EXTENDED PICARD COMPLEXES AND LINEAR ALGEBRAIC GROUPS

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ABSTRACT. For a smooth geometrically integral variety X over a field k of characteristic 0, we introduce and investigate the extended Picard complex $\mathrm{UPic}(X)$. It is a certain complex of Galois modules of length 2, whose zeroth cohomology is $\overline{k}[X]^\times/\overline{k}^\times$ and whose first cohomology is $\mathrm{Pic}(\overline{X})$, where $\overline{k}$ is a fixed algebraic closure of k and $\overline{X}$ is obtained from X by extension of scalars to $\overline{k}$. When X is a k-torsor of a connected linear k-group G, we compute $\mathrm{UPic}(X)=\mathrm{UPic}(G)$ (in the derived category) in terms of the algebraic fundamental group $\pi_1(G)$. As an application we compute the elementary obstruction for such X.

INTRODUCTION

Throughout the paper, k denotes a field of characteristic 0 and $\overline{k}$ is a fixed algebraic closure of k. By a k-variety we mean a geometrically integral k-variety. If X is a k-variety, we write $\overline{X}$ for $X\times_k\overline{k}$.

Let G be a connected reductive k-group. Let

$$\rho\colon G^{\mathrm{sc}}\twoheadrightarrow G^{\mathrm{ss}}\hookrightarrow G$$

be Deligne's homomorphism, where G^{ss} is the derived subgroup of G (it is semisimple) and G^{sc} is the universal covering of G^{ss} (it is simply connected). Let T be a maximal torus of G (defined over k) and let $T^{\mathrm{sc}}:=\rho^{-1}(T)$ be the corresponding maximal torus of G^{sc}. The 2-term complex of tori

$$T^{\mathrm{sc}}\xrightarrow{\ \rho\ }T$$

(with T^{sc} in degree -1) plays an important role in the study of the arithmetic of reductive groups. For example, the Galois hypercohomology $H^i(k,T^{\mathrm{sc}}\to T)$ of this complex is the abelian Galois cohomology of G (cf. [Bor98]). The corresponding Galois module

$$\pi_1(\overline{G}):=\mathbf{X}_*(\overline{T})/\rho_*\,\mathbf{X}_*(\overline{T}^{\mathrm{sc}})$$

(where $\mathbf{X}_*$ denotes the cocharacter group of a torus) is the algebraic fundamental group of $\overline{G}$ (*loc. cit.*). The related group of multiplicative type over $\mathbf{C}$ with holomorphic $\mathrm{Gal}(\overline{k}/k)$-action

$$Z(\hat{G}):=\mathrm{Hom}(\pi_1(\overline{G}),\mathbf{C}^\times)=\ker[\mathbf{X}^*(\overline{T})\otimes\mathbf{C}^\times\to\mathbf{X}^*(\overline{T}^{\mathrm{sc}})\otimes\mathbf{C}^\times]$$

(where $\mathbf{X}^*$ denotes the character group) is the center of a connected Langlands dual group $\hat{G}$ for G, considered by Kottwitz [Kot84].

Clearly, the above constructions rely on the linear algebraic group structure of G. However we show in this paper that in fact they are related to a very natural geometric/cohomological construction that works for an arbitrary smooth geometrically integral k-variety X. Namely, we consider the cone $\mathrm{UPic}(\overline{X})$ of the morphism

$$\mathbf{G}_{\mathrm{m}}(\overline{k})\to\tau_{\le 1}R\Gamma(\overline{X},\mathbf{G}_{\mathrm{m}})$$

in the derived category of discrete Galois modules. More explicitly, this cone is represented by the 2-term complex

$$\overline{k}(\overline{X})^\times/\overline{k}^\times\to\mathrm{Div}(\overline{X})$$

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(with $\overline{k}(\overline{X})^\times/\overline{k}^\times$ in degree 0), where $\overline{k}(\overline{X})$ denotes the field of rational functions on $\overline{X}$, and $\mathrm{Div}(\overline{X})$ is the divisor group of $\overline{X}$. It follows from the definitions that the cohomology groups $\mathscr{H}^i$ of the complex $\mathrm{UPic}(\overline{X})$ vanish for $i \neq 0,1$, and

$$\mathscr{H}^0(\mathrm{UPic}(\overline{X})) = U(\overline{X}) := \overline{k}[\overline{X}]^\times/\overline{k}^\times$$
$$\mathscr{H}^1(\mathrm{UPic}(\overline{X})) = \mathrm{Pic}(\overline{X})$$

where $\overline{k}[\overline{X}]$ is the ring of regular functions on $\overline{X}$. We see that $\mathrm{UPic}(\overline{X})$ can be regarded as a 2-extension of the Picard group $\mathrm{Pic}(\overline{X})$ by $U(\overline{X})$. We shall call $\mathrm{UPic}(\overline{X})$ *the extended Picard complex of* X. The importance of the extended Picard complex lies in the fact that $\mathrm{UPic}(\overline{X})$ contains more information than $U(\overline{G})$ and $\mathrm{Pic}(\overline{G})$ separately.

Let G be an arbitrary connected linear k-group, not necessarily reductive. We write G^{u} for the unipotent radical of G, and set $G^{\mathrm{red}} = G/G^{\mathrm{u}}$ (it is reductive). We define $\pi_1(\overline{G}) := \pi_1(\overline{G}^{\mathrm{red}})$. This means the following. Let

$$\rho\colon G^{\mathrm{sc}} \twoheadrightarrow G^{\mathrm{ss}} \hookrightarrow G^{\mathrm{red}}$$

be Deligne's homomorphism, where G^{ss} is the derived subgroup of G^{red} and G^{sc} is the universal covering of G^{ss}. Let T be a maximal torus of G^{red} and let $T^{\mathrm{sc}} := \rho^{-1}(T)$ be the corresponding maximal torus of G^{sc}. Then $\pi_1(\overline{G}) = \mathbf{X}_*(\overline{T})/\rho_*\mathbf{X}_*(\overline{T}^{\mathrm{sc}})$.

Consider the derived dual complex to $\pi_1(\overline{G})$, which by definition is given by

$$\pi_1(\overline{G})^D = (\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})) \quad (\text{with } \mathbf{X}^*(\overline{T}) \text{ in degree 0}).$$

By Rosenlicht's lemma [Ros61] we have $\mathrm{U}(\overline{G}) = \mathbf{X}^*(\overline{G})$. By a formula of Voskresenskiĭ [Vos69], Fossum–Iversen [FI73] and Popov [Pop74], we have $\mathrm{Pic}(\overline{G}) \simeq \mathbf{X}^*(\ker[\rho\colon \overline{G}^{\mathrm{sc}} \to \overline{G}^{\mathrm{red}}])$. From these results one can easily obtain that

$$\mathscr{H}^i(\mathrm{UPic}(\overline{G})) \simeq \mathscr{H}^i(\pi_1(\overline{G})^D) \text{ for } i = 0,1.$$

The central result of this paper is that $\mathrm{UPic}(\overline{G})$ and $\pi_1(\overline{G})^D$ themselves are isomorphic in the derived category.

Theorem 1 (Theorem 4.8)**.** *For a connected group G over a field k of characteristic 0, there is a canonical isomorphism, functorial in G,*

$$\mathrm{UPic}(\overline{G}) \xrightarrow{\sim} \pi_1(\overline{G})^D$$

in the derived category of discrete Galois modules.

Both Rosenlicht's lemma and the vanishing of $U(\overline{G})$ and $\mathrm{Pic}(\overline{G})$ for a semisimple simply connected group G are used in the proof.

We also prove a version of Theorem 1 for torsors.

Proposition 2 (Lemma 5.2(iii))**.** *Let G be a connected group over a field k of characteristic 0, and let X be a k-torsor under G. There is a canonical isomorphism, functorial in G and X,*

$$\mathrm{UPic}(\overline{X}) \xrightarrow{\sim} \mathrm{UPic}(\overline{G})$$

in the derived category of discrete Galois modules.

Corollary 3. *Let G be a connected group over a field k of characteristic 0, and let X be a k-torsor under G. There is a canonical isomorphism, functorial in G and X,*

$$\mathrm{UPic}(\overline{X}) \xrightarrow{\sim} \pi_1(\overline{G})^D$$

in the derived category of discrete Galois modules.

This central result gives a good conceptual explanation of many existing results in the literature concerning the striking relationship between the arithmetic of a linear algebraic group G and the Galois modules $\mathbf{X}^*(\overline{G})$ and $\mathrm{Pic}(\overline{G})$.

Picard group and Brauer group.

Proposition 4 (Corollary 2.20(i))**.** *Let X be a smooth geometrically integral variety over k. Then there is a canonical injection*

$$\mathrm{Pic}(X) \hookrightarrow H^1(k, \mathrm{UPic}(\overline{X}))$$

which is an isomorphism if $X(k) \neq \emptyset$ or if $\mathrm{Br}(k) = 0$.

Corollary 5. *For a connected linear algebraic group G over k we have a canonical isomorphism*

$$\mathrm{Pic}(G) \xrightarrow{\sim} H^1(k, \pi_1(\overline{G})^D).$$

Proof. The corollary follows immediately from Proposition 4 and Theorem 1. □

Let X be a smooth geometrically integral variety over k. Let $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ be the Brauer group of X, and let $\mathrm{Br}_1(X)$ be the kernel of the map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$. We write $\mathrm{Br}_{\mathrm{a}}(X)$ for the cokernel of the canonical homomorphism $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$.

Proposition 6 (Corollary 2.20(ii))**.** *Let X be a smooth geometrically integral variety over k. There is a canonical injection*

$$\mathrm{Br}_{\mathrm{a}}(X) \hookrightarrow H^2(k, \mathrm{UPic}(\overline{X}))$$

which is an isomorphism if $X(k) \neq \emptyset$ or $H^3(k, \overline{k}^\times) = 0$.

Corollary 7. *For a connected linear algebraic group over k we have a canonical isomorphism*

$$\mathrm{Br}_{\mathrm{a}}(G) \xrightarrow{\sim} H^2(k, \pi_1(\overline{G})^D)$$

Proof. This follows immediately from Proposition 6 and Theorem 1. □

Note that Corollaries 5 and 7 are versions of results of Kottwitz [Kot84, 2.4]. Kottwitz proved that for a connected reductive k-group G we have

$$\mathrm{Pic}(G) = \pi_0(Z(\hat{G})^{\mathfrak{g}}), \quad \mathrm{Br}_{\mathrm{a}}(G) = H^1(k, Z(\hat{G})),$$

where $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$.

UPic and smooth compactifications.

Proposition 8 (Proposition 2.13)**.** *Let Y be a smooth compactification of a smooth geometrically integral k-variety X. Then we have a distinguished triangle*

$$\mathrm{Pic}(\overline{Y})[-1] \xrightarrow{j^*} \mathrm{UPic}(\overline{X}) \to \mathscr{Z}^1_{Y-X} \to \mathrm{Pic}(\overline{Y})$$

where the morphism j^* is induced by the inclusion map $j\colon X \to Y$, and $\mathscr{Z}^1_{Y-X}$ is the permutation module of divisors in the complement of $\overline{X}$ in $\overline{Y}$.

We see that $\mathrm{Pic}(\overline{Y})$ is very close to $\mathrm{UPic}(\overline{X})$: up to translation, the difference between them is a permutation module.

If C is a complex of $\mathrm{Gal}(\overline{k}/k)$-modules, we write

$$Ш^i_\omega(k, C) = \ker\left[H^i(k, C) \to \prod_\gamma H^i(\gamma, C)\right]$$

where $H^i(k, C)$ is the corresponding Galois hypercohomology group, and γ runs over all closed procyclic subgroups of $\mathrm{Gal}(\overline{k}/k)$.

Proposition 9 (Corollary 2.16)**.** *Let Y be a smooth compactification of a smooth k-variety X. Then there is a canonical isomorphism*

$$Ш^1_\omega(k, \mathrm{Pic}(\overline{Y})) \xrightarrow{\sim} Ш^2_\omega(k, \mathrm{UPic}(\overline{X})).$$

Proposition 9 follows easily from Proposition 8.

Corollary 10. *Let Y be a smooth compactification of a k-torsor X under a connected linear k-group G. There is a canonical isomorphism*

$$Ш^1_\omega(k,\mathrm{Pic}(\overline{Y})) \simeq Ш^2_\omega(k,\pi_1(\overline{G})^D)$$

Proof. The corollary follows immediately from Proposition 9 and Corollary 3. □

Note that we have $H^1(k,\mathrm{Pic}(\overline{Y})) = Ш^1_\omega(k,\mathrm{Pic}(\overline{Y}))$ (see [CTK98, Prop. 3.2], [BK04, Cor. 3.4]). Thus we have a new proof of the fact that

$$H^1(k,\mathrm{Pic}(\overline{Y})) \simeq Ш^2_\omega(k,\pi_1(\overline{G})^D),$$

cf. [BK00, Thm. 2.4].

