K-stability of constant scalar curvature Kähler manifolds

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Abstract

We show that a polarised manifold with a constant scalar curvature Kähler metric and discrete automorphisms is K-stable. This refines the K-semistability proved by S. K. Donaldson.

1 Introduction

Let (X, L) be a polarised manifold. One of the more striking realisations in Kähler geometry over the past few years is that if one can find a constant scalar curvature Kähler (cscK) metric g on X whose $(1,1)$-form ω_g belongs to the cohomology class $c_1(L)$ then (X, L) is *semistable*, in a number of senses. The seminal references are Yau [16], Tian [13], Donaldson [5], [7].

In this note we are concerned with Donaldson's *algebraic K-stability* [7], see also Definition 2.5 below. This notion generalises Tian's K-stability for Fano manifolds [13]. It should play a role similar to Mumford-Takemoto slope stability for bundles. The necessary general theory is recalled in Section 2.

Asymptotic Chow stability (which implies K-semistability, see e.g. [12] Theorem 3.9) for a cscK polarised manifold was first proved by Donaldson [6] in the absence of continous automorphisms. Important work in this connection was also done by Mabuchi, see e.g. [10]. From the analytic point of view the fundamental result is the lower bound on the K-energy proved by Chen-Tian [4].

The neatest result in the algebraic context seems to be Donaldson's *lower bound on the Calabi functional*, which we now recall.

For a Kähler form ω let $S(\omega)$ denote the scalar curvature, $\widehat{S}$ its average (a topological quantity). Denote by F the Donaldson-Futaki invariant of a test configuration (Definitions 2.1, 2.2). The precise definition of the norm $\|\mathcal{X}\|$ appearing below will not be important for us.

Theorem 1.1 (Donaldson [8]) *For a polarised manifold* (X, L)

$$\inf_{\omega \in c_1(L)} \int_X (S(\omega) - \widehat{S})^2 \omega^n \geq -\frac{\sup_{\mathcal{X}} F(\mathcal{X})}{\|\mathcal{X}\|}. \tag{1.1}$$

where the supremum is taken with respect to all test configurations $(\mathcal{X}, \mathcal{L})$ *for* (X, L).

Thus if $c_1(L)$ *admits a cscK representative* (X, L) *is K-semistable.*

There is a strong analogy here with Hermitian Yang-Mills metrics on vector bundles. By the celebrated results of Donaldson and Uhlenbeck-Yau these are known to exist if and only if the bundle is slope polystable, namely a semistable direct sum of slope stable vector bundles.

In particular a simple vector bundle endowed with a HYM metric is slope stable. In this note we will prove the corresponding result for polarised manifolds.

Theorem 1.2 *If* $c_1(L)$ *contains a cscK metric and* $\mathrm{Aut}(X, L)$ *is discrete then* (X, L) *is K-stable.*

Theorem 1.2 fits in a more general well known conjecture.

Conjecture 1.3 (Donaldson [7]) If $c_1(L)$ contains a cscK metric then (X, L) is K-polystable (Definition 2.6).

Thus our result confirms this expectation when the group $\mathrm{Aut}(X, L)$ is discrete. From a differential-geometric point of view this means that X has no nontrivial Hamiltonian holomorphic vector fields - holomorphic fields that vanish somewhere.

Remark 1.4 Conjecture 1.3 and its converse are known as Yau - Tian - Donaldson Conjecture, and sometimes called the Hitchin-Kobayashi correspondence for manifolds.

For the rest of the note we will assume $\dim(X) > 1$ in all our statements.

K-stability for Riemann surfaces is completely understood thanks to the work of Ross and Thomas [12] Section 6. In particular Conjecture 1.3 is known to hold for Riemann surfaces.

Our proof of Theorem 1.2 rests on the general principle that one should be able to *perturb* a semistable object (in the sense of geometric invariant theory) to make it unstable - altough this necessarily involves perturbing the GIT problem too, since the locus of semistable points for an action on a fixed variety is open. Conversely in the absence of continous automorphisms, the cscK property is open - at least in the sense of small deformations - so cscK should imply stability. Of course we need to make this rigorous; in particular testing small deformations is not enough to prove K-stability.

