ and declare a unitary representation π_λ of G on $\mathcal{H}$ via

$$[\pi_\lambda(g)f](x) = |cx+d|^{-1+i\lambda} f\left(\frac{ax+b}{cx+d}\right) \tag{5.2}$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $f \in \mathcal{H}$ and $x \in \mathbb{R}$. In the literature one finds π_λ under the term *spherical unitary principal series*. This representation is K-*spherical*, i.e. the space of K-fixed vectors $\mathcal{H}^K$ is non-zero. More precisely, $\mathcal{H}^K = \mathbb{C}v_K$ with

$$v_K(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(1+x^2)^{\frac{1}{2}(1-i\lambda)}}$$

being a normalized representative. With v_K we form the matrix coefficient

$$\phi_\lambda(g) := \langle \pi_\lambda(g) v_K, v_K \rangle \qquad (g \in G)\,.$$

The function ϕ_λ is K-invariant from both sides, in particular descends to an analytic function on $X = G/K$, also denoted by ϕ_λ. We record the integral representation for ϕ_λ:

$$\phi_\lambda(x) = \int_K a(kx)^{\rho(1+i\lambda)} \, dk \qquad (x \in X)$$

where dk is a normalized Haar measure on X, and the other notation standard too: for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in A$ we let $a^\mu := e^{\mu(\log a)}$ and $\rho \in \mathfrak{a}^*$

is fixed by $\rho\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 1$. Now in view of Proposition 4.9(iii), this implies that ϕ_λ extends to a holomorphic function on Ξ given by

$$\phi_\lambda(z) = \int_K a_{\mathbb{C}}(kz)^{\rho(1+i\lambda)}\ dk \qquad (z\in\Xi)\,.$$

With a little bit of functional analysis one then gets that the orbit map f_{v_K} extends holomorphically to $\Xi K_{\mathbb{C}}$. Since $\mathcal{H}_K = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})v_K$ we thus deduce that f_v extends to $\Xi K_{\mathbb{C}}$ for all $v\in\mathcal{H}_K$. For $v\neq 0$, this is actually a maximal domain, but that would require more work. We summarize the discussion:

Proposition 5.3. *Let π_λ be a unitary spherical principal series, then for all $v\in\mathcal{H}_K$, the orbit map $f_v : G\to\mathcal{H}$ extends to a holomorphic function on $\Xi K_{\mathbb{C}}$.*

Remark 5.4. *Observe that the above proposition implies that ϕ_λ extends holomorphically to Ξ.*

5.2. **A complex geometric classification of $\hat G$**

5.2.1. *More geometry.* Before we turn to the subject proper we have to introduce two more geometric objects. We define two G-invariant domains in $X_{\mathbb{C}}$ by

$$\Xi^+ = X\times\mathbb{P}^1(\mathbb{C})\backslash\,\mathrm{diag}\,,$$
$$\Xi^- = \mathbb{P}^1(\mathbb{C})\times\overline{X}\backslash\,\mathrm{diag}\,.$$

We immediately observe that both Ξ^+ and Ξ^- feature the following properties:

- G acts properly on Ξ^+ and Ξ^-,
- Both Ξ^+ and Ξ^- are maximal G-domains in $X_{\mathbb{C}}$ with proper actions,
- Both Ξ^+ and Ξ^- are Stein,
- $\Xi^+\cap\Xi^- = \Xi$.

In terms of structure theory one can define Ξ^+ and Ξ^- as follows. Let us denote by $Q^\pm$ the stabilizer of $\pm i$ in $G_{\mathbb{C}}$. Note that $Q^\pm = K_{\mathbb{C}}\rtimes P^\pm$ with

$$P^\pm = \left\{\begin{pmatrix}1+z & \mp iz\\ \mp iz & 1-z\end{pmatrix} \mid z\in\mathbb{C}\right\}.$$

We easily obtain:

Lemma 5.5. *The following assertions hold:*

(i) $\Xi^+ K_{\mathbb{C}} = G K_{\mathbb{C}} P^+$,
(ii) $\Xi^- K_{\mathbb{C}} = G K_{\mathbb{C}} P^-$.

5.2.2. *The classification theorem.* In this section $(\pi, \mathcal{H})$ denotes an irreducible unitary representation of G. We call π a *highest weight*, resp. *lowest weight*, representation if $\mathrm{Lie}(P^+)$, resp. $\mathrm{Lie}(P^-)$, acts finitely on $\mathcal{H}_K$. We state the main result (cf. [25] for $\mathrm{Sl}(2, \mathbb{R})$ and [21] in general).

Theorem 5.6. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Let $0 \neq v \in \mathcal{H}_K$ be a K-finite vector. Then a maximal $G \times K_{\mathbb{C}}$-invariant domain D_v to which*

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

extends as a holomorphic function is given as follows:

(i) *$G_{\mathbb{C}}$, if π is the trivial representation;*
(ii) *$\Xi^+ K_{\mathbb{C}}$, if π is a non-trivial highest weight representation;*
(iii) *$\Xi^- K_{\mathbb{C}}$, if π is a non-trivial lowest weight representation;*
(iv) *$\Xi K_{\mathbb{C}}$ in all other cases.*

It is our desire to explain how to prove this theorem. We found out that there is an intimate relation of this theorem with proper actions of G on $X_{\mathbb{C}}$.

5.2.3. *Proper actions and representations.* The material in this section is taken from [25], Section 4. It holds for a general semisimple group. We begin with a simple reformulation of the Riemann-Lebesgue Lemma for representations.

Lemma 5.7. *Let $(\pi, \mathcal{H})$ be a unitary representation of G which does not contain the trivial representation. Then G acts properly on $\mathcal{H} - \{0\}$.*

Proof. Let $C \subset \mathcal{H} - \{0\}$ be a compact subset and $C_G = \{g \in G \mid \pi(g)C \cap C \neq \emptyset\}$. Suppose that C_G is not compact. Then there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in C_G and a sequence $(v_n)_{n \in \mathbb{N}}$ in C such that $\pi(g_n)v_n \in C$ and $\lim_{n \to \infty} g_n = \infty$. As C is compact we may assume that $\lim_{n \to \infty} v_n = v$ and $\lim_{n \to \infty} \pi(g_n)v_n = w$ with $v, w \in C$. We claim that

$$\lim_{n \to \infty} \langle \pi(g_n)v, w \rangle \neq 0 \,. \tag{5.3}$$

In fact $\|\pi(g_n)v_n - \pi(g_n)v\| = \|v_n - v\| \to 0$ and thus $\pi(g_n)v \to w$ as well. As $w \in C$, it follows that $w \neq 0$ and our claim is established.

Finally we observe that (5.3) contradicts the Riemann-Lebesgue lemma for representations which asserts that the matrix coefficient vanishes at infinity. $\square$

From Lemma 5.7 we deduce the following result.

Theorem 5.8. *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G which is not trivial. Let $v \in \mathcal{H}_K$, $v \neq 0$, be a K-finite vector. Let $\tilde{D}$ be a $G \times K_{\mathbb{C}}$-invariant domain in $G_{\mathbb{C}}$ with respect to the property that the orbit map $F_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$ extends to a G-equivariant holomorphic map $\tilde{\Xi} \to \mathcal{H}$. Then G acts properly on $\tilde{D}/K_{\mathbb{C}} \subset X_{\mathbb{C}}$.*

Proof. We argue by contradiction and assume that G does not act properly on $D = \tilde{D}/K_{\mathbb{C}}$. We obtain sequences $(z_n')_{n\in\mathbb{N}} \subset D$ and $(g_n)_{n\in\mathbb{N}} \subset G$ such that $\lim_{n\to\infty} z_n' = z' \in D$, $\lim_{n\to\infty} g_n z_n' = w' \in D$ and $\lim_{n\to\infty} g_n = \infty$. We select preimages z_n, z and w of z_n', z' and w' in $\tilde{D}$. We may assume that $\lim_{n\to\infty} z_n = z$ and find a sequence $(k_n)_{n\in\mathbb{N}}$ in $K_{\mathbb{C}}$ such that $\lim_{n\to\infty} g_n z_n k_n = w$.

Before we continue we claim that

$$(\forall z \in \tilde{D}) \qquad \pi(z)v \neq 0 \tag{5.4}$$

In fact assume $\pi(z)v = 0$ for some $z \in \tilde{D}$. Then $\pi(g)\pi(z)v = 0$ for all $g \in G$. In particular the map $G \to \mathcal{H}, \quad g \mapsto \pi(g)v$ is constantly zero. However this map extends to a holomorphic map to a G-invariant neighborhood in $G_{\mathbb{C}}$. By the identity theorem for holomorphic functions this map has to be zero as well. We obtain a contradiction to $v \neq 0$ and our claim is established.

Write $V = \mathrm{span}\{\pi(K)v\}$ for the finite dimensional space spanned by the K-translates of v. In our next step we claim that

$$(\exists c_1, c_2 > 0) \qquad c_1 < \|\pi(k_n)v\| < c_2\,. \tag{5.5}$$

In fact from

$$\lim_{n\to\infty} \pi(g_n z_n k_n)v = \pi(w)v \quad \text{and} \quad \|\pi(g_n z_n k_n)v\| = \|\pi(z_n)\pi(k_n)v\|$$

we conclude with (5.4) that there are positive constants $c_1', c_2' > 0$ such that $c_1' < \|\pi(z_n)\pi(k_n)v\| < c_2'$ for all n. We use that $\lim_{n\to\infty} z_n = z \in \tilde{D}$ to obtain $\pi(z_n)|_V - \pi(z)|_V \to 0$ and our claim follows.

We define C to be the closure of the sequences $(\pi(z_n k_n)v)_{n\in\mathbb{N}}$ and $(\pi(g_n z_n k_n)v)_{n\in\mathbb{N}}$ in $\mathcal{H}$. With our previous claims (5.4) and (5.5) we obtain that $C \subset \mathcal{H} - \{0\}$ is a compact subset. But $C_G = \{g \in G \mid \pi(g)C \cap C \neq \emptyset\}$ contains the unbounded sequence $(g_n)_{n\in\mathbb{N}}$ and hence is not compact - a contradiction to Lemma 5.7. $\square$

5.2.4. *Remarks on the proof of Theorem 5.6.* We are going to discuss the various cases in the Theorem.

Case 1: π is trivial. This is clear.

Case 2: π is a non-trivial highest weight representation. In this case all orbit maps $f_v : G \to \mathcal{H}$ of K-finite vectors v extend to $GK_{\mathbb{C}}P^+$. As $GK_{\mathbb{C}}P^+/K_{\mathbb{C}} = \Xi^+$ and $\Xi^+ \subset X_{\mathbb{C}}$ is maximal for proper G-action, the assertion follows from Theorem 5.8.

Case 3: π is a non-trivial lowest weight representation. Argue as in case 2.

