SOME EQUIVARIANT CONSTRUCTIONS IN NONCOMMUTATIVE ALGEBRAIC GEOMETRY

ZORAN ŠKODA

To Mamuka Jibladze on occasion of his 50th birthday

Abstract. We here present rudiments of an approach to geometric actions in noncommutative algebraic geometry, based on geometrically admissible actions of monoidal categories. This generalizes the usual (co)module algebras over Hopf algebras which provide affine examples. We introduce a compatibility of monoidal actions and localizations which is a distributive law. There are satisfactory notions of equivariant objects, noncommutative fiber bundles and quotients in this setup.

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Modern mathematics is ever creating new kinds of geometries, and again viewpoints of unification emerge. Somehow category theory seems to be very effective in making this new order. Grothendieck taught us how important for geometry the *relative* method is and the emphasis on general maps rather than just on the incidence hierarchy of subspaces, intersections and so on. Important properties of maps are often just categorical properties of morphisms in a category (possibly with a structure). Various spectral constructions in algebra and category theory use valuations, ideals, special kinds of modules, coreflective subcategories, and so on, to single out genuine "underlying sets" of points, or of subschemes – to rings, algebras and categories – which appear as objects representing 'spaces'. Abstract localization enables us to consider local properties of objects in categorical setup; sheaf theory and generalizations enable passage between local and global. It is always enjoyable listening about a rich vision of categorical and any other geometry from Mamuka, due to his enthusiasm, width of interests and knowledge.

1. Noncommutative Algebraic Geometry

1.1. Descriptively, a *noncommutative space* X is a geometric entity which is determined by a structure (algebra A_X, category $C_X \ldots$) carried by the collection of objects (functions, cocycles, modules, sheaves ...) which are heuristically, or in a genuine model, living over X. In this article, our primary interest will be spaces represented by abelian categories "of quasicoherent sheaves". Gabriel–Rosenberg theorem says that every scheme can be reconstructed up to isomorphism of schemes from its category of quasicoherent sheaves. This involves spectral constructions [26]: from an abelian category, Rosenberg constructs a genuine set, its spectrum (many different spectra have been defined for various purposes), which can be equipped with a natural induced topology and a stack of local categories.

1.2. Noncommutative analogues of group actions, quotients and principal bundles have been abundantly studied earlier, particularly within quantum group renaissance [12, 22], in the context of study of noncommutative algebras and graded algebras representing noncommutative affine or projective varieties. As known from commutative geometry, it is easy to get out of these categories when performing the most basic constructions, e.g., the quotient spaces. The Tannakian reconstruction points out the correspondence between group-like objects and categories of representations, and it is natural to try to extend this principle not only to symmetry objects but also to actions themselves, considering thus the actions of monoidal categories of modules over symmetry objects to some other categories of quasicoherent sheaves. However, not every action qualifies.

1.3. (Affine morphisms.) Given a ring R, denote by $R-\mathrm{Mod}$ the category of left R-modules. To a morphism of rings $f^\circ : R \to S$ (which is thought of as a dual morphism to $f : \mathrm{Spec}\, S \to \mathrm{Spec}\, R$) one associates

- **extension of scalars** $f^* : R-\mathrm{Mod} \to S-\mathrm{Mod}$, $M \mapsto S \otimes_R M$;
- **restriction of scalars** (forgetful functor) $f_* : S-\mathrm{Mod} \to R-\mathrm{Mod}$, ${}_SM \mapsto {}_RM$;
- $f^! : R-\mathrm{Mod} \to S-\mathrm{Mod}$, $M \mapsto \mathrm{Hom}_R({}_RS, M)$.

Denote $F \dashv G$ when a functor F is left adjoint to a functor G. An easy fact: $f^* \dashv f_* \dashv f^!$. In particular, f^* is left exact, $f^!$ right exact and f_* exact. Moreover, f_* is faithful. As maps of commutative rings correspond precisely to maps of affine schemes, one says that an (additive) functor f^* is **almost affine** if it has a right adjoint f_* which is faithful and that f^* is **affine** if, in addition, f_* has a right adjoint as well (another motivation for this definition: Serre's affinity criterion, Éléments de géométrie algébrique, II 5.2.1, IV 1.7.18).

1.4. (Pseudogeometry of functors) Given two abelian categories $\mathcal{A}$, $\mathcal{B}$, (equivalent to small categories) a **morphism** $f : \mathcal{B} \to \mathcal{A}$ (viewed as a categorical analogue of a map of spectra or rings) is an isomorphism class of right exact additive functors from $\mathcal{A}$ to $\mathcal{B}$. An **inverse image functor** $f^* : \mathcal{A} \to \mathcal{B}$ of f is a chosen representative of f. If it has a right adjoint, then it will be referred to the **direct image functor** of the morphism f. An inverse image functor f^* is said to be flat (resp. coflat; biflat) if it has a right adjoint and it is exact (resp. if f_* is exact; if both f^* and f_* are exact). A morphism is flat (resp. coflat, biflat, almost affine, affine) if its inverse image functor is such.

1.4.1. Grothendieck topologies and their noncommutative generalizations ([28, 17]) may be used to talk about *locally affine* noncommutative spaces. For instance, localization functors and exactness properties of functors may be used to define relative noncommutative schemes, see **6.3** and [27].

2. Actegories, Biactegories, Distributive Laws

2.1. To fix the notation we recall that a **monoidal category** is given by a 6-tuple $\tilde{\mathcal{C}} = (\mathcal{C}, \otimes, \mathbf{1}, a, \rho, \lambda)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ the monoidal product, with unit object $\mathbf{1}$, associativity coherence $a : \_\otimes(\_\otimes\_) \Rightarrow (\_\otimes\_)\otimes\_$ and $\rho : \mathrm{Id}_\mathcal{C} \Rightarrow \mathrm{Id}_\mathcal{C} \otimes \mathbf{1}$ and $\lambda : \mathrm{Id}_\mathcal{C} \Rightarrow \mathbf{1} \otimes \mathrm{Id}_\mathcal{C}$ are the right and left unit coherences satisfying the usual coherence diagrams. A (strong) **monoidal functor** $F : (\mathcal{C}, \otimes, \mathbf{1}, a, \lambda, \rho) \to (\mathcal{D}, \otimes', \mathbf{1}', a', \lambda', \rho')$ is given by a triple (F, χ, ξ) where $F : \mathcal{C} \to \mathcal{D}$ is a functor, and $\chi : F(\_) \otimes' F(\_) \Rightarrow F(\_\otimes\_)$ and $\xi : F(\_) \otimes' \mathbf{1}' \Rightarrow F(\_\otimes \mathbf{1})$ are invertible natural transformations satisfying the coherence conditions as in [21]. An action of monoidal category $\tilde{\mathcal{C}}$ on a category $\mathcal{A}$ is a monoidal functor from $\tilde{\mathcal{C}}$ to the strict monoidal category of endofunctors $\mathrm{End}\,\mathcal{A}$; these data are also said to form a (left) $\tilde{\mathcal{C}}$-**actegory**. Right actegories correspond to reversing the order of the tensor product in $\mathrm{End}\,\mathcal{A}$. If $\mathcal{L} : \mathcal{C} \to \mathrm{End}\,\mathcal{A}$ is an action, then one often describes it in terms of the bifunctor $\triangleright : \mathcal{C} \times \mathcal{A} \to \mathcal{C}$ given by $c \triangleright a = \mathcal{L}(c)(a)$. Then the coherences $\chi^\mathcal{L}, \xi^\mathcal{L}$ for $\mathcal{L}$ are replaced by the coherences Ψ, u with components $\Psi^{X,Y}_M : (X \otimes Y) \triangleright M \mapsto X \triangleright (Y \triangleright M)$ and $u_M : M \to M \triangleright \mathbf{1}$ for $\triangleright$. Thus a $\tilde{\mathcal{C}}$-actegory can be described as a 4-tuple $(X, \triangleright, \Psi, u)$.

