The Continuing Story of Zeta

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We can only guess at the number of careers in mathematics that have been launched by the sheer wonder of Euler's formula from 1734,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}. \quad (1)$$

Euler further obtained the generalization that for integral $k \geq 1$ the inverse $2k$-th powers of the natural numbers sum to a rational multiple of π^{2k}, and identified that rational multiple. This identification involves the sequence of *Bernoulli numbers* (B_n), which is defined via the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \quad (2)$$

The first few Bernoulli numbers are shown below:

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Euler showed that for $k \geq 1$

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \frac{1}{5^{2k}} + \cdots = \frac{(-1)^{k+1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}. \quad (3)$$

In particular, $\zeta(2k)$ is irrational for $k \geq 1$. Little is known about $\zeta(2k+1)$ for $k \geq 1$; indeed it is only relatively recently that $\zeta(3)$ was shown to be irrational by Apéry. In his lovely paper [10], van der Poorten refers to Apéry's theorem as "A proof that Euler missed". What follows is an even more stunning formula[1] than (1) which Euler certainly found (in 1740, see [2, Section 7]),

$$1 - 2^2 + 3^2 - 4^2 + 5^2 + \cdots = 0. \quad (4)$$

Readers doubting the validity of formula (4) will be reassured to note that it follows from

$$1 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots = 0, \quad (5)$$

after multiplying (5) by $-7 = 1 - 2.2^2$.

The concept of analytic continuation was developed partly in order to make sense of formulae such as (4) and (5). Here the concept is applied to Riemann's zeta function ζ, which is defined for complex s with $\Re(s) > 1$ by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (6)$$

In this article we will report on recent work that allows ζ to be evaluated to the left of the line $\Re(s) = 1$ in an extremely elementary and natural way. If Euler's ghost is sensed, then it is with good reason. In [2, Section 7], Ayoub comments on Euler's article of 1740 in which he boldly evaluates divergent series to obtain formulae such as (4). The methods we espouse are in the same tradition, only taking care to articulate the convergence issues.

The most interesting values of the zeta function occur outside the domain of convergence of the series in (6). For one thing, the formula describing $\zeta(-k)$ is simpler than that for $\zeta(2k)$, let alone the mysteries surrounding $\zeta(2k+1)$. For another, the location of zeros of $\zeta(s)$ other than those for $s = -2k$ (see Corollary 3) has a just claim to be one of the most important unsolved problems in Mathematics. Indeed, the Clay Mathematics Institute offers a prize of one million dollars for a proof that all these "non-trivial" zeros lie on the line $\Re(s) = \frac{1}{2}$ – the famous Riemann Hypothesis.

Special values of the zeta function, of interest in themselves, also hint at a possible route into the functional

[1] At least, when one of us showed it to a final-year class in Analytic Number Theory, they were (to their credit) stunned.

equation. Knowledge of the values of $\zeta(k)$ for *all* integers k might enable one to predict the shape of the functional equation of the zeta function. A comparison of (3) and (18) suggests that the function

$$s \mapsto \zeta(s)/\zeta(1-s)$$

can be represented by a simple combination of factorials (or Gamma functions) and exponentials. The extent to which Riemann might have been aware of Euler's work on this subject is not clear; the interested reader might begin by consulting Ayoub's paper.

Taking the Low Road

Analytic continuation is easily illustrated using a simple example. Consider the power series

$$f(s) = 1 + s + s^2 + \cdots, \tag{7}$$

which converges absolutely for $|s| < 1$. The series diverges for $|s| \geq 1$, thus one could never evaluate f in that region using the definition (7). Nonetheless, inside the domain $|s| \leq 1$ we have

$$f(s) = \frac{1}{1-s}, \tag{8}$$

and the right-hand side can be evaluated everywhere on $\mathbb{C} \setminus \{1\}$. In light of this, Euler would have no compunction in using the definition of the left-hand side of (7) to describe the behaviour of the right-hand side of (8). On these grounds, it would be natural for him to write