Elementary obstruction. Let X be a smooth geometrically integral k-variety. We have an extension of complexes of Galois modules

$$0 \to \overline{k}^\times \to \left(\overline{k}(\overline{X})^\times \to \mathrm{Div}(\overline{X})\right) \to \left(\overline{k}(\overline{X})^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X})\right) \to 0.$$

It defines an element $e(X) \in \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times)$. If X has a k-point, then this extension splits (in the derived category), hence $e(X) = 0$. We shall call $e(X)$ the *elementary obstruction* to the existence of a k-point in X, since it is a variant of the original elementary obstruction of Colliot-Thélène and Sansuc [CTS87, Déf. 2.2.1] which lives in $\mathrm{Ext}^1(\overline{k}(\overline{X})^\times/\overline{k}^\times,\overline{k}^\times)$.

Now let G be a connected linear k-group and let X be a k-torsor under G. By Corollary 3 we have $\mathrm{UPic}(\overline{X}) = \pi_1(\overline{G})^D$. Using Lemma 1.5 below, we obtain

$$\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times) = H^1(k,\mathrm{Hom}(\pi_1(\overline{G})^D,\overline{k}^\times)) = H^1(k,T^{\mathrm{sc}} \to T)$$

(where T^{sc} is in degree -1). Recall that the first abelian Galois cohomology group of G is by definition the abelian group $H^1_{\mathrm{ab}}(k,G) := H^1(k,T^{\mathrm{sc}} \to T)$, so the above identification gives us $e(X) \in H^1_{\mathrm{ab}}(k,G)$. Here we compare the elementary obstruction $e(X) \in H^1_{\mathrm{ab}}(k,G)$ with the image of the cohomology class $[X] \in H^1(k,G)$ of the torsor X under the abelianization map $\mathrm{ab}^1\colon H^1(k,G) \to H^1_{\mathrm{ab}}(k,G)$ constructed in [Bor98].

Theorem 11 (Theorem 5.5)**.** *Let X be a k-torsor under a connected linear k-group G. With notation as above, we have $e(X) = \mathrm{ab}^1([X])$.*

The theorem allows us to translate existing results on abelian Galois cohomology of connected k-groups to results on the elementary obstruction for torsors. We simultaneously obtain results on smooth compactifications of torsors, since Proposition 8 implies that the elementary obstruction $e(Y)$ for a smooth compactification Y of a smooth variety X vanishes if and only if the elementary obstruction $e(X)$ for X vanishes.

Proposition 12 (Proposition 5.7)**.** *For (a smooth compactification of) a torsor under a connected linear algebraic group G over a p-adic field k, the elementary obstruction is the only obstruction to the existence of k-rational points.*

Proposition 13 (Proposition 5.8)**.** *For (a smooth compactification of) a torsor under a connected linear algebraic group G over a number field k, the elementary obstruction is the only obstruction to the Hasse principle.*

Corollary 14 (Sansuc [San81], Cor. 8.7)**.** *For a smooth compactification Y of a torsor X under a connected linear algebraic group G over a number field k, the Brauer–Manin obstruction is the only obstruction to the Hasse principle.*

Proof. Assume that Y has points over all the completions of k. By [Sko01, Prop. 6.1.4] the vanishing of the Brauer–Manin obstruction implies that the elementary obstruction vanishes, and we see from Proposition 13 that Y has a k-point. □

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1. Preliminaries

Throughout this paper, k will be a field of characteristic zero. Let $\overline{k}$ denote a fixed algebraic closure of k. For a variety X over k we denote by $D^b(X_{\text{ét}})$ the derived category of complexes of sheaves on the (small) étale site over X with bounded cohomology. We write

$$R\Gamma_{X/k} := R\varphi_*\colon D^+(X_{\text{ét}}) \to D^+(k_{\text{ét}}).$$

where $\varphi\colon X \to \operatorname{Spec} k$ denotes the structure morphism. We shall not distinguish between the category of étale sheaves on $\operatorname{Spec} k$ and the category of discrete Galois modules. We shall always assume our varieties to be geometrically integral.

Let $\mathbf{G}_{\mathrm{m}}$ be the multiplicative group. We shall denote an étale sheaf represented by a group scheme by the same symbol as the group scheme itself. For a variety X over k write $\overline{X} = X \times_k \overline{k}$. We define the following Galois modules:

$$\mathrm{U}(\overline{X}) := (\Gamma_{X/k}\mathbf{G}_{\mathrm{m}})/\mathbf{G}_{\mathrm{m}} = \overline{k}[\overline{X}]^\times/\overline{k}^\times$$
$$\mathrm{Pic}(\overline{X}) := R^1\Gamma_{X/k}\mathbf{G}_{\mathrm{m}} = H^1(\overline{X}, \mathbf{G}_{\mathrm{m}}).$$

These Galois modules are contravariantly functorial in X.

In this paper we shall be mostly interested in a *complex* of Galois modules that combines $\mathrm{U}(\overline{X})$ and $\mathrm{Pic}(\overline{X})$. For this we want to take the object $\tau_{\le 1}R\Gamma_{X/k}\mathbf{G}_{\mathrm{m}}$ in $D^b(k_{\text{ét}})$ modulo $\mathbf{G}_{\mathrm{m}}$ (i.e. modulo $\overline{k}^\times$), where $R\Gamma_{X/k}$ is the derived functor, and $\tau_{\le 1}$ is the truncation functor. To make this precise, we shall introduce some terminology and notation. For definitions of derived categories, triangulated categories, derived functors, truncation functors etc. we refer to original works [Ver77], [Ver96], [BBD82], and textbooks [Ive86], [GM96], [Wei94] (see also [GM99]).

1.1. *Cones and fibres.* Let $f\colon P \to Q$ be a morphism of complexes of objects of an abelian category $\mathscr{A}$. We denote by

$$\langle P \to Q]$$

the *cone* of f, i.e., the complex with the object in degree i equal to

$$P^{i+1} \oplus Q^i,$$

and differential given by the matrix

$$\begin{pmatrix} -d_P & 0 \\ -f & d_Q \end{pmatrix},$$

which denotes the homomorphism $(p,q) \mapsto (-d_P(p), -f(p) + d_Q(q))$. We adopt the convention that the diagrams of the form

$$P \xrightarrow{f} Q \xrightarrow{\binom{0}{\mathrm{id}}} \langle P \to Q] \xrightarrow{(\mathrm{id}\quad 0)} P[1] \tag{1}$$

are distinguished triangles.

Similarly, we denote by

$$[P \to Q\rangle$$

the complex with the object in degree i equal to

$$P^i \oplus Q^{i-1},$$

and differential given by the matrix

$$\begin{pmatrix} d_P & 0 \\ f & -d_Q \end{pmatrix}, \tag{2}$$

which denotes the homomorphism $(p,q) \mapsto (d_P(p), f(p) - d_Q(q))$. We call $[P \to Q\rangle$ the *fibre* (or co-cone) of f. Then

$$\langle P \to Q] = [P \to Q\rangle[1],$$

and we have a distinguished triangle

$$[P \to Q\rangle \xrightarrow{\begin{pmatrix} -\mathrm{id} & 0 \end{pmatrix}} P \xrightarrow{f} Q \xrightarrow{\begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix}} [P \to Q\rangle[1]. \tag{3}$$

We have $[P \to 0\rangle = P$, and $\langle 0 \to Q] = Q$.

Remark 1.2. Note that our sign convention for the differentials in the cone corresponds to [Ive86, I.4], but is different from other sources, such as [GM96, III.3.2]. For example, in the latter the cone has differential

$$\begin{pmatrix} -d_P & 0 \\ f & d_Q \end{pmatrix}.$$

The choice of signs also has an influence on the class of distinguished triangles. Indeed, consider the following diagram

$$P \xrightarrow{f} Q \xrightarrow{\begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix}} C_{\mathrm{GM}}(f) \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \end{pmatrix}} P[1] \tag{4}$$

where we write $C_{\mathrm{GM}}(f)$ for the cone as defined in [GM96]. Then this diagram is a distinguished triangle in $D(\mathscr{A})$ in the convention of [GM96] (cf. Def. III.3.4 and Lemma III.3.3 in *loc. cit.*). However, in our convention we would need to change the last homomorphism of diagram (4) to $\begin{pmatrix} -\mathrm{id} & 0 \end{pmatrix}$ in order to have a distinguished triangle.

1.3. Let $f\colon P \to Q$ be a morphism in the derived category $D^b(\mathscr{A})$. We define a cone $\langle P \to Q]$ as the third vertex of a distinguished triangle (1). Similarly, we define a fibre $[P \to Q\rangle$ as the third vertex of a distinguished triangle (3). It is well known that in general in a derived category (or in a triangulated category) a cone and a fibre are defined only up to a non-canonical isomorphism. However we shall prove, that all the cones and fibres that we shall consider, will be defined up to a *canonical* isomorphism (we shall use [BBD82, Prop. 1.1.9]).

1.4. Ext *and Galois cohomology.* In order to compute the elementary obstruction to the existence of a rational point in k-variety X, we need the following lemma, which is probably well-known (compare for example the closely related result [Mil86, Theorem 0.3 and Example 0.8]). We are grateful for J. Bernstein for proving this lemma.

Lemma 1.5. *Let $M^\bullet$ be a bounded complex of torsion free finitely generated (over $\mathbf{Z}$) discrete* $\mathrm{Gal}(\overline{k}/k)$*-modules. Then for all integers i we have canonical isomorphisms*

$$\mathrm{Ext}^i(M^\bullet, \overline{k}^\times) = H^i(k, \mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet, \overline{k}^\times)).$$

Let $\mathfrak{g}$ be a profinite group. By a $\mathfrak{g}$-module we mean a discrete $\mathfrak{g}$-module. By a torsion free finitely generated $\mathfrak{g}$-module we mean a $\mathfrak{g}$-module which is torsion free and finitely generated over $\mathbf{Z}$. Lemma 1.5 follows from the following Lemma 1.6.

Lemma 1.6. *Let A be a $\mathfrak{g}$-module, B a $\mathfrak{g}$-module, and let $M^\bullet$ be a complex of torsion free finitely generated $\mathfrak{g}$-modules. Then there are canonical isomorphisms*

$$\mathrm{Ext}^i_{\mathfrak{g}}(A, \mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet, B)) = \mathrm{Ext}^i_{\mathfrak{g}}(A \underset{\mathbf{Z}}{\otimes} M^\bullet, B).$$

To obtain Lemma 1.5 we just take $A = \mathbf{Z}$, $B = \overline{k}^\times$ in Lemma 1.6.

Proof of Lemma 1.6. First let M be a finitely generated $\mathfrak{g}$-module. We have a canonical isomorphism

$$\mathrm{Hom}_{\mathbf{Z}}(A, \mathrm{Hom}_{\mathbf{Z}}(M,B)) = \mathrm{Hom}_{\mathbf{Z}}(A \underset{\mathbf{Z}}{\otimes} M, B).$$

Taking $\mathfrak{g}$-invariants, we obtain

$$\mathrm{Hom}_{\mathfrak{g}}(A, \mathrm{Hom}_{\mathbf{Z}}(M,B)) = \mathrm{Hom}_{\mathfrak{g}}(A \underset{\mathbf{Z}}{\otimes} M, B).$$

If $M^\bullet$ is a complex of torsion free finitely generated $\mathfrak{g}$-modules, we obtain similarly

$$\mathrm{Hom}^\bullet_{\mathfrak{g}}(A, \mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,B)) = \mathrm{Hom}^\bullet_{\mathfrak{g}}(A \underset{\mathbf{Z}}{\otimes} M^\bullet, B).$$

Now let $I^\bullet$ be an injective resolution of B in the category of discrete $\mathfrak{g}$-modules. Again

$$\mathrm{Hom}^\bullet_{\mathfrak{g}}(A, \mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,I^\bullet)) = \mathrm{Hom}^\bullet_{\mathfrak{g}}(A \underset{\mathbf{Z}}{\otimes} M^\bullet, I^\bullet).$$

By a definition of Ext^i we have

$$\mathscr{H}^i(\mathrm{Hom}^\bullet_{\mathfrak{g}}(A \underset{\mathbf{Z}}{\otimes} M^\bullet, I^\bullet)) = \mathrm{Ext}^i_{\mathfrak{g}}(A \underset{\mathbf{Z}}{\otimes} M^\bullet, B).$$

To prove Lemma 1.6 it suffices to prove that

$$\mathscr{H}^i(\mathrm{Hom}^\bullet_{\mathfrak{g}}(A, \mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,I^\bullet))) = \mathrm{Ext}^i_{\mathfrak{g}}(A, \mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,B)).$$

This follows from the next lemma. □

Lemma 1.7. $\mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,I^\bullet)$ *is an injective resolution of* $\mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,B)$.