Thus suppose that (X, L) is properly K-semistable (Definition 2.7). We will find a natural way to construct from this a family of K-unstable small perturbations $(X_\varepsilon, L_\varepsilon)$ for small $\varepsilon > 0$. Our choice for X_ε is actually constant, the blowup $\widehat{X} = \mathrm{Bl}_q X$ at a very special point q with exceptional divisor E. Only the polarisation changes, and quite naturally $L_\varepsilon = \pi^* L - \varepsilon \mathcal{O}(E)$. This would involve taking $\varepsilon \in \mathbb{Q}^+$ and working with $\mathbb{Q}$-divisors, but in fact we rather take tensor powers and work with $\widehat{X}$ polarised by $L_\gamma = \pi^* L^\gamma - \mathcal{O}(E)$ for integer $\gamma \gg 0$. K-(semi, poly, in)stability is unaffected by Definition 2.2.

Proposition 1.5 *Let (X, L) be a properly K-semistable polarised manifold. Then there exists a point $q \in X$ such that the polarised blowup $(\mathrm{Bl}_q X, \pi^* L^\gamma \otimes \mathcal{O}(-E))$ is K-unstable for $\gamma \gg 0$.*

Remark 1.6 It is interesting to note that the corresponding result for vector bundles follows from Buchdahl [3]. Let (X, L) be a polarised manifold and $E \to X$ a properly slope semistable vector bundle. Then the pullback $\pi^* E$ to the blowup $\mathrm{Bl}_{q_1, \dots q_m} X$ in a finite number of suitably chosen points is slope unstable with respect to the polarisation $\pi^* L^\gamma \otimes \mathcal{O}_{\mathrm{Bl}_{\{q_i\}} X}(1)$ for $\gamma \gg 0$.

Assume now that a properly semistable (X, L) also admits a cscK metric $\omega \in c_1(L)$. If $\mathrm{Aut}(X, L)$ is discrete the blowup perturbation problem for ω is unobstructed by a theorem of Arezzo and Pacard [1], so we would get cscK metrics in $c_1(\pi^* L^\gamma \otimes \mathcal{O}(-E))$ for $\gamma \gg 0$, a contradiction.

Remark 1.7 This perturbation strategy for proving 1.3 is very general, and was first pointed out to the author by S. Donaldson and G. Székelyhidi. Different choices for $(X_\varepsilon, L_\varepsilon)$ lead to different perturbation problems for ω, which may settle Conjecture 1.3 in the presence of continous automorphisms. A possible variant is to perturb the cscK equation with ε at the same time, but one would then need to develop the relevant K-stability theory for a more general equation.

To sum up the main ingredients for our proof (besides Theorem 1.1) are:

1. A well known embedding result for test configurations (Proposition 2.9), together with the algebro-geometric estimate Proposition 3.3;

2. A blowup formula for the Donaldson-Futaki invariant proved by the author [14] Theorem 1.3;

3. A special case of the results of Arezzo and Pacard on blowing up cscK metrics [1].

Aknowledgements. I am grateful to S. K. Donaldson, G. Székelyhidi and my advisor R. Thomas for many useful discussions. The reference [3] was pointed out to me by J. Keller.

2 Some general theory

Let n denote the complex dimension of X.

Definition 2.1 (Test configuration.) A *test configuration* for a polarised manifold (X, L) is a polarised flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ with $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L)$ and which is $\mathbb{C}^*$-equivariant with respect to the natural action of $\mathbb{C}^*$ on $\mathbb{C}$.

Given a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) denote by A_k the matrix representation of the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$. By (equivariant) Riemann-Roch we can find expansions

$$h^0(\mathcal{X}_0, \mathcal{L}_0^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \tag{2.1}$$

$$\mathrm{tr}(A_k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}). \tag{2.2}$$

Definition 2.2 (Donaldson-Futaki invariant.) This is the rational number

$$F(\mathcal{X}) = a_0^{-2}(b_0 a_1 - a_0 b_1) \tag{2.3}$$

which is independent of the choice of a lifting of the action to $\mathcal{L}_0$.

Equivalently $F(\mathcal{X})$ is the coefficient of k^{-1} in the Laurent series expansion of the quotient

$$\frac{\mathrm{tr}(A_k)}{k h^0(\mathcal{X}_0, \mathcal{L}_0^k)}.$$

Note moreover that F is invariant under taking tensor powers, i.e.

$$F(\mathcal{X}, \mathcal{L}) = F(\mathcal{X}, \mathcal{L}^r).$$

Therefore for the rest of this note we will assume without loss of generality that $\mathcal{L}$ is *relatively very ample*.

