A NONCOMMUTATIVE BOHNENBLUST–SPITZER IDENTITY FOR ROTA–BAXTER ALGEBRAS SOLVES BOGOLIUBOV'S RECURSION

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Abstract. The Bogoliubov recursion is a particular procedure appearing in the process of renormalization in perturbative quantum field theory. It provides convergent expressions for otherwise divergent integrals. We develop here a theory of functional identities for noncommutative Rota–Baxter algebras which is shown to encode, among others, this process in the context of Connes–Kreimer's Hopf algebra of renormalization. Our results generalize the seminal Cartier–Rota theory of classical Spitzer-type identities for commutative Rota–Baxter algebras. In the classical, commutative, case, these identities can be understood as deriving from the theory of symmetric functions. Here, we show that an analogous property holds for noncommutative Rota–Baxter algebras. That is, we show that functional identities in the noncommutative setting can be derived from the theory of noncommutative symmetric functions. Lie idempotents, and particularly the Dynkin idempotent play a crucial role in the process. Their action on the pro-unipotent groups such as those of perturbative renormalization is described in detail along the way.

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Contents

1. Introduction

Spitzer identities first appeared in fluctuation theory, together with the notion of Baxter relations —now called Rota–Baxter (RB) relations [48, 3, 2]. It was soon realized by Rota, Cartier, and others, that the theory could be founded on purely algebraic grounds and had many other applications [44, 6, 45] —appearing retrospectively as one of the many striking successes of Rota's approach to algebra, combinatorics and their applications.

The purpose of the present article is to extend the theory to the noncommutative setting. Indeed, the classical Spitzer identities involve commutative Rota–Baxter operators and algebras. However, to consider the noncommutative case is natural. For example, the integration operator acting on matrix algebras is a RB operator (in that particular case, the Rota–Baxter relation identifies with the integration by parts rule), so that the Magnus or Strichartz identities for the solutions of first order linear differential equations can be viewed as particular examples of RB-type identities (see [5, 32, 49] and our account in section 8 of the present article).

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Actually, a striking application of the RB formalism in the noncommutative setting emerged very recently in the context of the Connes–Kreimer Hopf algebra approach to renormalization in perturbative quantum field theory (pQFT) [11, 17, 18, 19, 34], and motivated the present article. The Bogoliubov recursion is a purely combinatorial recursive process that allows to give a meaning to the divergent integrals appearing in pQFT [9]. Using the RB point of view, the process finds a very compact and simple formulation. Abstractly, the recursion takes place in a particular noncommutative RB algebra and writes

$$X = 1 - R\big(X \star a\big),$$

where X is the quantity to be computed recursively, R is the RB operation, $\star$ is the algebra product, and a is a divergent series naturally associated to the physical quantities to be computed (the regularized Feynman rules in, say, the dimensional regularization scheme [9]). We refer the reader to the last section of the present article for definitions and further details.

The above functional identity lies in fact at the heart of Baxter's original work and, for a commutative RB algebra, its solution is given by nothing but Spitzer's classical identity [48]. However, as we mentioned, in renormalization the very RB structure one has to deal with is noncommutative, so that these results do not apply. In [17] it was shown that one can prove a first noncommutative Spitzer (also known as Pollaczeck–Spitzer) identity, and that this identity was related to a so-called Baker–Campbell–Hausdorff (BCH) recursion, which is another way, besides Bogoliubov's, to perform recursively the renormalization process [18, 20].

Here, we show that one can actually prove more. That is, we derive noncommutative generalizations of the Bohnenblust–Spitzer identity (Thm. 6.1, Thm. 6.2, Thm. 7.1). They allow us to solve completely the renormalization problem, in the sense that they lead to a closed formula for X —as opposed to the BCH or Bogoliubov recursions—, and also to the Pollaczeck–Spitzer identity. The new formula is also completely different from the celebrated Zimmermann forest formula [52], that relies on particular combinatorial properties of Feynman diagrams.

However, although our first motivation was renormalization theory in the Connes–Kreimer approach, one should be aware that the existence of noncommutative Bohnenblust–Spitzer identities, as well as the ideas developed to prove the identities, are of general interest and should lead to a noncommutative approach in the many fields where commutative Rota–Baxter algebras have been a useful tool.

To understand our approach, recall one of the main events in the study of RB algebras. In their seminal 1972 article [45], Rota and Smith showed that Spitzer-type identities for commutative Rota–Baxter algebras could be understood as deriving from the theory of symmetric functions. Here, we actually show that the same is true for noncommutative RB algebras. That is, we show that functional identities for these algebras can be derived from the theory of noncommutative symmetric functions [25] or, equivalently, from the theory of descent algebras of bialgebras —a cornerstone of the modern approach to the theory of free Lie algebras [42, 39]. In the process, we establish a connection between noncommutative RB algebras and quasi-symmetric functions in noncommutative variables.

Eventually, as already alluded at, our findings lead to noncommutative generalizations of the classical Bohnenblust–Spitzer identity. Moreover, these new identities are derived from a functional equation (Thm. 2.1) for the classical Dynkin operator (the iteration of the Lie bracket in the theory of free Lie algebras). As an application we present a closed formula for the Bogoliubov recursion in the context of Connes–Kreimer's Hopf algebra approach to perturbative renormalization. This last finding is complementary to the main result of the recent article [21], where two of the present authors together with J.M. Gracia-Bondía proved that the mathematical properties of locality and the so-called beta-function in pQFT could be derived from the properties of the Dynkin operator. Our new findings reinforce the idea that the Dynkin operator and its algebraic properties have to be considered as one of the building blocks of the modern mathematical theory of renormalization.

Some partial results were announced in the Letter [22]. We give here their complete proofs and develop the general theory of noncommutative functional identities for Rota–Baxter algebras as well as their applications to perturbative renormalization which were alluded at also in [22].

Let us briefly outline the organization of this paper. The second section develops the theory of pro-unipotent groups and pro-unipotent Lie algebras from the point of view of Lie idempotents. The properties of the Dynkin idempotent are recalled. The next section surveys the classical Bogoliubov recursion, emphasizing the Rota–Baxter approach and the connections with Atkinson's recursion and Spitzer identities. Definition and basic properties of Rota–Baxter algebras are recalled in the process, i.e. the Rota–Baxter double and pre-Lie product. We then extend in the fourth section Rota–Smith's construction of the free commutative Rota–Baxter algebra (in an arbitrary number of generators) to the noncommutative case. This leads naturally to the link between free Rota–Baxter algebras and noncommutative symmetric and quasi-symmetric functions that are explored afterwards. Section 6 concentrates on the Bohnenblust–Spitzer identity in the noncommutative setting, whereas the following one features an extension of the Magnus recursion and of Strichartz' solution thereof to arbitrary noncommutative RB algebras. Applications to perturbative renormalization are considered in the last section; the relevance of our general results in this setting is emphasized, since they provide a non recursive solution to the computation of counterterms in pQFT together with new theoretical and computational insights on the subject.

2. Lie idempotents actions on pro-unipotent groups

The ground field $\mathbb{K}$ over which all algebraic structures are defined is of characteristic zero.

Lie idempotents are well known to be one of the building blocks of the modern theory of free Lie algebras. This includes the modern approaches to the Baker–Campbell–Hausdorff formula, the Dynkin formula for the Hausdorff series, the Zassenhaus formula, and continuous versions of the same formulas such as Magnus' continuous Baker–Campbell–Hausdorff formula [42, Chap. 3].

One of the purposes of the present article is to show that the same result holds for general Rota–Baxter algebras. That is, functional identities for Rota–Baxter algebras can be derived from the theory of Lie idempotents. In a certain sense, the result is not so surprising: after all, in their seminal work, Rota and Smith [45] explained the classical Spitzer identities for commutative RB algebras by means of the Waring formula, which holds in the algebra of symmetric functions. Lie idempotents appear (very generally) as soon as one moves from the commutative algebra setting to the noncommutative one, where phenomena such as the Baker–Campbell–Hausdorff formula require, for their solution and combinatorial expansion, free Lie algebraic tools. For that purpose, the algebra of symmetric functions, which encodes many of the main functional identities for commutative algebras, has to be replaced by the descent algebra, which is the algebra in which Lie idempotents live naturally. We refer to Reutenauer's standard reference [42] for further details on the subject and a general picture of Lie idempotents, descent algebras, and their applications to the study of Lie and noncommutative algebras. The definitions which are necessary for our purposes are given below.

To start with, let us first recall from [42, 25, 40] and [21] some definitions and properties relating the classical Dynkin operator to the fine structure theory of Hopf algebras. Some details are needed, since the results gathered in the literature are not necessarily stated in a form convenient for our purposes.

Recall first the classical definition of the Dynkin operator D. The Dynkin operator is the linear map from $A := T(X)$, the tensor algebra over a countable set X, into itself defined as the left-to-right iteration of the associated Lie bracket, so that, for any sequence $y_1, \dots, y_n$ of elements of X:

$$D(y_1 \dots y_n) := [\dots[[y_1, y_2], y_3] \dots, y_n]$$

where $[x, y] := xy - yx$. We also write D_n for the action of D on $T_n(X)$, the component of degree n of the tensor algebra (the linear span of words $y_1 \dots y_n$, $y_i \in X$, $i = 1, \dots, n$). Notice, for further use, the iterated structure of the definition of D; it will appear below that the Dynkin operator is a natural object to understand advanced properties of the Spitzer algebra [22]. The Dynkin operator can be shown to be a quasi-idempotent. That is, its action on an homogeneous element of degree n of the tensor algebra satisfies: $D^2 = n \cdot D$ and, moreover, the associated projector $\frac{D}{n}$ is a projection from $T_n(X)$ onto the component of degree n, $Lie_n(X)$, of the free Lie algebra over X, see [42].