$$1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2}.$$

For the example in (7), the analyticity of $f(s)$ on $\mathbb{C} \setminus \{1\}$ comes as a by-product; we simply recognize that the function in (8) can be differentiated by the usual rules of calculus. Nonetheless, the differentiability is important because it guarantees that the continuation of $f(s)$ is unique in $\mathbb{C} \setminus \{1\}$ (the region where it is analytic). In a similar fashion, the analyticity of the continuation of the zeta function will be understated in the text that follows. Actually, it is no harder than proving the analyticity in the half-plane $\Re(s) > 1$. What is needed is the concept of uniform convergence, and we refer the reader to any of the standard texts for a full account of this topic.

To obtain the analytic continuation of ζ to the left of its natural half-plane of convergence requires more guile than for $f(s)$ above. However, the principle is the same: an expression needs to be found for $\zeta(s)$ valid in a half-plane strictly containing $\Re(s) > 1$. The high road, Riemann's own [13], uses contour integration at an early stage, and leads directly to the functional equation. Many authors ([1, 4, 5, 9, 11, 16], and [17]) use this method, or variants of it. Other methods are known ([16, Chap. 2] lists seven) but a toll seems inevitable on any route ending with the functional equation.

There are lower roads that give both the continuation to the whole plane and the evaluation at nonpositive integers but stop short of proving the functional equation. Our purpose in this article is to draw wider attention to these, often very scenic, roads. For example, Sondow [14] notes one way in which Euler's argument can be made rigorous. Mináč [7] showed how to evaluate ζ at negative integers in an extremely simple and elegant way, by integrating a polynomial on [0, 1]. Other authors [3, 8, 12, 15], have shown how the continuation and evaluation of the Hurwitz zeta function can be obtained in a down-to-earth way that is applicable to the zeta function and to many L-functions. Their method, which uses little more than the binomial theorem and seems to be new, is presented here for the archetypal case of the zeta function itself. The main point of the article is to highlight how easily the continuation and evaluation of ζ can be obtained. The workhorse is (19), which can be viewed as the truncation of a formula of Landau [6, p. 274].

A Journey of a Thousand Miles..

Throughout, we use the standard notation $s = \sigma + \mathrm{i}\, t$ with $\sigma, t \in \mathbb{R}$. Notice that for $\sigma > 1$,

$$\int_1^\infty x^{-s}\, dx = \frac{-1}{1-s} = \frac{1}{s-1}. \tag{9}$$

The formula (9) yields a second example of analytic continuation. Clearly the integral in (9) can only be evaluated

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for $\sigma > 1$. However, the right-hand side is analytic everywhere apart from a simple pole at $s = 1$. Thus we obtain the continuation to $\mathbb{C} \setminus \{1\}$ of the function represented by the integral for $\sigma > 1$.

In the half-plane $\sigma > 1$,

$$\frac{1}{s-1} = \int_1^\infty x^{-s}\,dx = \sum_{n=1}^\infty \int_n^{n+1} x^{-s}\,dx = \sum_{n=1}^\infty \int_0^1 (n+x)^{-s}\,dx = \sum_{n=1}^\infty \frac{1}{n^s}\int_0^1 \left(1+\frac{x}{n}\right)^{-s} dx. \tag{10}$$

All the sums converge absolutely for $\sigma > 1$. In the text that follows, we assume that $\sigma > 1$ and that $|s|$ is bounded by K, a fixed arbitrary constant. The binomial expansion of the integrand in (10) yields

$$\left(1+\frac{x}{n}\right)^{-s} = 1 - \frac{sx}{n} + sE_1(s,x,n). \tag{11}$$

In (11) the function E_1 satisfies

$$|E_1(s,x,n)| \le \frac{C_1 x^2}{n^2} \le \frac{C_1}{n^2}, \tag{12}$$

for all $x \in [0,1]$ and all $n \ge 1$, with $C_1 = C_1(K)$ (since E_1 is the error term of the Taylor series in x/n). Substituting (11) into the sum (10) and integrating with respect to x gives

$$\frac{1}{s-1} = \zeta(s) - \frac{s}{2}\zeta(s+1) + sA_1(s). \tag{13}$$

The function $A_1(s)$ is analytic for $\sigma > -1$, and the proof of this, which we do not detail here, uses no more than uniform convergence alongside (12).