Case 4: The remaining cases. Here we restrict ourselves to spherical principal series π_λ. We have already seen that $D_v \supset \Xi K_{\mathbb{C}}$. The remaining inclusion will follow from the following Theorem, cf. [11] Th. 5.1.

Theorem 5.9. *The crown is a maximal G-invariant domain on $X_{\mathbb{C}}$ to which a spherical function ϕ_λ, $\lambda \in \mathbb{R}$, extends holomorphically.*

In order to prove this result we need some preparation first. We recall the domain $X_{\mathbb{C}}(\Omega)$ from Subsection 4.2.2. Likewise one defines

$$X_{\mathbb{C}}(2\Omega) = p^{-1}p(A\exp(2i\Omega)\cdot x_0)\,.$$

Here is the first Lemma.

Lemma 5.10. *ϕ_λ extends to a $K_{\mathbb{C}}$-invariant holomorphic function on $X_{\mathbb{C}}(2\Omega)$.*

Proof. Recall that ϕ_λ can be written as a matrix coefficient

$$\phi_\lambda(x) = \langle \pi_\lambda(x)v_K, v_K\rangle\,.$$

For $x = a\exp(2iY)\cdot x_0$ with $a \in A$ and $Y \in \Omega$ we now set

$$\phi_\lambda(a\exp(2iY)\cdot x_0) = \langle \pi_\lambda(a\exp(iY))v_K, \pi_\lambda(\exp(iY)v_K\rangle\,. \tag{5.6}$$

It is easy to see that this is well defined and holomorphic on $A\exp(2i\Omega)\cdot x_0$. Extend by $K_{\mathbb{C}}$-invariance. □

Remark 5.11. *We will show below that $X_{\mathbb{C}}(2\Omega)$ is the largest $K_{\mathbb{C}}$-domain to which ϕ_λ extends holomorphically.*

Explicitly the $K_{\mathbb{C}}$-domains $X_{\mathbb{C}}(\Omega)$ and $X_{\mathbb{C}}(2\Omega)$ are given by

$$X_{\mathbb{C},\Omega} = \{z \in X_{\mathbb{C}} : \operatorname{Re} P(z) > 0\}$$
$$X_{\mathbb{C},2\Omega} = \{z \in X_{\mathbb{C}} : P(z) \in \mathbb{C}\backslash]-\infty,-2]\}\,.$$

We have to understand the inclusion $\Xi \subset X_{\mathbb{C}}(\Omega) \subset X_{\mathbb{C}}(2\Omega)$ better. It turns out that Ξ cannot be enlarged. Here is the precise result.

Lemma 5.12. *Let $G = \mathrm{Sl}(2, \mathbb{R})$. Then for $Y \in 2\Omega \backslash \overline{\Omega}$,*

$$G \exp(iY) \cdot x_0 \not\subseteq X_{\mathbb{C}, 2\Omega} .$$

More precisely, there exists a curve $\gamma(s)$, $s \in [0,1]$, in G such that the assignment

$$s \mapsto \sigma(s) = P(\gamma(s) \exp(iY) \cdot x_o)$$

is strictly decreasing with values in $[-2, 2]$ such that $\sigma(0) = P(x_o) = 2$ and $\sigma(1) = -2$.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in \exp(2i\Omega) \backslash \exp(i\overline{\Omega})$. This means $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and $\frac{\pi}{4} < |\phi| < \frac{\pi}{2}$ for $\phi \in \mathbb{R}$. Thus

$$\begin{aligned} p(gz \cdot x_o) &= p \begin{pmatrix} ae^{i\phi} & be^{-i\phi} \\ ce^{i\phi} & de^{-i\phi} \end{pmatrix} = a^2 e^{2i\phi} + b^2 e^{-2i\phi} + c^2 e^{2i\phi} + d^2 e^{-2i\phi} \\ &= \cos(2\phi)(a^2 + b^2 + c^2 + d^2) + i \sin 2\phi (a^2 - b^2 + c^2 - d^2) \end{aligned}$$

Using that $G = KAN$ and that p is left K-invariant, we may actually assume that $g \in AN$, i.e.

$$g = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$$

for some $a > 0$ and $b \in \mathbb{R}$. Then

$$p(gz \cdot x_o) = \cos(2\phi)(a^2 + \frac{1}{a^2} + b^2) + i \sin 2\phi (a^2 - \frac{1}{a^2} - b^2).$$

We now show that $p(gz \cdot x_o) = -2$ has a solution for fixed $\frac{\pi}{4} < |\phi| < \frac{\pi}{2}$. This is because $p(gz \cdot x_o) = -2$ forces $\operatorname{Im} p(gz \cdot x_o) = 0$ and so $b^2 = a^2 - \frac{1}{a^2}$. Thus

$$p(gz \cdot x_o) = 2a^2 \cos(2\phi) = -2.$$

Thus if we choose $a = \frac{1}{\sqrt{-\cos 2\phi}}$ we obtain a solution. The desired curve $\gamma(s)$ is now given by

$$\gamma(s) = \begin{pmatrix} a(s) & b(s) \\ 0 & \frac{1}{a(s)} \end{pmatrix}$$

with

$$a(s) = \frac{1}{\sqrt{-\cos 2\phi}} (\sqrt{-\cos 2\phi} + s(1 - \sqrt{-\cos 2\phi}))$$

and

$$b(s) = \sqrt{a(s)^2 - \frac{1}{a(s)^2}} .$$

$\square$

We are ready for the

Proof of Theorem 5.9. We first observe from our previous discussion that there exists a holomorphic function Φ_λ on $\mathbb{C}\backslash(-\infty, 2] = p(X_{\mathbb{C},2\Omega})$ such that

$$\phi_\lambda(z) = \Phi_\lambda(P(z)) \qquad (z \in X_{\mathbb{C},2\Omega}). \tag{5.7}$$

Let $Y \in 2\Omega\backslash\overline{\Omega}$. Let $\gamma \subset G$ and $\sigma \subset [-2, 2]$ be curves as in the previous lemma.

Note that $\gamma(s)\exp(iY) \cdot x_o \subset G$ for all $s \in [0, 1)$. Hence (5.7) gives

$$\varphi_\lambda(\gamma(s)\exp(iY) \cdot x_o) = \Phi_\lambda(\sigma(s)) \qquad (s \in [0, 1)\,.$$

Now recall that $s \mapsto \Phi_\lambda(\sigma(s))$ is positive by (5.6) and tends to infinity for $s \nearrow 1$ (cf. [24], Th. 2.4). Let now $\Xi \subset \Xi'$ be a G-domain in $X_{\mathbb{C}}$ which strictly contains Ξ. Thus $\partial\Xi \cap \Xi' \neq \emptyset$. We recall that $\partial\Xi = \partial_d\Xi \cup \partial_u\Xi$ and distinguish two cases.

Case 1: $\partial_d\Xi \cap \Xi' \neq \emptyset$. In this case Ξ' contains a point $\exp(i2\Omega\backslash\overline{\Omega}) \cdot x_0$ and we arrive at a contradiction.

Case 2: $\partial_n\Xi \cap \Xi' \neq \emptyset$. This means that $\begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix} \in \Xi'$ for some t with absolute value sufficiently close to 1 by (4.2).

With $a_r = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \in A$, $r > 0$, and $-1 < t < 1$ that

$$p\left(a_r \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix} .x_0\right) = r^2 + \frac{1}{r^2} - t^2 r^2\,.$$

In particular, if $|t| > 1$, then there would exist a sequence $r_n \to r_0$ such that $p\left(a_{r_t}\begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}\right) \to -2^+$. We argue as before. $\square$

5.3. **Holomorphic H-spherical vectors**

To begin with I want to explain a few things on spherical representations first. Throughout this section we let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. For a subgroup $L < G$ we write $\mathcal{H}^L \subset \mathcal{H}$ for the subspace of L-fixed elements. As a consequence of the Riemann-Lebesgue Lemma for representations we obtain:

Lemma 5.13. *If $L < G$ is closed and non-compact and π is non-trivial, then $\mathcal{H}^L = \{0\}$.*

So why is this of interest. In case of finite groups, Frobenius reciprocity tells us that π can be realized in functions on G/L if and only if

$\mathcal{H}^L \neq \{0\}$. For non-compact continuous groups we need a more sophisticated version of Frobenius reciprocity: the Hilbert space $\mathcal{H}$ is simply too small for carrying L-fixed elements. We enlarge $\mathcal{H}$. Recall the space of analytic vectors $\mathcal{H}^\omega$ of π. This is a locally convex topological vector space of compact type, i.e. a Hausdorff direct limit space with compact inclusion maps. We form $\mathcal{H}^{-\omega}$, the strong anti-dual of $\mathcal{H}^\omega$, i.e. the space of continuous anti-linear functionals $\mathcal{H}^\omega \to \mathbb{C}$ endowed with the strong topology. As a topological vector space $\mathcal{H}^{-\omega}$ is nuclear Fréchet. In particular it is reflexive, i.e. its strong anti-dual gives us $\mathcal{H}^\omega$ back. We note that $\mathcal{H}$ is naturally included in $\mathcal{H}^{-\omega}$ via $v \mapsto \langle \cdot, v\rangle$ and obtain the reflexive sandwiching

$$\mathcal{H}^\omega \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}$$

with all inclusions G-equivariant and continuous. Sometimes one calls $(\mathcal{H}^\omega, \mathcal{H}, \mathcal{H}^{-\omega})$ a *Gelfand triple.*

Now for $G = \mathrm{Sl}(2,\mathbb{R})$ and $H = \mathrm{SO}(1,1)$ there is the dimension bound

$$\dim(\mathcal{H}^{-\omega})^H \leq 2\,.$$

To be more precise, for highest or lowest weight representations the dimension is zero or 1 depending on the parity of the smallest K-type. For the principal series the dimension is 2.

Example 5.14. *For a principal series representation π_λ the space of H-fixed hyperfunction vectors is given by $(\mathcal{H}^{-\omega})^H = \mathrm{span}_{\mathbb{C}}\{\eta_1, \eta_2\}$ with*

$$\eta_1(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(1-x^2)^{\frac{1}{2}(1-i\lambda)}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1; \end{cases}$$

and

$$\eta_2(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(x^2-1)^{\frac{1}{2}(1-i\lambda)}} & \text{for } |x| > 1, \\ 0 & \text{for } |x| \leq 1. \end{cases}$$

We take a closer look at the basis $\{\eta_1, \eta_2\}$ in the previous example. For what follows it is useful to compactify $\mathbb{R}$ to $\mathbb{P}^1(\mathbb{R}) = G/MAN$ and view $\mathcal{H}$ as a function space on $\mathbb{P}^1(\mathbb{R})$. Then both η_1 and η_2 are supported on the two open H-orbits in $\mathbb{P}^1(\mathbb{R})$, namely $(-1,1)$ and $\mathbb{P}^1(\mathbb{R})\backslash[-1,1]$. Thus η_1, η_2 appear to be natural in view of the natural H-action on the flag variety. However, we claim that it is not the natural basis for $(\mathcal{H}^{-\omega})^H$. Why? Simply because it is not invariant under intertwining operators – intertwiners here are pseudo-differential operators which do not preserve supports. So it is our aim to provide a natural basis for the H-sphericals. For that our theory of holomorphic extension of representations comes handy.