The tensor algebra is a graded cocommutative connected Hopf algebra: the coalgebra structure is entirely specified by the requirement that the elements $x \in X$ are primitive elements in $T(X)$. It is therefore naturally provided with an antipode S and with a grading operation Y —the map Y acting as the multiplication by n on $T_n(X)$. One can then show that the Dynkin operator can be rewritten in purely Hopf algebraic terms as $D = S \star Y$, where $\star$ stands for the convolution product in $\mathrm{End}(T(X))$. Recall, for further use, that, writing Δ and π for the coproduct and the product, respectively, in an arbitrary Hopf algebra H, the convolution product of two endomorphisms f and g of H is defined by:

$$f \star g := \pi \circ (f \otimes g) \circ \Delta.$$

The definition of the Dynkin map as a convolution product can be extended to any graded connected cocommutative or commutative Hopf algebra [40], as a particular case of a more general phenomenon, namely the possibility to define an action of Σ_n, the classical Solomon's algebra of type A_n (resp. the opposite algebra) on any graded connected commutative (resp. cocommutative) Hopf algebra [39]. In particular, if we call *descent algebra* and write $\mathcal{D} := \bigoplus_{n\in\mathbb{N}} \mathcal{D}_n$ for the convolution subalgebra of $\mathrm{End}(T(X))$ generated by the graded projections, $p_n : T(X) \longrightarrow T_n(X)$, then Σ_n^{op}, the opposite algebra to Solomon's algebra of type A_n, identifies naturally with $\mathcal{D}_n$, which inherits an associative algebra structure from the composition product in $\mathrm{End}(T(X)$ [42, 39].

In the present article we call *Lie idempotents* the projectors from $T_n(X)$ to $Lie_n(X)$ that belong to Solomon's algebra Σ_n (one sometimes calls Lie idempotents the more general projectors belonging to the symmetric group algebra $\mathbb{Q}[S_n]$, in which Solomon's algebra is naturally embedded —however the latter idempotents can not be generalized naturally to idempotents acting on bialgebras and are therefore not relevant for our purposes).

A Lie idempotent series is a sequence of Lie idempotents or, equivalently, a projection map γ from $T(X)$ to $Lie(X)$ belonging to the descent algebra the graded components γ_n of which are Lie idempotents. Besides the Dynkin idempotent series, $\frac{D_n}{n}$, the list of Lie idempotents series include Solomon's Eulerian idempotent series, the Klyachko idempotent series, the Zassenhaus idempotent series etc. We refer to [42, 25] for further details on the subject.

Proposition 2.1. *The descent algebra is a free graded associative algebra freely generated by the p_n. Any Lie idempotent series generates freely the descent algebra.*

The first part of the Proposition is Corollary 9.14 of [42]. The second part follows e.g. from Theorem 5.15 in [25].

As already mentioned, it follows from [39] that all these idempotent series act naturally on any graded connected commutative or cocommutative bialgebra. Here, we will restrict our attention to the cocommutative case, we refer to [21] for applications of the Dynkin operator formalism to the commutative but noncocommutative Hopf algebras appearing in the Connes–Kreimer Hopf algebraic theory of renormalization in pQFT, see [10, 11, 12, 19].

Theorem 2.1. *Let $H = \bigoplus_{n\in\mathbb{N}} H_n$ be an arbitrary graded connected cocommutative Hopf algebra over a field of characteristic zero. Any Lie idempotent series induces an isomorphism between the pro-unipotent group $G(H)$ of group-like elements of $\hat{H} := \prod_{n\in\mathbb{N}} H_n$ and the pro-nilpotent Lie algebra $Prim(H)$ of primitive elements in $\hat{H}$.*

When the Lie series is the Eulerian idempotent series, the isomorphism is simply the exponential/logarithm isomorphism between a pro-unipotent group and its pro-nilpotent Lie algebra. When the Lie series is the Dynkin series, the inverse morphism is given by $\Gamma : Prim(H) \to G(H)$, mapping $h = \sum_{n\geq 0} h_n$ to:

$$\Gamma(h) = \sum_{n\geq 0} \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \frac{h_{i_1}\cdots h_{i_k}}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}.$$

This is a result dual to Theorem 4.1 in [21], which established the same formula for characters and infinitesimal characters of graded connected commutative Hopf algebras when the Lie series was the Dynkin series. The assertion on pro-unipotency and pro-nilpotency follows e.g. from the classical equivalence between group schemes and commutative Hopf algebras. It is simply a way to recall that the group and Lie algebra we consider come from graded connected Hopf algebras, and inherit from them the usual nilpotence and completeness properties of graded connected algebras (see e.g. [10, 11, 12, 13, 15], where the pro-unipotent group schemes point of view is put to use systematically instead of the Hopf algebraic one to deal with similar questions).

We sketch the proof. The descent algebra carries naturally a Hopf algebra structure [25, 33]. Since $\mathcal{D}$ is freely generated by the p_n, the coproduct Δ is entirely defined by the requirement that the p_n form a sequence of divided powers (that is, $\Delta(p_n) = \sum_{i+j=n} p_i \otimes p_j$) or, equivalently, that any Lie idempotent is a primitive element, see e.g. Corollary 5.17 in [25] or Corollary 3 in [40].

It follows from [40] that there is a compatibility relation between this coproduct and the descent algebra natural action on an arbitrary graded connected cocommutative Hopf algebra H. Namely, for any element f in the descent algebra, we have

$$\Delta(f) \circ \delta = \delta \circ f$$

where δ stands for the coproduct in H, and where the action of f on H is induced by the convolution algebra morphism that maps p_n, viewed as an element of $\mathcal{D}$, to the graded projection (also written abusively p_n) from H to H_n. In particular, for any Lie idempotent l_n acting on H_n and any $h \in H_n$, we have:

$$\begin{aligned}\delta(l_n(h)) &= \Delta(l_n)(\delta(h)) \\ &= (l_n \otimes \epsilon + \epsilon \otimes l_n)(h \otimes 1 + 1 \otimes h + h' \otimes h'') \\ &= l_n(h) \otimes 1 + 1 \otimes l_n(h)\end{aligned}$$

where ϵ stands for the counit of H (the natural projection from H to H_0 with kernel $H^+ := \bigoplus_{n>0} H_n$) and $h' \otimes h''$ belongs to $H^+ \otimes H^+$; the identity follows from l_n being primitive in the descent algebra. So that, in particular, $l_n(h) \in Prim(H)$. This implies that the action of any Lie series on H and, in particular, on $G(H)$, the set of group-like elements in H, induces a map to $Lie(H)$.

The particular example of the Eulerian idempotent is interesting. The Eulerian idempotent e is the logarithm of the identity in the endomorphism algebra of $T(X)$ and belongs to the descent algebra. It acts on H as the logarithm of the identity of H in the convolution algebra $End(H)$, see [47, 42, 38, 39]. For any $h \in G(H)$, we get:

$$e(h) = \log(Id_H)(h) = \sum_{n\in\mathbb{N}} \frac{(-1)^{n-1}}{n}(Id_H - \epsilon)^{\star n}(h) = \sum_{n\in\mathbb{N}} \frac{(-1)^{n-1}}{n}(h - \epsilon(h))^n = \log(h),$$

since, h being group-like: $(f \star g)(h) = f(h)g(h)$ for any $f, g \in End(H)$, so that $(Id_H - \epsilon)^{\star n}(h) = (h - \epsilon(h))^n$. This proves the assertion on the Eulerian idempotent in the Theorem 2.1.

Now, according to Proposition 2.1, the graded components of any Lie series l generate freely the descent algebra as an associative algebra. It follows in particular that the identity of $T(X)$ can be written as a noncommutative polynomial in the l_n. That is, for any Lie series l there exist a unique family of coefficients $\alpha_{n_1,\dots,n_k}$ such that:

$$Id_{T(X)} = \sum_{n=0}^{\infty} \sum_{n_1+\cdots+n_k=n} \alpha_{n_1,\dots,n_k} l_{n_1} \star \cdots \star l_{n_k}.$$

In particular, for any $g \in G(H)$, we get, since the element $Id_{T(X)}$ of the descent algebra acts as the identity on H [39]:

$$g = \sum_{n=0}^{\infty} \sum_{n_1+\cdots+n_k=n} \alpha_{n_1,\dots,n_k} l_{n_1}(g) \star \cdots \star l_{n_k}(g)$$

so that the map from $Lie(H)$ to $G(H)$:

$$Lie(H) = \bigoplus_{n=0}^{\infty} Lie_n(H) \ni \sum_{n=0}^{\infty} \lambda_n \longmapsto \sum_{n=0}^{\infty} \sum_{n_1+\cdots+n_k=n} \alpha_{n_1,\dots,n_k} \lambda_{n_1} \star \cdots \star \lambda_{n_k}$$

is a right inverse (and in fact also a left inverse) to l. The particular formula for the Dynkin idempotent follows from [25] or from Lemma 2.1 in [21].