Remark 2.3 (Coverings) Given a test configuration $(\mathcal{X}, \mathcal{L})$ we can construct a new test configuration for (X, L) by pulling $\mathcal{X}$ and $\mathcal{L}$ back under the d-fold ramified covering of $\mathbb{C}$ given by $z \mapsto z^d$. This changes A_k to $d \cdot A_k$ and consequently F to $d \cdot F$.

Definition 2.4 A test configuration $(\mathcal{X}, \mathcal{L})$ is called a *product* if it is $\mathbb{C}^*$-equivariantly isomorphic to the product $(X \times \mathbb{C}, p_X^* L)$ endowed with the composition of a $\mathbb{C}^*$-action on (X, L) with the natural action of $\mathbb{C}^*$ on $\mathbb{C}$.

A product test configuration is called *trivial* if the associated action on (X, L) is trivial.

The Donaldson-Futaki invariant $F(\mathcal{X})$ in this case coincides with the classical Futaki invariant for holomorphic vector fields.

Definition 2.5 (K-stability) A polarised manifold (X, L) is *K-semistable* if for all test configurations $(\mathcal{X}, \mathcal{L})$

$$F(\mathcal{X}) \geq 0.$$

It is *K-stable* if the strict inequality holds for nontrivial test configurations.

In particular if (X, L) is K-stable $\mathrm{Aut}(X, L)$ must be discrete. The correct notion to take care of continous automorphisms is K-polystability.

Definition 2.6 A polarised manifold (X, L) is *K-polystable* if it is K-semistable and moreover any test configuration $(\mathcal{X}, \mathcal{L})$ with $F(\mathcal{X}) = 0$ is a product.

Definition 2.7 A polarised manifold (X, L) is *properly K-semistable* if it is K-semistable and it admits a nonproduct test configuration with vanishing Donaldson-Futaki invariant.

Remark 2.8 The terminology *strictly K-semistable* is also found in the literature with the same meaning.

Test configurations are well known to be equivalent to 1-parameter flat families induced by projective embeddings.

Proposition 2.9 (see e.g. Ross-Thomas [12] 3.7) *A test configuration for (X, L) is equivalent to a 1-parameter subgroup of* $\mathrm{GL}(H^0(X, L)^*)$.

In [14] the author proved a blowup formula for the Donaldson-Futaki invariant. The statement involves some more terminology.

Definition 2.10 (Hilbert-Mumford weight.) Let α be a 1-parameter subgroup of $\mathrm{SL}(N+1)$, inducing a $\mathbb{C}^*$-action on $\mathbb{P}^N$. Choose projective coordinates $[x_0 : ... : x_N]$ such that α is given by $\mathrm{Diag}(\lambda^{m_0}, ...\lambda^{m_N})$. The Hilbert-Mumford weight of a closed point $q \in \mathbb{P}^N$ is defined by

$$\mu(q, \alpha) = -\min\{m_i : q_i \neq 0\}.$$

Note that this coincides with the weight of the induced action on the fibre of the hyperplane line bundle $\mathcal{O}(1)$ over the specialisation $\lim_{\lambda \to 0} \lambda \cdot q$.

Definition 2.11 (Chow weight.) Let (Y, L) be a polarised scheme, $y \in Y$ a closed point, and α a $\mathbb{C}^*$-action on (Y, L). Suppose that L is very ample and $\alpha \hookrightarrow \mathrm{SL}(H^0(Y, L)^*)$. The Chow weight $\mathcal{CH}_{(Y,L)}(q, \alpha)$ is defined to be the Hilbert-Mumford weight of $y \in \mathbb{P}(H^0(Y, L)^*)$ with respect to the induced action. The definition extends to 0-dimensional cycles on Y, that is effective linear combinations of closed points.

Theorem 2.12 (S. [14] 1.3) *For points $q_i \in X$ and integers $a_i > 0$ let $Z \subset X$ be the 0-dimensional closed subscheme $Z = \cup_i a_i q_i$. Let Λ be the 0-cycle on X given by $\sum_i a_i^{n-1} q_i$.*

A 1-parameter subgroup $\alpha \hookrightarrow \mathrm{Aut}(X, L)$ induces a test configuration $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ for $(\mathrm{Bl}_Z X, \pi^* L^\gamma \otimes \mathcal{O}_{\mathrm{Bl}_Z X}(1))$, where $\mathcal{O}_{\mathrm{Bl}_Z X}(1)$ denotes the exceptional invertible sheaf. More precisely let $O(Z)^-$ be the closure of the orbit of Z. Then $\widehat{\mathcal{X}} = \mathrm{Bl}_{O(Z)^-} \mathcal{X}$ and $\widehat{\mathcal{L}} = \pi^* \mathcal{L}^\gamma \otimes \mathcal{O}_{\widehat{\mathcal{X}}}(1)$.