2.2. (Restriction for actegories). Let $(J, \zeta, \xi) : (\mathcal{B}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{G}, \otimes', \mathbf{1}', a', l', r')$ be a monoidal functor, then the bifunctor $\triangleright_\mathcal{B} = J \otimes' \mathrm{Id}_\mathcal{G} : \mathcal{B} \times \mathcal{G} \to \mathcal{G}$ is an action making $\mathcal{G}$ into a left $\mathcal{B}$-actegory with obvious coherence. More generally, let $\mathcal{N}$ be a left $\mathcal{G}$-actegory. Then it becomes a left $\mathcal{B}$-actegory as follows. The action functor is of course $\mathcal{B} \triangleright_\mathcal{B} \mathcal{N} := J \triangleright_\mathcal{G} \mathcal{N} : \mathcal{B} \times \mathcal{N} \to \mathcal{N}$. The action coherence component $\Psi^\mathcal{B}_{b,b',n}$ is the composition

$$\begin{aligned} b \triangleright_\mathcal{B} (b' \triangleright_\mathcal{B} n) = Jb \triangleright_\mathcal{G} (Jb' \triangleright_\mathcal{G} n) &\xrightarrow{\Psi^\mathcal{G}_{Jb,Jb',n}} (Jb \otimes' Jb') \triangleright_\mathcal{G} n \xrightarrow{\zeta_{b,b'} \otimes n} J(b \otimes b') \triangleright_\mathcal{G} n \\ &= (b \otimes b') \triangleright_\mathcal{B} n \end{aligned}$$

We say that $\mathcal{N}$ carries the **restricted action** of $\mathcal{B}$ along (J, ζ, ξ); we obtain $\mathcal{B}$-actegory $J_*(\mathcal{N}) = (J, \zeta, \xi)_*(\mathcal{N})$ (sometimes written simply ${}_\mathcal{B}\mathcal{N}$). It is easy to check that any $\mathcal{G}$-equivariant functor $(K, \gamma) : \mathcal{N} \to \mathcal{P}$ of $\mathcal{G}$-actegories restricts to the $\mathcal{B}$-equivariant functor $J_*(K, \gamma) := (K, \gamma^J) : {}_\mathcal{B}\mathcal{N} \to {}_\mathcal{B}\mathcal{P}$, where the restricted coherence $\gamma^J : \mathcal{B} \triangleright_\mathcal{B} K \to K \circ (\mathcal{B} \triangleright_\mathcal{B} \mathcal{N})$ has components $(\gamma^J)^b_n = \gamma^{Jb}_n : b \triangleright_\mathcal{B} Kn = Jb \triangleright_\mathcal{G} Kn \to K(Jb \triangleright_\mathcal{G} n) = K(b \triangleright_\mathcal{B} n)$.

2.3. Given two monoidal categories, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ acting on the same category $\mathcal{A}$, from the left and right via bifunctors $\triangleleft$ and $\triangleright$, respectively, a distributive law is a transformation $l : \mathcal{C} \triangleright (\mathcal{A} \triangleleft \mathcal{D}) \Rightarrow (\mathcal{C} \triangleright \mathcal{A}) \triangleleft \mathcal{D}$ satisfying two coherence pentagons and two triangles, generalizing the coherences for the usual distributive laws between monads ([1, 5, 38]). For clarity, we draw one of the pentagons:

$$\begin{tikzcd}
(c\otimes c')\triangleright(a\triangleleft d) \arrow[r, "\Psi"] \arrow[d, "l"] & c\triangleright(c'\triangleright(a\triangleleft d)) \arrow[r, "c\triangleright l"] & c\triangleright((c'\triangleright a)\triangleleft d) \arrow[d, "l"] \\
((c\otimes c')\triangleright a)\triangleleft d) \arrow[rr, "\Psi\triangleleft d"] & & (c\triangleright(c'\triangleright a))\triangleleft d
\end{tikzcd}$$

commutes for all objects $a \in \mathcal{A}$, $c, c' \in \mathcal{C}$, $d \in \mathcal{D}$ (subscripts on l, Ψ omitted).

We say that such data form a **$\tilde{\mathcal{C}}$-$\tilde{\mathcal{D}}$-biactegory** if the components of the distributive law involved are invertible. Biactegories are a categorification of bimodules (over monoids).

2.4. There exists a hierarchy in generality: distributive laws for two actions of two different monoidal categories are more general than between a monad and an actegory [32], which is in turn more general than between two monads, all provided we allow not only strong actions but also general (lax) monoidal and (colax) (co)monoidal functors into $\mathrm{End}\,\mathcal{A}$ (e.g., a monad interpreted as a lax monoidal functor from the trivial monoidal category $\tilde{1}$ or alternatively, as the action of a PRO for monoids on $\mathcal{A}$). For fixed $\tilde{\mathcal{C}}$, $\tilde{\mathcal{C}}$-actegories, colax $\tilde{\mathcal{C}}$-equivariant functors and transformations of equivariant functors form a 2-category $\tilde{\mathcal{C}} - \mathrm{act}_c$, and a monad in that 2-category (in the sense of formal theory of monads [40]) is precisely the usual monad in $\mathcal{A}$ equipped with the distributive law between the action of $\tilde{\mathcal{C}}$ and the monads. The Eilenberg-Moore construction exists for such $\mathcal{C}$-equivariant monads; this existence is an abstract consequence of a theorem on limits for lax morphisms in [18]; we have given a direct proof and the concrete formulas for the Eilenberg-Moore 2-isomorphism from a 2-category of $\mathcal{C}$-equivariant monads to $\tilde{\mathcal{C}} - \mathrm{act}_c$ in [36].

2.4.1. (Remark.) One sometimes needs more general 2-categorical symmetry objects than monoidal categories and bicategories; hence the distributive laws between the *actions* of two such 2-symmetries (each given by a pseudomonad) on the same object may be of interest. To this aim we sketch in [36] a new concept of *relative distributive law* (which is of course different form the notion of a distributive law between two pseudomonads).

2.5. (Tensor product of actegories) It is the basic observation in our work on *biactegories* (in preparation) that one can define a tensor product of a left $\tilde{\mathcal{C}}$-actegory $\mathcal{M}$ and a right $\tilde{\mathcal{C}}$-actegory $\mathcal{N}$ as the vertex $\mathcal{M}\otimes_{\tilde{\mathcal{C}}}\mathcal{N}$ of the pseudo-coequalizer $(\mathcal{M}\otimes_{\tilde{\mathcal{C}}}\mathcal{N}, p, \sigma)$ of the functors $\triangleright\times\mathcal{N}, \mathcal{M}\times\triangleleft : \mathcal{M}\times\mathcal{C}\times\mathcal{N}\to\mathcal{M}\times\mathcal{N}$ in **Cat** with projection $p : \mathcal{M}\times\mathcal{N}\to\mathcal{M}\otimes\mathcal{N}$ and the invertible 2-cell part $\sigma : p\circ(\triangleright\times\mathcal{N}) \Rightarrow p\circ(\mathcal{M}\times\triangleleft)$. It suffices actually to consider the pseudo-coequalizer as a representative of a bicoequalizer. If one or both actegories underly biactegories, then the unused actions get inherited by this tensor product; if both, a distributive law will be induced as well, yielding therefore the tensor product of biactegories.

For the tensor product of biactegories it is essential that we require that the distributive laws in the definition of biactegories be indeed invertible.

2.5.1. Claim. *The monoidal categories, biactegories, colax biequivariant functors, natural transformations of colax biequivariant functors with the tensor product of biactegories (using pseudocoequalizers) form a tricategory* Biact_c*.*

This tricategory is an analogue of the bicategory of rings and bimodules. Though straightforward, the proof is extremely long, and left out for a future article. The result of course generalizes to actions of bicategories instead of monoidal categories. If the monoidal category is in fact a categorical group G, then one may want to restrict to biactegories whose left and right actegories are in fact G-2-torsors; this way we get bi-2-torsors, and a tricategory relevant to the categorified geometrical Morita theory (I. Baković has studied 2-torsors with structure bigroupoid in [2] and is now thinking further on bi-2-torsors).

2.6. The most important special case of the tensor product of biactegories is the construction of the *induction for actegories*, which supplies the left pseudoadjoint to the restriction 2-functor from **2.2** (in the setup of functors between appropriate 2-categories of actegories over fixed monoidal categories). In the theory of categorified bundles this sort of induction may be used to define the associated 2-vector bundles to the 2-torsors over categorical groups.

In the setting of **2.2**, and with $\mathcal{M}$ a right $\tilde{\mathcal{B}}$-actegory, the pseudocoequalizer

$$\mathcal{M}\times\mathcal{B}\times\mathcal{G} \underset{\mathcal{M}\times\triangleright}{\overset{\triangleleft\times\mathcal{G}}{\rightrightarrows}} \mathcal{M}\times\mathcal{G} \xrightarrow{\quad p \quad} \mathcal{M}\otimes_{\mathcal{B}}\mathcal{G} = \mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}\mathcal{M}$$

with $\sigma : p\circ(\mathcal{M}\times\triangleright) \Rightarrow p\circ(\triangleleft\times\mathcal{G})$ is equipped with a canonical right $\tilde{\mathcal{G}}$-action, defining the induction 2-functor. It takes a considerable work to prove the coherence pentagon for the induced $\tilde{\mathcal{G}}$-action.