Two particular applications of the theorem are well-known. Consider first the case where H is the Hopf algebra of noncommutative symmetric functions. Then, H is generated as a free associative algebra by the *complete homogeneous noncommutative symmetric functions* S_k, $k \in \mathbb{N}$, which form a sequence of divided powers, that is, their sum is a group-like element in $\hat{H}$. The graded components of the corresponding primitive elements under the action of the Dynkin operator are known as the *power sums noncommutative symmetric functions of the first kind* [25]. Second, consider the classical descent algebra viewed as a Hopf algebra. Then, the Dynkin operator (viewed as the convolution product $S \star Y$ acting on the Hopf algebra $\mathcal{D}$) sends the identity of $T(X)$, which is a group-like element of the descent algebra to the classical Dynkin operator. This property was put to use in Reutenauer's monograph to rederive all the classical functional Lie-type identities in the tensor algebra —for example the various identities related to the Baker–Campbell–Hausdorff formula.

As it will appear, a surprising conclusion of the present article is that *the same machinery* can be used to derive the already known formulas for commutative Rota–Baxter algebras but, moreover, can be used to prove new formulas in the noncommutative setting.

3. Rota–Baxter algebras and Bogoliubov's recursion

In this section we first briefly recall the definition of Rota–Baxter (RB) algebra and its most important properties. For more details we refer the reader to the classical papers [2, 3, 6, 44, 45], as well as for instance to the references [19, 20].

Let A be an associative not necessarily unital nor commutative algebra with $R \in \mathrm{End}(A)$. The product of a and b in A is written $a \cdot b$ or simply ab when no confusion can arise. We call a tuple (A, R) a *Rota–Baxter algebra* of weight $\theta \in \mathbb{K}$ if R satisfies the *Rota–Baxter relation*

$$R(x)R(y) = R\big(R(x)y + xR(y) + \theta xy\big). \tag{1}$$

Changing R to $R' := \mu R$, $\mu \in \mathbb{K}$, gives rise to a RB algebra of weight $\theta' := \mu\theta$, so that a change in the θ parameter can always be achieved, at least as long as weight non-zero RB algebras are considered.

The definition generalizes to other types of algebras than associative algebras: for example one may want to consider RB Lie algebra structures. Further below we will encounter examples of such structures.

In the following we denote the particular argument of the map R on the right hand side of (1) by

$$x *_\theta y := R(x)y + xR(y) + \theta xy,$$

and will come back to it further below.

Let us recall some classical examples of RB algebras. First, consider the integration by parts rule for the Riemann integral map. Let $A := C(\mathbb{R})$ be the ring of real continuous functions with pointwise product. The indefinite Riemann integral can be seen as a linear map on A

$$I : A \to A, \qquad I(f)(x) := \int_0^x f(t)\,dt. \tag{2}$$

Then, integration by parts for the Riemann integral can be written compactly as

$$I(f)(x)I(g)(x) = I\big(I(f)g\big)(x) + I\big(fI(g)\big)(x), \tag{3}$$

dually to the classical Leibniz rule for derivations. Hence, we found our first example of a weight zero Rota–Baxter map. Correspondingly, on a suitable class of functions, we define the following Riemann summation operators

$$R_\theta(f)(x) := \sum_{n=1}^{[x/\theta]} \theta f(n\theta) \qquad \text{and} \qquad R'_\theta(f)(x) := \sum_{n=1}^{[x/\theta]-1} \theta f(n\theta). \tag{4}$$

We observe readily that

$$\begin{aligned}
\Big(\sum_{n=1}^{[x/\theta]} \theta f(n\theta)\Big)\Big(\sum_{m=1}^{[x/\theta]} \theta g(m\theta)\Big) &= \Big(\sum_{n>m=1}^{[x/\theta]} + \sum_{m>n=1}^{[x/\theta]} + \sum_{m=n=1}^{[x/\theta]}\Big)\theta^2 f(n\theta)g(m\theta)\\
&= \sum_{m=1}^{[x/\theta]} \theta^2\Big(\sum_{k=1}^{m} f\big(k\theta\big)\Big)g(m\theta) + \sum_{n=1}^{[x/\theta]} \theta^2\Big(\sum_{k=1}^{n} g\big(k\theta\big)\Big)f(n\theta) - \sum_{n=1}^{[x/\theta]} \theta^2 f(n\theta)g(n\theta)\\
&= R_\theta\big(R_\theta(f)g\big)(x) + R_\theta\big(fR_\theta(g)\big)(x) + \theta R_\theta(fg)(x).
\end{aligned} \tag{5}$$

Similarly for the map R'_θ. Hence, the Riemann summation maps R_θ and R'_θ satisfy the weight $-\theta$ and the weight θ Rota–Baxter relation, respectively.

Another classical example, and the reason why RB algebras first appeared in fluctuation theory, comes from the operation that associates to the characteristic function of a real valued random variable X the characteristic function of the random variable $max(0, X)$. It is worth pointing out that all these classical examples involve *commutative* RB algebras.

One readily verifies that $\tilde{R} := -\theta \mathrm{id}_A - R$ is a Rota–Baxter operator. Note that

$$R(a)\tilde{R}(b) = \tilde{R}(R(a)b) + R(a\tilde{R}(b)),$$

and similarly exchanging R and $\tilde{R}$. In the following we denote the image of R and $\tilde{R}$ by A_- and A_+, respectively.

Proposition 3.1. *Let (A, R) be a Rota–Baxter algebra. $A_\pm \subseteq A$ are subalgebras in A.*

We omit the proof since it follows directly from the Rota–Baxter relation. A *Rota–Baxter ideal* of a Rota–Baxter algebra (A, R) is an ideal $I \subset A$ such that $R(I) \subseteq I$.

The Rota–Baxter relation extends to the Lie algebra L_A corresponding to A

$$[R(x), R(y)] = R\big([R(x), y] + [x, R(y)]\big) + \theta R\big([x, y]\big)$$

making (L_A, R) into a Rota–Baxter Lie algebra of weight θ. Let us come back to the product we defined after equation (1).

Proposition 3.2. *The vector space underlying A equipped with the product*

$$x *_\theta y := R(x)y + xR(y) + \theta xy \tag{6}$$

is again a Rota–Baxter algebra of weight θ with Rota–Baxter map R. We denote it by (A_θ, R) and call it double Rota–Baxter algebra.

Proof. Let $x, y, z \in A$. We first show associativity

$$\begin{aligned} x *_\theta (y *_\theta z) &= xR\big(yR(z) + R(y)z + \theta yz\big) + R(x)\big(yR(z) + R(y)z + \theta yz\big) \\ &\quad + \theta x\big(yR(z) + R(y)z + \theta yz\big) \\ &= xR(y)R(z) + R(x)yR(z) + \theta xyR(z) + R(x)R(y)z \\ &\quad + \theta xR(y)z + \theta R(x)yz + \theta^2 xyz \\ &= (x *_\theta y) *_\theta z. \end{aligned}$$

Now we show that the original Rota–Baxter map $R \in \mathrm{End}(A)$ also fulfills the Rota–Baxter relation with respect to the $*_\theta$-product.

$$\begin{aligned} R(x) *_\theta R(y) - \theta R(x *_\theta y) &= R^2(x)R(y) + R(x)R^2(y) + \theta R(x)R(y) - \theta R(x)R(y) \\ &= R\big(x *_\theta R(y) + R(x) *_\theta y\big). \end{aligned}$$

□

We used the following homomorphism property of the Rota–Baxter map between the algebras A_θ and A.

Lemma 3.1. *Let (A, R) be a Rota–Baxter algebra of weight θ. The Rota–Baxter map R becomes a (not necessarily unital even if A is unital) algebra homomorphism from the algebra A_θ to A*

$$R\big(a *_\theta b\big) = R(a)R(b). \tag{7}$$

For $\tilde{R} := -\theta \mathrm{id}_A - R$ we find

$$\tilde{R}\big(a *_\theta b\big) = -\tilde{R}(a)\tilde{R}(b). \tag{8}$$

We remark here that if R is supposed to be idempotent, (A, R) must be a Rota–Baxter algebra of unital weight $\theta = -1$. Now we introduce the notion of an associator for $x, y, z \in B$, where B is an arbitrary algebra:

$$a_{.}(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z).$$

Recall that a left *pre-Lie algebra* P is a vector space, together with a bilinear pre-Lie product $\triangleright : P \otimes P \to P$, satisfying the left pre-Lie relation

$$a_\triangleright(x, y, z) = a_\triangleright(y, x, z).$$

With an obvious analog notion of a right pre-Lie product $\triangleleft : P \otimes P \to P$ and right pre-Lie relation

$$a_\triangleleft(x, y, z) = a_\triangleleft(x, z, y).$$

See [7] for more details. Let P be a left (or right) pre-Lie algebra. The commutator $[a, b]_\triangleright := a \triangleright b - b \triangleright a$ for $a, b \in P$ satisfies the Jacobi identity. Hence, the vector space P together with this commutator is a Lie algebra, denoted by L_P. Of course, every associative algebra is also pre-Lie.

Lemma 3.2. *Let (A, R) be an associative Rota–Baxter algebra of weight θ. The binary compositions*

$$a \triangleright_\theta b := R(a)b - bR(a) - \theta ba = [R(a), b] - \theta ba = R(a)b - \big(-b\tilde{R}(a)\big), \tag{9}$$

$$a \triangleleft_\theta b := aR(b) - R(b)a - \theta ba = [a, R(b)] - \theta ba = aR(b) - \big(-\tilde{R}(b)a\big), \tag{10}$$

define a left respectively right pre-Lie structure on A.