Suppose that α acts through $\mathrm{SL}(H^0(X, L)^*)$ with Futaki invariant $F(X)$. Then the following expansion holds as $\gamma \to \infty$

$$F(\widehat{\mathcal{X}}) = F(X) - \mathcal{CH}_{(X,L)}(\Lambda, \alpha) \frac{\gamma^{1-n}}{2(n-1)!} + O(\gamma^{-n}).$$

We will need a slight generalisation of this result, covering blowups of non-product test configurations.

Proposition 2.13 *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for (X, L), $Z = \cup_i a_i q_i$ as above. There is a test configuration $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ for $(\mathrm{Bl}_Z X, \pi^* L^\gamma \otimes \mathcal{O}_{\mathrm{Bl}_Z X}(1))$ with total space $\widehat{\mathcal{X}}$ given by the blowup of $\mathcal{X}$ along $O(Z)^-$. The linearisation is the natural one induced on $\widehat{\mathcal{L}} = \pi^* \mathcal{L}^\gamma \otimes \mathcal{O}_{\widehat{\mathcal{X}}}(1)$.*

Let $q_{i,0} = \lim_{\lambda \to 0} \lambda \cdot q_i$ be the specialisation, Λ_0 the 0-cycle on $\mathcal{X}_0$ given by $\sum_i a_i^{n-1} q_{i,0}$.

Let α denote the induced action on $(\mathcal{X}_0, \mathcal{L}_0)$ and suppose that α acts through $\mathrm{SL}(H^0(\mathcal{X}_0, \mathcal{L}_0)^*)$. Then the expansion

$$F(\widehat{\mathcal{X}}) = F(\mathcal{X}) - \mathcal{CH}_{(\mathcal{X}_0, \mathcal{L}_0)}(\Lambda_0, \alpha) \frac{\gamma^{1-n}}{2(n-1)!} + O(\gamma^{-n})$$

holds as $\gamma \to \infty$.

We emphasise that the relevant Chow weight is computed *on the central fibre* $(\mathcal{X}_0, \mathcal{L}_0)$ with its induced $\mathbb{C}^*$-action.

Proof. The argument of [14] Section 4 goes over verbatim to non-product test configurations, with only two exceptions:

1. The proof of flatness of the composition $\widehat{\mathcal{X}} \to \mathcal{X} \to \mathbb{C}$;
2. The identification of the weight $\mathcal{CH}_{(\mathcal{X}_0,\mathcal{L}_0)}(\Lambda_0)$ (with respect to the induced action on $\mathcal{X}_0$) with $\mathcal{CH}_{(X,L)}(\Lambda, \alpha)$.

We do not need the latter identification, and indeed it does not make sense in this case since the general fibre is not preserved by the $\mathbb{C}^*$-action.

To prove flatness we use the criterion [9] III Proposition 9.7. Thus we need to prove that all associated points of $\widehat{\mathcal{X}}$ (i. e. irreducible components and their thickenings) map to the generic point of Spec($\mathbb{C}$).

By flatness this is true for the morphism $\mathcal{X} \to \mathbb{C}$, and blowing up $O(\Lambda)^-$ does not contribute new associated points, only the Cartier exceptional divisor $\pi^{-1}O(\Lambda)^-$.

More precisely let $\mathcal{I}$ denote the ideal sheaf of $O(q)^- \subset \mathcal{X}$, and recall $\widehat{\mathcal{X}}$ is defined as Proj $\bigoplus_{d\geq 0} \mathcal{I}^d$. Any homogeneous zero divisor in the graded sheaf $\bigoplus_{d\geq 0} \mathcal{I}^d$ is already a zero divisor when regarded as an element of $\mathcal{O}_{\mathcal{X}}$. On the other hand an associated point $\widehat{x} \in \widehat{\mathcal{X}}$ is by definition (following [9] III Corollary 9.6) a point for which every element of $\mathfrak{m}_{\widehat{x}}$ is a zero divisor. The natural map $\widehat{\mathcal{X}} \to \mathcal{X}$ maps $\mathfrak{m}_{\widehat{x}}$ to its degree 0 piece. Thus by the above remark $\widehat{x}$ necessarily maps to an associated point $x \in \mathcal{X}$. But x maps to the generic point of Spec($\mathbb{C}$) by flatness, so the same is true for $\widehat{x}$.