2.7. Proposition. *Every biactegory in* $\mathrm{Biact}_c(\tilde{\mathcal{G}},\tilde{\mathcal{G}}')$ *is biequivalent to a biactegory with the identity as a distributive law.*

Indeed, one replaces $\mathcal{M}$ by $\mathcal{G}\otimes_{\mathcal{G}}\mathcal{M}$. After consideration of the standard construction of pseudocoequlizers in **Cat**, one easily realizes that for the distributive law on $\mathcal{G}\otimes_{\mathcal{G}}\mathcal{M}$ one should choose identity. $\mathcal{M}$ and $\mathcal{G}\otimes_{\mathcal{G}}\mathcal{M}$ are, of course, biequivalent biactegories.

3. Actions of Monoidal Categories in Noncommutative Geometry

3.1. (Free versus tensor product). In commutative algebraic geometry, the category of commutative Hopf algebras over a field $\mathbf{k}$ is antiequivalent to the category of affine group $\mathbf{k}$-schemes, and essentially the only non-affine examples of group $\mathbf{k}$-schemes are abelian varieties. Thus extending the view that the affine noncommutative schemes make a category $\mathrm{NAff}_{\mathbf{k}}$ dual to the category of noncommutative rings, V. Drinfeld in the 1980-s took the viewpoint that the noncommutative Hopf algebras are the (duals to) affine group schemes in the noncommutative world ([12, 22, 25]). This viewpoint seemed very successful in view of many examples, many of which are called quantum groups. However, the drawback of this point of view is that for many geometric constructions the product of 'spaces' which behaves well for the development of advanced constructions is the categorical product. In $\mathrm{NAff}_{\mathbf{k}}$, the latter corresponds to the coproduct of noncommutative $\mathbf{k}$-algebras, in other words the free product $\star$ of $\mathbf{k}$-algebras, hence *not* the tensor product $\otimes_{\mathbf{k}}$. Examples of cogroup objects in the category of $\mathbf{k}$-algebras exist as well; the prime example is the noncommutative GL_n, corepresenting the functor $R \mapsto GL_n(R)$ which to $\mathbf{k}$-algebra R assigns the set of all $n \times n$ invertible matrices over R. However such examples are obviously very big, close to the free algebras ([3, 13]), and far from the "quantum" examples which are deformations of and hence closer in size and ring-theoretic properties to commutative algebras.

3.2. (Bimodules as morphisms). Another peculiarity of noncommutative geometry, the Morita equivalence, comes partly at rescue for Hopf algebras. Indeed, geometrically and physically meaningful constructions usually do not distinguish algebras in the same Morita equivalence class. Thus one can compose a usual morphism of rings with a Morita equivalence and still have a valid morphism in the noncommutative world. In other words, one considers bimodules as morphisms, and more generally, allowing for nonaffine schemes, the pairs of adjoint functors between 'categories of quasicoherent sheaves'. Working over a fixed base category (typically: category of modules over a possibly noncommutative ring $\mathbf{k}$) sometimes restores a distinguished element in the Morita equivalence class, namely the inverse image functor of the morphism to the base scheme, applied to the distinguished generator in the base.

In this setting of 'spaces' represented by categories over a fixed category $\mathrm{Spec}\,\mathbf{k}$, and with adjoint pairs as morphisms, one reintroduces the Hopf $\mathbf{k}$-algebra H (where $\mathbf{k}$ is commutative) in the disguise of the monoidal category ${}_H\mathcal{M}$ of left H-modules equipped with the inverse and direct image functors of a morphism to ${}_{\mathbf{k}}\mathcal{M}$; the direct image functor is the forgetful functor; it is crucial that this functor is *strict monoidal*. A distinguished action of the monoidal category ${}_H\mathcal{M}$ on ${}_{\mathbf{k}}\mathcal{M}$, is given by applying the direct image functor in the first component and then tensoring in ${}_{\mathbf{k}}\mathcal{M}$. This is natural because the Hopf algebra H lives in ${}_{\mathbf{k}}\mathcal{M}$, and the actions have to respect the $\mathbf{k}$-structure. Thus if ${}_H\mathcal{M}$ is acting on any other category $\mathcal{C}$ over ${}_{\mathbf{k}}\mathcal{M}$ the square

$$\begin{tikzcd} {}_H\mathcal{M} \times \mathcal{C} \arrow[r, "\triangleright"] \arrow[d] & \mathcal{C} \arrow[d] \\ {}_H\mathcal{M} \times {}_{\mathbf{k}}\mathcal{M} \arrow[r, "\triangleright_0"] & {}_{\mathbf{k}}\mathcal{M} \end{tikzcd}$$

commutes, where $\triangleright_0$ is the distinguished action. This picture where the monoidal action represents the action of group schemes may also be found in commutative geometry, where one essentially takes the action of $\mathfrak{Qcoh}_G$ on $\mathfrak{Qcoh}_X$, where G is a group and X is a scheme, and the categorical action is induced from a usual group action $\nu : G \times X \to X$ via the formula on objects $F \triangleright L = \nu_*(F \boxtimes L)$ where $\boxtimes$ denotes the external tensor product of sheaves.

Leaving apart the difficult question on which monoidal categories $\mathcal{G}$ generalizing ${}_H\mathcal{M}$ for Hopf algebra H, should qualify as representing the 'noncommutative group schemes', we set the convention that $\mathcal{G}$ will be equipped with a distinguished action on the base category (which in general does not need to be monoidal); we consider only the actions respecting (via direct image functors) the distinguished action on the base, and call such actions **(geometrically) admissible** (or strictly compatible with the distinguished action in the base). This innocent condition is in fact very central to our approach! The following result is now a geometric restatement of our simple earlier result in [32]:

3.3. Proposition. *If $\mathcal{C}$ is monadic over the base $\mathcal{B}$, that is $\mathcal{C} \cong \mathcal{B}^{\mathbf{T}}$, where the monad $\mathbf{T}$ on the base is the composition of the inverse and direct image functors of the geometric morphism, then for a monoidal category $\mathcal{G}$ representing a symmetry object over the base $\mathcal{B}$, the distributive laws between the distinguished action of $\mathcal{G}$ on the base $\mathcal{B}$ and the monad $\mathbf{T}$ are in a bijective correspondence with the geometrically admissible monoidal actions of $\mathcal{G}$ on $\mathcal{C}$.*

This simple analogue (secretly, a generalization) of the classical Beck's theorem ([5, 1]) on the bijection between the distributive laws of monads and lifts of one monad to the Eilenberg–Moore category of another monad explains many appearances of "entwining structures" in noncommutative fiber bundle theories.

3.4. Recall that the category of left modules over any bialgebra is monoidal.

Definition. Let B be a bialgebra. A **right B-comodule algebra** E is a right B-comodule for which the coaction $\rho : E \to E \otimes B$ is an algebra map.

Proposition. *Every right B-comodule algebra E canonically induces a geometrically admissible action of ${}_B\mathcal{M}$ on ${}_E\mathcal{M}$.*

Proof. It is sufficient to write down a canonical distributive law enabling the lifting of the geometrically admissible ${}_B\mathcal{M}$-action from ${}_{\mathbf{k}}\mathcal{M}$ to ${}_E\mathcal{M}$. Let $\Diamond : {}_{\mathbf{k}}\mathcal{M} \times {}_B\mathcal{M} \to {}_{\mathbf{k}}\mathcal{M}$ be the "trivial" tensor product action of ${}_B\mathcal{M}$ on ${}_{\mathbf{k}}\mathcal{M}$. The monad in question is of course $E \otimes_{\mathbf{k}}$ and the distributive law has the components $l_{E,M,Q} : E \otimes (M \Diamond Q) \to (E \otimes M) \otimes Q$ which are given by the $\mathbf{k}$-linear extension of formulas $e \otimes (m \otimes q) \mapsto \sum e_{(0)} \otimes m \otimes e_{(1)} q$, where $e \in E$, $m \in M$, $q \in Q$, $\rho(e) = \sum e_{(0)} \otimes e_{(1)}$ is the formula for coaction in the extended Sweedler notation ([22]). Easy calculations show that $l_{E,M,Q}$ is indeed a distributive law.

This way, the comodule algebras supply examples of geometrically admissible actions; hence we view them as (a class of) noncommutative G-spaces.

3.5. Definition. A B-**module algebra** is an algebra A with a B-action $\triangleright$ satisfying the "Leibniz rule" $b \triangleright (aa') = \sum (b_{(1)} \triangleright a)(b_{(2)} \triangleright a)$.

Proposition. *Let A be a left B-module algebra. Then the monoidal category of right B-comodules acts on ${}_A\mathcal{M}$.*

Again, for all $\mathbf{k}$-modules M and B-comodules Q, one needs to write the components of the distributive law $l_{M,Q} : A \otimes (M \otimes Q) \to (A \otimes M) \otimes Q$. Indeed, the formula $a \otimes (m \otimes q) \mapsto \sum (q_{(1)} \triangleright_A a \otimes m) \otimes q_{(0)}$ does the job.