Proof. The Lemma follows by direct inspection. It may also be deduced from deeper properties of Rota–Baxter algebras related to dendriform di- and trialgebras [1, 16, 29, 30]. That is, the identification $a \prec b := aR(b)$, $a \bullet b = \theta ab$ and $a \succ b := R(a)b$ in a Rota–Baxter algebra (B, R) defines a dendriform trialgebra, hence also a dialgebra structure. We refer the reader to [23] for more. □

Recall that anti-symmetrization of a pre-Lie product gives a Lie bracket. In the case of the Rota–Baxter pre-Lie compositions (9,10), we find

$$\begin{aligned} [a,b]_{\rhd_\theta} &:= a \rhd_\theta b - b \rhd_\theta a = a \lhd_\theta b - b \lhd_\theta a = [a,b]_{\lhd_\theta} \\ &= [R(a),b] + [a,R(b)] + \theta[a,b] = [a,b]_{*_\theta}. \end{aligned}$$

Hence, the double Rota–Baxter product and the left as well as right Rota–Baxter pre-Lie products define the same Lie bracket on (A, R).

Lemma 3.3. *Let (A, R) be an associative Rota–Baxter algebra of weight θ. The left pre-Lie algebra $(A, \rhd_\theta)$ is a Rota–Baxter left pre-Lie algebra of weight θ, with Rota–Baxter map R. Similarly for $(A, \lhd_\theta)$ being a Rota–Baxter right pre-Lie algebra of weight θ.*

Proof. We prove only the statement for the left RB pre-Lie algebra. Let $x, y \in A$.

$$\begin{aligned} R(x) \rhd_\theta R(y) &= R\big(R(x)\big)R(y) - R(y)R\big(R(x)\big) - \theta R(y)R(x) \\ &= R\big(R(R(x))y + R(x)R(y) + \theta R(x)y\big) \\ &\quad -R\big(yR\big(R(x)\big) + R(y)R(x) + \theta yR(x)\big) - \theta R\big(R(y)x + yR(x) + \theta yx\big) \\ &= R\big(R(x) \rhd_\theta y + x \rhd_\theta R(y) + \theta x \rhd_\theta y\big). \end{aligned}$$

□

Let us now turn to the Bogoliubov recursion. Briefly, this recursion provides an elaborate procedure to give a sense (or to renormalize, that is, to associate a finite quantity, called the renormalized amplitude) to divergent integrals appearing in perturbative high-energy physics calculations. The renormalization process and in particular the Bogoliubov recursion can be reformulated in purely algebraic terms inside the Connes–Kreimer paradigmatic Hopf algebraic approach to perturbative renormalization. We follow this point of view and refer the reader to Collins' monograph [9] and the by now standard references [10, 11, 12, 24, 34] for further information on the subject. Further details on the physical meaning of the recursion will be given in the last section of the article, we concentrate for the time being on its mathematical significance.

Let us outlay the general setting, following [22]. Let H be a graded connected commutative Hopf algebra (in the physical setting H would be a Hopf algebra of Feynman diagrams), and let A be a commutative unital algebra. Assume further that A splits into a direct sum of subalgebras, $A = A_+ \bigoplus A_-$, with $1 \in A_+$. The projectors to $A_\pm$ are written $R_\pm$ respectively. The pair (A, R_-) is then a weight $\theta = -1$ commutative Rota–Baxter algebra, whereas the algebra $Lin(H, A)$ with the idempotent operator defined by $\mathcal{R}_-(f) := R_- \circ f$ for $f \in Lin(H, A)$ is a (in general noncommutative) unital Rota–Baxter algebra. Here, the algebra structure on $Lin(H, A)$ is induced by the convolution product, that is, for any $f, g \in Lin(H, A)$ and any $h \in H$:

$$f \star g(h) := f(h^{(1)})g(h^{(2)}),$$

with Sweedler's notation for the action of the coproduct δ of H on h: $\delta(h) = h^{(1)} \otimes h^{(2)}$.

The essence of renormalization is contained in the existence of a decomposition of the group $G(A)$ of algebra maps from H to A into a (set theoretic) product of the groups $G_-(A)$ and $G_+(A)$ of algebra maps from H^+ to A_-, respectively from H to A_+. We view $G_-(A)$ as a subgroup of $G(A)$ by extending maps γ from H^+ to A_- to maps from H to A by requiring that $\gamma(1) = 1$. In concrete terms, any element γ of $G(A)$ can be rewritten uniquely as a product $\gamma_-^{-1} \star \gamma_+$, where $\gamma_- \in G_-(A)$ and $\gamma_+ \in G_+(A)$.

The Bogoliubov recursion is a process allowing the inductive construction of the elements γ_- and γ_+ of the aforementioned decomposition. Writing e_A for the unit of $G(A)$ (the counit map of H composed with the unit map of A, $e_A = \eta_A \circ \epsilon$), γ_- and γ_+ solve the equations:

$$\gamma_\pm = e_A \pm \mathcal{R}_\pm\big(\gamma_- \star (\gamma - e_A)\big). \tag{11}$$

The recursion process is by induction on the degree n components of γ_- and γ_+ viewed as elements of the (suitably completed) graded algebra $Lin(H, A)$. The fact that the recursion defines elements of $G_-(A)$ and $G_+(A)$ is not obvious from the definition: since the recursion takes place in $Lin(H, A)$, one would expect $\gamma_- - e_A$ and γ_+ to belong to $Lin(H, A_-)$ and $Lin(H, A_+)$, respectively. The fact that γ_- and γ_+ do belong to $G_-(A)$ and $G_+(A)$, respectively, follows from the Rota–Baxter algebra structure of A, as has been shown by Kreimer and Connes-Kreimer, see e.g. [27, 11] and the references therein. The map

$$\bar{\gamma} := \gamma_- \star (\gamma - e_A) \tag{12}$$

is called *Bogoliubov's preparation* or $\bar{R}$*-operation*. Hence, on H^+ we see that $\gamma_\pm = \pm\mathcal{R}_\pm(\bar{\gamma})$.

Setting $a := -(\gamma - e_A)$, the recursion can be rewritten:

$$\gamma_\pm = e_A \mp \mathcal{R}_\pm(\gamma_- \star a), \tag{13}$$

and can be viewed as an instance of results due to F.V. Atkinson, who, following Baxter's work [3], has made an important observation [2] when he found a multiplicative decomposition for associative unital Rota–Baxter algebras. We will state his result for the ring of power series $B[[t]]$, (B,R) an arbitrary Rota–Baxter algebra. Inductively define in a general RB algebra (B,R),

$$(Ra)^{[n+1]} := R\big((Ra)^{[n]}a\big) \quad \text{and} \quad (Ra)^{\{n+1\}} := R\big(a(Ra)^{\{n\}}\big). \tag{14}$$

with the convention that $(Ra)^{[1]} := R(a) =: (Ra)^{\{1\}}$ and $(Ra)^{[0]} := 1_B =: (Ra)^{\{0\}}$.

Theorem 3.1. *Let (B,R) be an associative unital Rota–Baxter algebra. Fix $a \in B$ and let F and G be defined by $F := \sum_{n\in\mathbb{N}} t^n (Ra)^{[n]}$ and $G := \sum_{n\in\mathbb{N}} t^n (\tilde{R}a)^{\{n\}}$. Then, they solve the equations*

$$F = 1_B + tR(F\ a) \quad \textit{and} \quad G = 1_B + t\tilde{R}(a\ G), \tag{15}$$

in $B[[t]]$ and we have the following factorization

$$F\big(1_B + at\theta\big)G = 1_B, \qquad \textit{so that} \qquad 1_B + at\theta = F^{-1}G^{-1}. \tag{16}$$

For an idempotent Rota–Baxter map this factorization is unique.

Proof. The proof follows simply from calculating the product FG. Uniqueness for idempotent Rota–Baxter maps is easy to show, see for instance [20]. □

One may well ask what equations are solved by the inverses F^{-1} and G^{-1}. We answer this question, the solution of which will be important in forthcoming developments, in the following corollary.

Corollary 3.1. *Let (B,R) be an associative unital Rota–Baxter algebra. Fix $a \in B$ and assume F and G to solve the equations in the foregoing theorem. The inverses F^{-1} and G^{-1} solve the equations*

$$F^{-1} = 1_B - tR(a\ G) \quad \textit{and} \quad G^{-1} = 1_B - t\tilde{R}(F\ a), \tag{17}$$

in $B[[t]]$.

Proof. Let us check this for F and F^{-1}. Recall that $G = 1_B + t\tilde{R}(a\ G)$.

$$\begin{aligned}
FF^{-1} &= 1_B - tR(a\ G) + tR(F\ a) - t^2R(F\ a)R(a\ G)\\
&= 1_B - tR(a\ G) + tR(F\ a) - t^2R\big(R(F\ a)a\ G\big) - t^2R\big(F\ aR(a\ G)\big) - t^2\theta R\big(F\ a^2\ G\big)\\
&= 1_B - tR\big((1_B + tR(F\ a))a\ G\big) + tR(F\ a) + t^2R\big(F\ a\tilde{R}(a\ G)\big)\\
&= 1_B - tR\big((1_B + tR(F\ a))a\ G\big) + tR\big(F\ a(1_B + t\tilde{R}(a\ G))\big)\\
&= 1_B + tR\big(F\ a\ G\big) - tR\big(F\ a\ G\big) = 1_B
\end{aligned}$$

□

Going back to Bogoliubov's recursions (11) we see that γ_- corresponds to the first equation in (15), whereas γ_+ corresponds to the inverse of the second equation in (15), see (17).