Proof. Since $M^\bullet$ is a bounded complex of torsion free finitely generated modules, we see that $\mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,I^\bullet)$ is a resolution of $\mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,B)$. This is an injective resolution, since for any torsion-free finitely generated $\mathfrak{g}$-module M and an injective $\mathfrak{g}$-module I, the $\mathfrak{g}$-module $\mathrm{Hom}_{\mathbf{Z}}(M,I)$ is injective (see for example [Mil86, Lemma 0.5]). This completes the proofs of Lemmas 1.7, 1.6, and 1.5. □

2. The extended Picard complex

2.1. Let X be a geometrically integral k-variety. Consider the cone

$$\mathrm{UPic}(\overline{X}) := \langle \mathbf{G}_{\mathrm{m}} \to \tau_{\leq 1} R\Gamma_{X/k} \mathbf{G}_{\mathrm{m}}].$$

In more detail: we can represent $\tau_{\leq 1} R\Gamma_{X/k} \mathbf{G}_{\mathrm{m}}$ as a complex in degrees 0 and 1. We have a homomorphism $i\colon \mathbf{G}_{\mathrm{m}} \to H^0(\overline{X}, \mathbf{G}_{\mathrm{m}})$, which induces a morphism $i_*\colon \mathbf{G}_{\mathrm{m}} \to \tau_{\leq 1} R\Gamma_{X/k} \mathbf{G}_{\mathrm{m}}$. Then $\mathrm{UPic}(\overline{X})$ is a cone of this map. Note that the map i is injective, hence $\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})) = 0$, and $\mathrm{UPic}(\overline{X})[-1] \in \mathrm{Ob}(D^b(k_{\text{ét}})^{\geq 1})$. It follows that $\mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, \mathrm{UPic}(\overline{X})[-1]) = 0$, so by [BBD82, Prop. 1.1.9] $\mathrm{UPic}(\overline{X})$ is defined up to a canonical isomorphism. We call $\mathrm{UPic}(\overline{X})$ the *extended Picard complex* of a variety X. We have a canonical distinguished triangle

$$\mathbf{G}_{\mathrm{m}} \to \tau_{\leq 1} R\Gamma_{X/k} \mathbf{G}_{\mathrm{m}} \to \mathrm{UPic}(\overline{X}) \to \mathbf{G}_{\mathrm{m}}[1]. \tag{5}$$

Note that

$$\begin{aligned} \mathscr{H}^0(\mathrm{UPic}(\overline{X})) &= \mathrm{U}(\overline{X}), \\ \mathscr{H}^1(\mathrm{UPic}(\overline{X})) &= \mathrm{Pic}(\overline{X}), \\ \mathscr{H}^i(\mathrm{UPic}(\overline{X})) &= 0 \text{ for } i \neq 0,1. \end{aligned}$$

Hence $\mathrm{UPic}(\overline{X})$ is indeed a combination of $\mathrm{Pic}(\overline{X})$ and $U(\overline{X})$. In particular, if X is projective, then $\mathrm{UPic}(\overline{X}) = \mathrm{Pic}(\overline{X})[-1]$.

The construction of the complex $\mathrm{UPic}(\overline{X})$ is functorial in X in the derived category. Indeed, a morphism of k-varieties $f\colon X \to Y$ induces a pull-back morphism $f^*\colon \tau_{\leq 1} R\Gamma_{Y/k} \mathbf{G}_{\mathrm{m}} \to \tau_{\leq 1} R\Gamma_{X/k} \mathbf{G}_{\mathrm{m}}$, hence by [BBD82, Prop. 1.1.9] a canonical morphism

$$f^*\colon \mathrm{UPic}(\overline{Y}) \to \mathrm{UPic}(\overline{X}).$$

2.2. *An explicit presentation of UPic.* Assume X to be nonsingular. We write $\mathrm{Div}(\overline{X})$ for the Galois module of divisors on $\overline{X}$, and $\overline{k}(\overline{X})$ for the rational function field of $\overline{X}$. The divisor map

$$\overline{k}(\overline{X})^{\times} \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})$$

has kernel equal to $\overline{k}[\overline{X}]^{\times}$ and cokernel equal to $\mathrm{Pic}(\overline{X})$. We write $\mathrm{KDiv}(\overline{X})$ for the complex of Galois modules $[\overline{k}(\overline{X})^{\times} \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\rangle$. We show below that $\mathrm{UPic}(\overline{X}) \simeq \langle \overline{k}^{\times} \to \mathrm{KDiv}(\overline{X})]$

For this, we need the following fact, which should be well-known to experts, but for which we do not have an explicit reference.

Lemma 2.3. *There is a canonical isomorphism*

$$\mathrm{KDiv}(\overline{X}) \xrightarrow{\sim} \tau_{\leq 1} R\Gamma_{X/k}\mathbf{G}_{\mathrm{m}}\,.$$

To prove Lemma 2.3 we need a construction.

Construction 2.4. Let K be a complex of sheaves on X, $K = K^0 \to K^1 \to \dots$. We write $\Gamma_{X/k}K = \Gamma_{X/k}K^0 \to \Gamma_{X/k}K^1 \to \dots$. By definition of a right derived functor (see for example [GM96, Def. III.6.6]), we have a homomorphism

$$\Gamma_{X/k}K \to R\Gamma_{X/k}K$$

Now assume that we have a morphism $A \to B$ of sheaves on X. Then we have a distinguished triangle

$$[A \to B\rangle \to A \to B \to [A \to B\rangle[1],$$

a morphism of triangles

$$\begin{array}{ccccccc}
[\Gamma_{X/k}A \to \Gamma_{X/k}B\rangle & \longrightarrow & \Gamma_{X/k}A & \longrightarrow & \Gamma_{X/k}B & \longrightarrow & [\Gamma_{X/k}A \to \Gamma_{X/k}B\rangle[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\Gamma_{X/k}[A \to B\rangle & \longrightarrow & R\Gamma_{X/k}A & \longrightarrow & R\Gamma_{X/k}B & \longrightarrow & R\Gamma_{X/k}[A \to B\rangle[1]
\end{array}$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathscr{H}^0[\Gamma_{X/k}A \to \Gamma_{X/k}B\rangle & \to & \Gamma_{X/k}A & \longrightarrow & \Gamma_{X/k}B & \to & \mathscr{H}^1[\Gamma_{X/k}A \to \Gamma_{X/k}B\rangle[1] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
R^0\Gamma_{X/k}[A \to B\rangle & \longrightarrow & R^0\Gamma_{X/k}A & \to & R^0\Gamma_{X/k}B & \longrightarrow & R^1\Gamma_{X/k}[A \to B\rangle & \longrightarrow & R^1\Gamma_{X/k}A
\end{array}$$

Proof of Lemma 2.3. By [Gro68, II.1] we have a resolution

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathscr{K}_X^{\times} \to \mathscr{D}iv_X \to 0$$

of the sheaf $\mathbf{G}_{\mathrm{m}}$ by the sheaf $\mathscr{K}_X^{\times}$ of invertible rational functions and the sheaf $\mathscr{D}iv_X$ of Cartier divisors. Hence we get a canonical isomorphism

$$R\Gamma_{X/k}\mathbf{G}_{\mathrm{m}} \simeq R\Gamma_{X/k}[\mathscr{K}_X^{\times} \to \mathscr{D}iv_X\rangle. \tag{6}$$

We have $R^0\Gamma_{X/k}\mathscr{K}_X^{\times} = \overline{k}(\overline{X})^{\times}$ and $R^0\Gamma_{X/k}\mathscr{D}iv_X = \mathrm{Div}(\overline{X})$. Applying Construction 2.4 to the morphism of sheaves $\mathscr{K}_X^{\times} \to \mathscr{D}iv_X$, we obtain a canonical morphism

$$[\overline{k}(\overline{X})^{\times} \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\rangle \to R\Gamma_{X/k}\mathbf{G}_{\mathrm{m}} \tag{7}$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{H}^0(\mathrm{KDiv}(\overline{X})) & \longrightarrow & \overline{k}(\overline{X})^{\times} & \longrightarrow & \mathrm{Div}(\overline{X}) & \longrightarrow & \mathscr{H}^1(\mathrm{KDiv}(\overline{X})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\simeq & & \downarrow\simeq & & \downarrow & & \\
0 & \longrightarrow & R^0\Gamma_{X/k}\mathbf{G}_{\mathrm{m}} & \longrightarrow & R^0\Gamma_{X/k}\mathscr{K}_X^{\times} & \to & R^0\Gamma_{X/k}\mathscr{D}iv_X & \longrightarrow & R^1\Gamma_{X/k}\mathbf{G}_{\mathrm{m}} & \longrightarrow & R^1\Gamma_{X/k}\mathscr{K}_X^{\times}
\end{array}$$

(we use the isomorphism (6)). By Hilbert 90 in Grothendieck's form we have $R^1\Gamma_{X/k}\mathscr{K}_X^\times = 0$ (cf. [Gro68, II, Lemme 1.6]). Hence the five lemma gives us that the vertical arrows $\mathscr{H}^i(\mathrm{KDiv}(\overline{X})) \to R^i\Gamma_{X/k}\mathbf{G}_\mathrm{m}$ for $i = 0, 1$ are isomorphisms. In other words, the morphism (7) induces an isomorphism

$$[\overline{k}(\overline{X})^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\rangle \xrightarrow{\sim} \tau_{\leq 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m} \tag{8}$$

in the derived category. □

Corollary 2.5. *There is a canonical isomorphism*

$$\langle\overline{k}^\times \to \mathrm{KDiv}(\overline{X})] \xrightarrow{\sim} \mathrm{UPic}(\overline{X}).$$

Proof. We have a natural commutative diagram in the derived category of Galois modules

$$\begin{array}{ccc} \mathbf{G}_\mathrm{m} & \longrightarrow & \tau_{\leq 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m} \\ \downarrow & & \downarrow \\ \overline{k}^\times & \longrightarrow & \mathrm{KDiv}(\overline{X}) \end{array}$$

of which the vertical arrows are isomorphisms. The map $\overline{k}^\times \to \overline{k}[\overline{X}]^\times = \mathscr{H}^0(\mathrm{KDiv}(\overline{X}))$ is injective. Now our corollary follows from [BBD82, Prop. 1.1.9] (similar to the argument in 2.1). □

Remark 2.6. Observe that

$$\langle\overline{k}^\times \to \mathrm{KDiv}(\overline{X})] \simeq [\overline{k}(\overline{X})^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X})\rangle.$$

We shall write $\mathrm{KDiv}(\overline{X})/\overline{k}^\times$ for $[\overline{k}(\overline{X})^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X})\rangle$. Then by Corollary 2.5 we have $\mathrm{KDiv}(\overline{X})/\overline{k}^\times \simeq \mathrm{UPic}(\overline{X})$.

Remark 2.7. The complex $\mathrm{KDiv}(\overline{X})/\overline{k}^\times$ is not functorial in X in the category of complexes. Indeed, neither $\overline{k}(\overline{X})^\times/\overline{k}^\times$ nor $\mathrm{Div}(\overline{X})$ are functorial in X.

2.8. *Splitting.*

Let X be a nonsingular k-variety. Assume that X has a k-point x. We set

$$\begin{aligned} \mathrm{Div}(\overline{X})_x &= \{D \in \mathrm{Div}(\overline{X}) |\ x \notin \mathrm{supp}(D)\} \\ \overline{k}(\overline{X})_x^\times &= \{f \in \overline{k}(\overline{X})^\times |\ \mathrm{div}(f) \in \mathrm{Div}(\overline{X})_x\} \\ \mathrm{KDiv}(\overline{X})_x &= [\overline{k}(\overline{X})_x^\times \to \mathrm{Div}(\overline{X})_x\rangle \end{aligned}$$

By a well-known moving lemma, the composed map

$$\mathrm{Div}(\overline{X})_x \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X})$$

is surjective. It follows that the morphism of complexes

$$\mathrm{KDiv}(\overline{X})_x \to \mathrm{KDiv}(\overline{X})$$

is a quasi-isomorphism.

Set

$$\begin{aligned} \overline{k}(\overline{X})_{x,1}^\times &= \{f \in \overline{k}(\overline{X})^\times |\ f(x) = 1\} \\ \mathrm{KDiv}(X)_{x,1} &= [\overline{k}(\overline{X})_{x,1}^\times \to \mathrm{Div}(\overline{X})_x\rangle \end{aligned}$$

We have an isomorphism

$$\overline{k}^\times \oplus \overline{k}(\overline{X})_{x,1}^\times \xrightarrow{\sim} \overline{k}(\overline{X})_x^\times$$

given by

$$(c, f) \mapsto cf \quad \text{where} \quad c \in \overline{k}^\times, f \in \overline{k}(\overline{X})_{x,1}^\times.$$

Hence we obtain an isomorphism

$$\overline{k}^\times \oplus \mathrm{KDiv}(\overline{X})_{x,1} \xrightarrow{\sim} \mathrm{KDiv}(\overline{X})_x\,.$$

We see that the cone $\langle \overline{k}^\times \to \mathrm{KDiv}(\overline{X})_x]$ is canonically quasi-isomorphic to $\mathrm{KDiv}(\overline{X})_{x,1}$. Thus $\mathrm{UPic}(\overline{X}) \simeq \mathrm{KDiv}(\overline{X})_{x,1}$.