4. Equivariant Sheaves in Noncommutative Geometry

4.1. Mumford ([24], 1.3) defines equivariant sheaves using an explicit cocycle condition. We start with a conceptually simple definition of an equivariant object in a fibered category ([41]), which is easy to generalize.

Given a category $\mathcal{C}$ and an internal group G in $\mathcal{C}$, the Yoneda embedding induces a presheaf of groups h_G on $\mathcal{C}$. Given any presheaf of groups $\hat{G} : \mathcal{C}^\circ \to \underline{\mathrm{Group}}$ over $\mathcal{C}$, an action of $\hat{G}$ on an object X in $\mathcal{C}$ is given by a natural transformation of functors $\nu : \hat{G} \times h_X \to h_X$ such that for each object U in $\mathcal{C}$ the component $\nu_U : \hat{G}(U) \times \mathrm{hom}(U, X) \to \mathrm{hom}(U, X)$ is a group action of the group $G(U)$ on a set $\mathrm{hom}(U, X)$. One obtains the category G-$\mathcal{C}$ of G-actions in $\mathcal{C}$.

Let now $\pi : \mathcal{F} \to \mathcal{C}$ be a fibered category and ν an action of $\hat{G}$ on the fixed object X. The composition $\pi \circ \hat{G}$ is a presheaf of groups in $\mathcal{F}$ so one can form the category of $\pi \circ \hat{G}$-actions in $\mathcal{F}$, and this category clearly projects via naturally induced projection π' to the category $G - \mathcal{C}$ (in fact this projection, for cartesian closed $\mathcal{C}$ and $\mathcal{F}$ is a fibered category as well). The fiber $(\pi')^{-1}(\mathrm{id}_{(X,\nu)})$ is **the category of equivariant objects** in $\mathcal{F}$ over (X, ν).

4.2. Cartesian product with G is in fact a monad in $\mathcal{C}$ and $h_G \times h_X \cong h_{G\times X}$. Thus one in fact induces a presheaf over $\mathcal{C}$ of monads in **Set** and for any presheaf of monads $\mathbf{T}$ one can do similar trick as in **4.1** to define the Eilenberg–Moore fibered category $\mathcal{F}^{\mathbf{T}} \to \mathcal{C}^{\mathbf{T}}$, which may be viewed as the fibered category of equivariant objects. Unfortunately, few monads in $\mathcal{C}$ can be replaced using Yoneda by a presheaf of monads in **Set**.

Similarly, for a functor of V-enriched categories $\pi : \mathcal{F} \to \mathcal{C}$ one uses the enriched Yoneda lemma to define, for any presheaf of (co)monads $\mathbf{T}$ in V (e.g., tensoring with a (co)group in the monoidal category V), the $\mathbf{T}$-equivariant objects in $\mathcal{F}$ over a $\mathbf{T}$-module (X, ν) in the base $\mathcal{C}$.

4.3. One can enrich categories to get 2-categories, and apply the above mechanism. But Yoneda lemma should be better replaced by the pseudo-Yoneda lemma for 2-functors in a pseudo sense (contravariant version: 2-presheaves). C. Hermida studied in [15] a concept of *2-fibered 2-category*. For 1-fibrations one requires that every arrow has a (strongly) Cartesian lift. Hermida requires 2 universal properties for liftings of 1-cells (1-Cartesian and 2-Cartesian 1-cells) and 2 universal properties for 2-cells in order to call a 2-functor a 2-fibered 2-category. Given a weak (in pseudo sense) 3-functor from the base 2-category $\mathcal{C}^{\mathrm{op}}$ to 2**Cat** one can perform a 2-categorical analogue of Grothendieck construction to obtain a 2-fibered 2-category in the sense of Hermida. Vice versa, using a 2-categorical analogue of a cleavage one may represent 2-fibered 2-categories by pseudo-2-functors. Recall that if Y is an internal (monoidal) category in a 2-category $\mathcal{C}$, then $\mathrm{hom}_{\mathcal{C}}(-, Y)$ is a usual internal (monoidal) category.

4.3.1. Definition. Let G be an internal monoidal category in the base 2-category $\mathcal{C}$ of a 2-fibered category $\pi : \mathcal{F} \to \mathcal{C}$. Then $\mathrm{hom}(-, G)$ gives a representable 2-presheaf with values in the 2-category of (usual) monoidal categories

MonCat. Let $\hat{G} = \hom(-, G) \circ \pi : \mathcal{F} \to \mathbf{Cat}$ be a 2-presheaf of monoidal categories over $\mathcal{F}$. A G**-equivariant (or 2-equivariant) object** ρ **in** $\mathcal{F}$ over an internal G-actegory $(X, \triangleright, \Psi, u)$ (where $\triangleright : G \times X \to X$ is a monodial action internal functor in $\mathcal{C}$, and Ψ, u are the coherences) is a natural transformation of 2-functors $\tilde{\triangleright} : \hat{G} \times \hom(-, \rho) \to \hom(-, \rho)$ together with modifications $\tilde{\Psi}$, $\tilde{u}$ of natural transformations of 2-functors $\tilde{\Psi} : (- \otimes -)\tilde{\triangleright} - \Rightarrow -\tilde{\triangleright}(-\tilde{\triangleright}-) : \hat{G} \times \hat{G} \times \hom(-, \rho) \Rightarrow \hom(-, \rho)$, $\tilde{u} : (- \otimes \mathbf{1}) \to -$ and satisfying the action pentagon and unit triangle for every fixed argument $U \in \mathcal{F}$, and such that there the action of $\hat{G}$ on $\hom(-\rho)$ is compatible via projection with the action of G on X, i.e., $\pi(\tilde{\triangleright}) = \hom_{\mathcal{C}}(-, \triangleright)$, $\pi(\tilde{\Psi}) = \hom_{\mathcal{C}}(-, \Psi)$, $\pi(\tilde{u}) = \hom_{\mathcal{C}}(-, u)$.

The last part is just symbolic and may need clarification. Given $U \in \mathcal{F}$, the U-component of the equality $\pi(\tilde{\triangleright}) = \hom_{\mathcal{C}}(-, \triangleright)$ means that for all $g \in \hat{G}(U)$, $z \in \hom(U, \rho)$, $\pi(g \tilde{\triangleright} z) = \pi(g) \triangleright \pi(z)$ and alike for the modifications $\tilde{\Psi}, \tilde{u}$.

This clear definition of 2-equivariant objects is new (we anounced further results involving this definition at WAGP06 conference "Gerbes, groupoids and QFT" at ESI, Vienna, in May 2006). One can write explicit cocycle descriptions of 2-equivariant 2-objects in $\mathcal{F}$ in the style of Mumford. In physics, actions of groups on gerbes, involve a special case of 2-(categorical) equivariance, which are discussed in [30]. The detailed treatment and applications will appear elsewhere.

4.4. (Comonad for the relative Hopf modules). Let B be a $\mathbf{k}$-bialgebra. To any right B-comodule algebra (E, ρ_E) (**3.4**), we associate an endofunctor $G : {}_E\mathcal{M} \to {}_E\mathcal{M}$ in the category ${}_E\mathcal{M}$ of left E-modules on objects M in ${}_E\mathcal{M}$ given by the formula $G : M \mapsto M \otimes B$, where the left E-module structure on $M \otimes B$ is given by $e(m \otimes b) := \rho_E(e)(m \otimes b)$, or in an extended Sweedler notation ([22]), $e(m \otimes b) = \sum e_{(0)} m \otimes e_{(1)} b$ ($e \in E, m \in M, b \in B$). In calculations we will often write just $e_{(0)} \otimes e_{(1)}$, omiting even the summation sign $\sum$ in the Sweedler notation. The comultiplication $\Delta = \Delta^B$ on B induces the comultiplication $\delta = \mathrm{id} \otimes \Delta : G \to GG$ on G with counit $\epsilon^G = \mathrm{id} \otimes \epsilon$ making $\mathbf{G} = (G, \delta, \epsilon^G)$ a comonad (cf. the coring picture in [8]).

4.4.1. A left-right **relative** (E, B)**-Hopf module** is a triple (N, ρ_N, ν_N) such that $\nu_N : E \otimes N \to N$ is a left E-action, $\rho_N : N \to N \otimes B$ is a right B-coaction and $\rho_N(\nu(e, n)) = (\nu \otimes \mu_B)(\mathrm{id} \otimes \tau_{B,N} \otimes \mathrm{id})(\rho_E(e) \otimes \rho_N(n))$ for all $e \in E$, $n \in N$, where $\tau_{B,N} : B \otimes N \to N \otimes B$ is the flip of tensor factors. Maps of relative Hopf modules are morphisms of underlying $\mathbf{k}$-modules, which are maps of E-modules and B-comodules.