The solution to Atkinson's recursion can be simply expressed as follows: the coefficient of t^n in the expansion of F is $(Ra)^{[n]}$. When the Rota–Baxter algebra (A,R) is commutative, the classical Spitzer formulas allow to reexpress and expand these terms, giving rise to non-recursive expansions. The first Spitzer identity, or Pollaczeck–Spitzer identity reads:

$$\sum_{m\in\mathbb{N}} t^m (Ra)^{[m]} = \exp\big(\theta^{-1}R\log(1+at\theta)\big),$$

where a is an arbitrary element in a weight θ Rota–Baxter algebra A [3, 48]. In the framework of the Rota–Smith presentation [45] of the free commutative RB algebra on one generator (the “standard” RB algebra), this becomes the Waring formula relating elementary and power sum symmetric functions [46]. In fact, comparing the coefficient of t^n on both sides, Spitzer's identity says that:

$$n!(Ra)^{[n]} = \sum_{\sigma}(-\theta)^{n-k(\sigma)}R(a^{|\tau_1|})\cdots R(a^{|\tau_{k(\sigma)}|}),$$

where the sum is over all permutations on $[n]$ and $\sigma = \tau_1 \dots \tau_{k(\sigma)}$ is the decomposition of σ into disjoint cycles [45]. We denote by $|\tau_i|$ the number of elements in τ_i. The second Spitzer formula, or *Bohnenblust–Spitzer formula*, follows by polarization [45]:

$$\sum_{\sigma} R\big(R(\dots(R(a_{\sigma_1})a_{\sigma_2}\dots)a_{\sigma_n})\big) = \sum_{\pi\in\mathcal{P}_n}(-\theta)^{n-|\pi|}\prod_{\pi_i\in\pi}(m_i-1)!R(\prod_{j\in\pi_i}a_j)$$

for an arbitrary sequence of elements $a_1,\dots,a_n$ in A. Here, π runs over unordered set partitions $\mathcal{P}_n$ of $[n]$; by $|\pi|$ we denote the number of blocks in π; and $m_i := |\pi_i|$ is the size of the particular block π_i.

In the sequel of the article, we will show how these identities can be generalized to arbitrary (i.e. noncommutative) Rota–Baxter algebras, giving rise to closed formulas for the terms in the Bogoliubov, i.e. Atkinson recursion.

4. Free Rota–Baxter algebras and NCQSym

In the present section, we introduce a model for noncommutative (NC) weight one free RB algebras that extends to the noncommutative setting the notion of *standard Baxter algebra* [45].

Let $X = (x_1,\dots,x_n,\dots)$ be an ordered set of variables, or *alphabet* and $T(X)$ be once again the tensor algebra or free associative algebra over X. Recall that the elements of $T(X)$ are linear combinations of noncommutative products $x_{i_1}\dots x_{i_k}$ of elements of X, or words over X. We shall also consider finite ordered families of alphabets $X^1,\dots,X^n$ and write x^i_n for the elements in $X^i = (x^i_1,\dots,x^i_n,\dots)$. The tensor algebra over $X^1\coprod\cdots\coprod X^n$ is written $T(X^1,\dots,X^n)$.

We write A (resp. $A^{(n)}$) for the algebra of countable sequences $Y = (y_1,\dots,y_n,\dots)$ of elements of $T(X)$ (resp. $T(X^1,\dots,X^n)$) equipped with pointwise addition and products: $(y_1,\dots,y_n,\dots)+(z_1,\dots,z_n,\dots) := (y_1+z_1,\dots,y_n+z_n,\dots)$ and $(y_1,\dots,y_n,\dots)\cdot(z_1,\dots,z_n,\dots) := (y_1\cdot z_1,\dots,y_n\cdot z_n,\dots)$. We also write Y_i for the i-th component of the sequence Y. By a slight abuse of notation, we view X (resp. X^i, $i\le n$) as a sequence, and therefore also as an element of A (resp. of $A^{(n)}$).

Lemma 4.1. *The operator $R\in\mathrm{End}(A)$ (resp. $R\in\mathrm{End}(A^{(n)})$)*

$$R(y_1,\dots,y_n,\dots) := (0, y_1, y_1+y_2,\dots,y_1+\dots+y_n,\dots)$$

defines a weight one RB algebra structure on A (resp. $A^{(n)}$).

Proof. Let us check the formula —in the sequel we will omit some analogous straightforward verifications.

$$\begin{aligned}
&R\big((y_1,\dots,y_n,\dots)\cdot R(z_1,\dots,z_n,\dots)\big) + R\big(R(y_1,\dots,y_n,\dots)\cdot(z_1,\dots,z_n,\dots)\big)\\
&= \Big(0,0,y_2z_1,\dots,\sum_{i=1}^{n-1}y_i(z_1+\dots+z_{i-1}),\dots\Big) + \Big(0,0,y_1z_2,\dots,\sum_{i=1}^{n-1}(y_1+\dots+y_{i-1})z_i,\dots\Big)\\
&= \big(0, y_1z_1-y_1z_1, (y_1+y_2)(z_1+z_2)-(y_1z_1+y_2z_2),\dots\\
&\qquad \dots,(y_1+\dots+y_{n-1})(z_1+\dots+z_{n-1})-(y_1z_1+\dots+y_{n-1}z_{n-1}),\dots\big)\\
&= R(y_1,\dots,y_n,\dots)\cdot R(z_1,\dots,z_n,\dots) - R\big((y_1,\dots,y_n,\dots)\cdot(z_1,\dots,z_n,\dots)\big)
\end{aligned}$$

□

Recall the notation introduced in (14), where we defined inductively $(Ra)^{[n]}$ and $(Ra)^{\{n\}}$. Let us recall also Hivert's notion of quasi-symmetric functions over a set of noncommutative variables from [4, 36]. Let f be a surjective map from $[n]$ to $[k]$, and let X be a countable set of variables, as above. Then, the quasi-symmetric function over X associated to f, written M_f is, by definition,

$$M_f := \sum_{\phi} x_{\phi^{-1}\circ f(1)}\dots x_{\phi^{-1}\circ f(n)},$$

where ϕ runs over the set of increasing bijections between subsets of $\mathbb{N}$ of cardinality k and $[k]$. It is often convenient to represent f as the sequence of its values, $f = (f(1), \dots, f(n))$ or $f = f(1), \dots, f(n)$ in the notation M_f. The definition is best understood by means of an example:

$$M_{1,3,3,2} = x_1x_3x_3x_2 + x_1x_4x_4x_2 + x_1x_4x_4x_3 + x_2x_4x_4x_3 + \dots$$

The linear span $NCQSym$ of the M_f's is a subalgebra of the algebra of noncommutative polynomials over X (up to classical completion arguments that we omit, and that allow to deal with infinite series such as the M_f as if they were usual noncommutative polynomials). It is related to various fundamental objects such as the Coxeter complex of type A_n or the corresponding Solomon–Tits and twisted descent algebras. We refer to [41, 4, 36] for further details on the subject.

We also introduce, for further use, the notation, M_f^n for the image of M_f under the map sending x_i to 0 for $i > n$ and x_i to itself else. For the above example for instance we find

$$M^3_{1,3,3,2} = x_1x_3x_3x_2,$$

$$M^4_{1,3,3,2} = x_1x_3x_3x_2 + x_1x_4x_4x_2 + x_1x_4x_4x_3 + x_2x_4x_4x_3$$

At last, we write $[n]$ for the identity map on $[n]$ and ω_n for the endofunction of $[n]$ reversing the ordering, so that $\omega_n(i) := n - i + 1$ and $M_{\omega_n} = M_{n,n-1,\dots,1}$.

Proposition 4.1. *In the RB algebra A, we have:*

$$(RX)^{[n]} = (0, M^1_{[n]}, M^2_{[n]}, \dots, M^{k-1}_{[n]}, \dots),\ n > 1,$$

where $M^{k-1}_{[n]}$ is at the kth position in the sequence, and

$$(RX)^{\{n\}} = (0, M^1_{\omega_n}, M^2_{\omega_n}, \dots, M^{k-1}_{\omega_n}, \dots),\ n > 1,$$

where $M^{k-1}_{\omega_n}$ is at the kth position in the sequence.

Indeed, let us assume that $M^{k-1}_{[n]}$ is at the kth position in the sequence $(RX)^{[n]}$. Then, the kth component of $(RX)^{[n]} \cdot X$ reads $\sum_{0<i_1<\dots<i_n<k} x_{i_1} \dots x_{i_n} x_k$ and the kth component of $R((RX)^{[n]} \cdot X)$ reads $\sum_{i=1}^{k-1} \sum_{0<j_1<\dots<j_n<i} x_{j_1} \dots x_{j_n} x_i = M^{k-1}_{[n+1]}$. The identity for $(RX)^{\{n\}}$ follows by symmetry.

Corollary 4.1. *The elements $(RX)^{[n]}$ generate freely an associative subalgebra of A. Similarly, the elements $(RX^i)^{[n]}$ generate freely an associative subalgebra of $A^{(n)}$.*

Let us sketch the proof. Noncommutative monomials over X such as the ones appearing in the expansion of $M_{1,3,3,2}$ are naturally ordered by the lexicographical ordering $<_L$, so that, for example, $x_2x_6x_6x_4 <_L x_2x_7x_7x_5$. Let us write, for any noncommutative polynomial P in $T(X)$, $Sup(P)$ for the highest noncommutative monomial for the lexicographical ordering appearing in the expansion of P, so that, for example, for $k \geq n$, $Sup(M^k_{[n]}) = x_{k-n+1} \dots x_k$, or $Sup(x_2x_6x_7 + x_2x_7x_5) = x_2x_7x_5$. For two such polynomials P and Q, we write $P <_L Q$ when $Sup(P) <_L Sup(Q)$.