Our motivation comes from finite dimensional representations.

5.3.1. *Finite dimensional spherical representations.* Let (ρ, V) be a representation of G on a finite dimensional complex vector space V. Then ρ naturally extends to a holomorphic representation of V, also denoted by ρ, and observe:

$$V^K = V^{K_\mathbb{C}} \qquad \text{and} \qquad V^H = V^{H_\mathbb{C}}\,.$$

Here is the punch line: While H and K are not conjugate in G (one is non-compact, one is compact), their complexifications $H_\mathbb{C}$ and $K_\mathbb{C}$ are conjugate in $G_\mathbb{C}$. With

$$z_H = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

there is the identity:

$$z_H H_\mathbb{C} z_H^{-1} = K_\mathbb{C}\,.$$

Therefore the map

$$V^K \to V^H, \quad v \mapsto \rho(z_H)v \tag{5.8}$$

is an isomorphism.

5.3.2. *Construction of the holomorphic H-spherical vector.* Our goal here is to find an analogue of (5.8) for infinite dimensional representations. For what follows we assume in addition that $(\pi, \mathcal{H})$ is K-spherical and fix a normalized generator $v_K \in \mathcal{H}^K$. Now, observe that $z_H \cdot x_0 \in \partial_d \Xi = Y = G/H$. For $\epsilon > 0$ we set

$$a_\epsilon := \begin{pmatrix} e^{i(\pi/4-\epsilon)} & 0 \\ 0 & e^{-i(\pi/4-\epsilon)} \end{pmatrix}$$

and remark:

$$\lim_{\epsilon\to 0} a_\epsilon = z_H \qquad \text{and} \qquad a_\epsilon \in \Xi K_\mathbb{C}\,.$$

In particular $\pi(a_\epsilon)v_K$ exists for all $\epsilon > 0$ small. It is no surprise that the limit exists in $\mathcal{H}^{-\omega}$ and is H-fixed. In fact it is a matter of elementary functional analysis to establish the following theorem, see [11], Th. 2.1.3 for a result in full generality.

Theorem 5.15. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Then the map*

$$\mathcal{H}^K \to (\mathcal{H}^{-\omega})^H, \quad v_K \mapsto v_H := \lim_{\epsilon\to 0} \pi(a_\epsilon)v_K$$

is defined and injective.

We call the vector v_H the *H-spherical holomorphic hyperfunction vector* of π. It is natural in the sense that it is preserved by intertwining (observe that intertwiners commute with analytic continuation). We will return to this topic later when we discuss the most continuous spectrum of $L^2(Y)$.

We wish to make v_H explicit for the principal series π_λ. A simple calculation gives

$$v_H = e^{-i\frac{\pi}{4}(1-\lambda)}\eta_1 + e^{i\frac{\pi}{4}(1-\lambda)}\eta_2 \,.$$

Upon conjugating the coefficients we get a second, linearly independent vector

$$\overline{v_H} = e^{i\frac{\pi}{4}(1-\lambda)}\eta_1 + e^{-i\frac{\pi}{4}(1-\lambda)}\eta_2 \,.$$

which we call the anti-holomorphic H-spherical vector. Likewise one obtains $\overline{v_H}$ by using $\overline{z_H} = z_H^{-1}$ instead of z_H. It features the same invariance properties as v_H. We therefore arrive at a basis

$$\{v_H, \overline{v_H}\}$$

of $(\mathcal{H}^{-\omega})^H$ which is invariant under intertwining, i.e. a canonical diagonalization of scattering in the affine symmetric space Y.

6. Growth of holomorphically extended orbit maps

Throughout this section $(\pi, \mathcal{H})$ is a unitary irreducible representation of G and $v = v_K \in \mathcal{H}^K$ a normalized K-finite vector. Our objective of this section is to discuss the growth of the orbit map

$$f_v : \Xi \to \mathcal{H}, \quad zK_\mathbb{C} \mapsto \pi(z)v$$

for z approaching the boundary of Ξ. We are interested in two quantitities:

- The norm of $\|\pi(z)v\|$ for $z \to \partial\Xi$.
- The invariant Sobolev norms $S_k^G(\pi(z)v)$ for $z \to \partial\Xi$.

The invariant Sobolev norms were introduced by Bernstein and Reznikov in [4] as a powerful tool to give growth estimates for analytically continued automorphic forms. We will comment more on that in the subsections below.

We notice that

$$\|f_v(g\exp(iY)\cdot x_0)\| = \|\pi(\exp(iY))v\|$$

for all $g \in G$ and $Y \in \Omega$. Thus for our growth- interest for $z \mapsto \partial\Xi$ we may assume that $z = \exp(iY) \cdot x_0$ for $Y \to \partial\Omega$, or with our previous notation with $Z = a_\epsilon \cdot x_0$ for $\epsilon \to 0$.

6.1. **Norm estimates**

Here we determine the behaviour of

$$\|\pi(a_\epsilon)v\| \qquad \text{for } \epsilon \to 0\,.$$

For $G = \mathrm{Sl}(2,\mathbb{R})$ this is a simple matter - for general G this is a serious and difficult problem; it was settled in [25].

Proposition 6.1. *Let $(\pi, \mathcal{H})$ be a unitary K-spherical representation of G and v a normalized K-fixed vector. Then*

$$\|\pi(a_\epsilon)v\| \asymp \sqrt{|\log \epsilon|}$$

for $\epsilon \to 0$.

Proof. It is no big loss of generality to assume that $\pi = \pi_\lambda$. Within the non-compact realization we determine:

$$\begin{aligned}
\|\pi(a_\epsilon)v\|^2 &= \frac{1}{\pi} e^{\lambda\pi/2} \int_{\mathbb{R}} \left| \frac{1}{(1 + e^{-i(\pi - 4\epsilon)} x^2)^{\frac{1}{2}(1+i\lambda)}} \right|^2 \, dx\,, \\
&\asymp \int_{-2}^{2} \left| \frac{1}{(1 + (-1 + i\epsilon)x^2)} \right| \, dx\,, \\
&\asymp \int_0^1 \frac{1}{(|u| + \epsilon)} \, du\,, \\
&\asymp |\log \epsilon|\,.
\end{aligned}$$

$\square$

I want to pose the following

Problem: *Fix $\sigma \in \hat{K}$ and let $\mathcal{H}(\sigma)$ be the corresponding K-type. Determine optimal bounds for*

$$\|\pi(a_\epsilon)v\| \qquad (v \in \mathcal{H}(\sigma))$$

for $\epsilon \to 0$. Possibly generalize to all semi-simple groups.

6.2. **Invariant Sobolev norms**

We first recall some definitions from [4].

Definition 6.2. (Infimum of seminorms; cf. [4], Appendix A) *Let V be a complex vector space and $N_{i\in I}$ a family of semi-norms. Then the prescription*

$$\inf_{i\in I} N_i(v) := \inf_{v=\sum_{i\in I} v_i} \sum_{i\in I} N_i(v_i)$$

defines a semi-norm. It is the largest seminorm with respect to the property of being dominated by all N_i.

Remark 6.3. *To get an idea of the nature of the definition of the infimum seminorm* $\inf N_i$ *it is good to think in the following analogy: Think of V as a function space, say on $\mathbb{R}$ and think of N_i as a semi-norm with support on a certain interval, say J_i. such that $\cup J_i = \mathbb{R}$. Further $v = \sum_{i\in I} v_i$ should be considered as breaking the function v into functions v_i with smaller support in J_i.*

We want to bring in a symmetry group G which acts linearly on the vector space V. We start with one seminorm $N : V \to \mathbb{R}_{\geq} 0$ and produce others: for $g \in G$ we let

$$N_g(v) := N(g(v))\,.$$

In this way we obtain a seminorm

$$N^G := \inf_{g\in G} N_g(v)$$

which is uniquely characterized as being the largest G-invariant semi-norm on V which is dominated by N.

We come to specific choices for V and N. For V we use the Fréchet-space of smooth vectors $\mathcal{H}^\infty$ for the representation π; the seminorm N will be Sobolev norm. We briefly recall their construction. Recall that the derived representation $d\pi$ of $\mathfrak{g}$ is defined as

$$d\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty), \quad d\pi(Z)(v) := \frac{d}{dt}\Big|_{t=0} \pi(\exp(tZ))v\,.$$

We fix a basis Z_1, Z_2, Z_3 of $\mathfrak{g}$ and an integer $k \in \mathbb{N}_0$. Then the k-th Sobolev norm S_k of π is defined as

$$S_k(v) := \sum_{k_1+k_2+k_3\leq k} \|d\pi(Z_1)^{k_1} d\pi(Z_2)^{k_2} d\pi(Z_3)^{k_3} v\| \qquad (v \in \mathcal{H}^\infty)\,.$$

Let us emphasize that S_k depends on the chosen basis Z_1, Z_2, Z_3, but a different basis yields an equivalent norm. Our interest is now with S_k^G the G-invariant Sobolev norm. Notice that $S_0^G = \|\cdot\|$ is the Hilbert norm, as we assume that π is unitary. In view of our preceding remark it is natural to view S_k^G as some Besov-type norm for the representation.

We wish to understand the nature of S_k^G. For that it is useful to introduce the following notation: For a closed subgroup $L < G$ we write $S_{k,L}$ for the k-th Sobolev norm for the restricted representation $\pi|_L$. We make a first simple observation:

Lemma 6.4. *Let $(\pi, \mathcal{H})$ be a unitary representation of G and $v \in \mathcal{H}^\infty$. Then for all $k \geq 0$:*

(i) $S_{k,N}^{A^+}(v) = \|v\|$.
(ii) $S_{k,AN}^{G}(v) = S_k^G(v)$.

Proof. Easy; see [23], Lemma 6.5 for the general statement. □

The following Theorem is fundamental ([23], Prop. 6.6).