It is precisely now that the crunch comes. The functions at both ends of (13) are defined for $\sigma > 0$, provided $s \ne 1$. Also, since $\zeta(s)$ is defined by a sum for $\sigma > 1$, it follows that $\zeta(s+1)$ is defined by a sum for $\sigma > 0$. Therefore (13) may be taken as the definition of $\zeta(s)$ in this larger half-plane. Moreover (13) shows that the extended function is analytic in the half-plane $\sigma > 0$, apart from a simple pole at $s = 1$ with residue 1. In other words, (13) implies that

$$\lim_{s\to 1}(s-1)\zeta(s) = 1 \text{ and therefore } \lim_{s\to 0} s\zeta(s+1) = 1. \tag{14}$$

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Letting $s \to 0^+$ in (13), and noting the second part of (14), we obtain

$$-1 = \zeta(0) - \frac{1}{2},$$

which yields the value $\zeta(0) = -\frac{1}{2}$.

The preceding argument begins with the binomial estimate (11), finds the analytic continuation of the zeta function to the half-plane $\sigma > 0$, and evaluates $\zeta(0)$ by a limiting process. What happens if more terms of the binomial expansion are included? An additional term in the binomial expansion gives

$$\left(1+\frac{x}{n}\right)^{-s} = 1 - \frac{sx}{n} + \frac{s(s+1)x^2}{2n^2} + (s+1)E_2(s,x,n);$$

the higher binomial coefficients all include a factor $(s+1)$. Here, E_2 is a function that satisfies

$$|E_2(s,x,n)| \le \frac{C_2 x^3}{n^3} \le \frac{C_2}{n^3}$$

for all $x \in [0,1]$ and all n, where $C_2 = C_2(K)$. Substituting this into (10) and integrating as before yields

$$\frac{1}{s-1} = \zeta(s) - \frac{s}{2}\zeta(s+1) + \frac{s(s+1)}{6}\zeta(s+2) + (s+1)A_2(s), \tag{15}$$

where A_2 is analytic for $\sigma > -2$. Thus, (15) may be used to continue ζ to the half-plane $\sigma > -1$. As before, letting $s \to -1^+$ and using (14) with $s \to s+2$, we obtain

$$-\frac{1}{2} = \zeta(-1) + \frac{1}{2}\zeta(0) - \frac{1}{6} = \zeta(-1) - \frac{1}{4} - \frac{1}{6},$$

yielding $\zeta(-1) = -\frac{1}{12}$.

General Method

This method can be repeated in order to continue the zeta function further to the left in the complex plane. The method also yields the explicit evaluation at the nonpositive integers in terms of the Bernoulli numbers. To describe this, we record two well-known properties of these fascinating numbers in the following lemma.

Lemma 1 *With B_n defined by (2),*

$$\sum_{n=0}^{N-1} \binom{N}{n} B_n = 0 \qquad \text{for all } N \ge 1, \tag{16}$$

and

$$B_n = 0 \qquad \text{for all odd } n \ge 3. \tag{17}$$

Proof. The defining relation (2) can be written

$$(e^x - 1)\sum_{n=0}^\infty B_n \frac{x^n}{n!} = x.$$

For $N > 1$ the coefficient of x^N in the left-hand side is

$$\sum_{m=0}^{N-1} \frac{1}{(N-m)!\,m!} B_m = 0,$$

which gives (16) after multiplying by $N!$. The second statement follows from the fact that

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x(1 + e^x)}{2(e^x - 1)}$$

is an even function.

Either (2) or (16) determines the Bernoulli numbers, but the latter allows them to be readily computed inductively.