Let us consider now a noncommutative polynomial Q in the $(RX)^{[n]}$ with non trivial coefficients, and let us prove that it is not equal to 0 in A. For degree reasons, we may first assume that Q is homogeneous, that is, that Q can be written

$$Q = \sum_k \sum_{n_1+\dots+n_k=p} \alpha_{n_1,\dots,n_k} (RX)^{[n_1]} \cdot \dots \cdot (RX)^{[n_k]},$$

Then, the corollary follows from the observation that, for $l \gg p$, $Sup(M^l_{[n_1]} \cdot \dots \cdot M^l_{[n_k]}) > Sup(M^l_{[m_1]} \cdot \dots \cdot M^l_{[m_j]})$ with $n_1 + \dots + n_k = m_1 + \dots + m_j$ if and only if the sequence $(n_1, \dots, n_k)$ is less than the sequence $(m_1, \dots, m_j)$ in the lexicographical ordering. Indeed, let us assume that the two sequences are distinct, and let j be the lowest index such that $m_j \neq n_j$, then $Sup(M^l_{[n_j]}) = x_{l-n_j+1} \dots x_l$ whereas $Sup(M^l_{[m_j]}) = x_{l-m_j+1} \dots x_l$, so that, in particular $l - n_j + 1 > l - m_j + 1$ if and only if $m_j > n_j$. We can then conclude using the following obvious but useful lemma.

Lemma 4.2. *For any x, y homogeneous noncommutative polynomials in $T(X)$ and z, t in $T(X)$, due to the properties of the lexicographical ordering, we have:*

$$x <_L y \Rightarrow x \cdot z <_L y \cdot z \qquad \text{and} \qquad z <_L t \Rightarrow x \cdot z <_L x \cdot t.$$

The proof goes over to $A^{(n)}$, provided one chooses a suitable ordering on the elements of $X^1 \coprod \cdots \coprod X^n$, for example the order extending the order on the X^i and such that $x_m^i < x_n^j$ whenever $i < j$.

Corollary 4.2. *The elements $(RX)^{[n]}$ generate freely an associative subalgebra of A for the double Rota–Baxter product $*$.*

Here, we abbreviate the notation for the weight one double product $*_1$ to $*$. The same assertion (and its proof) holds *mutatis mutandis* for the $(RX^i)^{[n]}$ and $A^{(n)}$.

The corollary follows from the previous lemma and the observation that, for any sequence $(n_1, \ldots, n_k)$, $Sup(\{(RX)^{[n_1]} \cdot \cdots \cdot (RX)^{[n_k]}\}_l) = Sup(\{(RX)^{[n_1]} * \cdots * (RX)^{[n_k]}\}_l)$, where the lower l indicates the order of the component in the sequences.

Let us, once again, sketch the proof that relies on the usual properties of the lexicographical ordering and the definition of $(RX)^{[n]}$. Notice first that, for $k \gg n$, $Sup(R((RX)^{[n]})_k) < Sup((RX)^{[n]}_k)$. Indeed, $(RX)^{[n]}_k = M^{k-1}_{[n]}$, whereas

$$R((RX)^{[n]})_k = (RX)^{[n]}_1 + \cdots + (RX)^{[n]}_{k-1} = M^1_{[n]} + \cdots + M^{k-2}_{[n]}$$

so that

$$\begin{aligned} Sup\big(R((RX)^{[n]})_k\big) &= Sup\big(M^{k-2}_{[n]}\big) \\ &= x_{k-n-1} \ldots x_{k-2} <_L x_{k-n} \ldots x_{k-1} = Sup\big(M^{k-1}_{[n]}\big) = Sup\big(((RX)^{[n]})_k\big) \end{aligned}$$

The same argument shows that, more generally, for any element $Y = (y_1, \ldots, y_n, \ldots)$ in A satisfying the lexicographical growth condition $Sup(y_i) <_L Sup(y_{i+1})$ for all $i \in \mathbb{N}^*$, we have $Sup(y_{i-1}) = Sup(R(Y)_i) <_L Sup(Y_i) = Sup(y_i)$ for all $i \in \mathbb{N}^*$. This property is therefore stable under the map R and is (up to neglecting the zero entries in the sequences) common to all the elements we are going to consider. It applies in particular to $(RX)^{[n_1]} \cdot \cdots \cdot (RX)^{[n_k]}$ and $(RX)^{[n_1]} * \cdots * (RX)^{[n_k]}$; the verification follows from the same line of arguments and is left to the reader.

In the end, we have:

$$\begin{aligned} (RX)^{[n_1]} * \cdots * (RX)^{[n_k]} &= (RX)^{[n_1]} \cdot \big((RX)^{[n_2]} * \cdots * (RX)^{[n_k]}\big) \\ &\quad + R\big((RX)^{[n_1]}\big) \cdot ((RX)^{[n_2]} * \cdots * (RX)^{[n_k]}) \\ &\quad + (RX)^{[n_1]} \cdot R\big((RX)^{[n_2]} * \cdots * (RX)^{[n_k]}\big) \end{aligned}$$

from which we deduce by recursion on k and for $l \gg 0$:

$$\begin{aligned} Sup\big(((RX)^{[n_1]} * \cdots * (RX)^{[n_k]})_l\big) &= Sup\big(((RX)^{[n_1]} \cdot ((RX)^{[n_2]} * \cdots * (RX)^{[n_k]}))_l\big) \\ &= Sup\big(((RX)^{[n_1]} \cdot \cdots \cdot (RX)^{[n_k]})_l\big) \end{aligned}$$

Let us conclude by proving that these constructions give rise to a model for free Rota–Baxter algebras.

Theorem 4.1. *The RB subalgebra $\mathcal{R}$ of A generated by X is a free RB algebra on one generator. More generally, the RB subalgebra $\mathcal{R}^{(n)}$ of $A^{(n)}$ generated by $X^1, \ldots, X^n$ is a free RB algebra on n generators.*

Our proof is inspired by the one in Rota–Smith [45], but the adaptation to the noncommutative setting requires some care.

Let us write $\mathcal{F}$ for the free noncommutative RB algebra on one generator Y, so that, by the universal properties of free algebras, the map sending Y to X induces a surjective RB map from $\mathcal{F}$ to $\mathcal{R}$ (recall that the latter is generated by X).

Let us call End-algebra any associative algebra V provided with a linear endomorphism $T_V \in \mathrm{End}(V)$, with the obvious notion of morphisms of End-algebras, so that a End-algebra morphism f from V to W satisfies $f \circ T_V = T_W \circ f$. Let us write $\mathcal{L}$ for the free End-algebra on one generator. We now write Z for the generator: the elements of $\mathcal{L}$ are linear combinations of all the symbols obtained from Z by iterative applications of the endomorphism T and of the associative product. The elements look like $ZT^2(T(Z)T^3(Z))$, and so on. We write M for the set of these symbols and call them $\mathcal{L}$-monomials.

A RB algebra B is a End-algebra together with extra (Rota–Baxter) relations on $T_B = R$. In particular, there is a unique natural End-algebra map from $\mathcal{L}$ to an arbitrary RB algebra on one generator (mapping

Z to that generator) and in particular a unique map to $\mathcal{F}$ and $\mathcal{R}$ sending Z to Y, resp. Z to X. The map to $\mathcal{F}$ factorizes the map to $\mathcal{R}$.

Proving that $\mathcal{F}$ and $\mathcal{R}$ are isomorphic as RB algebras, that is, that $\mathcal{R}$ is a free RB algebra on one generator amounts to prove that the kernel —say $Ker(F)$— of the map from $\mathcal{L}$ to $\mathcal{F}$ is equal to the kernel —say $Ker(U)$— of the map from $\mathcal{L}$ to $\mathcal{R}$.

For any $l \in \mathcal{L}$, which can be written uniquely as a linear combination of $\mathcal{L}$-monomials, we write $Max(l)$ for the maximal number of Ts occurring in the monomials (with the obvious conventions for the powers of T, so that for example: $Max(ZT^2(ZT(Z)) + Z^3T^2(Z)Z) = 3$).

We say that an element α of M is *elementary* if and only if it can be written either Z^i, $i \geq 0$ or as a product $Z^{i_1} \cdot T(b_1) \cdot Z^{i_2} \cdot \dots \cdot T(b_k) \cdot Z^{i_{k+1}}$, where the b_i's are elementary and $i_2, \dots, i_k$ are strictly positive integers (i_1 and i_{k+1} may be equal to zero); the definition of elementariness makes sense by induction on $Max(\alpha)$.

Lemma 4.3. *Every $l \in \mathcal{L}$ is of the form $l = r + s$ where $F(s) = 0$ and r is a sum of elementary monomials.*

It is enough to prove the lemma for $l = t \in M$. If t is not elementary, then t has at least in its expansion a product of two consecutive factors of the form $T(c) \cdot T(d)$. However, since $\mathcal{L}$ is a Rota–Baxter algebra, the relation

$$T(c \cdot T(d) + T(c) \cdot d + c \cdot d) - T(c) \cdot T(d) \in Ker(L)$$

holds, and t can be rewritten, up to an element in $Ker(L)$, by substituting $T(c \cdot T(d) + T(c) \cdot d + c \cdot d)$ to $T(c) \cdot T(d)$ in its expansion. Notice that $Max(c \cdot T(d) + T(c) \cdot d + c \cdot d) < Max(T(c) \cdot T(d))$. The proof follows by a joint induction on the number of such consecutive factors and on $Max(T(c) \cdot T(d))$. In other words, products $T(c) \cdot T(d)$ can be iteratively cancelled from the expression of t using the RB fundamental relation.