Theorem 6.5. *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Let $k \in \mathbb{Z}_{\geq 0}$. Then there exists a constant $C = C(k, \pi)$ such that*

$$S_k^G(v) \leq C \cdot S_{k,A}^G(v) \qquad (v \in \mathcal{H}^\infty).$$

Proof. We will only treat the case of $\pi = \pi_\lambda$. We remark that

$$\mathcal{H}^\infty = \{ f \in C^\infty(\mathbb{R}) \, |x|^{i\lambda - 1} f(\frac{1}{x}) \in C^\infty(\mathbb{R}) \}$$

and introduce some standard notation

We use a usual basis for the Lie algebra of γ

$$\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathfrak{a} = \mathbb{R}\mathbf{h}$, $\mathfrak{n} = \mathbb{R}\mathbf{e}$ and $\overline{\mathfrak{n}} = \mathbb{R}\mathbf{f}$. With $\mathbf{u} = \mathbf{e} - \mathbf{f}$ we have $\mathfrak{k} = \mathbb{R}\mathbf{u}$. Differentiating the action (5.2) one obtains the formulas

$$d\pi_\lambda(\mathbf{h}) = (i\lambda - 1) - 2x\frac{d}{dx}, \tag{6.1}$$

$$d\pi_\lambda(\mathbf{e}) = -\frac{d}{dx}, \tag{6.2}$$

$$d\pi_\lambda(\mathbf{f}) = (1 - i\lambda)x + x^2\frac{d}{dx}, \tag{6.3}$$

$$d\pi_\lambda(\mathbf{u}) = (i\lambda - 1) - (1 + x^2)\frac{d}{dx}, \tag{6.4}$$

$$d\pi_\lambda(\mathbf{e} + \mathbf{f}) = (1 - i\lambda)x - (1 - x^2)\frac{d}{dx}. \tag{6.5}$$

We also define the *radial operators* by

$$(R_j f)(x) = (x^j \frac{d^j}{dx^j} f)(x)$$

and define the *radial Sobolev norms* by

$$S_{k,\mathrm{rad}}(f) = \sum_{j=0}^{k} \|R_j f\|.$$

From the action of $d\pi_\lambda(\mathbf{h})$ and R^j it is clear that there exists a constant $C > 0$, depending on k and λ, such that for all $f \in \mathcal{S}(\mathbb{R})$

$$\frac{1}{C} S_{k,\mathrm{rad}}(f) \leq S_{k,A}(f) \leq C S_{k,\mathrm{rad}}(f)\,. \tag{6.6}$$

We wish to point out that in (6.1) and (6.3) the coefficient of the derivative term has a zero, consequently $S_k(v)$ can not be majorized by $S_{k,A\overline{N}}(v)$ or by $S_{k,A}(v)$ in general. However, we shall show in the next Proposition that there is such a relationship for the $G-$invariant Sobolev norms.

The A action on $K/M \cong S^1$ has two fixed points, corresponding to the two Bruhat cells. In the non-compact realization N they become the origin and the point at infinity. We shall estimate $S_k^G(f)$ by using first a cutoff function at infinity, $\overline{\mathfrak{n}}$, and an elementary estimate there. Near the origin a dilated cutoff localizes sufficiently high derivatives of f to get an estimate. Away from the fixed points, motivated by an argument in [4] and classical Littlewood-Paley theory, we use a family of suitably dilated cutoff functions which compress the $\mathfrak{n}$ derivatives in the definition of G-invariant norm to *radial* derivatives thereby obtaining the desired estimate.

For $j \in \mathbb{Z}$ we denote by I_j the set $\{x \in \mathbb{R}\, 2^{-j-1} \leq |x| \leq 2^{-j+1}\}$. For a function ψ on $\mathbb{R}$ we write $\psi_j(x) = \psi(2^j x)$. Notice that if ψ is supported in I_0 then ψ_j is supported in I_j, and

$$\mathrm{supp}(\psi_j) \ \cap \ \mathrm{supp}(\psi_{j+1}) \subseteq [\frac{1}{2^{j+1}}, \frac{1}{2^j}].$$

We take a smooth, non-negative function ϕ supported in I_0 and such that for every $m \in \mathbb{N}_0$ we have

$$\sum_{j=0}^{m} \phi_j(x) = \begin{cases} 0 & \text{if } |x| \leq 2^{-m-1}\,, \\ 1 & \text{if } 2^{-m} \leq |x| \leq 1\,, \\ 0 & \text{if } 2 \leq |x| \end{cases}$$

Choose a nonnegative function $\tau \in C^\infty(\mathbb{R})$ with support in $\{x \in \mathbb{R}\, 1 \leq |x|\}$ such that $(\tau + \phi)(x) = 1$ for $|x| \geq 1$. Finally for each $m \in \mathbb{N}$

define the function $\tau_m \in C_c^\infty(\mathbb{R})$ by $\tau_m = \mathbf{1} - \tau - \sum_{j=0}^m \phi_j$. Notice that $\operatorname{supp}\tau_m \subset \{x \in \mathbb{R} \mid |x| \leq 2^{-m}\}$ and $\tau_m(x) = 1$ for $|x| \leq 2^{-m-1}$. From the properties of the ϕ_j and τ it is easy to see that for any $l \geq 1$, $\tau_m^{(l)}(x) = -2^{lm}\phi^{(l)}(2^m x)$.

Let $f \in \mathcal{H}^\infty$. Since

$$\begin{aligned}\mathbf{1} &= \tau + \mathbf{1} - \tau\\ &= \tau + \tau_m + \sum_{j=0}^m \phi_j\\ &= \tau + \phi + \tau_m + \sum_{j=1}^m \phi_j\,,\end{aligned}$$

then

$$f = (\tau+\phi)f + \tau_m f + \sum_{j=1}^m \phi_j f.$$

For any choices of $g, g_1, \ldots, g_m \in G$, using the definition of S_k^G, we get

$$S_k^G(f) \leq S_k((\tau+\phi)f) + S_k(\pi_\lambda(g)(\tau_m f)) + \sum_{j=1}^m S_k(\pi_\lambda(g_j)(\phi_j f))\,. \tag{6.7}$$

First we consider the term $S_k((\tau+\phi)f)$. From an examination of formulas (6.1) - (6.3) one sees that $S_k((\tau+\phi)f) \leq CS_{k,\overline{N}}((\tau+\phi)f)$ for all $f \in \mathcal{H}^\infty$. (Throughout this proof C will denote a constant depending only on k, τ, ϕ and λ.) Hence we have

$$S_k((\tau+\phi)f) \leq CS_{k,\overline{N}}((\tau+\phi)f) \leq CS_{k,\overline{N}}(f)$$

for all $f \in \mathcal{H}^\infty$. Majorizing this term in (6.8) we get

$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + S_k((\pi_\lambda(g)\tau_m f)) + \sum_{j=1}^m S_k(\pi_\lambda(g_j)(\phi_j f)) \tag{6.8}$$

for all $f \in \mathcal{H}^\infty$.

Next we specify a good choice of the elements $g, g_1, \ldots, g_m \in G$. For every $t > 0$ denote by b_t the element

$$b_t = \begin{pmatrix} \frac{1}{\sqrt{t}} & 0 \\ 0 & \sqrt{t} \end{pmatrix} \in A.$$

From (5.2) it follows that

$$(\pi_\lambda(b_t)f)(x) = t^{\frac{1}{2}(1-\lambda)} f(tx)$$

for all $t > 0$ and $x \in \mathbb{R}$. Take $g_j = b_{2^{-j}}$ for all $1 \leq j \leq m$ and $g = b_{2^{-(m+1)}}$. Notice that for every m all the $\pi_\lambda(g_j)(\phi_j f)$ are supported

in $[-2,2]$, as is $\pi_\lambda(g)(\tau_m f)$. For any smooth function h supported in $[-2,2]$ we can conclude from the formulas (6.1) - (6.4) that $S_k(h) \leq CS_{k,N}(h)$. Using this in (6.9) we get
(6.10)

$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + CS_{k,N}(\pi_\lambda(g)(\tau_m f)) + C\sum_{j=1}^{m} S_{k,N}(\pi_\lambda(g_j)(\phi_j f))$$

for all $f \in \mathcal{H}^\infty$.

We estimate $S_{k,N}(\pi_\lambda(g)(\tau_m f))$. For this we use Leibniz on $\tau_m f$ and L^∞ estimates on $\tau_m^{(j)} = -2^{jm}\phi^{(j)}(2^m x)$. From (6.2) one sees that $S_{k,N}(h) = \sum_{l=0}^{k} \|h^{(l)}\|$. Then

$$\begin{aligned}
S_{k,N}(\pi_\lambda(g)(\tau_m f)) &= \sum_{l=0}^{k} \|\frac{d^l}{dx^l} 2^{-\frac{(m+1)}{2}(1-\lambda)}(\tau_m f)(2^{-(m+1)}\cdot)\| \\
&= \sum_{l=0}^{k} |2^{-\frac{(m+1)}{2}(1-\lambda)}| \Big[\int \Big| \sum_{n=0}^{l} 2^{-(m+1)l}\binom{l}{l-n} \cdot \\
&\qquad \cdot \tau_m^{(l-n)}(2^{-(m+1)}x) f^{(n)}(2^{-(m+1)}x)\Big|^2 \, dx\Big]^{\frac{1}{2}} \\
&\leq \sum_{l=0}^{k} |2^{-\frac{(m+1)}{2}(1-\lambda)}| \sum_{n=0}^{l} \Big[\int_{|x|\leq 2} \Big| 2^{-(m+1)l}\binom{l}{l-n} \cdot \\
&\qquad \cdot \tau_m^{(l-n)}(2^{-(m+1)}x) f^{(n)}(2^{-(m+1)}x)\Big|^2 \, dx\Big]^{\frac{1}{2}} \\
&= \sum_{l=0}^{k} \Big|2^{\frac{(m+1)}{2}\lambda}\Big| \sum_{n=0}^{l} \Big[\int_{|y|\leq \frac{1}{2^m}} \Big|2^{-(m+1)l}\binom{l}{l-n}\tau_m^{(l-n)}(y) f^n(y)\Big|^2 \, dy\Big]^{\frac{1}{2}} \\
&\leq \sum_{l=0}^{k} |2^{\frac{(m+1)}{2}\lambda}| \sum_{n=0}^{l} \binom{l}{l-n} \frac{\|2^{(l-n)m}\phi^{(l-n)}\|_\infty}{2^{(m+1)l}} \Big[\int_{|y|\leq\frac{1}{2^m}} |f^{(n)}(y)|^2 \, dy\Big]^{\frac{1}{2}} \\
&= \sum_{n=0}^{k} |2^{\frac{(m+1)}{2}\lambda}| \frac{1}{2^{mn}} \sum_{l=n}^{k} \binom{l}{l-n} \frac{\|\phi^{(l-n)}\|_\infty}{2^l} \Big[\int_{|y|\leq\frac{1}{2^m}} |f^{(n)}(y)|^2 \, dy\Big]^{\frac{1}{2}} \\
&= \sum_{n=0}^{k} |2^{\frac{(m+1)}{2}\lambda}| \frac{1}{2^{(m+1)n}} \sum_{j=0}^{k-n} \binom{j+n}{n} \frac{\|\phi^j\|_\infty}{2^j} \Big[\int_{|y|\leq\frac{1}{2^m}} |f^{(n)}(y)|^2 \, dy\Big]^{\frac{1}{2}} \\
&\leq (\sum_{j=0}^{k} \frac{\|\phi^{(j)}\|_\infty}{j!2^j}) \sum_{n=0}^{k} \frac{k!}{n!2^{(m+1)n}} \Big[\int_{|y|\leq\frac{1}{2^m}} |f^{(n)}(y)|^2 \, dy\Big]^{\frac{1}{2}}.
\end{aligned}$$