Let $f\colon X \to Y$ be a morphism of nonsingular k-varieties, and let $x \in X(k)$. Set $y = f(x) \in Y(k)$. Then we have a morphism of complexes

$$f^*\colon \mathrm{KDiv}(\overline{Y})_{y,1} \to \mathrm{KDiv}(\overline{X})_{x,1}\ .$$

We see that the complex $\mathrm{KDiv}(\overline{X})_{x,1}$ is functorial in (X,x) in the category of complexes.

Lemma 2.9. *Let X be a nonsingular k-variety having a k-point x. Then the triangle* (5) *of 2.1 splits, i.e. the third morphism* $\mathrm{UPic}(\overline{X}) \to \mathbf{G}_\mathrm{m}[1]$ *in this triangle is 0.*

Proof. If X has a k-point x, then the triangle (5) is isomorphic to the split triangle

$$\overline{k}^\times \to \overline{k}^\times \oplus \mathrm{KDiv}(\overline{X})_{x,1} \to \mathrm{KDiv}(\overline{X})_{x,1} \to \overline{k}^\times[1]$$

with obvious morphisms, where the third morphism is 0. Hence the third morphism in the triangle (5) is 0. □

The lemma shows that the triangle (5) can provide a cohomological obstruction to the existence of a k-rational point.

Definition 2.10. Let X be a nonsingular variety over k. We define the *elementary obstruction*

$$e(X) \in \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \mathbf{G}_\mathrm{m})$$

to be the class $e(X)$ of the triangle (5).

2.11. We call $e(X)$ the elementary obstruction, because it is closely related to the elementary obstruction $\mathrm{ob}(X) \in \mathrm{Ext}^1(\overline{k}(\overline{X})^\times/\overline{k}^\times, \overline{k}^\times)$ of Colliot-Thélène and Sansuc [CTS87, Déf. 2.2.1]. Indeed, by definition $\mathrm{ob}(X)$ is the class of the extension

$$0 \to \overline{k}^\times \to \overline{k}(\overline{X})^\times \to \overline{k}(\overline{X})^\times/\overline{k}^\times \to 0,$$

whereas under the identification $\mathrm{UPic}(\overline{X}) \simeq \mathrm{KDiv}(\overline{X})/\overline{k}^\times$ of Corollary 2.5, $e(X)$ is the extension class of the triangle associated to the short exact sequence of complexes

$$0 \to \overline{k}^\times \to \mathrm{KDiv}(\overline{X}) \to \mathrm{KDiv}(\overline{X})/\overline{k}^\times \to 0.$$

Hence $e(X)$ is the image of the class $\mathrm{ob}(X)$ under the homomorphism

$$\mathrm{Ext}^1(\overline{k}(\overline{X})^\times/\overline{k}^\times, \overline{k}^\times) \to \mathrm{Ext}^1(\mathrm{KDiv}(\overline{X})/\overline{k}^\times, \overline{k}^\times) \tag{9}$$

induced by the natural map $\mathrm{KDiv}(\overline{X})/\overline{k}^\times \to \overline{k}(\overline{X})^\times/\overline{k}^\times$.

Lemma 2.12. *For a nonsingular k-variety X, we have $e(X) = 0$ if and only if* $\mathrm{ob}(X) = 0$.

Proof. The homomorphism (9) fits into an exact sequence of Ext-groups

$$\mathrm{Ext}^1(\mathrm{Div}(\overline{X}), \overline{k}^\times) \to \mathrm{Ext}^1(\overline{k}(\overline{X})^\times/\overline{k}^\times, \overline{k}^\times) \to \mathrm{Ext}^1\left([\overline{k}(\overline{X})^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X})\rangle, \overline{k}^\times\right). \tag{10}$$

induced by the exact sequence of complexes

$$0 \to \mathrm{Div}(\overline{X})[-1] \to [\overline{k}(\overline{X})^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X})\rangle \to \overline{k}(\overline{X})^\times/\overline{k}^\times \to 0$$

Since $\mathrm{Div}(\overline{X})$ is a direct sum of permutation modules, Lemma 1.5 gives that $\mathrm{Ext}^1(\mathrm{Div}(\overline{X}), \overline{k}^\times)$ is a direct product of the H^1-groups of quasi-trivial tori, hence $\mathrm{Ext}^1(\mathrm{Div}(\overline{X}), \overline{k}^\times) = 0$, so we see from the exact sequence (10) that the homomorphism (9) is injective, from which the statement follows. □

Now we investigate how UPic changes under open embeddings.

Proposition 2.13. *Let $X \subset Y$ be an open k-subvariety of a nonsingular k-variety Y. Let $j\colon X \hookrightarrow Y$ denote the inclusion map. Then we have a distinguished triangle*

$$\mathrm{UPic}(\overline{Y}) \xrightarrow{j^*} \mathrm{UPic}(\overline{X}) \to \mathscr{Z}^1_{Y-X} \to \mathrm{UPic}(\overline{Y})[1]$$

where $\mathscr{Z}^1_{Y-X}$ is the permutation module of divisors in the complement of $\overline{X}$ in $\overline{Y}$.

Proof. Clearly we have a short exact sequence of complexes

$$0 \to \mathscr{Z}^1_{Y-X}[-1] \to \mathrm{KDiv}(\overline{Y})/\overline{k}^\times \xrightarrow{j^*} \mathrm{KDiv}(\overline{X})/\overline{k}^\times \to 0,$$

whence we obtain distinguished triangles

$$\mathscr{Z}^1_{Y-X}[-1] \to \mathrm{KDiv}(\overline{Y})/\overline{k}^\times \xrightarrow{j^*} \mathrm{KDiv}(\overline{X})/\overline{k}^\times \to \mathscr{Z}^1_{Y-X}$$

and

$$\mathrm{KDiv}(\overline{Y})/\overline{k}^\times \xrightarrow{j^*} \mathrm{KDiv}(\overline{X})/\overline{k}^\times \longrightarrow \mathscr{Z}^1_{Y-X} \longrightarrow (\mathrm{KDiv}(\overline{Y})/\overline{k}^\times)[1]$$

□

Remark 2.14. Let $X \subset Y$ be an open k-subvariety of a nonsingular complete k-variety Y. Proposition 2.13 implies that $\mathrm{UPic}(\overline{X})$ is non-canonically isomorphic to the fibre $\left[\mathscr{Z}^1_{Y-X} \to \mathrm{Pic}(\overline{Y})\right\rangle$. Skorobogatov actually gave a canonical isomorphism in the derived category $\mathrm{UPic}(\overline{X}) \xrightarrow{\sim} \left[\mathscr{Z}^1_{Y-X} \to \mathrm{Pic}(\overline{Y})\right\rangle$ (cf. [CT06, Rem. B.2.1(2)]).

By $Ш^i_\omega(k,M)$ we denote the subgroup of $H^i(k,M)$ of elements that map to zero in $H^i(\gamma,M)$ for every closed procyclic subgroup $\gamma \subset \mathrm{Gal}(\overline{k}/k)$. Recall that for a permutation module P we have $H^1(k,P) = 0$ and $Ш^2_\omega(k,P) = 0$ (cf. [BK00, 1.2.1]).

Corollary 2.15. *Let $X \subset Y$ be an open k-subvariety of a nonsingular k-variety Y. Then the restriction map $\mathrm{UPic}(\overline{Y}) \to \mathrm{UPic}(\overline{X})$ induces an injection*

$$H^2(k, \mathrm{UPic}(\overline{Y})) \hookrightarrow H^2(k, \mathrm{UPic}(\overline{X}))$$

and an isomorphism

$$Ш^2_\omega(k, \mathrm{UPic}(\overline{Y})) \xrightarrow{\sim} Ш^2_\omega(k, \mathrm{UPic}(\overline{X})).$$

Proof. By Proposition 2.13 we have an exact sequence

$$H^1(k, \mathscr{Z}^1_{Y-X}) \to H^2(k, \mathrm{UPic}(\overline{Y})) \to H^2(k, \mathrm{UPic}(\overline{X})) \to H^2(k, \mathscr{Z}^1_{Y-X}),$$

where $\mathscr{Z}^1_{Y-X}$ is a permutation Galois module. Now the injectivity of the two maps follows from the vanishing of $H^1(k, \mathscr{Z}^1_{Y-X})$. The surjectivity of the $Ш^2_\omega$-map follows from the vanishing of $Ш^2_\omega(k, \mathscr{Z}^1_{Y-X})$ and an easy diagram chase. □

Corollary 2.16. *Let $X \subset Y$ be an open k-subvariety of a nonsingular complete k-variety Y. Then the restriction map $\mathrm{Pic}(\overline{Y})[-1] = \mathrm{UPic}(\overline{Y}) \to \mathrm{UPic}(\overline{X})$ induces an injection*

$$H^1(k, \mathrm{Pic}(\overline{Y})) \hookrightarrow H^2(k, \mathrm{UPic}(\overline{X}))$$

and an isomorphism

$$Ш^1_\omega(k, \mathrm{Pic}(\overline{Y})) \xrightarrow{\sim} Ш^2_\omega(k, \mathrm{UPic}(\overline{X})).$$

Corollary 2.17. *Let $X \subset Y$ be an open k-subvariety of a nonsingular k-variety Y. Let $j\colon X \hookrightarrow Y$ denote the inclusion map. Then the induced map*

$$j_*\colon \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \mathbf{G}_\mathrm{m}) \to \mathrm{Ext}^1(\mathrm{UPic}(\overline{Y}), \mathbf{G}_\mathrm{m})$$

is injective. In particular, the elementary obstruction $e(X)$ vanishes if and only if $e(Y)$ vanishes.

Proof. Applying the functor Ext to the distinguished triangle of Proposition 2.13, we obtain an exact sequence

$$\mathrm{Ext}^1(\mathscr{Z}^1_{Y-X}, \mathbf{G}_\mathrm{m}) \to \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \mathbf{G}_\mathrm{m}) \xrightarrow{j_*} \mathrm{Ext}^1(\mathrm{UPic}(\overline{Y}), \mathbf{G}_\mathrm{m}).$$

By Lemma 1.5 $\mathrm{Ext}^1(\mathscr{Z}^1_{Y-X}, \mathbf{G}_\mathrm{m}) = H^1(k,P)$, where P is the k-torus such that $\mathbf{X}^*(P) = \mathscr{Z}^1_{Y-X}$. Since $\mathscr{Z}^1_{Y-X}$ is a permutation module, we see that P is a quasi-trivial torus, hence $H^1(k,P) = 0$, and therefore the homomorphism j_* is injective. □

2.18. *UPic, the Picard group and the Brauer group.* Let X be a nonsingular variety over k. Let $\mathrm{Br}(X) = H^2(X, \mathbf{G}_\mathrm{m})$ denote the (cohomological) Brauer group of X, let $\mathrm{Br}_1(X)$ denote the kernel of the map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$, and let $\mathrm{Br}_\mathrm{a}(X)$ denote the cokernel of the map $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$.

We have equalities

$$\mathrm{Pic}(X) = H^1(X, \mathbf{G}_\mathrm{m}) = H^1(k, R\Gamma_{X/k}\mathbf{G}_\mathrm{m}) = H^1(k, \tau_{\le 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m})$$
$$\mathrm{Br}(X) = H^2(X, \mathbf{G}_\mathrm{m}) = H^2(k, R\Gamma_{X/k}\mathbf{G}_\mathrm{m}) = H^2(k, \tau_{\le 2}R\Gamma_{X/k}\mathbf{G}_\mathrm{m}).$$

From the distinguished triangle

$$\tau_{\le 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m} \to \tau_{\le 2}R\Gamma_{X/k}\mathbf{G}_\mathrm{m} \to R^2\Gamma_{X/k}\mathbf{G}_\mathrm{m}[-2] \to \tau_{\le 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m}[1]$$

we obtain a Galois cohomology exact sequence

$$0 \to H^2(k, \tau_{\le 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m}) \to H^2(k, \tau_{\le 2}R\Gamma_{X/k}\mathbf{G}_\mathrm{m}) \to H^0(k, R^2\Gamma_{X/k}\mathbf{G}_\mathrm{m}).$$

Since $H^2(k, \tau_{\le 2}R\Gamma_{X/k}\mathbf{G}_\mathrm{m}) = \mathrm{Br}(X)$, and $H^0(k, R^2\Gamma_{X/k}\mathbf{G}_\mathrm{m}) = \mathrm{Br}(\overline{X})^{\mathrm{Gal}(\overline{k}/k)}$, it follows that

$$\mathrm{Br}_1(X) = H^2(k, \tau_{\le 1}R\Gamma_{X/k}\mathbf{G}_\mathrm{m}).$$

Proposition 2.19. *Let X be a nonsingular variety over k. We have an exact sequence*

$$0 \to \mathrm{Pic}(X) \to H^1(k, \mathrm{UPic}(\overline{X})) \to \mathrm{Br}(k) \to \mathrm{Br}_1(X) \to H^2(k, \mathrm{UPic}(\overline{X})) \to H^3(k, \mathbf{G}_\mathrm{m}),$$

in which the homomorphisms $H^1(k, \mathrm{UPic}(\overline{X})) \to \mathrm{Br}(k)$ and $H^2(k, \mathrm{UPic}(\overline{X})) \to H^3(k, \mathbf{G}_\mathrm{m})$ are zero when $X(k) \ne \emptyset$.