Let us show now that, with the notation of the lemma, $U(r) = 0$ implies $F(r) = 0$, from which, since we already know that $Ker(F) \subset Ker(U)$, the freeness property will follow.

We actually claim the stronger property that, for p large enough and for $\mu \neq \mu'$ elementary, $Sup(U(\mu)_p) \neq Sup(U(\mu')_p)$, from which the previous assertion will follow. Indeed, if $\mu = (Z^{i_1} \cdot T(b_1) \cdot Z^{i_2} \cdot \dots \cdot T(b_k) \cdot Z^{i_{k+1}})$ with the b_i elementary, then

$$\begin{aligned} Sup(U(\mu)_p) &= x_p^{i_1}(Sup(U(T(b_1)))_p)x_p^{i_2} \dots x_p^{i_k}(Sup(U(T(b_k)))_p)x_p^{i_{k+1}} \\ &= \quad x_p^{i_1}(Sup(R(U(b_1)))_p)x_p^{i_2} \dots x_p^{i_k}(Sup(R(U(b_k)))_p)x_p^{i_{k+1}} \end{aligned}$$

the last identity follows since U is an End-algebra map. Since $U(\tau)$ for τ elementary satisfies the lexicographical growth condition (as may be checked by induction), we have $Sup(R(U(\tau))_i) = Sup(U(\tau)_{i-1})$, so that

$$Sup(U(\mu)_p) = x_p^{i_1}(Sup(U(b_1)))_{p-1})x_p^{i_2} \dots x_p^{i_k}(Sup((U(b_k))_{p-1})x_p^{i_{k+1}}.$$

The proof follows by induction on $Max(\mu)$.

The proof goes over to an arbitrary number of generators, provided one defines the suitable notion of elementary monomials in the free End-algebra $\mathcal{L}^{(n)}$ on n generators $Z_1, \dots, Z_n$: these are the elements of $\mathcal{L}^{(n)}$ that can be written either as noncommutative monomials in the Z_i, or as a product $a_1T(b_1)a_2 \dots T(b_n)a_{n+1}$, where the b_i are elementary and the a_i noncommutative monomials in the Z_i (nontrivial whenever $1 < i < n+1$).

Notice the following interesting corollary of our previous computations.

Corollary 4.3. *The images of the elementary monomials of $\mathcal{L}$ in $\mathcal{R}$ form a basis (as a vector space) of the free Rota–Baxter algebra on one generator.*

The same assertion holds for the free Rota–Baxter algebra on n generators.

5. Rota–Baxter algebras and NCSF

In the present section, we associate to the free RB algebra on one generator a Hopf algebra naturally isomorphic to the Hopf algebra of noncommutative symmetric functions or, equivalently, to the descent algebra. The reasons for the introduction of this Hopf algebra will become clear in the next section. Let us simply mention at this stage that these notions will provide the right framework to extend to noncommutative

RB algebras and noncommutative symmetric functions the classical results of Rota and Smith relating commutative RB algebras and symmetric functions [45].

Recall that the algebra $NCQSym$ of quasi-symmetric functions in noncommutative variables introduced in the previous section is naturally provided with a Hopf algebra structure [4]. On the elementary quasi-symmetric functions $M_{[n]}$, the coproduct Δ acts as on a sequence of divided powers

$$\Delta\big(M_{[n]}\big) = \sum_{i=0}^{n} M_{[i]} \otimes M_{[n-i]}.$$

The same argument as in the previous section shows that the $M_{[n]}$ generate a free subalgebra of $NCQSym$. In the end, the $M_{[n]}$s form a sequence of divided powers in a free associative sub-algebra of $NCQSym$, and this algebra is isomorphic to the algebra of NCSF (which is, by its very definition a Hopf algebra freely generated as an associative algebra by a sequence of divided powers [25], and therefore is naturally isomorphic to the descent algebra: see Prop. 2.1 and the description of the Hopf algebra structure on the descent algebra in the same section).

The same construction of a Hopf algebra structure goes over to the algebras introduced in the previous section, that is, to the free algebras over the $(RX)^{[n]}$ for the $\cdot$ and $*$ products. As a free algebra over the $(RX)^{[n]}$, the first algebra is naturally provided with a cocommutative Hopf algebra structure for which the $(RX)^{[n]}$s form a sequence of divided powers, that is

$$\Delta\big((RX)^{[n]}\big) = \sum_{0\leq m\leq n} (RX)^{[m]} \otimes (RX)^{[n-m]}. \tag{18}$$

This is the structure inherited from the Hopf algebra structure on $NCQSym$. We will be particularly interested in this Hopf algebra, that is the algebra freely generated by the $(RX)^{[n]}$ for the $\cdot$ product, viewed as a subalgebra of A and as a Hopf algebra. We call it the *free noncommutative Spitzer (Hopf) algebra on one generator* or, for short, the Spitzer algebra, and write it $\mathcal{S}$.

When dealing with the Rota–Baxter double product, $*$, the right subalgebra to consider, as will appear below, is not the free algebra generated by the $(RX)^{[n]}$ but the free algebra freely generated by the $(RX)^{[n]}\cdot X$. We may also consider this algebra as a Hopf algebra by requiring the free generators to form a sequence of divided powers, that is by defining the coproduct by

$$\Delta_*\big((RX)^{[n-1]}\cdot X\big) = (RX)^{[n-1]}\cdot X\otimes 1 + \sum_{m\leq n-2} (RX)^{[m]}\cdot X\otimes (RX)^{[n-m-2]}\cdot X + 1\otimes (RX)^{[n-1]}\cdot X \tag{19}$$

We will also investigate briefly this second structure, strongly related with the Hopf algebra structure on free dendriform dialgebras of [8, 43]. We call it the *double Spitzer algebra* —and write it $\mathcal{C}$.

To understand first the structure of the Spitzer algebra, $\mathcal{S}$, recall that we write $\tilde{R}$ for $-\theta\mathrm{id} - R$ and $(\tilde{R}a)^{\{n\}}$ respectively $(\tilde{R}a)^{[n]}$ for the corresponding iterated operator.

Lemma 5.1. *The action of the antipode S in the Spitzer algebra, $\mathcal{S}$, is given by*

$$S\big((RX)^{[n]}\big) = -R\big(X\cdot(\tilde{R}X)^{\{n-1\}}\big).$$

Indeed, the Spitzer algebra is naturally a graded Hopf algebra. The series $F := \sum_{n\geq 0}(RX)^{[n]}$ is a group-like element in the Hopf algebra. The inverse series follows from Atkinson's formula 3.1, and gives the action of the antipode on the terms of the series. Since

$$F^{-1} = 1 - R\big(X\cdot(\sum_{n\geq 0}(\tilde{R}X)^{\{n\}})\big),$$

the corollary follows.

Corollary 5.1. *The action of the antipode S in the double Spitzer algebra, $\mathcal{C}$, is given by*

$$S\big((RX)^{[n]}\cdot X\big) = -\big(X\cdot(\tilde{R}X)^{\{n\}}\big).$$

The proof will illustrate the links between the two Hopf algebras, $\mathcal{S}$ and $\mathcal{C}$. Recall that, on any RB algebra, we have, by the very definition of the $*_\theta$ product

$$R(x)\cdot R(y) = R(x *_\theta y) \quad \text{and} \quad \tilde{R}(x)\cdot\tilde{R}(y) = -\tilde{R}(x *_\theta y).$$

Recall also that, in the algebra of series A (in which the Spitzer algebra and the double Spitzer algebra can be embedded), the operator R can be inverted on the left —that is, if $Y = (0, y_1, y_2, \dots, y_n, \dots) = R(U)$, then $U = (y_1, y_2 - y_1, y_3 - y_2, \dots)$.

It follows from this observation that the RB operator R induces an isomorphism of free graded algebras between the double Spitzer algebra and the Spitzer algebra. That is, for any sequence of integers $i_1, \dots, i_k$, we have

$$R\Big(\big((RX)^{[i_1]} \cdot X\big) * \big((RX)^{[i_2]} \cdot X\big) * \cdots * \big((RX)^{[i_k]} \cdot X\big)\Big) = (RX)^{[i_1+1]} \cdot (RX)^{[i_2+1]} \cdot \dots \cdot (RX)^{[i_k+1]}$$

The isomorphism is extended by the identity to the scalar (that is to the zero degree components of the two Hopf algebras).

Since this isomorphism maps the generators of $\mathcal{C}$ to the generators of $\mathcal{S}$, and since both families of generators form a sequence of divided powers in their respective Hopf algebras, we obtain that, writing again S for the antipode in $\mathcal{C}$

$$\begin{aligned} R\Big(S\big(\sum_{n>0}(RX)^{[n-1]} \cdot X\big)\Big) &= R\Big(\big(\sum_{n>0}(RX)^{[n-1]} \cdot X\big)^{-1}\Big) \\ &= \big(\sum_{n>0}(RX)^{[n]}\big)^{-1} = \sum_{n>0} -R\big(X \cdot (\tilde{R}X)^{\{n-1\}}\big) \end{aligned}$$

so that, eventually

$$S\big((RX)^{[n]} \cdot X\big) = -X \cdot (\tilde{R}X)^{\{n\}}.$$

Corollary 5.2. *The free $*$ subalgebras of A generated by the $(RX)^{[n]} \cdot X$ and by the $X \cdot (\tilde{R}X)^{\{n\}}$ identify canonically. The antipode exchanges the two families of generators. In particular, the $X \cdot (\tilde{R}X)^{\{n\}}$ also form a sequence of divided powers in the double Spitzer algebra.*

6. The Bohnenblust–Spitzer formula and the Dynkin idempotent

As already alluded at, one surprising conclusion of the present article is that *the same machinery* that one uses to derive fundamental identities in the theory of free Lie algebras can be used to recover the already known formulas for commutative Rota–Baxter algebras but, moreover, can be used to prove new formulas in the noncommutative setting. These results rely on the computation of the action of the Dynkin operator on the generators of the Spitzer and of the double Spitzer Hopf algebras.