Q.E.D.

Remark 2.14 In both cases the assumption that α acts through SL is not really restrictive. This can always be achieved by replacing $\mathcal{L}$ by some power and pulling back $\mathcal{X}$ by $z \mapsto z^d$ for some d. This gives a new test configuration for which α can be rescaled to act through SL and for which the Futaki invariant is only multiplied by d, by Remark 2.3.

This property of the Futaki invariant turns out to be important in our proof of Theorem 1.2.

3 Proof of Theorem 1.2

It will be enough to prove Proposition 1.5 and to apply the result of Arezzo and Pacard recalled as Theorem 3.1 below.

Thus let

$$(\widehat{X}, L_\gamma) = (\mathrm{Bl}_q X, \pi^* L^\gamma \otimes \mathcal{O}(-E)).$$

We need to show that $(\widehat{X}, L_\gamma)$ is K-unstable for $\gamma \gg 0$. We will construct test configurations $(\mathcal{X}_\gamma, \mathcal{L}_\gamma)$ for $(\widehat{X}, L_\gamma)$ which have strictly negative Donaldson-Futaki invariant for $\gamma \gg 0$.

By assumption (X, L) is properly semistable, so it admits a nontrivial test configuration $(\mathcal{X}, \mathcal{L})$ with $F(\mathcal{X}) = 0$.

Moreover we can assume that the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ is special linear. Indeed this can be achieved by taking some power $\mathcal{L}^r$ and a ramified cover $z \mapsto z^d$. The new Futaki invariant F' still vanishes since $F' = d \cdot F = 0$.

We blow $\mathcal{X}$ up along the closure $O(q)^-$ of the orbit $O(q)$ of $q \in \mathcal{X}_1$ under the $\mathbb{C}^*$-action on $\mathcal{X}$, i.e. define

$$\mathcal{X}_\gamma = \widehat{\mathcal{X}} = \mathrm{Bl}_{O(q)^-}\mathcal{X}. \tag{3.1}$$

Let $\mathcal{O}_{\widehat{\mathcal{X}}}(1)$ denote the exceptional invertible sheaf on $\widehat{\mathcal{X}}$. We endow $\widehat{\mathcal{X}}$ with the polarisation

$$\mathcal{L}_\gamma = \pi^* \mathcal{L}^\gamma \otimes \mathcal{O}_{\widehat{\mathcal{X}}}(1). \tag{3.2}$$

Define the closed point $q_0 \in \mathcal{X}_0$ to be the specialisation

$$q_0 = \lim_{\lambda \to 0} \lambda \cdot q.$$

Applying the blowup formula 2.13 in this case gives

$$\begin{aligned} F(\widehat{\mathcal{X}}_0, \pi^* \mathcal{L}_0^\gamma \otimes \mathcal{O}_{\widehat{\mathcal{X}}_0}(1)) &= F(\mathcal{X}_0, \mathcal{L}_0) - \mathcal{CH}_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0) \frac{\gamma^{1-n}}{2(n-2)!} + O(\gamma^{-n}) \\ &= -\mathcal{CH}_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0) \frac{\gamma^{1-n}}{2(n-2)!} + O(\gamma^{-n}). \end{aligned}$$

In Proposition 3.3 below we will prove that for a very special $q \in \mathcal{X}_1 \cong X$,

$$\mathcal{CH}_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0) > 0.$$

This holds thanks to the assumption $F(\mathcal{X}) = 0$, or more generally $F(\mathcal{X}) \leq 0$. This is enough to settle Proposition 1.5.

The final step for Theorem 1.2 is to show that the perturbation problem is unobstructed provided $\mathrm{Aut}(X, L)$ is discrete. This is precisely the content of a beautiful result of C. Arezzo and F. Pacard.

Theorem 3.1 (Arezzo-Pacard [1]) *Let (X, L) be a polarised manifold with a cscK metric in the class $c_1(L)$. Suppose $\mathrm{Aut}(X, L)$ is discrete and let $q \in X$ be any point. Then the blowup $\mathrm{Bl}_q X$ with exceptional divisor E admits a cscK metric in the class $\gamma\pi^* c_1(L) - c_1(\mathcal{O}(E))$ for $\gamma \gg 0$.*

Remark 3.2 The Arezzo-Pacard theorem also holds in the Kähler case and, more importantly, even when $\mathfrak{aut}(X, L) \neq 0$, provided a suitable stability condition is satisfied. We refer to [2], [14] for further discussion.