Proof. We obtain the exact sequence by taking Galois cohomology of the triangle (5) of 2.1 and applying Hilbert's Theorem 90 to the term $H^1(k, \mathbf{G}_\mathrm{m})$. For the case $X(k) \ne \emptyset$ we apply Lemma 2.9. □

Corollary 2.20. *Let X be a smooth geometrically integral variety over k.*

(i) *There is a canonical injection*

$$\mathrm{Pic}(X) \hookrightarrow H^1(k, \mathrm{UPic}(\overline{X}))$$

which is an isomorphism if $X(k) \ne \emptyset$ or if $\mathrm{Br}(k) = 0$.

(ii) *There is a canonical injection*

$$\mathrm{Br}_\mathrm{a}(X) \hookrightarrow H^2(k, \mathrm{UPic}(\overline{X}))$$

which is an isomorphism if $X(k) \ne \emptyset$ or $H^3(k, \mathbf{G}_\mathrm{m}) = 0$.

3. Picard groups, invertible functions, and the algebraic fundamental group

3.1. Let G be a connected linear algebraic k-group. As in [Bor98] we write $G^\mathrm{u} \subset G$ for the unipotent radical of G, G^red for the reductive group G/G^u, G^ss for the derived group of G^red (it is semisimple), G^tor for the torus $G^\mathrm{red}/G^\mathrm{ss}$, and G^sc for the universal covering of G^ss (it is simply connected). The composed map

$$\rho\colon G^\mathrm{sc} \twoheadrightarrow G^\mathrm{ss} \hookrightarrow G^\mathrm{red}$$

has finite kernel

$$Z := \ker \rho,$$

which is central in G^sc, and the cokernel of ρ is equal to the torus G^tor. We write

$$\mathbf{X}^*(\overline{G}) = \mathrm{Hom}_{\overline{k}}(G, \mathbf{G}_\mathrm{m})$$

for the character group of G. We have

$$\mathbf{X}^*(\overline{G}) = \mathbf{X}^*(\overline{G}^\mathrm{tor}).$$

For a torus T we write

$$\mathbf{X}_*(\overline{T}) = \mathrm{Hom}_{\overline{k}}(\mathbf{G}_\mathrm{m}, T)$$

for the cocharacter group of T. Note that the underlying abelian groups of the Galois modules $\mathbf{X}^*(\overline{G})$ and $\mathbf{X}_*(\overline{T})$ are free.

As in [Bor98] we define the algebraic fundamental group $\pi_1(\overline{G})$ as follows. Let $T \subset G^{\mathrm{red}}$ be a maximal torus. Set $T^{\mathrm{sc}} = \rho^{-1}(T)$, it is a maximal torus in G^{sc}. Set

$$\pi_1(\overline{G}) = \mathbf{X}_*(\overline{T})/\rho_*\mathbf{X}_*(\overline{T}^{\mathrm{sc}}).$$

It is a Galois module; it does not depend on the choice of $T \subset G$; it does not change under inner twistings of G. It follows from the definition, that $\pi_1(\overline{G}) = \pi_1(\overline{G}^{\mathrm{red}})$.

We define the derived dual to $\pi_1(\overline{G})$ by

$$\pi_1(\overline{G})^D = [\mathbf{X}^*(\overline{T}) \xrightarrow{\rho^*} \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle.$$

In this section we shall recall classical results that give isomorphisms

$$\mathscr{H}^0(\mathrm{UPic}(\overline{G})) = U(\overline{G}) = \ker\rho^* = \mathscr{H}^0(\pi_1(\overline{G})^D),$$

and

$$\mathscr{H}^1(\mathrm{UPic}(\overline{G})) = \mathrm{Pic}(\overline{G}) = \mathrm{coker}\,\rho^* = \mathscr{H}^1(\pi_1(\overline{G})^D).$$

Lemma 3.2 (Rosenlicht)**.** *For a connected linear algebraic group G over a perfect field k, the obvious map $\mathbf{X}^*(\overline{G}) \to \mathrm{U}(\overline{G})$ is an isomorphism which is functorial in G.*

Proof. See [Ros61], or [FI73, Cor. 2.2], or [KKV89, Prop. 1.2] □

Corollary 3.3. *For a connected linear k-group G we have a canonical isomorphism*

$$U(\overline{G}) \simeq \mathscr{H}^0(\pi_1(\overline{G})^D).$$

Proof. Clearly, $\mathbf{X}^*(\overline{G}^{\mathrm{tor}}) \simeq \mathbf{X}^*(\overline{G})$, hence $\mathbf{X}^*(\overline{G}^{\mathrm{tor}}) \simeq U(\overline{G})$. On the other hand, the identification $G^{\mathrm{tor}} = T/\rho(T^{\mathrm{sc}})$ gives an isomorphism

$$\mathbf{X}^*(\overline{G}^{\mathrm{tor}}) \simeq \ker[\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})] = \mathscr{H}^0(\pi_1(\overline{G})^D).$$

□

We shall now consider the identification $\mathscr{H}^1(\mathrm{UPic}(\overline{G})) = \mathscr{H}^1(\pi_1(\overline{G})^D)$. We first make a reduction to G^{ss} using the following lemma of Fossum–Iversen and Sansuc.

Lemma 3.4. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of connected linear k-groups. Then we have an exact sequence*

$$0 \to \mathbf{X}^*(\overline{G}'') \to \mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{G}') \to \mathrm{Pic}(\overline{G}'') \to \mathrm{Pic}(\overline{G}) \to \mathrm{Pic}(\overline{G}') \to 0.$$

Proof. See [San81, (6.11.4)]. In the case when $H^1(K, G') = 0$ for any extension K of $\overline{k}$, this exact sequence was obtained in [FI73, Prop. 3.1]. □

Corollary 3.5. *Let G be a linear algebraic group over k. Then the canonical maps $r\colon G \to G^{\mathrm{red}}$ and $G^{\mathrm{ss}} \to G^{\mathrm{red}}$ induce a natural isomorphism*

$$\mathrm{Pic}(\overline{G}^{\mathrm{ss}}) \simeq \mathrm{Pic}(\overline{G}).$$

Proof. We first apply Lemma 3.4 to the short exact sequence

$$1 \to G^{\mathrm{u}} \to G \to G^{\mathrm{red}} \to 1,$$

and then to the short exact sequence

$$1 \to G^{\mathrm{ss}} \to G^{\mathrm{red}} \to G^{\mathrm{tor}} \to 1.$$

using the fact that $\mathbf{X}^*(\overline{G}^{\mathrm{u}}) = 0$, $\mathrm{Pic}(\overline{G}^{\mathrm{u}}) = 0$, and $\mathrm{Pic}(G^{\mathrm{tor}}) = 0$. □

We need the following construction of [Pop74, §2] (see also [FI73, p. 275] and [KKLV89, Example 2.1]).

Construction 3.6. Let G be a connected linear k-group. Let $H \subset G$ be a k-subgroup, not necessarily connected. Set $X = G/H$. We construct a morphism of Galois modules

$$c\colon\ \mathbf{X}(\overline{H}) \to \mathrm{Pic}(\overline{X})$$

as follows. Let $\chi \in \mathbf{X}(\overline{H})$. Consider the embedding

$$\overline{H} \hookrightarrow \overline{G} \times \overline{\mathbf{G}}_{\mathrm{m}}, \quad h \mapsto (h, \chi(h)^{-1}).$$

Set $\overline{Y} = (\overline{G} \times \overline{\mathbf{G}}_{\mathrm{m}})/\overline{H}$, this quotient exists by Chevalley's theorem, see for example [Spr98, Thm. 5.5.5]. We have a canonical map $\overline{Y} \to \overline{X} = \overline{G}/\overline{H}$. Clearly $\overline{Y}$ is a torsor under $\overline{\mathbf{G}}_{\mathrm{m}}$ over $\overline{X}$, which admits a local section (in Zariski topology) by Hilbert 90. Since the group $\overline{G} \times \overline{\mathbf{G}}_{\mathrm{m}}$ acts transitively on $\overline{Y}$ and $\overline{X}$, we conclude that the torsor $\overline{Y} \to \overline{X}$ is locally trivial in Zariski topology. From the principal $\overline{\mathbf{G}}_{\mathrm{m}}$-bundle $\overline{Y}$ we construct (using the transition functions of $\overline{Y}$) a *linear* bundle on $\overline{X}$ which we denote by $L(\chi)$. Alternatively, we can construct $L(\chi)$ directly as the quotient $(\overline{G} \times \overline{\mathbf{G}}_{\mathrm{a}})/\overline{H}$ of $\overline{G} \times \overline{\mathbf{G}}_{\mathrm{a}}$ under the right action of $\overline{H}$ given by

$$(g, a) \cdot h = (gh, a\chi(h)^{-1}),$$

where $g \in \overline{G},\ a \in \overline{\mathbf{G}}_{\mathrm{a}},\ h \in \overline{H}$.

We denote by $c(\chi)$ the class of $L(\chi)$ in $\mathrm{Pic}(\overline{X})$. In terms of divisor classes, this means the following. Let ψ_X be a rational section of $L(\chi)$. Set $D = \mathrm{div}(\psi_X)$. We set $c(\chi) = \mathrm{cl}(D) \in \mathrm{Pic}(\overline{X})$, where $\mathrm{cl}(D)$ denotes the class of the divisor D.

Note that a rational section ψ_X of $L(\chi)$ over $\overline{X}$ lifts canonically to a rational function ψ_G on $\overline{G}$. Namely, the graph of ψ_G in $\overline{G} \times \overline{\mathbf{G}}_{\mathrm{a}}$ is the preimage of the graph of ψ_X in $L(\chi)$ with respect to the quotient map $\overline{G} \times \overline{\mathbf{G}}_{\mathrm{a}} \to L(\chi)$.

Lemma 3.7. [Pop74, Thm. 4], [KKV89, Prop. 3.2] *Let G be a connected linear k-group, and let H be a k-subgroup of G (not necessarily connected). Then the sequence*

$$\mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{H}) \xrightarrow{c} \mathrm{Pic}(\overline{G}/\overline{H}) \to \mathrm{Pic}(\overline{G})$$

is exact.

Corollary 3.8. *Let G be a connected semisimple k-group. We regard G as a homogeneous space $G = G^{\mathrm{sc}}/Z$, where $Z = \ker\rho$. Then we have an isomorphism*

$$c\colon\ \mathbf{X}^*(\overline{Z}) \xrightarrow{\sim} \mathrm{Pic}(\overline{G}),$$

where c is the homomorphism of Construction 3.6.

Proof. The corollary follows from Lemma 3.7. We use the facts that $\mathbf{X}^*(\overline{G}^{\mathrm{sc}}) = 0$ and $\mathrm{Pic}(\overline{G}^{\mathrm{sc}}) = 0$. □

Remark 3.9. The equality $\mathrm{Pic}(\overline{G}^{\mathrm{sc}}) = 0$ and the existence of an isomorphism $\mathbf{X}^*(\overline{Z}) \simeq \mathrm{Pic}(\overline{G})$ for a semisimple k-group G were proved by Voskresenskiĭ [Vos69], Fossum and Iversen [FI73, Cor. 4.6], and Popov [Pop74] (see also [Vos98, 4.3]).

Corollary 3.10. *For any connected linear k-group G we have a canonical isomorphism*

$$\mathbf{X}^*(\overline{Z}) \xrightarrow{\sim} \mathrm{Pic}(\overline{G})$$

where $Z = \ker\rho$.

Proof. By Corollary 3.5, we have an isomorphism $\mathrm{Pic}(\overline{G}^{\mathrm{ss}}) \xrightarrow{\sim} \mathrm{Pic}(\overline{G})$. By Corollary 3.8 we have an isomorphism $\mathbf{X}^*(\overline{Z}) \xrightarrow{\sim} \mathrm{Pic}(\overline{G}^{\mathrm{ss}})$. □

Corollary 3.11. *For any connected linear k-group G we have a canonical isomorphism* $\mathrm{Pic}(\overline{G}) \simeq \mathscr{H}^1(\pi_1(\overline{G})^D)$.

Proof. Indeed,

$$\mathbf{X}^*(\overline{Z}) = \mathbf{X}^*(\ker\rho) = \mathscr{H}^1([\mathbf{X}^*(T) \xrightarrow{\rho^*} \mathbf{X}^*(T^{\mathrm{sc}})\rangle = \mathscr{H}^1(\pi_1(\overline{G})^D).$$

□

4. The Comparison Theorem

We shall now construct a canonical isomorphism

$$\varkappa_G\colon \mathrm{UPic}(\overline{G}) \to \pi_1(\overline{G})^D$$

for any connected linear k-group G.

We first make a reduction to the reductive case.