Theorem 2 *There is an analytic continuation of the zeta function to the entire complex plane, where it is analytic apart from a simple pole at $s = 1$ with residue 1. For all $k \geq 1$,*

$$\zeta(-k) = -\frac{B_{k+1}}{k + 1}. \tag{18}$$

Corollary 3 *The Riemann zeta function vanishes at negative even integers:*

$$\zeta(-2k) = 0, \quad k = 1, 2, \ldots.$$

The proof of the Corollary follows from (17) and (18). The relation (18) is not true for $k = 0$, but our method has already given us the special value $\zeta(0) = -\frac{1}{2}$. The case when $k = 1$ is an elegant interpretation of formula (5).

Proof of Theorem 2. The analytic continuation of the zeta function to the half-plane $\sigma > -k$ arises in exactly the same way as before, by extracting an appropriate number of terms of the binomial expansion and using induction. For integral $k \geq 0$ and $\sigma > 1$, this gives the relation

$$\begin{aligned} \frac{1}{s-1} &= \zeta(s) + \sum_{r=0}^{k} \frac{(-1)^{r+1} s(s+1)\ldots(s+r)}{(r+2)!} \zeta(s + r + 1) \\ &\quad + (s + k) A_{k+1}(s) \end{aligned} \tag{19}$$

where $A_{k+1}(s)$ is analytic in $\sigma > -(k + 1)$, again because all higher binomial coefficients include a factor $(s + k)$. Notice that $k = 0$ gives (13) and $k = 1$ gives (15).

By induction, we may assume that the zeta function has already been extended to the half-plane $\sigma > 1 - k$ so (19) is valid there, because the singularities at $s = 0, -1, \ldots$ are removable. All the functions in (19) except $\zeta(s)$ itself are defined at least for $\sigma > -k$, which gives the analytic continuation of the zeta function to that half-plane. Let $s \to -k^+$ in (19) and use (14), suitably translated, for the term with $r = k$ to obtain

$$\begin{aligned} -\frac{1}{k+1} &= \zeta(-k) + \sum_{r=0}^{k-1} \binom{k}{r+1} \frac{\zeta(-k + r + 1)}{r + 2} \\ &\quad - \frac{1}{(k+1)(k+2)}. \end{aligned}$$

Writing r for $r + 1$ simplifies this to

$$0 = \zeta(-k) + \frac{1}{k+2} + \sum_{r=1}^{k} \binom{k}{r} \frac{\zeta(-k + r)}{r + 1}.$$

The term with $r = k$ is known. Using the inductive hypothesis on the other terms gives

$$\begin{aligned} 0 &= \zeta(-k) + \frac{1}{k+2} - \sum_{r=1}^{k-1} \binom{k}{r} \frac{B_{k-r+1}}{(r+1)(k-r+1)} \\ &\quad - \frac{1}{2(k+1)}. \end{aligned} \tag{20}$$

A simple manipulation of factorials gives

$$\frac{(k+1)(k+2)}{(r+1)(k-r+1)} \binom{k}{r} = \binom{k+2}{r+1} = \binom{k+2}{k-r+1},$$

which transforms (20) to

$$\begin{aligned} 0 &= \zeta(-k) + \frac{k}{2(k+1)(k+2)} \\ &\quad - \frac{1}{(k+1)(k+2)} \sum_{r=1}^{k-1} \binom{k+2}{k-r+1} B_{k-r+1}. \end{aligned} \tag{21}$$

Now multiply by $(k + 1)(k + 2)$ and apply (16) with $N = k + 2$. Only the terms for $r = 0$, k, $k + 1$ – missing in (21) – survive, yielding

$$\begin{aligned} 0 &= (k+1)(k+2)\zeta(-k) + \frac{k}{2} + (k+2)B_{k+1} + (k+2)B_1 + B_0 \\ &= (k+1)(k+2)\zeta(-k) + (k+2)B_{k+1}, \end{aligned}$$

and this completes the induction argument.

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