Thus the following Proposition will complete our proof(s). We believe it may also be of some independent interest.

Proposition 3.3 *Let $(\mathcal{X}, \mathcal{L})$ be a nonproduct test configuration for a polarised manifold (X, L) with nonpositive Donaldson-Futaki invariant and suppose the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ is special linear. Then there exists $q \in \mathcal{X}_1 \cong X$ such that $\mathcal{CH}_{(\mathcal{X}_0,\mathcal{L}_0)}(q_0) > 0$.*

Proof. By the embedding Theorem 2.9 we reduce to the case of a nontrivial $\mathbb{C}^*$ acting on $\mathbb{P}^N$ for some N, of the form $\mathrm{Diag}(\lambda^{m_0} x_0, \ldots \lambda^{m_N} x_N)$, ordered by

$$m_0 \leq m_1 \ldots \leq m_N.$$

Let $\{Z_i\}_{i=1}^k$ be the distinct projective weight spaces, where Z_i has weight m_i (i.e. the induced action on Z_i is trivial with weight m_i). Each Z_i is a projective subspace of $\mathbb{P}^N$, and the central fibre with its reduced induced structure $\mathcal{X}_0^{\mathrm{red}}$ is a contained in $\mathrm{Span}(Z_{i_1}, \ldots Z_{i_l})$ for some minimal flag $0 = i_1 < i_2 \ldots < i_l$.

The case $1 < l$. In this case the induced action on closed points of $\mathcal{X}_0$ is nontrivial. Let $q \in \mathcal{X}_1$ be any point with

$$\lim_{\lambda \to 0} \lambda \cdot q = q_0 \in Z_{i_l}.$$

Such a point exists by minimality and because the specialisation of every point must lie in some Z_j. Since the action on $\mathcal{X}_0$ is induced from that on $\mathbb{P}^N$, q_0 belongs to the totally repulsive fixed locus $R = \mathcal{X}_0 \cap Z_{i_l} \subset \mathcal{X}_0$. By this we mean that every closed point in $\mathcal{X}_0 \backslash R$ specialises to a closed point in $\mathcal{X}_0 \backslash R$. In particular the natural birational morphism $\mathcal{X}_0 \dashrightarrow \mathrm{Proj}(\bigoplus_d H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes d})^{\mathbb{C}^*})$ blows up along R. So $q_0 \in R$ is an unstable point for the $\mathbb{C}^*$-action in the sense of geometric invariant theory. By the Hilbert-Mumford criterion the weight of the induced action on the line $\mathcal{L}_0|_{q_0}$ must be strictly positive. Since we are assuming that the induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ is special linear this weight coincides with the Chow weight, so $\mathcal{CH}_{(\mathcal{X}_0,\mathcal{L}_0)(q_0)} > 0$.

Degenerate case. In the rest of the proof we will show that in the degenerate case $\mathcal{X}_0^{\mathrm{red}} \subset Z_0$ the Donaldson-Futaki invariant is strictly positive. Note that since by assumption the original $\mathbb{C}^*$-action on $\mathbb{P}^N$ is nontrivial, $Z_0 \subset \mathbb{P}^N$ is a proper projective subspace.

We digress for a moment to make the following observation: for any $\mathbb{C}^*$-action on $\mathbb{P}^N$ with ordered weights $\{m_i\}$, and a smooth nondegenerate manifold $Y \subset \mathbb{P}^N$, the map $\rho : Y \ni y \mapsto y_0 = \lim_{\lambda\to 0} \lambda \cdot y$ is *rational*, defined on the open dense set $\{y \in Y : \mu(y) = m_0\}$ of points with minimal Hilbert-Mumford weight. Indeed, in the above notation, generic points specialise to some point in the lowest fixed locus Z_0. In any case the map ρ blows up exactly along loci where the Hilbert-Mumford weight jumps.