Now k is fixed and each of the at most k derivatives $f^{(n)}$ is in L^2, hence the integrals can be made uniformly small. So for each f we can choose an m so that the last line above is at most $\|f\|$. Then we have

$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + C\|f\| + C\sum_{j=1}^{m} S_{k,N}(\pi_\lambda(g_j)(\phi_j f))$$

for any $f \in \mathcal{H}^\infty$. Thus we obtain that

(6.9)
$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + C\|f\| + C\sum_{l=0}^{k}\sum_{j=1}^{m} \|\frac{d^l}{dx^l}(2^{-\frac{j}{2}(1-i\lambda)}\phi f(2^{-j}\cdot))\| \,.$$

As in the long computation above, using Leibniz on ϕf, L^∞ estimates on $\phi^{(j)}$, and majorizing the binomial coefficients we get

$$\begin{aligned}
\sum_{l=0}^{k}\sum_{j=1}^{m} \|\frac{d^l}{dx^l}(2^{-\frac{j}{2}}\phi f(2^{-j}\cdot))\| &\leq C\sum_{l=0}^{k}\sum_{j=1}^{m}\Big(\int_{I_0} 2^{-j-2l}|f^{(l)}(2^{-j}x)|^2 \, dx\Big)^{\frac{1}{2}} \\
&= C\sum_{l=0}^{k}\sum_{j=1}^{m}\Big(\int_{I_j} 2^{-2l}|f^{(l)}(x)|^2 \, dx\Big)^{\frac{1}{2}} \\
&\leq 4C\sum_{l=0}^{k}\sum_{j=1}^{m}\Big(\int_{I_j} |x^l f^{(l)}(x)|^2 \, dx\Big)^{\frac{1}{2}} \\
&\leq 4CS_{k,\mathrm{rad}}(f) \leq 4CS_{k,A}(f),
\end{aligned}$$

where the last inequality follows from (6.6) and again C depends only on τ, ϕ, k and λ. Thus we get from (6.9) and (**??**) that

$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + C\|f\| + CS_{k,A}(f) \leq C\|f\| + CS_{k,A\overline{N}}(f)$$

for all $f \in \mathcal{H}^\infty$. Thus

$$S_k^G \leq CS_{k,A\overline{N}}^G$$

and, using Lemma 6.4(ii), $S_k^G \leq CS_{k,A}^G$ as was to be shown. $\square$

With regard to the above theorem I want to pose the following

Problem: *Formulate and possibly prove the above result for all semisimple groups.*

We come to the main result of this section, see [23], Th. 6.7: the estimate for $S_k^G(\pi(a_\epsilon)v)$. We will only explain the idea and refer to [23] for a discussion in full detail. We fix on the case $\pi = \pi_\lambda$ and observe that, up to constant:

$$[\pi(a_\epsilon)v](x) = \frac{1}{(1 + e^{i\pi(1-\epsilon)}x^2)^{\frac{1}{2}(1-i\lambda)}} \qquad (x \in \mathbb{R})$$

Hence $\pi(a_\epsilon)v(x)$ develops singularities at $x = \pm 1$ which are logarithmic in the L^2-sense, see Proposition 6.1 from above. Taking the k-th Sobolev norm increases the singularity accordingly; one verifies for $k \geq 1$ that

$$S_k(\pi(a_\epsilon)v) \asymp \epsilon^{-k}\,.$$

It is so remarkable that the situation is much different for $S_k^G(\pi(a_\epsilon)v$. Why? Observe that

$$S_{k,H}(\pi(a_\epsilon)v) \asymp \|\pi(a_\epsilon)v\| \tag{6.10}$$

as the fixed points of H are precisely $x = \pm 1$, the loci where the function $\pi(a_\epsilon)v$ develops singularities (cf. with (6.5)). Now with

$$k_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in K$$

there is an element which rotates $\mathfrak{a}$ to $\mathfrak{h}$. Hence

$$S_{k,A}(\pi(k_0)\pi(a_\epsilon)v) = S_{k,H}(\pi(a_\epsilon)v)$$

and combined with (6.10) we arrive at the hardest result in this article.

Theorem 6.6. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G and $v \in \mathcal{H}$ a K-fixed vector. Let $k \in \mathbb{Z}_{\geq 0}$. Then there exists a constant $C = C(\pi, k)$ such that*

$$S_k^G(\pi(a_\epsilon)v) \leq C\|\pi(a_\epsilon)v\|$$

for all $\epsilon > 0$ small.

I expect the theorem from above to be true for all K-finite vectors v with the reservation that $C = C(\pi, K)$ depends on the occuring K-types in the support of v in addition. In [23] we conjecture (Conjecture C) that the the estimate holds even for arbitrary semisimple Lie groups. This is very difficult. For real rank one we could establish this for the K-fixed vector in [23].

7. Harmonic analysis on the crown

7.1. Holomorphic extension of eigenfunctions

Let

$$\Delta = -y^2(\partial_x^2 + \partial_y^2)$$

be the Laplace-Beltrami operator on X. For $\mu \in \mathbb{C}$ we consider the eigenvalue problem

$$\Delta\phi = \mu(1-\mu)\phi\,.$$

We observe that solutions ϕ are necessarily analytic functions as Δ is an elliptic operator. Analytic functions admit holomorphic extensions to some complex neighborhood of X in $X_\mathbb{C}$. Further, as G commutes with Δ, the resulting domain $D_\phi \subset X_\mathbb{C}$ attached to ϕ is G-invariant. By now it should be no surprise that $D_\phi = \Xi$ for generic choices of ϕ. In fact it is just a disguise of the non-unitary version of Theorem 5.6, see [24], Th. 1.1 and Prop. 1.3.

Theorem 7.1. *All Δ-eigenfunctions on X extend to holomorphic functions on Ξ.*

Proof. At this point it would better to switch from X to its bounded realization: the unit disc. It has the advantage of circular symmetry on a compact boundary and results in a good grip concerning convergence problems of boundary value issues on X. However, I do not want to do that and thus certain convergence issues will remain untreated below.

To begin with we recall the Poisson-kernel P on X:

$$P(z) = \frac{1}{\pi}\frac{\operatorname{Im} z}{z\cdot\overline{z}} \qquad (z \in X)\,.$$

Now if $\Delta\phi = \mu(1-\mu)\phi$ with $\mu \neq 0$, then there is a generalized function $\phi_\mathbb{R}$ on $\mathbb{R}$ as boundary value of ϕ from which we can reconstruct ϕ via Poisson integration:

$$\phi(z) = \int_\mathbb{R} \phi_\mathbb{R}(x) P^\mu(z-x)\; dx\,.$$

Now observe that P admits a holomorphic extension $P^\sim$ to $\Xi = X \times \overline{X}$ obtained by polarization:

$$P^\sim(z,w) = \frac{1}{2\pi i}\frac{z-w}{z\cdot w} \qquad ((z,w)\in\Xi)\,.$$

Thus ϕ admits a holomorphic extension $\phi^\sim$ to Ξ by setting

$$\phi^\sim(z,w) = \int_\mathbb{R} \phi_\mathbb{R}(x)(P^\sim)^\mu(z-x, w-x)\; dx\,.$$

□

7.2. **Paley-Wiener revisited**

Let us begin with a short disgression into history: the theorems of Paley and Wiener [28] on the restriction of the Fourier transform to various meaningful function spaces.

When dealing with Fourier analysis on $\mathbb{R}^n$ one often identifies $\mathbb{R}^n$ with its dual space. However, it is better not to do it in order to avoid confusion between the geometric and spectral features.

Let V be a finite dimensional real vector space V. Its dual space shall be denoted V^*. We fix an Euclidean structure on V (and hence on V^*) and normalize the resulting Lebesgue measures dv, $d\alpha$ such that the Fourier transform

$$\mathcal{F} : L^1(V) \to C_0(V^*), \quad f \mapsto \mathcal{F}(f) := \hat{f};\ \hat{f}(\alpha) := \int_V e^{i\alpha(v)} f(v)\ dv$$

extends to an isometry $L^2(V) \to L^2(V^*)$.

Actually we wish to view V^* as $\hat{V}$, the unitary dual of the abelian group $(V, +)$. The isomorphism is given by

$$V^* \to \hat{V}, \quad \alpha \mapsto \chi_\alpha;\ \chi_\alpha(v) = e^{i\alpha(v)}\ .$$

For a general, say reductive, group G, we know from the work of Segal that there is a Fourier transform from $L^1(G)$ to a Hilbert-valued fiber bundle $\mathcal{V} \to \hat{G}_{\rm temp}$ over the tempered unitary dual $\hat{G}_{\rm temp}$ of G which extends to an isometry $\mathcal{F} : L^2(G) \to \Gamma^2(\mathcal{V})$. Here Γ^2 stands for the L^2-sections of the bundle with respect to the Plancherel measure which was determined explicitly by Harish-Chandra, [15].

Back to our original setup of V and V^*. In the context of Fourier transform one might ask about the image of certain function spaces, for instance test functions, Schwartz functions, their duals, or of functions on V which extend holomorphically to some tube domain in $V_\mathbb{C} = V + iV$. Paley-Wiener theory is concerned with the first and last mentioned examples in the uplisting. For a more serious discussion we need more precision.

The image of test functions. We want to characterize $\mathcal{F}(C_c^\infty(V))$. For that we define for every $R > 0$ the subspace $C_R^\infty(V)$ of those test functions which are supported in the Euclidean ball of radius R. Likewise we define $\mathrm{PW}_R(V_\mathbb{C}^*)$ to be the space of those holomorphic functions f on $V_\mathbb{C}^*$, the complexification of V^*, which satisfy the growth condition

$$|f(\alpha + i\beta)| \ll e^{R\|\beta\|}(1 + \|\alpha\| + \|\beta\|)^{-N} \qquad (\alpha, \beta \in V^*)$$

for all $N > 0$. Then the smooth version of Theorem X of Paley and Wiener (cf. [28]) asserts that

$$\mathcal{F}(C_R^\infty(V)) = PW_R(V_\mathbb{C}^*) \qquad \text{(PW-I)}. \tag{7.1}$$

The image of strip functions. For $R > 0$ we let B_R be the ball of radius R centered at the origin and define a tube domain in $V_\mathbb{C}$ by

$$S_R = V + iB_R .$$

Further we define

$$\mathcal{S}_R(V) := \{f \in \mathcal{O}(S_R) \mid \sup_{w \in B_R} \int_V |f(v + iw)|^2 \, dv < \infty\}$$

and simply call them strip functions. Then Theorem IV of Paley and Wiener [28] specializes to

$$\mathcal{F}(\mathcal{S}_R(V)) = \mathcal{E}_R(V^*) \qquad \text{(PW-II)} \tag{7.2}$$

with

$$\mathcal{E}_R(V^*) = \{f \in L^2(V^*) \mid \int_{V^*} |f(\alpha)|^2 e^{2R\|\alpha\|} \, d\alpha < \infty\}$$

the space of exponentially decaying functions L^2-functions on V^* with decay exponent R.