4.4.2. Proposition. *The category* $({}_E\mathcal{M})_{\mathbf{G}}$ *of* $\mathbf{G}$*-comodules (coalgebras) is equivalent to the category* ${}_E\mathcal{M}^B$ *of left-right relative* (E, B)*-Hopf modules.*

This is one of our basic observations in a collaboration with V. Lunts (2002), and is independently observed and used for similar purposes in coring theory about at the same time (and generalizations for entwined modules and so on).

4.5. P. Deligne in [9] notes that the category of G-equivariant sheaves naturally embeds into the category of simplicial sheaves over the Borel construction considered as a simplicial space. In Autumn 2002, we noticed with V. Lunts that a parallel construction exists for relative (E, B)-Hopf modules.

4.5.1. Recall that any comonad $\mathbf{G}$ on a category $\mathcal{A}$ induces an augmented *simplicial* endofunctor $\mathbf{G}_\bullet \xrightarrow{\epsilon} \mathrm{Id}_{\mathcal{A}}$ and dually a monad induces an augmented cosimplicial endofunctor. Starting with the comonad $\mathbf{G}$ from **4.4** on ${}_E\mathcal{M}$, we can form the category of $\mathbf{G}$-comodules $({}_E\mathcal{M})_{\mathbf{G}} \cong {}_E\mathcal{M}^B$, and form a *monad* $\mathbf{T}_{\mathbf{G}}$ on $({}_E\mathcal{M})_{\mathbf{G}}$ obtained from the adjoint pair of the forgetful and cofree functors $U_G : ({}_E\mathcal{M})_{\mathbf{G}} \leftrightarrow {}_E\mathcal{M} : F_G$. Thus this *monad* $\mathbf{T}_{\mathbf{G}}$ on $({}_E\mathcal{M})_{\mathbf{G}}$ induces a *cosimplicial* endofunctor $\mathbf{T}^\bullet_{\mathbf{G}}$ on ${}_E\mathcal{M}^B$. As (E, ρ) is a monoid in ${}_E\mathcal{M}^B$, $\mathbf{G}^\bullet(E, \rho) := U_{\mathbf{G}}\mathbf{T}^\bullet_{\mathbf{G}}(E, \rho)$ is a cosimplicial algebra in ${}_E\mathcal{M}$, the **coborel construction** on (E, ρ).

Let $f : \mathbf{n} \to \mathbf{m}$ be a morphism in the category of nonempty finite ordinals (simplices) Δ, and $G_f : G^n E \to G^m E$ the corresponding map in $G^\bullet E$. The following idea is due to V. Lunts:

4.5.2. Definition. A **simplicial module** $M_\bullet$ over the coborel construction $\mathbf{G}^\bullet(E, \rho)$ is a sequence $(M_n)_{n=0,1,2,\ldots}$ of $\mathbf{k}$-modules, with the structure of a left E_n-module on M_n, together with structure maps of left E_n-modules $\beta_f : G^*_f M_m \to M_n$, for all morphisms $f : \mathbf{n} \to \mathbf{m}$ in Δ (where G^*_f is the extension of scalars along G_f), and such that for all $\mathbf{n} \xrightarrow{f} \mathbf{m} \xrightarrow{g} \mathbf{r}$ in Δ the cocycle condition holds:

$$\begin{tikzcd} G^*_{g\circ f} M_r \arrow[r, "\cong"] \arrow[rrr, bend right, "\beta_{g\circ f}"'] & G^*_f G^*_g M_r \arrow[r, "G^*_f(\beta_g)"] & G^*_f M_m \arrow[r, "\beta_f"] & M_n \end{tikzcd}$$

The morphisms of simplicial modules are ladders of maps $M_n \to N_n$ of E_n-modules $n = 0, 1, 2, \ldots$ commuting with the structure maps β_f. This way we get a category ${}_E\mathbf{Sim}^B$ of simplicial modules over $G^\bullet(E, \rho)$.

4.5.3. Theorem (with V. Lunts). *The category of relative Hopf modules ${}_E\mathcal{M}^B$ is equivalent to the full subcategory of ${}_E\mathbf{Sim}^B$ of those objects for which all β_f are isomorphisms.*

We found several interesting proofs of this theorem (our forthcoming paper with V. Lunts: *Hopf modules,* `Ext`*-groups and descent*). One of the proofs is via an intermediate construction of independent interest:

4.5.4. Given a right B-comodule algebra (E, ρ), denote by $p = p_E : E \to E \otimes B$ the map $e \mapsto e \otimes 1_B$ and by $p_{12} : E \otimes B \to E \otimes B \otimes B$ the map $e \otimes b \mapsto e \otimes b \otimes 1$. A **right B-coequivariant left E-module** is a pair (M, θ) where M is a left E-module and $\theta : \rho^* M \to p^* M$ is an isomorphism of left E-modules which satisfies the following Mumford-style cocycle condition:

$$\begin{tikzcd}
(\mathrm{id}\otimes\Delta)^*\rho^*M \arrow[r, "(\mathrm{id}\otimes\Delta)^*\theta"] \arrow[d, "\cong"] & (\mathrm{id}\otimes\Delta)^*p^*M \arrow[r, "\cong"] & p_{12}^*p^*M \\
(\rho\otimes\mathrm{id})^*\rho^*M \arrow[r, "(\rho\otimes\mathrm{id})^*\theta"] & (\rho\otimes\mathrm{id})^*p^*M \arrow[r, "\cong"] & p_{12}^*\rho^*M \arrow[u, "p_{12}^*\theta"']
\end{tikzcd} \qquad (1)$$

Notice that in our notation $f \mapsto f^*$ is a *covariant* functor, and that the three canonical isomorphisms denoted by $\cong$ in the diagram are nontrivial when written in terms of tensor products. A **morphism of pairs** $f : (M, \theta_M) \to (N, \theta_N)$ is a morphism of left E-modules $f : M \to N$ such that $p^*f \circ \theta_M = \theta_N \circ \rho^* f$. This way we obtain a category ${}_E\mathcal{M}^{\mathrm{coeq}B}$ of right B-coequivariant left E-modules.

4.5.5. Theorem (with V. Lunts). *There is a canonical equivalence of categories ${}_E\mathcal{M}^B \cong {}_E\mathcal{M}^{\mathrm{coeq}B}$.*

Sketch of the proof. If $f : E \to E \otimes B$ then the class in f^*M with representative $e \otimes m \otimes b$ in $(E \otimes B) \otimes M$ will be denoted $[e \otimes m \otimes b]_{f^*M}$.

The equivalence of categories needs to produce θ from ρ_M and viceversa (the underlying $M \in {}_E\mathcal{M}$ does not change). Given coaction ρ_M, define $\theta : \rho_E^* M \to p^*M$ by the $\mathbf{k}$-linear extension of the formula

$$\theta([e \otimes b \otimes m]_{\rho^*M}) := [\sum e \otimes bm_{(1)} \otimes m_{(0)}]_{p^*M}.$$

Given θ, define $\rho_M : M \to M \otimes B$ by $\rho_M(m) := (\mathrm{nat} \circ \theta)[1 \otimes 1 \otimes m]_{\rho_E^*M}$. Map $\mathrm{nat} : p^*M \to M \otimes B$ is the isomorphism of $\mathbf{k}$-linear spaces which is the composition in the bottom line of the commutative diagram

$$\begin{array}{ccccc}
E \otimes B \otimes M & \xrightarrow{E\otimes\tau_{M,B}} & E \otimes M \otimes B & \xrightarrow{\nu\otimes B} & M \otimes B \\
\downarrow & & \downarrow & & \downarrow \\
p^*M & \longrightarrow & \frac{E\otimes M\otimes B}{\langle e\otimes m\otimes b - 1\otimes em\otimes b\rangle} & \longrightarrow & M \otimes B
\end{array}$$

where ν is the action $E \otimes M \to M$ and $\tau_{M,B} : B \otimes M \to M \otimes B$ is the flip of tensor factors and the vertical lines are the natural projections.

Now all the verifications (the correspondences are well defined and mutually inverse, the cocycle condition for the new theta, the coaction axiom for new ρ_M) are just calculations with classes $[e \otimes m \otimes b]_{f^*M}$ (and "longer" versions). Finally, in the same terms, one checks that the map of left E-modules $h : M \to N$ is a morphism $(M, \rho_M) \to (N, \rho_N)$ iff it is morphism between the corresponding coequivariant modules $(M, \theta_M) \to (N, \theta_N)$. This finishes the proof.