Going back to our discussion of the case $\mathcal{X}_0^{\mathrm{red}} \subset Z_0$, we see that this means precisely that all points of $\mathcal{X}_1$ have minimal Hilbert-Mumford weight m_0, so there is a well defined *morphism*

$$\rho : \mathcal{X}_1 \to Z_0.$$

Moreover ρ is a finite map: the pullback of $\mathcal{L}_0$ under ρ is L which is ample, therefore ρ cannot contract a positive dimensional subscheme. If ρ were an isomorphism on its image, it would fit in a $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \cong X \times \mathbb{C}$. Therefore ρ cannot be injective, either on closed points or tangent vectors. If, say, ρ identifies distinct points x_1, x_2, this means that the x_i specialise to the same x under the $\mathbb{C}^*$-action; by flatness then the local ring $\mathcal{O}_{\mathcal{X}_0,x}$ contains a nontrivial nilpotent pointing outwards of Z_0, i.e. the sheaf $\mathcal{I}_{\mathcal{X}_0\cap Z_0}/\mathcal{I}_{\mathcal{X}_0}$ is nonzero. In other words $\mathcal{X}_0$ is not a closed subscheme of Z_0. The case when ρ annihilates a tangent vector produces the same kind of nilpotent in the local ring of the limit, by specialisation.

To sum up, the central fibre $\mathcal{X}_0$ is nonreduced, containing nontrivial Z_0-orthogonal nilpotents. Equally important, the induced action on the closed subscheme $\mathcal{X}_0 \cap Z_0 \subset \mathcal{X}_0$ is trivial. The proof will be completed by a weight computation.

Donaldson-Futaki invariant. Suppose $Z_0 \subset \mathbb{P}^N$ has projective coordinates $[x_1 : ... : x_r]$, i.e. it is cut out by $\{x_{r+1} = ... = x_N = 0\}$. We change the linearisation by changing the representation of the $\mathbb{C}^*$-action, to make it of the form

$$[x_0 : ...x_r : x_{r+1} : ... : x_N] \mapsto [x_0 : ...x_r : \lambda^{m_{r+1}-m_0} x_{r+1} : ... : \lambda^{m_N - m_0} x_N], \tag{3.3}$$

and recall $m_{r+i} > m_0$ for all $i > 0$. It is possible that the induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ will not be special linear anymore, however this does not affect the Donaldson-Futaki invariant.

Note that for all large k,

$$H^0(\mathbb{P}^N, \mathcal{O}(k)) \to H^0(\mathcal{X}_0, \mathcal{L}_0^k) \to H^1(\mathcal{I}_{\mathcal{X}_0}(k)) = 0. \tag{3.4}$$

By 3.4, our geometric description of $\mathcal{X}_0$ and the choice of linearisation 3.3 we see that any section $\xi \in H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ has nonnegative weight under the induced $\mathbb{C}^*$-action. The section ξ can only have strictly positive weight if it is of the form $x_{r+i} \cdot f$ for some $i > 0$. Moreover we know there exists an integer $a > 0$ such that $x_{r+i}^a|_{\mathcal{X}_0} = 0$ for all $i > 0$. Let $w(k)$ denote the total weight of the action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$, i.e. the induced weight on the line $\Lambda^{P(k)} H^0(\mathcal{X}_0, \mathcal{L}_0^k)$, where $P(k) = h^0(\mathcal{X}_0, \mathcal{L}_0^k)$ is the Hilbert polynomial. Our discussion implies the upper bound

$$w(k) \leq C(P(k-1) + ... + P(k-a)) \tag{3.5}$$

for some $C > 0$, independent of k. In particular,

$$w(k) = O(k^n). \tag{3.6}$$

On the other hand we can look at just one section x_{r+i}, $i > 0$ with $x_{r+i}|_{\mathcal{X}_0} \neq 0$. This gives a lower bound

$$w(k) \geq C \cdot P(k-1) \tag{3.7}$$

for some $C > 0$, independent of k. So we see that

$$\frac{w(k)}{kP(k)} \geq \frac{C'}{k}. \tag{3.8}$$

holds for $k \gg 0$ and some $C' > 0$ independent of k. Together with

$$\frac{w(k)}{kP(k)} = O(k^{-1}) \tag{3.9}$$

which follows from 3.6 this implies

$$\frac{w(k)}{kP(k)} = \frac{C''}{k} + O(k^{-2}) \tag{3.10}$$

for some $C'' > 0$ independent of k.

By definition of Donaldson-Futaki invariant, this immediately implies

$$F(\mathcal{X}) \geq C'' > 0,$$

a contradiction.

Q.E.D.

Remark 3.4 One can characterise the degenerate case in the above proof more precisely.