We move from V to G. As we remarked earlier, we have to be careful because of the symmetry break between G and $\hat{G}$. So there are in fact four different types of Paley-Wiener theorems which are of interest: (PW-I) and (PW-II) and as well their inverse versions for $\mathcal{F}^{-1}$.

Arthur did a case of (PW-I) in [3] when he characterized the image of the $K \times K$-finite test functions $C_c^\infty(G)_{K\times K}$ under $\mathcal{F}$. We emphasize the subspace

$$C_c^\infty(G)_{K\times\mathbf{1}}^{\mathbf{1}\times K} \subset C_c^\infty(G)_{K\times K}$$

of functions which are fixed under right K-displacements. These functions naturally realize as K-finite functions on X. [1] Then Arthur's Paley Wiener result gives us the image of $C_c^\infty(X)_K$ as certain entire sections over the complexification of the spherical unitary dual, i.e. $\mathfrak{a}_\mathbb{C}^*/\mathcal{W}$. It became the bad habit to restrict even further to K-fixed functions on X – this makes the sections scalar valued and matters reduced to some "Euclidean" Harmonic analysis with respect to a specific weighted measure space. In this simplified context a Paley-Wiener theorem for the inverse of (PW-I) was established for some class of examples [29]. A fully geometric version of the inverse of (PW-I) was recently obtained by Thangavelu in [31], when he showed that sections

[1] As for analysis on X one should think of it as K-invariant analysis on G.

with compact support in a ball correspond to holomorphic functions on the crown with a certain growth condition related to the size of the support. We will not further delve into that but focus on (PW-II) instead.

So far the discussion was general, but now I wish to return – for the sake of the exposition – to $G = \mathrm{Sl}(2,\mathbb{R})$ and the upper halfplane X where very concrete formulas hold. For $0 < R \leq \pi/4$ we define a G-domain in Ξ by

$$\Xi_R = G \exp(i\pi/4(-R,R)\mathbf{h}) \cdot x_0\,.$$

For $R = \pi/4$ we obtain the crown and in general $(\Xi_R)_R$ is a filtration of Ξ of G-invariant Stein domains (see [10] for the general fact). We think of Ξ_R as a strip domain around X and define the analogue of the space of strip functions by

$$\mathcal{S}_R(X) := \{f \in \mathcal{O}(\Xi_R) \mid \sup_{r\in(0,R)} \int_G |f(g\exp(ir\mathbf{h})\cdot x_0)|^2 \; dg < \infty\}$$

By a theorem of Harish-Chandra , $\mathcal{F}$ identifies $L^2(X)$ with

$$L^2\left(K/M \times i\mathfrak{a}^*/\mathcal{W}, d(kM) \otimes \lambda \tanh(\pi\lambda) d\lambda\right)$$

where we have identified $i\mathfrak{a}^*$ linearly with $\mathbb{R}$ subject to the normalization that the functional $ci\mathbf{h} \to c$ corresponds to $1 \in \mathbb{R}$. As $\mathcal{W} = \mathbb{Z}_2$ acts as the flip on $\mathbb{R}$ we may safely identify $i\mathfrak{a}^*/\mathcal{W}$ with $[0,\infty)$. Obviously K/M identifies with the unit circle. The Fourier transform on G, restricted to K-invariants is then given by

$$\mathcal{F}f(kM,\lambda) = \int_X f(z)\phi_\lambda(k^{-1}z)\; dz$$

The Parseval identity for G reduced to X then states that:

$$\int_X |f(z)|^2 \; dz = \int_{K/M}\int_0^\infty |(\mathcal{F}f)(kM,\lambda)|^2 \; d(kM) \; \lambda \tanh(\pi\lambda/2) d\lambda\,.$$

If we want to extend this identity by moving the G-orbit X into Ξ_R, i.e. a contour shift, then we need the Plancherel theorem for G (and not only of X). For a function $f \in S_R(X)$, we then get for all $r < R$:

$$\begin{aligned}\int_G &|f(g\exp(ir\mathbf{h})\cdot x_0)|^2 \; dg \\ &= \int_{K/M}\int_0^\infty |(\mathcal{F}f)(kM,\lambda)|^2 \phi_\lambda(\exp(i2r\mathbf{h}) \; \lambda \tanh(\pi\lambda/2) d\lambda\,.\end{aligned}$$

In [6] Faraut named this equality *Gutzmer identity* in the honour of Gutzmer who, in the 19th century, investigated growth of Fourier coefficients with respect to analytic continuation , [6]. We emphasize that $\phi_\lambda(\exp(i2r\mathbf{h})$ is a positive quantity as we know from the doubling identity (5.6).

Let us define the analogue of $\mathcal{E}_R(V^*)$ to be

$$\mathcal{E}_R(\hat{G}) = \Big\{ f \in L^2\left(K/M \times i\mathfrak{a}^*/\mathcal{W}, d(kM) \otimes \lambda \tanh(\pi\lambda/2)d\lambda\right) \mid \\ \sup_{0\leq r<R} \int_{K/M} \int_0^\infty |(\mathcal{F}f)(kM,\lambda)|^2 \phi_\lambda(\exp(i2r\mathbf{h})\ \lambda \tanh(\pi\lambda/2)d\lambda < \infty \Big\}$$

and state the analogue of Theorem IV of Paley and Wiener.

Theorem 7.2. *For all* $0 \leq R \leq \pi/4$

$$f \in S_R(X) \iff \mathcal{F}(f) \in \mathcal{E}_R(\hat{G}) .$$

To end this section I want to pose the following

Problem: *Formulate and possibly prove geometric Paley-Wiener theorems, i.e. (PW2) and inverse of (PW1), for* G*.*

7.3. **Hard estimates on extended Maaß cusp forms**

Let $\Gamma < G$ be a lattice. Then, an analytic function $\phi : X \to \mathbb{C}$ is called a Maaß automorphic form if

- ϕ is Γ-invariant,
- ϕ is a Δ-eigenfunction,
- ϕ is of moderate growth at the cusps of $\Gamma\backslash X$.

We note that the third bulleted item is automatic if Γ is co-compact, i.e. $\Gamma\backslash X$ is compact.

A Maaß form ϕ is called a cusp form if it vanishes at all cusps of $\Gamma\backslash X$, i.e.

$$\int_{N'\cap\Gamma\backslash N'} \phi(n'x)dn' = 0 \qquad (x \in X)$$

for all unipotent groups $N' < G$ with $\Gamma \cap N' \neq \emptyset$.

From now on we assume that ϕ is a cusp form. Frobenius reciprocity (see [8] and [5] for a quantitative version) tells us that

$$\phi(gK) = (\pi(g)v_K, \eta) \qquad (g \in G)$$

for $(\pi, \mathcal{H})$ a unitary irreducible representation of G, $v_K \in \mathcal{H}^K$ a normalized K-fixed vector and $\eta \in (\mathcal{H}^{-\infty})^\Gamma$ a Γ-invariant distribution vector (

in [5]it is, perhaps more appropriately, called *automorphic functional*). It is useful to allow arbitrary smooth vectors $v \in \mathcal{H}^\infty$ and build Γ-invariant smooth functions ϕ_v on G by

$$\phi_v(g) = (\pi(g)v_K, \eta) \qquad (g \in G)\,.$$

Langland's modification of the Sobolev Lemma for cusp forms then reads as:

$$\|\phi_v\|_\infty = \sup_{g\in G} |\phi_v(g)| \leq C \cdot S_2(v) \qquad (v \in \mathcal{H}^\infty) \tag{7.3}$$

for $C > 0$ a constant only depending on the geometry of $\Gamma\backslash G$ (see [4], Appendix B for an exposition). As $\|\cdot\|_\infty$ is G-invariant, we deduce from 7.3 that

$$\|\phi_v\|_\infty = \sup_{g\in G} |\phi_v(g)| \leq C \cdot S_2^G(v) \qquad (v \in \mathcal{H}^\infty) \tag{7.4}$$

(cf. [4], Section 3). One deep observations in [4] was that $S_2^G(v)$ can be considerably smaller as $S_2(v)$, for instance if $v = \pi(a_\epsilon)v_K$. We combine with Theorem 6.6 and Proposition 6.1 (cf. [23], Th. 6.7)

Theorem 7.3. *Let ϕ be a Maaß cusp form. Then there exist constants $C, C' > 0$ such that*

$$\sup_{g\in G} |\phi(ga_\epsilon \cdot x_0)| \leq C\|\phi(\cdot a_\epsilon)\|_{L^2(\Gamma\backslash G)} \leq C' \cdot \sqrt{|\log\epsilon|}$$

Remark 7.4. *In* [4] *a slightly weaker bound was established, namely:*

$$\sup_{g\in G} |\phi(ga_\epsilon \cdot x_0)| \leq C \cdot |\log\epsilon|,$$

see [4] *Sect.1, Proposition part (3).*

8. **Automorphic cusp forms**

In this section we explain how one can use the unipotent model for the crown domain in the theory of automorphic functions on the upper half plane.

To avoid extra notation we will stick to

$$\Gamma = \mathrm{Sl}(2, \mathbb{Z})$$

for our choice of lattice.

In the sequel we let ϕ be a Maaß cusp form. Let us fix $y > 0$ and consider the 1-periodic function

$$F_y : \mathbb{R} \to \mathbb{C}, \quad u \mapsto \phi(n_u a_y(i)) = \phi(u+iy)\,.$$

This function being smooth and periodic admits a Fourier expansion

$$F_y(u) = \sum_{n\neq 0} A_n(y) e^{2\pi i n x}\,.$$

Here, $A_n(y)$ are complex numbers depending on y. Now observe that

$$n_u a_y = a_y a_y^{-1} n_u a_y = a_y n_{u/y}$$

and so

$$F_y(u) = \phi(a_y n_{u/y}.x_0)\,.$$

As ϕ is a $\mathcal{D}(X)$-eigenfunction, it admits a holomorphic continuation to $\Xi = X \times \overline{X}$. So we employ the crown model and conclude that F_y admits a holomorphic continuation to the strip domain

$$S_y = \{w = u + iv \in \mathbb{C} \mid |v| < y\}\,.$$

Let now $\epsilon > 0$, ϵ small. Then, for $n > 0$, we proceed with Cauchy

$$\begin{aligned} A_n(y) &= \int_0^1 F_y(u - i(1-\epsilon)y) e^{-2\pi i n(u - i(1-\epsilon)y)}\, du \\ &= e^{-2\pi n(1-\epsilon)y} \int_0^1 F_y(u - i(1-\epsilon)y) e^{-2\pi i n u}\, du \\ &= e^{-2\pi n(1-\epsilon)y} \int_0^1 \phi(a_y n_{u/y} n_{-i(1-\epsilon)}.x_0) e^{-2\pi i n u}\, du\,. \end{aligned}$$

Thus we get, for all $\epsilon > 0$ and $n \neq 0$ the inequality

$$|A_n(y)| \leq e^{-2\pi |n| y(1-\epsilon)} \sup_{\Gamma g \in \Gamma\backslash G} |\phi(\Gamma g n_{\pm i(1-\epsilon)}.x_0)| \tag{8.1}$$

We need an estimate.