Theorem **4.5.3** can now be proved along the following lines: starting with (M, ρ_M) we first form (M, θ), then we set $M_0 = M$, $M_1 = p^*M$, $M_2 = p_{12}^*p^*M$ and so on. Maps p, p_{12}, p_{123} etc. are the 0-th coface maps of the coborel construction, and the corresponding structure morphisms can be taken identities. A general structure map can be computed easily if we know the structure maps corresponding just to cofaces and codegeneracies. But those can be easily found from comparing domains f^*M_m with $p_{12..}^*M_m$. For example, $\rho^*M_0 \to M_1$ is the composition $\rho^* M_0 \xrightarrow{\theta} p^* M_0 \overset{=}{\to} M_1$, that is simply θ. After similar simple formulas are proposed for all generators, one can check the properties.

4.5.6. Above considerations propose the following more general definition applicable to a large class of cases, including relative Hopf modules and classical equivariant sheaves.

Let $\mathbf{G}$ be a (co)monad in a base category $\mathcal{C}$ of a fibered category $\pi : \mathcal{F} \to \mathcal{C}$ and (E, ρ) a $\mathbf{G}$-(co)module. The **category of equivariant objects over** E is the category of Cartesian functors from Δ^0 (respectively Δ) considered as a discrete fibered category into $\mathcal{F}$, whose bottom part is the (co)bar construction for (E, ρ). In other words, it is the fiber (the category of Cartesian sections) over the (co)bar construction considered as a functor. For Hopf modules the base category is the category of $\mathbf{k}$-algebras, the fiber over A is ${}_A\mathcal{M}$, the pullback is the extension of scalars, and the cobar construction is our coborel construction.

Similar constructions can be made for (co)lax actions of monoidal categories, generalizing the (co)monad case.

5. Compatible Localizations

5.1. Here we consider flat localizations of rings (e.g., Ore localizations), and also (additive) localization functors Q^* (between Abelian categories) possesing a right adjoint Q_* (we call them continuous localization functors); equivalently, Q_* is a fully faithful functor having a left adjoint; or counit of the adjunction is an isomorphism (equivalently, the multiplication of the corresponding monad is an isomorphism, i.e., the monad is idempotent) ([14, 38]).

The following concept has been introduced in my thesis (the thesis results are published in [37, 35, 31]).

5.2. (Compatibility of coactions and localizations). ([37]) Given a bialgebra B and a (say, right) B-comodule algebra (E, ρ), an Ore localization of rings $\iota : E \to S^{-1}E$ is ρ-**compatible** if there exists an (automatically unique) coaction $\rho_S : S^{-1}E \to S^{-1}E \otimes B$ making $S^{-1}E$ a B-comodule algebra, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\rho} & E \otimes B \\ \downarrow{\scriptstyle \iota_S} & & \downarrow{\scriptstyle \iota_S \otimes B} \\ S^{-1}E & \xrightarrow{\rho_S} & S^{-1}E \otimes B \end{array}$$

commutes; ρ_S is then called the **localized coaction**. We call the ρ_S-coinvariants in $S^{-1}E$ **localized coinvariants**. Even for compatible localizations, ι_S restricted to the subring $E^{\mathrm{co}B} \subset E$ is typically not underlying the ring localization $U^{-1}E^{\mathrm{co}B}$ with respect to any Ore subset U in $E^{\mathrm{co}B}$.

5.3. Theorem. *Let B be a $\mathbf{k}$-bialgebra, (E, ρ) a B-comodule algebra,* $\mathbf{G}$ *the corresponding comonad on the category of left E-modules, which is described in* **4.4**, *and let $\iota : E \to E_\mu$ be a perfect (e.g., Ore) localization of rings, which* *happens to be ρ-compatible. The* $\mathbf{k}$*-linear maps*

$$l_M : E_\mu \otimes_E (M \otimes B) \to (E_\mu \otimes_E M) \otimes B, \quad e \otimes (m \otimes b) \mapsto \sum (e_{(0)} \otimes m) \otimes e_{(1)} b,$$

where M runs through left E-modules, are well-defined morphisms of left E-modules and together they form a mixed distributive law $l : Q_* Q^* G \Rightarrow G Q_* Q^*$ between the localization monad $Q_* Q^*$ and the comonad $\mathbf{G}$ on ${}_E\mathcal{M}$.

Proof. Clearly, the ($\mathbf{k}$-linear extension of the) formula $l' : e \otimes (m \otimes b) \mapsto \sum (e_{(0)} \otimes m) \otimes e_{(1)} b$ gives a well-defined $\mathbf{k}$-linear map

$$E_\mu \otimes_{\mathbf{k}} M \otimes_{\mathbf{k}} B \to E_\mu \otimes_{\mathbf{k}} M \otimes_{\mathbf{k}} B.$$

To show that l' factors to a well-defined map $l : E_\mu \otimes_E (M \otimes_{\mathbf{k}} B) \to (E_\mu \otimes_E M) \otimes_{\mathbf{k}} B$ (where the E-module structure on $M \otimes B$ is from **4.4**), we need to show that if $e' \in E_\mu$ and $e \in E$, then the map l' sends $r = e'e \otimes m \otimes b$ and $s = \sum e' \otimes e_{(0)} m \otimes e_{(1)} b$ to the elements in the same class in $(E_\mu \otimes_E M) \otimes B$. This is easy: $l'(r) = \sum ((e'e)_{(0)} \otimes m) \otimes (e'e)_{(1)} b = \sum (e'_{(0)} e_{(0)} \otimes m) \otimes e'_{(1)} e_{(1)} b$ and $l'(s) = \sum (e'_{(0)} \otimes e_{(0)} m) \otimes e'_{(1)} e_{(1)} b$, which clearly becomes the same class when projected to $(E_\mu \otimes_E M) \otimes B$ (move $e_{(0)}$ along $\otimes_E$). Thus l_M is a well-defined $\mathbf{k}$-linear map. It is easy to see that l_M is also E-linear (and even E_μ-linear).

To check the first pentagon

$$\begin{tikzcd}
Q_*Q^*G \arrow[rr] \arrow[d] & & GQ_*Q^* \arrow[d] \\
Q_*Q^*GG \arrow[r] & GQ_*Q^*G \arrow[r] & GGQ_*Q^*
\end{tikzcd}$$

for some M in ${}_E\mathcal{M}$, we just directly calculate the two paths starting from a generic homogeneous element $e \otimes (m \otimes b) \in E_\mu \otimes_E (M \otimes B) = Q_*Q^*GM$.

$$\begin{tikzcd}
e \otimes (m \otimes b) \arrow[rr, mapsto] \arrow[d, mapsto] & & (e_{(0)} \otimes m) \otimes e_{(1)} b_{(1)} \arrow[d, mapsto] \\
e \otimes ((m \otimes b_{(1)}) \otimes b_{(2)}) \arrow[r, mapsto] & (e_{(0)} \otimes (m \otimes b_{(1)})) \otimes e_{(1)} b_{(2)} \arrow[r, mapsto] & e_{(0)} \otimes m \otimes e_{(1)} b_{(1)} \otimes e_{(2)} b_{(2)}
\end{tikzcd}$$

The second pentagon

$$\begin{tikzcd}
Q_*Q^*Q_*Q^* \arrow[r, "Q_*Q^*l"] \arrow[d, "Q_*\epsilon Q^*G"] & Q_*Q^*GQ_*Q^* \arrow[r, "lQ_*Q^*"] & GQ_*Q^*Q_*Q^* \arrow[d, "GQ_*\epsilon Q^*"'] \\
Q_*Q^*G \arrow[rr, "l"] & & GQ_*Q^*
\end{tikzcd}$$

is calculated similarly on $f \otimes (e \otimes (m \otimes b)) \in E_\mu \otimes_E (E_\mu \otimes_E (M \otimes B)) = Q_*Q^*Q_*Q^*GM$: the upper and right arrows compose

$$\begin{aligned}
&f \otimes (e \otimes (m \otimes b)) \longmapsto f \otimes ((e'_{(0)} \otimes m) \otimes e_{(1)} b) \\
&\longmapsto (f_{(0)} \otimes (e_{(0)} \otimes m)) \otimes f_{(1)} (e_{(1)} b) \longmapsto (f_{(0)} e_{(0)} \otimes m) \otimes f_{(1)} e_{(1)} b
\end{aligned}$$

while the left-below path gives

$$\begin{aligned}&f\otimes(e\otimes(m\otimes b))\longmapsto fe\otimes(m\otimes b)\\ &\longmapsto((fe)_{(0)}\otimes m)\otimes(fe)_{(1)}b=(f_{(0)}e_{(0)}\otimes m)\otimes f_{(1)}e_{(1)}b\end{aligned}$$

The component of the $\mathbf{G}$-counit triangle $(\epsilon^G Q_*Q^*)\circ l = Q_*Q^*\epsilon^G$ at object M in ${}_E\mathcal{M}$, can be computed at any tensor monomial $e\otimes m\otimes b\in E_\mu\otimes_E(M\otimes B)=Q_*Q^*GM$. The left-hand side computes as $e\otimes m\otimes b\mapsto e_{(0)}\otimes m\otimes e_{(1)}b\mapsto e_{(0)}\otimes m\epsilon(e_{(1)}b) = e\otimes m\epsilon(b)$ and the right-hand side gives the latter result immediately.