Lemma 4.1. *Let G be a connected linear algebraic group over k. Then the canonical homomorphism $r\colon G \to G^{\mathrm{red}}$ induces an isomorphism*

$$r^*\colon \mathrm{UPic}(\overline{G}^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{UPic}(\overline{G}).$$

Proof. By Lemma 3.4 we have an exact sequence

$$0 \to \mathbf{X}^*(\overline{G}^{\mathrm{red}}) \to \mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{G}^{\mathrm{u}}) \to \mathrm{Pic}(\overline{G}^{\mathrm{red}}) \to \mathrm{Pic}(\overline{G}) \to \mathrm{Pic}(\overline{G}^{\mathrm{u}}) \to 0$$

where $\mathbf{X}^*(\overline{G}^{\mathrm{u}}) = 0$ and $\mathrm{Pic}(\overline{G}^{\mathrm{u}}) = 0$. It follows that the map r induces isomorphisms $\mathbf{X}^*(\overline{G}^{\mathrm{red}}) \xrightarrow{\sim} \mathbf{X}^*(\overline{G})$ and $\mathrm{Pic}(\overline{G}^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{Pic}(\overline{G})$. We see that the morphism $r^*\colon \mathrm{UPic}(\overline{G}^{\mathrm{red}}) \to \mathrm{UPic}(\overline{G})$ induces isomorphisms on $\mathscr{H}^0$ and $\mathscr{H}^1$, hence it is an isomorphism. □

Lemma 4.2. *For any torus T over k we have* $\mathrm{UPic}(\overline{T}) \simeq \mathbf{X}^*(\overline{T})$.

Proof. This follows from Rosenlicht's lemma (Lemma 3.2), since $\mathrm{Pic}(\overline{T}) = 0$. □

Lemma 4.3. *Let G be a connected linear algebraic group over k such that G^{ss} is simply connected, then we have canonical isomorphisms* $\mathrm{UPic}(\overline{G}) = \mathbf{X}^*(\overline{G}) = \mathbf{X}^*(\overline{G}^{\mathrm{tor}})$. *In particular,* $\mathrm{UPic}(\overline{G}) = 0$ *if G is semi-simple and simply connected.*

Proof. We have $\mathbf{X}^*(\overline{G}) = \mathbf{X}^*(\overline{G}^{\mathrm{tor}})$ (for any connected G). By Lemma 3.10 $\mathrm{Pic}(\overline{G}) = 0$, hence $\mathrm{UPic}(\overline{G}) = \mathbf{X}^*(\overline{G})$. □

4.4. In this subsection and the next one we identify $\mathrm{UPic}(\overline{G})$ with $\mathrm{KDiv}(\overline{G})_{e,1}$ as in 2.8, where e is the unit element of G. We write $\mathrm{KDiv}(\overline{G})_1$ for $\mathrm{KDiv}(\overline{G})_{e,1}$. Note that $G \mapsto \mathrm{KDiv}(\overline{G})_1$ is a functor from the category of connected linear k-groups to the category of complexes of Galois modules.

For a maximal torus T in a connected reductive k-group G we have a commutative diagram

$$\begin{array}{ccc} T^{\mathrm{sc}} & \xrightarrow{\rho_T} & T \\ {\scriptstyle i^{\mathrm{sc}}}\downarrow & & \downarrow{\scriptstyle i} \\ G^{\mathrm{sc}} & \xrightarrow{\rho} & G \end{array}$$

(where i is the inclusion homomorphism), hence a commutative diagram of complexes

$$\begin{array}{ccc} \mathrm{KDiv}(\overline{G})_1 & \xrightarrow{\rho^*} & \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1 \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle (i^{\mathrm{sc}})^*} \\ \mathrm{KDiv}(\overline{T})_1 & \xrightarrow{\rho_T^*} & \mathrm{KDiv}(\overline{T}^{\mathrm{sc}})_1 \end{array} \tag{11}$$

and a morphism of complexes

$$\lambda = i^* \oplus (i^{\mathrm{sc}})^*\colon [\mathrm{KDiv}(\overline{G})_1 \to \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1\rangle \to [\mathrm{KDiv}(\overline{T})_1 \to \mathrm{KDiv}(\overline{T}^{\mathrm{sc}})_1\rangle.$$

Consider the fibre $[\mathrm{KDiv}(\overline{G})_1 \to \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1\rangle$. The canonical morphism

$$[\mathrm{KDiv}(\overline{G})_1 \to \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1\rangle \to \mathrm{KDiv}(\overline{G})_1$$

is an isomorphism in the derived category, because $\mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1 \simeq 0$ by Lemma 4.3.

Consider the fibre $[\mathrm{KDiv}(\overline{T})_1 \to \mathrm{KDiv}(\overline{T}^{\mathrm{sc}})_1\rangle$. The commutative diagram of complexes

$$\begin{array}{ccc} \mathbf{X}^*(\overline{T}) & \longrightarrow & \mathbf{X}^*(\overline{T}^{\mathrm{sc}}) \\ \downarrow & & \downarrow \\ \mathrm{KDiv}(\overline{T})_1 & \longrightarrow & \mathrm{KDiv}(\overline{T}^{\mathrm{sc}})_1 \end{array}$$

in which the vertical arrows are isomorphisms in the derived category, induces a morphism of complexes

$$[\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle \to [\mathrm{KDiv}(\overline{T})_1 \to \mathrm{KDiv}(\overline{T}^{\mathrm{sc}})_1\rangle$$

which is an isomorphism in the derived category.

Construction 4.5. For a reductive k-group G we define a morphism $\varkappa_G\colon \mathrm{UPic}(\overline{G}) \to \pi_1(\overline{G})^D$ as the composition

$$\begin{aligned} &\mathrm{UPic}(\overline{G}) \xrightarrow{\sim} [\mathrm{KDiv}(\overline{G})_1 \to \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1\rangle \xrightarrow{\lambda} [\mathrm{KDiv}(\overline{T})_1 \to \mathrm{KDiv}(\overline{T}^{\mathrm{sc}})_1\rangle \\ &\xrightarrow{\sim} [\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle = \pi_1(\overline{G})^D. \end{aligned}$$

For a general connected linear k-group G (not necessarily reductive) we define $\varkappa_G\colon \mathrm{UPic}(\overline{G}) \to \pi_1(\overline{G})^D$ as the composition

$$\mathrm{UPic}(\overline{G}) \xrightarrow{\sim} \mathrm{UPic}(\overline{G}^{\mathrm{red}}) \to \pi_1(\overline{G}^{\mathrm{red}})^D = \pi_1(\overline{G})^D.$$

Remark 4.6. Using [BBD82, Prop. 1.1.9], one can show that all the fibres and morphisms of fibres that we defined in 4.4 and 4.5 using an explicit representation by complexes and morphisms of complexes, do not depend on this representation. We use the fact that $\mathrm{Pic}(\overline{G})$ is a torsion group (by Corollary 3.10), hence

$$\mathrm{Hom}_{D^b(k_{\text{ét}})}(\mathrm{UPic}(\overline{G})[1], \mathrm{UPic}(\overline{T}^{\mathrm{sc}})) = \mathrm{Hom}(\mathrm{Pic}(\overline{G}), \mathbf{X}^*(\overline{T}^{\mathrm{sc}})) = 0,$$

and so [BBD82, Prop. 1.1.9] can be applied.

4.7. *Functoriality.* Let $\varphi\colon G_1 \to G_2$ be a homomorphism of connected linear k-groups. Consider the induced homomorphisms $\varphi^{\mathrm{red}}\colon G_1^{\mathrm{red}} \to G_2^{\mathrm{red}}$ and $\varphi^{\mathrm{sc}}\colon G_1^{\mathrm{sc}} \to G_2^{\mathrm{sc}}$. Choose maximal tori $T_1 \subset G_1^{\mathrm{red}}$ and $T_2 \subset G_2^{\mathrm{red}}$ such that $\varphi^{\mathrm{red}}(T_1) \subset T_2$. Let T_1^{sc} (resp. T_2^{sc}) be the preimage of T_1 in G_1^{sc} (resp. of T_2 in G_2^{sc}). We have homomorphisms $\varphi_*\colon T_1 \to T_2$ and $\varphi_*^{\mathrm{sc}}\colon T_1^{\mathrm{sc}} \to T_2^{\mathrm{sc}}$. We obtain a commutative diagram in the derived category

$$\begin{array}{ccc} \mathrm{KDiv}(\overline{G}_2)_1 & \xrightarrow{\varphi^*} & \mathrm{KDiv}(\overline{G}_1)_1 \\ {\scriptstyle \varkappa_{G_2}}\downarrow & & \downarrow{\scriptstyle \varkappa_{G_1}} \\ [\mathbf{X}^*(\overline{T}_2) \to \mathbf{X}^*(\overline{T}_2^{\mathrm{sc}})\rangle & \xrightarrow{\varphi^*} & [\mathbf{X}^*(\overline{T}_1) \to \mathbf{X}^*(\overline{T}_1^{\mathrm{sc}})\rangle \end{array}$$

Thus the morphism $\varkappa_G\colon \mathrm{UPic}(\overline{G}) \to \pi_1(\overline{G})^D$ is functorial in G.

The following theorem is the main result of this paper.

Theorem 4.8. *With notation as above, for a connected linear algebraic group G over a field k of characteristic 0, the canonical morphism $\varkappa_G\colon \mathrm{UPic}(\overline{G}) \to \pi_1(\overline{G})^D$ is an isomorphism.*

Before proving the theorem, let us first mention two corollaries.

Corollary 4.9. *The canonical isomorphism $\varkappa_G$ induces canonical isomorphisms*

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{G}), \mathbf{G}_{\mathrm{m}}) \simeq H^i_{\mathrm{ab}}(k, G),$$

where $H^i_{\mathrm{ab}}(k, G) := H^i(k, \langle T^{\mathrm{sc}} \to T])$.

Proof. By Theorem 4.8 $\mathrm{Ext}^i(\mathrm{UPic}(\overline{G}), \mathbf{G}_{\mathrm{m}}) = \mathrm{Ext}^i(\pi_1(\overline{G})^D, \mathbf{G}_{\mathrm{m}})$. By Lemma 1.5 $\mathrm{Ext}^i(\pi_1(\overline{G})^D, \mathbf{G}_{\mathrm{m}}) = H^i(k, \langle T^{\mathrm{sc}} \to T])$. □

Corollary 4.10. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of connected linear k-groups. Then we have a distinguished triangle*

$$\mathrm{UPic}(\overline{G}'') \to \mathrm{UPic}(\overline{G}) \to \mathrm{UPic}(\overline{G}') \to \mathrm{UPic}(\overline{G}'')[1].$$

Proof. Indeed, by [Bor98, Lemma 1.5] (see also [BK04, Lemma 3.7]) we have an exact sequence

$$0 \to \pi_1(\overline{G}') \to \pi_1(\overline{G}) \to \pi_1(\overline{G}'') \to 0,$$

hence a distinguished triangle

$$\pi_1(\overline{G}'')^D \to \pi_1(\overline{G})^D \to \pi_1(\overline{G}')^D \to \pi_1(\overline{G}'')^D[1],$$

and the assertion follows from Theorem 4.8. □

Remark 4.11. This triangle strengthens Lemma 3.4. Note that it does not give a new proof, since the lemma was used in the proof of Theorem 4.8, hence in the proof of this corollary.

4.12. *Proof of Theorem 4.8.* We may and shall assume that G is reductive. Recall that

$$\mathrm{KDiv}(\overline{G})_1 = [\overline{k}(\overline{G})^\times_{e,1} \to \mathrm{Div}(\overline{G})_e\rangle,$$

where

$$\overline{k}(\overline{G})^\times_{e,1} = \{f \in \overline{k}(\overline{G})^\times \mid f(e) = 1\},$$
$$\mathrm{Div}(\overline{G})_e = \{D \in \mathrm{Div}(\overline{G}) \mid e \notin \mathrm{supp}(D)\}.$$

From the diagram (11) we obtain a morphism of complexes

$$\lambda = i^* \oplus (i^{\mathrm{sc}})^* \colon C^\bullet_G \to C^\bullet_T\,,$$

where $C^\bullet_G$ and $C^\bullet_T$ are the complexes introduced in 4.4:

$$C^\bullet_G = [\mathrm{KDiv}(\overline{G})_1 \to \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1\rangle,$$
$$C^\bullet_T = [\mathrm{KDiv}(T)_1 \to \mathrm{KDiv}(T^{\mathrm{sc}})_1\rangle.$$

In other words,

$$C^\bullet_G = \mathrm{KDiv}(\overline{G})_1 \oplus \mathrm{KDiv}(\overline{G}^{\mathrm{sc}})_1[-1],$$
$$C^\bullet_T = \mathrm{KDiv}(T)_1 \oplus \mathrm{KDiv}(T^{\mathrm{sc}})_1[-1],$$

with differentials given by the matrix of formula (2) in 1.1.

To prove the theorem, it suffices to prove that λ is a quasi-isomorphism. We must prove that the maps

$$\lambda^0 \colon \mathscr{H}^0(C^\bullet_G) \to \mathscr{H}^0(C^\bullet_T) \text{ and } \lambda^1 \colon \mathscr{H}^1(C^\bullet_G) \to \mathscr{H}^1(C^\bullet_T)$$

are isomorphisms.