Lemma 8.1. *Let ϕ be a Maaß cusp form. Then there exists a constant C only depending on λ such that for all $0 < \epsilon < 1$*

$$\sup_{\Gamma g \in \Gamma\backslash G} |\phi(\Gamma g n_{i(1-\epsilon)}.x_0)| \leq C |\log \epsilon|^{\frac{1}{2}}$$

Proof. Let $-\pi/4 < t_\epsilon < \pi/4$ be such that $\pm(1-\epsilon) = \sin 2t_\epsilon$. Then, by Lemma 4.3 we have $Gn_{\pm i(1-\epsilon)}.x_0 = Ga_\epsilon.x_0$ with $a_\epsilon = \begin{pmatrix} e^{it_\epsilon} & 0 \\ 0 & e^{-it_\epsilon}\end{pmatrix}$.

Now note that $t_\epsilon \approx \pi/4 - \sqrt{2\epsilon}$ and thus Prop. 6.1 and Theorem 7.3 , give that

$$\sup_{\Gamma g \in \Gamma\backslash G} |\phi(ga_\epsilon.x_0)| \leq C|\log\epsilon|^{\frac{1}{2}}\,.$$

This concludes the proof of the lemma. □

We use the estimates in Lemma 8.1 in (8.1) and get

$$|A_n(y)| \leq Ce^{-2\pi|n|y(1-\epsilon)}|\log\epsilon|^{\frac{1}{2}}\,, \tag{8.2}$$

and specializing to $\epsilon = 1/y$ gives that

$$|A_n(y)| \leq Ce^{-2\pi|n|(y-1)}(\log y)^{\frac{1}{2}}\,. \tag{8.3}$$

This in turn yields for $y > 2$ that

$$\begin{aligned} |\phi(iy)| = |F_y(0)| &\leq \sum_{n\neq 0}|A_n(y)| \\ &\leq C(\log y)^{\frac{1}{2}}\sum_{n\neq 0} e^{-2\pi|n|(y-1)} \\ &\leq C(\log y)^{\frac{1}{2}}\cdot e^{-2\pi y} \end{aligned}$$

It is clear, that we can replace F_y by $F_y(\cdot + x)$ for any $x \in \mathbb{R}$ without altering the estimate. Thus we have proved:

Theorem 8.2. *Let ϕ be a Maaß cusp form. Then there exists a constant $C > 0$, only depending on λ, such that*

$$|\phi(x+iy)| \leq C(\log y)^{\frac{1}{2}}\cdot e^{-2\pi y} \qquad (y > 2)\,.$$

Remark 8.3. *It should be mentioned that this estimate is not optimal: one can drop the* log*-term by employing our knowledge about the coefficient functions $A_n(y)$. However the method presented above generalizes to all semi-simple Lie groups.*

9. G-innvariant Hilbert spaces of holomorphic functions on Ξ

Hilbert spaces of holomorphic functions are in particular reproducing kernel Hilbert spaces, cf. [2].

9.1. **General theory**

In this subsection G is a group and M is a second countable complex manifold. The compact-open topology turns $\mathcal{O}(M)$ into a Fréchet space.

We assume that G acts on M in a biholomorphic manner. This action induces an action of G on $\mathcal{O}(M)$ via:

$$G \times \mathcal{O}(M) \to \mathcal{O}(M), \quad (g, f) \mapsto f(g^{-1}\cdot)\,.$$

We assume that the action is continuous. By a G-invariant Hilbert space of holomorphic functions on M we understand a Hilbert space $\mathcal{H} \subset \mathcal{O}(M)$ such that

- The inclusion $\mathcal{H} \hookrightarrow \mathcal{O}(M)$ is continuous;
- G leaves $\mathcal{H}$ invariant and the action is unitary.

It follows that all point evaluations

$$\mathcal{K}_m : \mathcal{H} \to \mathbb{C}, \quad f \mapsto f(m); \quad (m \in M)$$

are continuous, i.e. $f(m) = \langle f, \mathcal{K}_m \rangle$. We obtain a kernel function

$$\mathcal{K} : M \times M \to \mathbb{C}, \quad (m, n) \mapsto \langle \mathcal{K}_n, \mathcal{K}_m \rangle = \mathcal{K}_n(m)$$

which is holomorphic in the first and anti-holomorphic in the second variable. The kernel $\mathcal{K}$ characterizes $\mathcal{H}$ completely. Moreover that G acts unitarily just means that $\mathcal{K}$ is G-invariant:

$$\mathcal{K} = \mathcal{K}(g\cdot, g\cdot) \qquad (g \in G)\,.$$

We denote by $\mathcal{C} = \mathcal{C}(M, G)$ the cone of all G-invariant holomorphic positive definite kernels (i.e reproducing kernels) on $M \times \overline{M}$. In the terminology of Thomas [32] is a conuclear cone in the Fréchet space $\mathcal{O}(M \times \overline{M})$ and as such admits a decomposition

$$\mathcal{K} = \int_{\mathrm{Ext}(\mathcal{C})} \mathcal{K}^\lambda \; d\mu(\lambda)\,, \tag{9.1}$$

see [19], Th. II.12 for a more general statement. In (9.1) the symbol $\mathrm{Ext}(\mathcal{C})$ denotes the equivalence classes (under $\mathbb{R}^+$-scaling) of extremal rays in $\mathcal{C}$ and

$$\lambda \mapsto \mathcal{K}^\lambda$$

is an appropriate assignment of representatives; furthermore μ is a Borel measure on $\mathrm{Ext}(\mathcal{C})$.

9.2. **Invariant Hilbert spaces on the crown**

We return to $G = \mathrm{Sl}(2,\mathbb{R})$ and $M = \Xi$. We write $\hat{G}_{\rm sph}$ for the K-spherical part of $\hat{G}$ and note that the map $\lambda \mapsto [\pi_\lambda]$ is a bijection from $(\mathbb{R} \cup (-i,i))/\mathcal{W}$ to $\hat{G}_{\rm sph}$. Morover for $[\pi] \in \hat{G}_{\rm sph}$ we define a positive definite holomorphic G-imvariant kernel $\mathcal{K}^\pi$ on Ξ via

$$\mathcal{K}^\pi(z,w) = \langle \pi(z)v, \pi(w)v\rangle \qquad (z,w \in \Xi)$$

where v is a unit K-fixed vector. Then each kernel $\mathcal{K}$ of a G-invariant Hilbert space $\mathcal{H} \subset \mathcal{O}(\Xi)$ can be written as

$$\mathcal{K}(z,w) = \int_{\hat{G}_{\rm sph}} \mathcal{K}^\lambda(z,w)\ d\mu(\lambda) \qquad (z,w\in\Xi) \tag{9.2}$$

where we simplified notation $\mathcal{K}^{\pi_\lambda}$ to $\mathcal{K}^\lambda$. The Borel measure μ satisfies the condition

$$(\forall 0 < c < 2) \qquad \int_{\hat{G}_{\rm sph}} e^{c|\mathrm{Re}\,\lambda|}\ d\mu(\lambda) < \infty \tag{9.3}$$

and conversely, a measure μ which satisfies (9.3) gives rise to a G-invariant Hilbert space of holomorphic functions on Ξ, see [24], Prop. 5.4.

9.3. **Hardy spaces for the most continuous spectrum of the hyperboloid**

A little bit of motivation upfront. We recall the splitting of square integrable functions on $\mathbb{R}$

$$L^2(\mathbb{R}) \simeq H^2(X) \oplus H^2(\overline{X})$$

into a sum of Hardy spaces:

$$H^2(X) = \left\{ f \in \mathcal{O}(X) \mid \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2\ dx < \infty \right\},$$

and

$$H^2(\overline{X}) = \left\{ f \in \mathcal{O}(\overline{X}) \mid \sup_{y<0} \int_{\mathbb{R}} |f(x+iy)|^2\ dx < \infty \right\}.$$

The isomorphism map from $H^2(X)$ to $L^2(\mathbb{R})$ is just the boundary value:

$$b: H^2(X) \to L^2(\mathbb{R});\ b(f)(x) = \lim_{y\to 0^+} f(\cdot + iy)$$

and likewise

$$\overline{b} : H^2(X) \to L^2(\mathbb{R});\ \overline{b}(f)(x) = \lim_{y\to 0^-} f(\cdot + iy)$$

In the sequel we replace the pair $(\mathbb{R}, X)$, i.e. Shilov boundary $\mathbb{R}$ of the complex manifold X, by (Y, Ξ) where $Y = G/H = \mathrm{Sl}(2,\mathbb{R})/\mathrm{SO}(1,1)$ is the distinguished boundary of Ξ. But now we have to be more careful with the space of square-integrable functions $L^2(Y)$. Recall the Casimir element, the generator of $\mathcal{Z}(\mathfrak{g}) := \mathcal{U}(\mathfrak{g})^G$:

$$\mathbf{C} = \mathbf{h}^2 + 4\mathbf{ef}\,.$$

Then

$$L^2(Y) = L^2(Y)_{\rm mc} \oplus L^2(Y)_{\rm disc}$$

accordingly whether $\mathbf{C}$ has continuous or discrete spectrum. Here our concern is only with the (most) continuous part $L^2(Y)_{\rm mc}$. So it is about to define the Hardy spaces $H^2(\Xi)$ and $H^2(\overline{\Xi})$. It was a result of [11] that $H^2(\Xi)$ actually exists and that the kernel is given (up to positive scale) by

$$\mathcal{K} = \int_0^\infty \mathcal{K}^\lambda\ \frac{\lambda\tanh(\pi\lambda/2)}{\cosh(\pi\lambda)}\cdot d\lambda\,. \tag{9.4}$$

There exists a well defined boundary value map

$$b : H^2(\Xi) \to L^2_{\rm mc}(Y);\ b(f)(g(1,-1)) = \lim_{e\to 0^+} f(ga_\epsilon\cdot x_0)$$

which is equivariant and isometric. Likewise one has a Hardy space $H^2(\overline{\Xi})$ on $\overline{\Xi}$ which is just the complex conjugate of $H^2(\Xi)$. The decomposition of the continuous spectrum then is [11]:

$$L^2_{\rm mc}(Y) = b(H^2(\Xi)) \oplus \overline{b}(H^2(\overline{\Xi})) \tag{9.5}$$

Remark 9.1. *(a) We caution the reader that $L^2_{\rm mc}(Y)$ is not exhausted by our Hardy spaces once the real rank of Y is larger then one.*
(b) We defined $H^2(\Xi)$ by its kernel and not by its norm. It is possible to give a geometric expression of the norm in $H^2(\Xi)$ in terms of certain G-orbital integrals on Ξ, see [13]*. This method was also quite useful in our work on the heat kernel transform* [22]*.*

10. Kähler structures on $\hat{G}_{\rm sph}$

Throughout this section $(\pi, \mathcal{H})$ denotes a non-trivial K-spherical unitary representation of G. We let $v_K \in \mathcal{H}$ be a K-fixed unit vector.