Finally, the unit triangle

$$\begin{tikzcd} & GM \arrow[dl, "G(\eta_M)"'] \arrow[dr, "\eta_{GM}"] & \\ E_\mu\otimes_E GM \arrow[rr, "l_M"] & & G(E_\mu\otimes_E M)\end{tikzcd}$$

is almost trivial to check: $m\otimes b\mapsto 1\otimes(m\otimes b)\mapsto(1\otimes m)\otimes b$ and the direct arrow $GM\to G(E_\mu\otimes_E M)$ is $m\otimes b\mapsto(1\otimes m)\otimes b$. $\Box$

5.4. Proposition. *If B is a Hopf algebra with antipode $S:B\to B$, then the formula $l_M^{-1}:(e\otimes m)\otimes b\mapsto e_{(0)}\otimes(m\otimes S(e_{(1)})b)$ defines a $\mathbf{k}$-linear map $l_M^{-1}:(E_\mu\otimes_E M)\otimes B\to E_\mu\otimes_E(M\otimes B)$ which is inverse to l_M and is a map of E_μ-modules.*

Proof. The inverse property is obvious: $e\otimes(m\otimes b)\overset{l_M}{\longmapsto}(e_{(0)}\otimes m)\otimes e_{(1)}b\overset{l_M^{-1}}{\longmapsto}(e_{(0)(0)}\otimes m)\otimes S(e_{(0)(1)})e_{(1)}b=e\otimes m\otimes b$. The map l_M^{-1} is indeed a homomorphism of E_μ-modules, because for $f\in E_\mu$ it sends $f((e\otimes m)\otimes b)=(f_{(0)}e\otimes m)\otimes f_{(1)}b$, by definition, to $(f_{(0)}e)_{(0)}\otimes(m\otimes S(f_{(1)}e_{(0)})f_{(1)}b)=f_{(0)}e_{(0)}\otimes(m\otimes Se_{(1)}Sf_{(1)}\cdot f_{(2)}b)=fe_{(0)}\otimes(m\otimes e_{(1)}b)=f(l_M^{-1}(e\otimes(m\otimes b)))$, as required. $\Box$

5.5. Lemma. *Any map of comonads* (*possibly in different categories*) *induces a functor between their respective categories of comodules* (*coalgebras*).

This follows from the dual statement for monads; the latter is a part of a stronger fact that taking the Eilenberg–Moore category extends to a (strict) 2-functor from the category of monads to the category of categories, see [40].

5.6. Theorem. *Given any continuous localization functor $Q^*:\mathcal{A}\to\mathcal{A}_\mu$ and a comonad $\mathbf{G}$ together with any mixed distributive law $l:Q_*Q^*G\Rightarrow GQ_*Q^*$,*

(i) *$G_\mu=Q^*GQ_*$ underlies a comonad $\mathbf{G}_\mu=(G_\mu,\delta^\mu,\epsilon^{G_\mu})$ on $\mathcal{A}_\mu$ with comultiplication δ^μ given by the composite*

$$Q^*GQ_*\xrightarrow{Q^*\delta^G Q_*}Q^*GGQ_*\xrightarrow{Q^*G\eta GQ_*}Q^*GQ_*Q^*GQ_*$$

and whose counit ϵ^{G_μ} is the composite

$$Q^*GQ_*\xrightarrow{Q^*\epsilon^G Q_*}Q^*Q_*\xrightarrow{\quad\epsilon\quad}\mathrm{Id}_{\mathcal{A}_\mu}$$

(*where the right-hand arrow ϵ is the counit of the adjunction $Q_*\dashv Q^*$*).

(ii) *the composite*

$$Q^*GM \xrightarrow{Q^*(\eta_{GM})} Q^*Q_*Q^*GM \xrightarrow{Q^*(l_M)} G_\mu Q^*M$$

defines a component of a natural transformation $\alpha = \alpha_l : Q^*G \Rightarrow G_\mu Q^*$ *for which the mixed pentagon diagram of transformations*

$$\begin{tikzcd} Q^*G \arrow[rr, "\alpha"] \arrow[d, "Q^*\delta^G"'] & & G_\mu Q^* \arrow[d, "\delta^\mu Q^*"] \\ Q^*GG \arrow[r, "\alpha G"] & G_\mu Q^* G \arrow[r, "G\alpha"] & G_\mu G_\mu Q^* \end{tikzcd}$$

commutes and $(\epsilon^{G_\mu}Q^*)\circ\alpha = Q^*\epsilon^G$. *In other words,* $(Q^*,\alpha_l) : (\mathcal{A},\mathbf{G}) \to (\mathcal{A}_\mu,\mathbf{G}_\mu)$ *is (up to orientation convention which depends on an author) a map of comonads* ([36, 40]).

Part (i) is a standard general nonsense on distributive laws once the continuous localization is replaced by the correspoding idempotent monad. The proof of (ii) is easy.

5.7. Proposition. *Suppose* $\iota : E \to E_\mu$ *is the* ρ*-compatible localization of a* B*-comodule algebra* E*,* $\mathcal{A} = {}_E\mathcal{M}$*,* $\mathcal{A}_\mu = {}_{E_\mu}\mathcal{M}$*,* $\mathbf{G}$*,* $\tilde{\mathbf{G}}_\mu$ *are the "comonads for Hopf modules" as in* **4.4** *and* G_μ*,* l *are constructed as in* **5.3**. *The comonad* $\mathbf{G}_\mu$ *is isomorphic to the comonad* $\tilde{\mathbf{G}}_\mu$*. Moreover* $Q_*\tilde{G}_\mu$*.* Q_*G_μ *and* GQ_* *are isomorphic endofunctors in* ${}_{E_\mu}\mathcal{M}$*.*

Proof. As $\mathbf{k}$-vector spaces, clearly both $Q_*\tilde{G}_\mu N$ and GQ_*N look for $N \in {}_{E_\mu}\mathcal{M}$ like $N \otimes B$. The E-module structure on $Q_*G_\mu N$ is restriction of the E_μ-module structure given by $f(n\otimes b) = \rho_{E_\mu}(f)(n\otimes b)$ for $f \in E_\mu$, that is $\iota(e)(n\otimes b) = \rho_{E_\mu}(\iota(e))(n\otimes b)$, while the E-module structure on GQ_*N is given by $e(n\otimes b) = ((\iota\otimes \mathrm{id}_B)\rho_E(e))(n\otimes b)$. By the ρ-compatibility of localization ι the two answers agree, i.e., $Q_*\tilde{G}_\mu = GQ_*$.

$\epsilon : Q^*Q_* \Rightarrow \mathrm{Id}$ is an isomorphism. Hence $G_\mu = Q^*GQ_* = Q^*Q_*\tilde{G}_\mu \cong \tilde{G}_\mu$.

One should further check that the comultiplications of $\mathbf{G}_\mu$ and $\tilde{\mathbf{G}}_\mu$ agree, that is the external square in the diagram

$$\begin{tikzcd} E_\mu \otimes_E ({}_E N \otimes B)) \arrow[r, "Q^*\delta^G Q_*"] \arrow[d, "="] & E_\mu \otimes_E (({}_E N \otimes B)\otimes B) \arrow[d, "="] \\ Q^*Q_*({}_{E_\mu}N \otimes B) \arrow[r, "Q^*Q_*\delta^{\tilde{G}_\mu}"] \arrow[d, "\epsilon"] & Q^*Q_*(({}_{E_\mu}N\otimes B)\otimes B) \arrow[d, "\epsilon"] \\ {}_{E_\mu}N \otimes B \arrow[r, "\delta^{\tilde{G}_\mu}"] & ({}_{E_\mu}N \otimes B)\otimes B \end{tikzcd}$$

commutes, where the upper vertical arrows are induced by isomorphisms $Q_*\tilde{G}_\mu \cong GQ_*$. The lower square commutes by naturality of ϵ and the upper square commutes as vertical arrows are identities and horizontal arrows are both $\mathrm{id}\otimes\Delta$ at the level of vector spaces. $\square$

5.8. Theorem. *Under the assumptions in* **5.3**, *there is a unique induced continuous localization functor* $Q^{B*} : {}_E\mathcal{M}^B \to {}_{E_\mu}\mathcal{M}^B$ *between the categories of relative Hopf modules such that* $U_\mu Q^{B*} = Q^*U$ *where* U *and* U_μ *are the forgetful functors from the category of relative Hopf modules to the categories of usual modules over* E *and* E_μ*, respectively.*

Proof. This follows from **5.3** by Lemma **5.5**, after applying the equivalences of categories ${}_E\mathcal{M}^B \cong ({}_E\mathcal{M})_{\mathbf{G}}$ and ${}_E\mathcal{M}^B \cong ({}_{E_\mu}\mathcal{M})_{\mathbf{G}_\mu}$. $\square$

In [34], we prove an interesting generalization of Theorem **5.8** for ρ-compatible localizations to *localization-compatible pairs of entwining structures* introduced therein.