We prove that λ^0 is an isomorphism. Using Rosenlicht's lemma, we see immediately that

$$\mathscr{H}^0(C^\bullet_G) = \ker[\rho^* \colon \mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{G}^{\mathrm{sc}})],$$
$$\mathscr{H}^0(C^\bullet_T) = \ker[\rho^*_T \colon \mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})],$$

and $\lambda^0 \colon \mathscr{H}^0(C^\bullet_G) \to \mathscr{H}^0(C^\bullet_T)$ is the map induced by the restriction map $i^* \colon \mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{T})$. Now it is clear that λ^0 is an isomorphism.

We prove that λ^1 is an isomorphism. Write $Z = \ker \rho$. Consider the composed map

$$\sigma_G \colon \mathbf{X}^*(\overline{Z}) \xrightarrow{c} \mathrm{Pic}(\overline{G}^{\mathrm{ss}}) \xrightarrow{\sim} \mathscr{H}^1(\mathrm{KDiv}(\overline{G})_1) \xrightarrow{\sim} \mathscr{H}^1(C^\bullet_G).$$

Here the last isomorphism is induced by the isomorphism $\mathrm{KDiv}(\overline{G})_1 \xrightarrow{\sim} C^\bullet_G$ in the derived category and the map c is the map of Construction 3.6. The map σ_G is an isomorphism, because it is a composition of isomorphisms. We compute σ_G explicitly.

We have

$$C^1_G = \mathrm{Div}(\overline{G})_e \oplus \overline{k}(\overline{G}^{\mathrm{sc}})^\times_{e,1}$$
$$\ker d^1_G = \{(D_G, f_{G^{\mathrm{sc}}}) \in \mathrm{Div}(\overline{G})_e \oplus \overline{k}(\overline{G}^{\mathrm{sc}})^\times_{e,1} \mid \rho^*(D_G) = \mathrm{div}(f_{G^{\mathrm{sc}}})\}$$

where $d_G^1\colon C_G^1 \to C_G^2$ is the differential in $C_G^\bullet$.

Let $\chi \in \mathbf{X}^*(\overline{Z})$. Define a right action of $\overline{Z}$ on $\overline{G}^{\mathrm{sc}} \times \overline{\mathbf{G}}_{\mathrm{a}}$ by $(g,a) * z = (gz, \chi(z)^{-1}a)$, where $g \in \overline{G}^{\mathrm{sc}}, a \in \overline{\mathbf{G}}_{\mathrm{a}}, z \in \overline{Z}$. Set $E^{\mathrm{ss}} = L(\chi) = (\overline{G}^{\mathrm{sc}} \times \overline{\mathbf{G}}_{\mathrm{a}})/\overline{Z}$, then E^{ss} is a linear bundle over $\overline{G}^{\mathrm{ss}} = \overline{G}^{\mathrm{sc}}/\overline{Z}$. By definition $\mathrm{cl}(E^{\mathrm{ss}}) = c(\chi) \in \mathrm{Pic}(\overline{G}^{\mathrm{ss}})$. Since we have a canonical isomorphism $\mathrm{Pic}(\overline{G}) \xrightarrow{\sim} \mathrm{Pic}(\overline{G}^{\mathrm{ss}})$, our E^{ss} comes from a unique (up to an isomorphism) line bundle E over $\overline{G}$. Let φ be a rational section of E such that $\varphi(e) \neq 0, \infty$. Set $D_G = \mathrm{div}(\varphi)$, then $\mathrm{cl}(D_G)$ is the image of χ in $\mathrm{Pic}(\overline{G}) = \mathscr{H}^1(\mathrm{KDiv}(\overline{G})_1)$. Set $D_{G^{\mathrm{sc}}} = \rho^*(D_G)$. Since $\mathrm{Pic}(\overline{G}^{\mathrm{sc}}) = 0$, there exists $f_{G^{\mathrm{sc}}}$ on $\overline{G}^{\mathrm{sc}}$ such that $D_{G^{\mathrm{sc}}} = \mathrm{div}(f_{G^{\mathrm{sc}}})$. Since $e_G \notin \mathrm{supp}(D_G)$, we see that $f_{G^{\mathrm{sc}}}(e_{G^{\mathrm{sc}}}) \neq 0, \infty$. Set $f'_{G^{\mathrm{sc}}} = f_{G^{\mathrm{sc}}}/f_{G^{\mathrm{sc}}}(e_{G^{\mathrm{sc}}})$. Then $(D_G, f'_{G^{\mathrm{sc}}}) \in \ker d_G^1$ and $\mathrm{cl}(D_G, f'_{G^{\mathrm{sc}}}) = \sigma_G(\chi) \in \mathscr{H}^1(C_G^\bullet)$.

We need a lemma.

Lemma 4.13. *The restriction of $f'_{G^{\mathrm{sc}}}$ to Z is χ^{-1}.*

Proof of Lemma 4.13. Consider the section $\rho^*(\varphi)$ of ρ^*E. By the construction of E^{ss} we have a canonical trivialization

$$\mu\colon \overline{G}^{\mathrm{sc}} \times \overline{\mathbf{G}}_{\mathrm{a}} \xrightarrow{\sim} \rho^*E$$

which maps $\rho^*(\varphi)$ to some $\psi = \mu^*(\rho^*(\varphi))$. We have

$$\psi(gz) = \chi(z)^{-1}\psi(g) \text{ for all } g \in \overline{G}^{\mathrm{sc}},\ z \in \overline{Z} \tag{12}$$

because $\varphi|_{\overline{G}^{\mathrm{ss}}}$ is a rational section of E^{ss}. But

$$D_{G^{\mathrm{sc}}} = \rho^*(D_G) = \mathrm{div}(\rho^*(\varphi)) = \mathrm{div}(\psi),$$

so we may take $f_{G^{\mathrm{sc}}} = \psi$. Since $U(\overline{G}^{\mathrm{sc}}) = \mathbf{X}^*(\overline{G}^{\mathrm{sc}}) = 0$, there exists, up to a constant, only one rational function $f_{G^{\mathrm{sc}}}$ on $\overline{G}^{\mathrm{sc}}$ such that $D_{G^{\mathrm{sc}}} = \mathrm{div}(f_{G^{\mathrm{sc}}})$. Using (12), we obtain that for any such $f_{G^{\mathrm{sc}}}$ we have

$$f'_{G^{\mathrm{sc}}}(z) = f_{G^{\mathrm{sc}}}(z)/f_{G^{\mathrm{sc}}}(e) = \psi(z)/\psi(e) = \chi(z)^{-1}.$$

□

4.14. *Proof of Theorem 4.8 (cont.).* We have

$$C_T^1 = \mathrm{Div}(\overline{T})_e \oplus \overline{k}(\overline{T}^{\mathrm{sc}})^\times_{e,1},$$
$$\ker d_T^1 = \{(D_T, f_{T^{\mathrm{sc}}}) |\ \rho_T^*(D_T) = \mathrm{div}(f_{T^{\mathrm{sc}}})\},$$

where $d_T^1\colon C_T^1 \to C_T^2$ is the differential in $C_T^\bullet$. The canonical isomorphism

$$C_T^\bullet \xrightarrow{\sim} [\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle \tag{13}$$

induces a composed map

$$\tau_T\colon \mathscr{H}^1(C_T^\bullet) \xrightarrow{\sim} \mathscr{H}^1([\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle) \xrightarrow{\sim} \mathbf{X}^*(\overline{Z})$$

where the latter isomorphism is defined as follows: $\mathrm{cl}(\kappa) \mapsto \kappa|_{\overline{Z}}$ for $\kappa \in \mathbf{X}^*(\overline{T}^{\mathrm{sc}})$. We compute τ_T explicitly.

Let $(D_T, f_{T^{\mathrm{sc}}}) \in \ker d_T^1$. Since $\mathrm{Pic}(\overline{T}) = 0$, there exists a rational function $f_T \in \overline{k}(\overline{T})^\times_{e,1}$ such that $\mathrm{div}(f_T) = -D_T$. Set $\tilde{f}_{T^{\mathrm{sc}}} = f_{T^{\mathrm{sc}}} \cdot \rho^*(f_T) \in \overline{k}(\overline{T}^{\mathrm{sc}})^\times_{e,1}$. Then $\mathrm{div}(\tilde{f}_{T^{\mathrm{sc}}}) = 0$ and $\tilde{f}_{T^{\mathrm{sc}}}(e) = 1$. By Rosenlicht's lemma $\tilde{f}_{T^{\mathrm{sc}}} \in \mathbf{X}^*(\overline{T}^{\mathrm{sc}})$. Moreover $(0, \tilde{f}_{T^{\mathrm{sc}}}) \in \ker d_T^1$ and $(0, \tilde{f}_{T^{\mathrm{sc}}}) = (D_T, f_{T^{\mathrm{sc}}}) + d_T^0(f_T)$. The construction of the isomorphism (13) then implies that $\mathrm{cl}(D_T, f_{T^{\mathrm{sc}}}) \in \mathscr{H}^1(C_T^\bullet)$ corresponds to $\mathrm{cl}(\tilde{f}_{T^{\mathrm{sc}}}) \in \mathscr{H}^1([\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle)$. The image of $\mathrm{cl}(\tilde{f}_{T^{\mathrm{sc}}})$ in $\mathbf{X}^*(\overline{Z})$ is

$$\tilde{f}_{T^{\mathrm{sc}}}|_Z = f_{T^{\mathrm{sc}}} \cdot \rho^*(f_T)|_Z = f_{T^{\mathrm{sc}}}|_Z\ .$$

Thus the map τ_T is given by $\mathrm{cl}(D_T, f_{T^{\mathrm{sc}}}) \mapsto f_{T^{\mathrm{sc}}}|_Z$.

Now we see that the composed map

$$\beta\colon\ \mathbf{X}^*(\overline{Z}) \xrightarrow{\sigma_G} \mathscr{H}^1(C_G^\bullet) \xrightarrow{\lambda^1} \mathscr{H}^1(C_T^\bullet) \xrightarrow{\tau_T} \mathbf{X}^*(Z)$$

is given by

$$\chi \mapsto (D_G, f'_{G^{\mathrm{sc}}}) \mapsto (i^*(D_G), (i^{\mathrm{sc}})^*(f'_{G^{\mathrm{sc}}})) \mapsto (i^{\mathrm{sc}})^*(f'_{G^{\mathrm{sc}}})|_Z\ .$$

Clearly we have $(i^{\mathrm{sc}})^*(f'_{G^{\mathrm{sc}}})|_Z = f'_{G^{\mathrm{sc}}}|_Z$. By Lemma 4.13 $f'_{G^{\mathrm{sc}}}|_Z = -\chi$ (with additive notation). We see that our composed map β is given by $\chi \mapsto -\chi$, hence it is an isomorphism. Since σ_G and τ_T are isomorphisms, we conclude that λ^1 is an isomorphism. This completes the proof of Theorem 4.8. □

Remark 4.15. A different proof of the existence of an isomorphism $\mathrm{UPic}(\overline{G}) \xrightarrow{\sim} \pi_1(\overline{G})^D$ was proposed in [CT06].

5. UPIC OF TORSORS AND THE ELEMENTARY OBSTRUCTION

Lemma 5.1. *Let X and Y be smooth geometrically integral k-varieties. Assume that Y is $\overline{k}$-rational. Then the canonical morphism*

$$\zeta\colon \mathrm{UPic}(\overline{X}) \oplus \mathrm{UPic}(\overline{Y}) \to \mathrm{UPic}(\overline{X} \times \overline{Y})$$

induced by the projections p_X and p_Y from $X \times Y$ to the corresponding factors, is a quasi-isomorphism.

Proof. By Rosenlicht's lemma [FI73, Lemma 2.1] the map

$$\mathscr{H}^0(\zeta)\colon U(\overline{X}) \oplus U(\overline{Y}) \to U(\overline{X} \times \overline{Y})$$

is an isomorphism. By a lemma of Colliot-Thélène and Sansuc [CTS77, Lemme 11 p. 188] the map

$$\mathscr{H}^1(\zeta)\colon \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X} \times \overline{Y})$$

is an isomorphism. Thus ζ is a quasi-isomorphism. □

For a k-torsor X under a connected linear algebraic k-group G it was shown by Sansuc that $U(\overline{X}) = U(\overline{G}) = \mathbf{X}^*(\overline{G})$ and $\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{G})$. Sansuc's result extends to UPic, and so does his proof.

Lemma 5.2. *Let $\varsigma\colon X \times G \to X$ be a k-morphism defining a right action of a connected linear algebraic k-group G on a smooth geometrically integral k-variety X. Then*

(i) *The canonical morphism*

$$\zeta\colon \mathrm{UPic}(\overline{X}) \oplus \mathrm{UPic}(\overline{G}) \to \mathrm{UPic}(\overline{X} \times \overline{G})$$

is a quasi-isomorphism.