We first recall that the projective space

$$\mathbb{P}(\mathcal{H}) = \mathcal{H}^\times/\mathbb{C}^*$$

of $\mathcal{H}$ is an infinite dimensional complex manifold which is complete under the Fubini-Study metric g_{FS}. We write

$$h_{FS} = g_{FS} + i\omega_{FS}$$

for the corresponding Hermitian structure on $\mathbb{P}(\mathcal{H})$.

Without proof we state two results, see [24], Prop. 3.1 and Th. 3.3, which hold in full generality.

Theorem 10.1. *The map*

$$F_\pi : \Xi \to \mathbb{P}(\mathcal{H}), \quad z \mapsto [\pi(z)v_K]$$

is proper. In particular $\mathrm{i}\, mF_\pi$ *is closed and the pull back* $h_\pi := F_\pi^* h_{FS}$ *defines a Hermitian Kähler structure on* Ξ *whose underlying Riemannian structure* g_π *is complete.*

Remark 10.2. *Elementary complex analysis shows that the map*

$$s_\pi : \Xi \to \mathbb{R}_{>0}, \quad z \mapsto \|\pi(z)v\|^2$$

is strictly plurisubharmonic. The Kähler form ω_π *from the previous theorem is then nothing else as*

$$\omega_\pi(z) = \frac{i}{2}\partial\overline{\partial}\log\|\pi(\cdot)v_K\|^2 .$$

The main result of this section then is, see [24], :

Theorem 10.3. *The map* $\pi \mapsto \omega_\pi$ *identifies* $\hat{G}_{\mathrm{sph}} \setminus \{\mathbf{1}\}$ *with positive Kähler forms on* Ξ *whose associated Riemannian metric is complete.*

The big problem then is to characterize the image of $\pi \mapsto \omega_\pi$.

11. Appendix: The hyperbolic model of the crown domain

The upper half plane $X = G/K$ does not depend on the isogeny class of G. Replacing G by its adjoint group $\mathrm{PSl}(2,\mathbb{R}) \simeq \mathrm{SO}_e(1,2)$ has essentially no consequences for the crown. Changing the perspective to $G = \mathrm{SO}_e(1,2)$ we obtain new view-points by realizing Ξ in the complex quadric. This is the topic of this section.

Let us fix the notation first. From now on $G = \mathrm{SO}_e(1,2)$ and we regard $K = \mathrm{SO}(2,\mathbb{R})$ as a maximal compact subgroup of G under the standard lower right corner embedding.

Let us define a quadratic form Q on $\mathbb{C}^3$ by

$$Q(\mathbf{z}) = z_0^2 - z_1^2 - z_2^2, \qquad \mathbf{z} = (z_0, z_1, z_2)^T \in \mathbb{C}^3 .$$

With Q we declare real and complex hyperboloids by

$$X = \{\mathbf{x} = (x_0, x_1, x_2)^T \in \mathbb{R}^3 \mid Q(\mathbf{x}) = 1, x_0 > 0\}$$

and

$$X_\mathbb{C} = \{\mathbf{z} = (z_0, z_1, z_2)^T \in \mathbb{C}^3 \mid Q(\mathbf{z}) = 1\} .$$

We notice that mapping

$$G_\mathbb{C}/K_\mathbb{C} \to X_\mathbb{C}, \quad gK_\mathbb{C} \mapsto g.\mathbf{x}_0 \qquad (\mathbf{x}_0 = (1, 0, 0))$$

is diffeomorphic and that X is identified with G/K.

At this point it is useful to introduce coordinates on $\mathfrak{g} = \mathfrak{so}(1, 2)$. We set

$$\mathbf{e_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} .$$

We notice that $\mathfrak{k} = \mathbb{R}\mathbf{e_3}$, $\mathfrak{p} = \mathbb{R}\mathbf{e_1} \oplus \mathbb{R}\mathbf{e_2}$ and make our choice of the flat piece $\mathfrak{a} = \mathbb{R}\mathbf{e_1}$. Then $\Omega = (-1, 1)\mathbf{e_1}$, $\Xi = G \exp(i(-\pi/2, \pi/2)\mathbf{e_1}).\mathbf{x_0}$ and we obtain Gindikin's favorite model of the crown

$$\Xi = \{\mathbf{z} = \mathbf{x} + i\mathbf{y} \in X_\mathbb{C} \mid x_0 > 0, Q(\mathbf{x}) > 0\} .$$

It follows that the boundary of Ξ is given by

$$\partial\Xi = \partial_s\Xi \amalg \partial_n\Xi \tag{11.1}$$

with semisimple part

$$\partial_s\Xi = \{i\mathbf{y} \in i\mathbb{R}^3 \mid Q(\mathbf{y}) = -1\} \tag{11.2}$$

and nilpotent part

$$\partial_n\Xi = \{\mathbf{z} = \mathbf{x} + i\mathbf{y} \in X_\mathbb{C} \mid x_0 > 0, Q(\mathbf{x}) = 0\} . \tag{11.3}$$

Notice that $\mathbf{z_1} = \exp(i\pi/2\mathbf{e_1}).\mathbf{x_0} = (0, 0, i)^T$ and that the stabilizer of $\mathbf{z}_1$ in G is the symmetric subgroup $H = \mathrm{SO}_e(1, 1)$, sitting inside of G as the upper left corner block. Hence

$$\partial_s\Xi = \partial_d\Xi = G.\mathbf{z_1} \simeq G/H \tag{11.4}$$

A first advantage of the hyperbolic model is a more explicit view on the boundary of Ξ: Proposition 4.6 becomes more natural in these coordinates. We allow ourselves to go over this topic again.

Write τ for the involution on G with fixed point set H and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the corresponding τ-eigenspace decomposition. Clearly, $\mathfrak{h} = \mathbb{R}\mathbf{e_2}$

and $\mathfrak{q} = \mathfrak{a} \oplus \mathfrak{k} = \mathbb{R}\mathbf{e_1} \oplus \mathbb{R}\mathbf{e_3}$. Notice that $\mathfrak{q}$ breaks as an $\mathfrak{h}$-module into two pieces

$$\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$$

with

$$\mathfrak{q}^\pm = \{Y \in \mathfrak{q} \mid [e_2, Y] = \pm Y\} = \mathbb{R}(\mathbf{e_1} \pm \mathbf{e_2})\,.$$

Let us define the H-stable pair of half lines

$$C = \mathbb{R}_{\geq 0}(\mathbf{e_1} \oplus \mathbf{e_3}) \cup \mathbb{R}_{\geq 0}(\mathbf{e_1} - \mathbf{e_3})$$

in $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$. We remark that C is the boundary of the H-invariant open cone

$$W = \mathrm{Ad}(H)(\mathbb{R}_{>0}\mathbf{e_1}) = \mathbb{R}_{>0}(\mathbf{e_1} + \mathbf{e_3}) \oplus \mathbb{R}_{>0}(\mathbf{e_1} - \mathbf{e_3})\,.$$

Recall that the tangent bundle $T(G/H)$ naturally identifies with $G \times_H \mathfrak{q}$ and let us mention that $\mathcal{C} = G \times_H C$ is a G-invariant subset thereof. Proposition 4.6 from before now reads as:

Proposition 11.1. *For $G = \mathrm{SO}_e(1,2)$, the mapping*

$$b : G \times_H C \to \partial\Xi, \quad [g, Y] \mapsto g\exp(-iY).\mathbf{z_1}$$

is a G-equivariant homeomorphism.

As a second application of the hyperbolic model we now prove the orbit-matching Lemma l=match from before.

Proof of Lemma 4.3. With $\mathfrak{a} = \mathbb{R}\mathbf{e_1}$ we come to our choice of $\mathfrak{n}$. For $z \in \mathbb{C}$ let

$$n_z = \begin{pmatrix} 1 + \frac{1}{2}z^2 & z & -\frac{1}{2}z^2 \\ z & 1 & -z \\ \frac{1}{2}z^2 & z & 1 - \frac{1}{2}z^2 \end{pmatrix}$$

and

$$N_\mathbb{C} = \{n_z \mid z \in \mathbb{C}\}\,.$$

Further for $t \in \mathbb{R}$ with $|t| < \frac{\pi}{2}$ we set

$$a_t = \begin{pmatrix} \cos t & 0 & -i\sin t \\ 0 & 1 & 0 \\ -i\sin t & 0 & \cos t \end{pmatrix} \in \exp(i\Omega)\,.$$

The statement of the lemma translates into the assertion

$$Gn_{i\sin t}.\mathbf{x}_0 = Ga_t.\mathbf{x}_0\,. \tag{11.5}$$

Clearly, it suffices to prove that

$$a_t.\mathbf{x}_0 = (\cos t, 0, -i\sin t)^T \in Gn_{i\sin t}.\mathbf{x}_0\,.$$

Now let $k \in K$ and $b \in A$ be elements which we write as

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \cosh r & 0 & \sinh r \\ 0 & 1 & 0 \\ \sinh r & 0 & \cosh r \end{pmatrix}$$

for real numbers r, θ. For $y \in \mathbb{R}$, a simple computation yields that

$$kbn_{iy}.\mathbf{x}_0 = \begin{pmatrix} \cosh r(1 - \frac{1}{2}y^2) - \frac{1}{2}y^2 \sinh r \\ iy\cos\theta + \sin\theta(\sinh r(1 - \frac{1}{2}y^2) - \frac{1}{2}y^2\cosh r) \\ -iy\sin\theta + \cos\theta(\sinh r(1 - \frac{1}{2}y^2) - \frac{1}{2}y^2\cosh r) \end{pmatrix}.$$

Now we make the choice of $\theta = \frac{\pi}{2}$ which gives us that

$$kbn_{iy}.\mathbf{x}_0 = \begin{pmatrix} \cosh r(1 - \frac{1}{2}y^2) - \frac{1}{2}y^2 \sinh r \\ \sinh r(1 - \frac{1}{2}y^2) - \frac{1}{2}y^2\cosh r \\ -iy \end{pmatrix}.$$

As $y = \sin t$ we only have to verify that we can choose r such that $\sinh r(1 - \frac{1}{2}y^2) - \frac{1}{2}y^2\cosh r = 0$. But this is equivalent to

$$\tanh r = \frac{\frac{1}{2}y^2}{1 - \frac{1}{2}y^2}.$$

In view of $-1 < y = \sin t < 1$, the right hand side is smaller than one and we can solve for r. □

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