5.9. The main reason why compatible localizations are needed in noncommutative geometry is that they are the analogues of G*-invariant* open sets (unions of G-orbits) in commutative geometry, where G is a group.

5.10. Lunts and Rosenberg studied ([20, 19]) the rings of differential operators for noncommutative rings, generalizing the commutative Grothendieck's definition in a nontrivial way (another approach yielding the same definition is in [23]). Their purpose was to generalize the Beilinson–Bernstein localization theorem in representation theory to quantum groups. The basic property of the differential operators is that they extend to exact localizations. Beilinson and Bernstein abstracted this to the compatibility of localization functors and monads; and prove that it is satisfied for their basic object in [19], *differential monads*. The D-affinity of Beilinson has its abstract and simple generalization in their general context.

The compatibility between a comonad $\mathbf{G}$ and a continuous localization functor $Q^* : \mathcal{A} \to \mathcal{B}$ of Lunts and Rosenberg is an isomorphism of functors of the form $Q^*G \cong G'Q^*$ where G' is some endofunctor in $\mathcal{B}$. This looks like our distributive law $Q^*G \Rightarrow G_\mu Q^*$. There are two differences: our map is a distributive law (satisfies two pentagons and two triangles, in which sense our definition is stronger), and they require an isomorphism while we have only a natural transformation (here their definition is stronger; in our main examples, induced by comodule algebras, we get invertibility for the Hopf algebra case, while not for bialgebras).

5.11. The notion of the compatibility of (co)monads and localization functors can easily be extended to the compatibility of actions of monoidal groups and localizations. For simplicity, we leave out this generalization (and present it in [34]). We implicitly (verbally in a definition) use it in the next section though.

6. Principal Bundles and Quotient Schemes

6.1. In commutative algebraic geometry, there is a notion of the descent along torsors: given a group scheme G, the category $\mathfrak{Qcoh}^G(Y)$ of G-equivariant quasicoherent sheaves on the total space of a G-torsor Y (in fpqc topology) over a scheme X is equivalent to the category $\mathfrak{Qcoh}(X)$ of usual quasicoherent sheaves over X. More generally, take any site $\mathcal{C}$, a group object G in $\mathcal{C}$, G-torsor Y over X, and any stack of categories over $\mathcal{C}$ (replacing the stack $\mathcal{F}$ of categories of quasicoherent sheaves over the site of schemes in fpqc topology); then G-equivariant fiber $\mathcal{F}^G_Y$ over Y is equivalent to the usual fiber $\mathcal{F}_X$ over X ([41]). The analogue holds for (E,B)-Hopf modules: the theorem of Schneider ([29]) states that, given a faithfully flat Hopf-Galois extension $U \hookrightarrow E$, the category of relative (E,B)-Hopf modules is equivalent to the category of left modules over U; this theorem has many generalizations for entwining modules and so on. A **Hopf-Galois extension** is the inclusion $U \hookrightarrow E$ of an algebra $U = E^{\mathrm{co}B}$ of B-coinvariants in a B-comodule algebra E into E, such that the canonical map $E \otimes_U E \to E \otimes B$, $e \otimes e' \mapsto \sum e_{(0)} e' \otimes e_{(1)}$ is an isomorphism.

6.2. However, already in the commutative geometry we know that it is a rare case that there are sufficiently many coinvariants to reconstruct the quotient of an affine torsor under a group action (the spectrum of the algebra of coinvariants is sometimes called the *affine quotient*). Thus I suggest below a globalization of Hopf-Galois extensions to the noncommutative schemes of Rosenberg ([27, 26]).

A flat **cover** $\{F_\mu : \mathcal{A} \to \mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ is a *conservative* family of flat functors (i.e., a morphism $a : A \to A'$ in $\mathcal{A}$ is invertible iff $F_\mu(a)$ is invertible for every $\lambda \in \Lambda$). Recall the terminology of **1.4**.

6.3. A **quasicompact relative noncommutative scheme** $(\mathcal{A}, \mathcal{O})$ over a category $\mathcal{V}$ as an abelian category $\mathcal{A}$ with a distinguished object $\mathcal{O}$, finite biflat affine cover by localizations $Q^*_\lambda : \mathcal{A} \to \mathcal{B}_\lambda$, with a continuous morphism g from $\mathcal{A}$ to $\mathcal{V}$ (think of it as $X \to \operatorname{Spec} \mathbf{k}$) such that each $g_* \circ Q_{\lambda *} : \mathcal{B}_\lambda \to \mathcal{V}$ is affine. If $\mathcal{V} = \mathbf{k}-\mathrm{Mod}$ then $\mathcal{O} = g^*(\mathbf{k})$ where $\mathbf{k}$ is viewed as an object in $\mathbf{k} - \mathrm{Mod}$. For more details and examples see [27].

6.4. Let me propose a globalization of the notion of Hopf-Galois extension:

Definition. Given a Hopf algebra B, a noncommutative scheme $\mathcal{A}$ over ${}_{\mathbf{k}}\mathcal{M}$ is a **noncommutative B-torsor** (over appropriate quotient) if there is a geometrically admissible action $\Diamond : {}_B\mathcal{M} \times \mathcal{A} \to \mathcal{A}$, and an affine flat cover of $\mathcal{A}$ by localizations $Q^*_\mu : \mathcal{A} \to \mathcal{A}_\mu$, with $\mathcal{A}_\mu \cong {}_{E^\mu}\mathcal{M}$; where each Q^*_μ is compatible with the comonad $\mathbf{G}$ induced by B (viewed as a comonoid in ${}_B\mathcal{M}$) and action $\Diamond$; and for each μ, the induced comonad $\mathbf{G}_\mu$ in $\mathcal{A}_\mu$, is induced by a B-comodule algebra on E^μ, and $(E^\mu)^{\mathrm{co}B} \hookrightarrow E^\mu$ is a faithfully flat Hopf-Galois extension.

Given a noncommutative B-torsor, algebras of localized coinvariants $(E^\mu)^{\mathrm{co}B}$ are local coordinate algebras of a cover of the quotient space represented by the Eilenberg-Moore category $\mathcal{A}^\Diamond$ of the monoidal action $\Diamond$ (${}_B\mathcal{M}$-equivariant sheaves on the space represented by $\mathcal{A}$). In other words, the definition explores a local version of the Hopf-Galois condition, which allows the local coordinatization of the "quotient stack" $\mathcal{A}^\diamondsuit$. My main example ([37]) is the quantum fibration $SL_q(n) \to SL_q(n)/B_q(n)$, (or the $GL_q(n)$ version $GL_q(n) \to GL_q(n)/\tilde{B}_q(n)$, which has multiparametric deformation generalizations), where the noncommutative spaces $SL_q(n), GL_q(n)$ are represented by the quantum linear groups $\mathcal{O}(SL_q(n)), \mathcal{O}(GL_q(n))$, and the quotient stack $SL_q(n)/B_q(n)$ is constructed as a noncommutative scheme in my earlier work [37]. This "quantum fibration" is a $\mathcal{O}(B_q(n))$-torsor with local trivialization given by $n!$ Hopf-Galois extensions in an interesting cover by $n!$ Ore localizations (defined in terms of quantum minors) $S_w^{-1}\mathcal{O}(SL_q(n))$ (w in the Weyl group), which is an analogue of the standard cover of SL_n by shifts of the main Bruhat cell. Our calculational proof shows existence of such a cover by formulas using the quantum Gauss decomposition, calculations with quantum minors, an explicit check of a nontrivial Ore property ([35]), a difficult check of the compatibility of coactions and localizations and, finally, the local Hopf-Galois property which is in this case implied by the stronger "local triviality", meaning that there is a smash product decomposition $S_w^{-1}\mathcal{O}(SL_q(n)) \cong U_w \sharp B_q$ (induced by quantum Gauss decomposition) as a right B_q-comodule algebra. Embedding of coinvariants $U_w \hookrightarrow U_w \sharp B_q$ into the smash product is a simple case of the Hopf-Galois extension. This picture was applied in ([31]) to develop a geometric theory of Perelomov coherent states for quantum groups with nontrivial resolution of unity formula obtained utilizing line bundles over the quantum coset space $SL_q(n)/B_q(n)$.

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Author's address:

Division of theoretical physics
Institute Rudjer Bošković
P.O.Box 180, Zagreb HR-10002
Croatia
E-mail: zskoda@irb.hr