(ii) *Denote by*

$$\pi_G\colon \mathrm{UPic}(\overline{X} \times \overline{G}) = \mathrm{UPic}(\overline{X}) \oplus \mathrm{UPic}(\overline{G}) \to \mathrm{UPic}(\overline{G})$$

the projection. Then

$$\varphi = \pi_G \circ \varsigma^*\colon \mathrm{UPic}(\overline{X}) \to \mathrm{UPic}(\overline{X} \times \overline{G}) \to \mathrm{UPic}(\overline{G})$$

is a canonical morphism, functorial in (X,G) and equal, for any $x_0 \in X(k)$ to $\alpha_{x_0}^*$, where $\alpha_{x_0}\colon G \to X$ is the k-morphism defined by $\alpha_{x_0}(g) = x_0 g$ for $g \in G$.

(iii) *If in addition X is a torsor of G over k, then φ is an isomorphism in the derived category.*

Proof. (i) Since G is $\overline{k}$-rational, by Lemma 5.1 ζ is a quasi-isomorphism.

(ii) Take $x_0 \in X(k)$. Let i_{x_0} be the k-morphism $G \to X \times G$ defined by $i_{x_0}(g) = (x_0, g)$. Then $p_G \circ i_{x_0} = \mathrm{id}$ and $p_X \circ i_{x_0}$ is the constant map $G \to \{x_0\} \in X$. Hence $i_{x_0}^* = \pi_G$ and, since $\alpha_{x_0} = \varsigma \circ i_{x_0}$, we get

$$\alpha_{x_0}^* = i_{x_0}^* \circ \varsigma^* = \pi_G \circ \varsigma^* = \varphi.$$

(iii) By [San81, Lemmes 6.4, 6.5(ii), 6.6(i)] the morphisms $\mathscr{H}^0(\varphi)\colon U(\overline{X}) \to \mathbf{X}^*(\overline{G})$ and $\mathscr{H}^1(\varphi)\colon \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{G})$ are isomorphisms, hence φ is an isomorphism in the derived category. □

As a corollary we obtain a canonical isomorphism between the target of the elementary obstruction (Definition 2.10) and the abelian Galois cohomology $H^i_{\mathrm{ab}}(k,G) := H^i(k, \langle T^{\mathrm{sc}} \to T]\rangle)$.

Corollary 5.3. *Let X be a k-torsor under a connected linear algebraic k-group G. We have a canonical isomorphism*

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}), \mathbf{G}_{\mathrm{m}}) \simeq H^i_{\mathrm{ab}}(k,G)$$

which is functorial in (G,X) and in k.

Proof. By Lemma 5.2(iii) $\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}), \mathbf{G}_{\mathrm{m}}) = \mathrm{Ext}^i(\mathrm{UPic}(\overline{G}), \mathbf{G}_{\mathrm{m}})$. By Corollary 4.9 $\mathrm{Ext}^i(\mathrm{UPic}(\overline{G}), \mathbf{G}_{\mathrm{m}}) = H^i_{\mathrm{ab}}(k,G)$. □

Let $\mathrm{ab}^1\colon H^1(k,G)\to H^1_{\mathrm{ab}}(k,G)$ be the abelianization map constructed in [Bor98]. For a k-torsor X of G, let $[X]$ denote its class in $H^1(k,G)$. We write $[X]_{\mathrm{ab}}:=\mathrm{ab}^1([X])\in H^1_{\mathrm{ab}}(k,G)$. We shall prove that the elementary obstruction $e(X)$ coincides up to sign with the $[X]_{\mathrm{ab}}$. For semisimple groups this was proved by Skorobogatov [Sko01, p. 54]. For tori this was proved by Sansuc [San81, (6.7.3), (6.7.4)] (see also Skorobogatov [Sko01, Lemma 2.4.3]). First we give Sansuc's proof for tori with details added.

Lemma 5.4 (Sansuc)**.** *Let T be a torus over k and let X be a k-torsor under T, determined by a cocycle $c\colon \sigma\mapsto c_\sigma\colon \mathrm{Gal}(\overline{k}/k)\to T(\overline{k})$. Consider the extension*

$$1\to\overline{k}^\times\to\overline{k}[X]^\times\to\mathbf{X}^*(\overline{T})\to 1 \tag{14}$$

The class $e(X)$ of this extension in $\mathrm{Ext}^1(\mathbf{X}^*(\overline{T}),\overline{k}^\times)$ corresponds under the isomorphism of Lemma 1.5

$$\mathrm{Ext}^1(\mathbf{X}^*(\overline{T}),\overline{k}^\times)=H^1(k,T)$$

to the class of the cocycle c^{-1}.

Proof. We regard $X(\overline{k})$ as $T(\overline{k})$ with the twisted Galois action $\sigma*t=c_\sigma\cdot\sigma t$, where $t\in T(\overline{k})$. Similarly we regard $\overline{k}[X]^\times$ as $\overline{k}[T]^\times$ with the twisted Galois action, etc. In all cases we use the notation $\sigma*$ to denote the twisted Galois action.

Let $\chi\in\mathbf{X}^*(\overline{T})$. We compute ${}^{\sigma*}\chi$. For $t\in T(\overline{k})$ we have

$$({}^{\sigma*}\chi)(\sigma*t)=\sigma(\chi(t)),$$

hence

$$({}^{\sigma*}\chi)(t)=\sigma(\chi(\sigma^{-1}*t))=\sigma(\chi(c_{\sigma^{-1}}\cdot\sigma^{-1}t))=$$
$$=\sigma(\chi(\sigma^{-1}(c_\sigma^{-1}t)))={}^{\sigma}\chi(c_\sigma^{-1}t)={}^{\sigma}\chi(c_\sigma^{-1})\cdot{}^{\sigma}\chi(t).$$

Thus

$${}^{\sigma*}\chi={}^{\sigma}\chi(c_\sigma^{-1})\cdot{}^{\sigma}\chi.$$

Now let $\varphi\colon \mathbf{X}^*(\overline{T})\to\overline{k}[\overline{X}]^\times$ be the standard (non-equivariant) splitting corresponding to the identification of $\overline{X}$ with $\overline{T}$. By abuse of notation we denote this splitting by $\chi\mapsto\chi$. Then the extension class $e(X)\in H^1(k,\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*(\overline{T}),\overline{k}^\times))$ is represented by the cocycle $\sigma\mapsto\sigma\varphi\cdot\varphi^{-1}$.

Since

$$(\sigma\varphi)(\chi)={}^{\sigma*}(\varphi({}^{\sigma^{-1}}\chi))={}^{\sigma*}({}^{\sigma^{-1}}\chi)=\chi(c_\sigma^{-1})\cdot\chi,$$

we see that $e(X)$ is represented by the cocycle

$$\sigma\mapsto\sigma\varphi\cdot\varphi^{-1}=(\chi\mapsto\chi(c_\sigma^{-1}))\in\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*(\overline{T}),\overline{k}^\times),$$

which corresponds to the cocycle $\sigma\mapsto c_\sigma^{-1}$ under the identification $T(\overline{k})\simeq\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*(\overline{T}),\overline{k}^\times)$ given by $t\mapsto(\chi\mapsto\chi(t))$ for $t\in T(\overline{k})$. □

Theorem 5.5. *Let X be a torsor under a linear algebraic group G over k. The elementary obstruction class $e(X)\in\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times)$ corresponds to $-[X]_{\mathrm{ab}}\in H^1_{\mathrm{ab}}(k,G)$ under the canonical isomorphism of Corollary 5.3*

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}),\mathbf{G}_{\mathrm{m}})\simeq H^i_{\mathrm{ab}}(k,G).$$

Proof. Without loss of generality we may assume that G is reductive.

As in [Kot86, p. 369] (compare [BK00, Lemma 1.1.4(i)]), we construct an epimorphism $\alpha\colon H\to G$, where H is a reductive k-group with H^{ss} simply connected, together with a k-torsor X_H under H such that $\alpha_*(X_H)\simeq X$. By functoriality of the isomorphism of Corollary 5.3, in order to prove the theorem for G and X, it is sufficient to prove it for H and X_H.

Since H^{ss} is simply connected, the homomorphism $H\to H^{\mathrm{tor}}$ induces an isomorphism $H^1_{\mathrm{ab}}(k,H)\simeq H^1_{\mathrm{ab}}(k,H^{\mathrm{tor}})=H^1(k,H^{\mathrm{tor}})$, cf. [Bor98, Example 2.12(2)]. We see that the functoriality of the isomorphism of Corollary 5.3 implies that it is sufficient to prove the theorem for torsors under tori, which was done in Lemma 5.4. □

Corollary 5.6. *Let Y be a smooth compactification of a torsor X under a connected linear algebraic group G over k. Let S be the Néron-Severi torus of Y, i.e. the k-torus S such that $\mathbf{X}^*(S) = \mathrm{Pic}(\overline{Y})$. Then $\mathrm{Ext}^1(\mathrm{UPic}(\overline{Y}), \mathbf{G}_\mathrm{m}) = H^2(k,S)$ and we have a canonical injection*

$$H^1_{\mathrm{ab}}(k,G) \hookrightarrow H^2(k,S)$$

sending $[X]_{\mathrm{ab}} \in H^1_{\mathrm{ab}}(k,G)$ to $-e(Y) \in H^2(k,S)$.

Proof. By Corollary 2.17 the open embedding $j\colon X \hookrightarrow Y$ induces an injection $j_*\colon \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \mathbf{G}_\mathrm{m}) \hookrightarrow \mathrm{Ext}^1(\mathrm{UPic}(\overline{Y}), \mathbf{G}_\mathrm{m})$. By Corollary 5.3 $\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \mathbf{G}_\mathrm{m}) = H^1_{\mathrm{ab}}(k,G)$. Since $\mathrm{UPic}(\overline{Y}) = \mathrm{Pic}(\overline{Y})[-1]$, we have $\mathrm{Ext}^1(\mathrm{UPic}(\overline{Y}), \mathbf{G}_\mathrm{m}) = \mathrm{Ext}^2(\mathrm{Pic}(\overline{Y}), \mathbf{G}_\mathrm{m}) = H^2(k,S)$ (we use Lemma 1.5). By Theorem 5.5 j_* takes $[X]_{\mathrm{ab}}$ to $-e(Y)$. □

Proposition 5.7. *For (a smooth compactification of) a torsor X under a connected linear algebraic group G over a p-adic field k, the elementary obstruction is the only obstruction to the existence of k-rational points.*

Proof. We will first show that the vanishing of $e(X)$ implies the existence of a k-rational point on X. Let $\mathrm{ab}^1\colon H^1(k,G) \to H^1_{\mathrm{ab}}(k,G)$ denote the abelianization map of [Bor98]. We have an exact sequence

$$H^1(k,G^{\mathrm{sc}}) \to H^1(k,G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(k,G),$$

see [Bor98, (3.10.1)]. By Theorem 5.5 $\mathrm{ab}^1([X]) = -e(X)$, and by assumption $e(X) = 0$. By Kneser's theorem $H^1(k,G^{\mathrm{sc}}) = 0$. We conclude that $[X] = 0$. Thus X has a k-point.

Now we will show that for a smooth compactification Y of X the vanishing of $e(Y)$ implies the existence of a k-rational point. By Corollary 2.17 the vanishing of $e(Y)$ implies the vanishing of $e(X)$. As above, this implies that $X(k) \neq \emptyset$, hence $Y(k) \neq \emptyset$. □

Proposition 5.8. *For (a smooth compactification of) a torsor X under a connected linear algebraic group G over a number field k, the elementary obstruction is the only obstruction to the Hasse principle.*

Proof. First assume that $X(k_v) \neq \emptyset$ for all places v of k, and assume that $e(X) = 0$. Clearly $[X] \in Ш^1(k,G)$, where $Ш^1(k,G)$ is the Tate-Shafarevich kernel for G. By Theorem 5.5 $\mathrm{ab}^1([X]) = -e(X)$. Clearly $e(X) \in Ш^1_{\mathrm{ab}}(k,G)$, where

$$Ш^1_{\mathrm{ab}}(k,G) := \ker\left[H^1_{\mathrm{ab}}(k,G) \to \prod_v H^1_{\mathrm{ab}}(k_v,G)\right].$$

By [Bor98, Thm. 5.12] the induced map $\mathrm{ab}^1_{Ш}\colon Ш^1(k,G) \to Ш^1_{\mathrm{ab}}(k,G)$ is bijective (here the Hasse principle for semisimple simply connected groups, due to Kneser, Harder and Chernousov, plays a major role in the proof). We see that $[X] = 0$, hence $X(k) \neq \emptyset$.

Now let Y be a smooth compactification of a k-torsor X. Assume that $e(Y) = 0$ and that $Y(k_v) \neq \emptyset$ for all v. By Corollary 2.17 $e(X) = 0$ because $e(Y) = 0$. Since Y is smooth, we have $X(k_v) \neq \emptyset$ for all v. As above we see that $X(k) \neq \emptyset$, hence $Y(k) \neq \emptyset$. □

For other proofs of Propositions 5.7 and 5.8 see [BCTS06].

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