As observed by Ross-Thomas [12] Section 3 a result of Mumford implies that any test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is a contraction of some blowup of $X \times \mathbb{C}$ in a flag of $\mathbb{C}^*$-invariant closed subschemes supported in some thickening of $X \times \{0\}$.

The existence of the map $\rho : \mathcal{X}_1 \to Z_0$ means precisely that in this Mumford representation of $\mathcal{X}$ no blowup occurs, i.e. $\mathcal{X}$ is a contraction of the product $X \times \mathbb{C}$.

Define a map $\nu : X \times \mathbb{C} \to \mathcal{X}$ by $\nu(x, \lambda) = \lambda \cdot x$ away from $X \times \{0\}$, $\nu = \rho$ on $X \times \{0\}$. This is a well defined morphism, and since ρ is finite, ν is precisely the *normalisation* of $\mathcal{X}$.

So in the degenerate case $\mathcal{X}_0^{\mathrm{red}} \subset Z_0$ the normalisation of $\mathcal{X}$ is $X \times \mathbb{C}$.

Ross-Thomas [12] Proposition 5.1 proved the general result that normalising a test configuration reduces the Donaldson-Futaki invariant. This already implies $F \geq 0$ in the degenerate case, since the induced action on $X \times \mathbb{C}$ must have vanishing Futaki invariant. In our special case our direct proof yields the strict inequality we need.

Remark 3.5 The result of Mumford mentioned above states more precisely that any test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is a contraction of the blowup of $X \times \mathbb{C}$ in an ideal sheaf

$$\mathbf{I}_r = \mathcal{I}_0 + t\mathcal{I}_1 + ... + t^{r-1}\mathcal{I}_{r-1} + (t^r)$$

where $\mathcal{I}_0 \subseteq ... \subseteq \mathcal{I}_{r-1} \subset \mathcal{O}_X$ correspond to a descending flag of closed subschemes $Z_0 \supseteq ... \supseteq Z_{r-1}$. The action on $(\mathcal{X}, \mathcal{L})$ is the natural one induced from the trivial action on $X \times \mathbb{C}$.

Suppose now that $F(\mathcal{X}) = 0$ and that *no contraction* occurs in Mumford's representation.

Then in Proposition 3.3 we can simply choose any closed point $q \in Z_{r-1}$. This is because the proper transform of $Z_{r-1} \times \mathbb{C}$ cuts $\mathcal{X}_0$ in the totally repulsive locus for the induced action, i.e. the action flows every closed point in $\mathcal{X}_0$ outside this locus to the proper transform of $X \times \{0\}$.

Conversely blowing up $q \in X \setminus Z_0$ only increases the Donaldson-Futaki invariant (at least asymptotically).

For example K-stability with respect to test configurations with $r = 1$ and no contraction is known as Ross-Thomas *slope stability* [12] and has found interesting applications to cscK metrics. In particular this discussion gives a simpler proof that a cscK polarised manifold with discrete automorphisms is slope stable.

Remark 3.6 A refinement of Conjecture 1.3 was proposed by G. Székelyhidi. If $\omega \in c_1(L)$ is cscK there should be a *strictly positive lower bound* for a suitable normalisation of F over all nonproduct test configurations. This condition is called *uniform K-polystability.* In [15] Section 3.1.1 it is shown that the correct normalisation in the case of algebraic surfaces coincides with that of Theorem 1.1, namely $\frac{F(\mathcal{X})}{\|\mathcal{X}\|}$. For toric surfaces K-polystability implies uniform K-polystability with respect to toric test configurations; this is shown in [15] Section 4.2. It seems clear however that the proof presented here cannot be refined to yield uniform K-stability for surfaces.

Example 3.7 (Del Pezzo surfaces) Del Pezzo surfaces played an important role in the development of the subject. By the work of Tian and others all smooth Del Pezzo surfaces V_d of degree $d \leq 6$ admit a Kähler-Einstein metric. For $d \leq 5$, V_d has discrete automorphism group. K-stability in the sense of Tian for V_d, $d \leq 5$ follows from [13] Theorem 1.2. K-stability with respect to "good" test configurations follows from [11] Theorem 2.

Our Theorem 1.2 refines this to K-stability in the sense of Donaldson.

Moreover Theorem 1.2 also applies to polarisations on V_d, $d \leq 5$ for which the exceptional divisors have sufficiently small volume, thanks to the results of Arezzo and Pacard [2].

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