Let us now introduce the definition of the iterated Rota–Baxter left and right pre-Lie brackets in a RB algebra (B, R) of weight θ.

$$\mathfrak{l}_\theta^{(n)}(a_1, \dots, a_n) := \Big(\cdots\big((a_1 \rhd_\theta a_2) \rhd_\theta a_3\big) \cdots \rhd_\theta a_{n-1}\Big) \rhd_\theta a_n \tag{20}$$

$$\mathfrak{r}_\theta^{(n)}(a_1, \dots, a_n) := a_1 \lhd_\theta \Big(a_2 \lhd_\theta \big(a_3 \lhd_\theta \cdots (a_{n-1} \lhd_\theta a_n)\big) \cdots\Big) \tag{21}$$

for $n > 0$ and $\mathfrak{l}_\theta^{(1)}(a) := a =: \mathfrak{r}_\theta^{(1)}(a)$. For fixed $a \in B$ we can write compactly for $n > 0$

$$\mathfrak{l}_\theta^{(n+1)}(a) = \big(\mathfrak{l}_\theta^{(n)}(a)\big) \rhd_\theta a \quad \text{and} \quad \mathfrak{r}_\theta^{(n+1)}(a) = a \lhd_\theta \big(\mathfrak{r}_\theta^{(n)}(a)\big). \tag{22}$$

We call those expressions left respectively right RB pre-Lie words. Let us now define

$$\mathfrak{L}_\theta^{(n+1)}(a) := R\big(\mathfrak{l}_\theta^{(n+1)}(a)\big) \quad \text{and} \quad \mathfrak{R}_\theta^{(n+1)}(a) := R\big(\mathfrak{r}_\theta^{(n+1)}(a)\big). \tag{23}$$

For B commutative $\mathfrak{L}_\theta^{(n)}(a) = (-\theta)^{n-1}R(a^n)$ and $\mathfrak{R}_\theta^{(n)}(a) = (-\theta)^{n-1}R(a^n)$. For (B, R) being of weight $\theta = 0$ the left (right) pre-Lie product (9) reduces to $a \rhd_0 b = ad_{R(a)}(b)$ (and $a \lhd_0 b = -ad_{R(b)}(a)$), so that

$$\mathfrak{L}_0^{(n+1)}(a) = R\Big(\big[R(\cdots\big[R([R(a), a]), a\big]\cdots), a\big]\Big) = -R\Big(ad_a(\mathfrak{L}_0^{(n)}(a))\Big) \tag{24}$$

for $n > 0$ and analogously for the right RB pre-Lie words. Now, in the context of the weight $\theta = 1$ Rota–Baxter algebra (A, R) we find the following proposition.

Proposition 6.1. *The action of the Dynkin operator on the generators $(RX)^{[n]}$ of the Spitzer algebra $\mathcal{S}$ (respectively on the generators of the double Spitzer algebra $\mathcal{C}$) is given by:*

$$D\big((RX)^{[n]}\big) = \mathfrak{L}_1^{(n)}(X) = R(\mathfrak{l}_1^{(n)}(X))$$

(respectively by $D((RX)^{[n]} \cdot X) = \mathfrak{l}_1^{(n+1)}(X)$).

Due to the existence of the Hopf algebra isomorphism induced by the map R between the Spitzer algebra and the double Spitzer algebra, the two assertions are equivalent. Let us prove the proposition by induction on n for the double Spitzer algebra. We denote by π_* the product on $\mathcal{C}$. Using $Y(1)=0$ we find for $n=0$

$$D(X)=(S\star Y)(X)=\pi_*\circ(S\otimes Y)(X\otimes 1+1\otimes X)=X=\mathfrak{l}_1^{(1)}(X).$$

Recall that $Y(X)=X$ and $Y((RX)^{[n-1]}X)=n\ (RX)^{[n-1]}X=Y((RX)^{[n-1]})\cdot X+(RX)^{[n-1]}\cdot X$. Let us also introduce a useful notation and write $(RX)^{[n-1]}\cdot X=:w^{(n)}$, so that $w^{(n+1)}=R(w^{(n)})\cdot X$, $w^{(1)}=1$ and $w^{(0)}=1$. We obtain

$$\begin{aligned}
D(w^{(n)}) &= D\big((RX)^{[n-1]}\cdot X\big)=(S\star Y)\big(w^{(n)}\big)\\
&= \pi_*\circ(S\otimes Y)\sum_{p=0}^{n}w^{(p)}\otimes w^{(n-p)}=\sum_{p=0}^{n-1}S\big(w^{(p)}\big)*Y\big(R(w^{(n-1-p)})\cdot X\big)\\
&= \sum_{p=0}^{n-1}S\big(w^{(p)}\big)*\big(Y\big(R(w^{(n-1-p)})\big)\cdot X\big)+\sum_{p=0}^{n-1}S\big(w^{(p)}\big)*\big(R(w^{(n-1-p)})\cdot X\big)\\
&= \sum_{p=0}^{n-1}S\big(w^{(p)}\big)*\big(Y\big(R(w^{(n-1-p)})\big)\cdot X\big)+\sum_{p=0}^{n}S\big(w^{(p)}\big)*w^{(n-p)}-S\big(w^{(n)}\big)\\
&= \sum_{p=1}^{n-1}S\big(w^{(p)}\big)*\big(Y\big(R(w^{(n-1-p)})\big)\cdot X\big)+Y\big(R(w^{(n-1)})\big)\cdot X-S\big(w^{(n)}\big)
\end{aligned}$$

where, since the antipode S is the convolution inverse of the identity, the term $(S\star\mathrm{id})(w^{(n)})$ on the right hand side cancels. Hence, using the general RB identity

$$a*\big(R(b)c\big)=R(a)R(b)c-a\tilde{R}\big(R(b)c\big)=R\big(a*b\big)c-a\tilde{R}\big(R(b)c\big)$$

we find immediately

$$\begin{aligned}
D\big(w^{(n)}\big) &= \sum_{p=1}^{n-1}R\Big(S\big(w^{(p)}\big)*Y(w^{(n-1-p)})\Big)\cdot X-\sum_{p=1}^{n-1}S\big(w^{(p)}\big)\cdot\tilde{R}\Big(R\big(Y(w^{(n-1-p)})\big)\cdot X\Big)\\
&\quad +Y\big(R(w^{(n-1)})\big)\cdot X-S\big(w^{(n)}\big)\\
&= \sum_{p=0}^{n-1}R\Big(S\big(w^{(p)}\big)*Y(w^{(n-1-p)})\Big)\cdot X-\sum_{p=1}^{n-1}S\big(w^{(p)}\big)\cdot\tilde{R}\Big(R\big(Y(w^{(n-1-p)})\big)\cdot X\Big)-S\big(w^{(n)}\big)\\
&= R\big((S\star Y)(w^{(n-1)})\big)\cdot X-S\big(w^{(n)}\big)-\sum_{p=1}^{n-1}X\cdot\tilde{R}\Big(S\big(w^{(p-1)}\big)\Big)\cdot\tilde{R}\Big(R\big(Y(w^{(n-1-p)})\big)\cdot X\Big)
\end{aligned}$$

where we used that

$$S\big(w^{(p)}\big)=S\big((RX)^{[p-1]}\cdot X\big)=-X\cdot(\tilde{R}X)^{\{p-1\}}=-X\cdot\tilde{R}\big(X\cdot(\tilde{R}X)^{\{p-2\}}\big)=X\cdot\tilde{R}\big(S(w^{(p-1)})\big).$$

Recall that $\tilde{R}$ is a Rota–Baxter operator as well, such that $\tilde{R}(a*_\theta b)=-\tilde{R}(a)\tilde{R}(b)$. This leads to

$$\begin{aligned}
D\big(w^{(n)}\big) &= R\big(D(w^{(n-1)})\big)\cdot X-S\big(w^{(n)}\big)+\sum_{p=1}^{n-1}X\cdot\tilde{R}\Big(S\big(w^{(p-1)}\big)*\big(Y\big(R(w^{(n-1-p)})\big)\cdot X\big)\Big)\\
&= R\big(\mathfrak{l}_1^{(n-1)}\big)\big)\cdot X-X\cdot\tilde{R}\big(S\big(w^{(n-1)}\big)+\sum_{p=1}^{n-1}X\cdot\tilde{R}\Big(S\big(w^{(p-1)}\big)*\big(Y\big(R(w^{(n-1-p)})\cdot X\big)\Big)\\
&\quad -\sum_{p=1}^{n-1}X\cdot\tilde{R}\Big(S\big(w^{(p-1)}\big)*\big(R(w^{(n-1-p)})\cdot Y(X)\big)\Big)\\
&= R\big(\mathfrak{l}_1^{(n-1)}(X)\big)\cdot X+\sum_{p=0}^{n-2}X\cdot\tilde{R}\Big(S\big(w^{(p)}\big)*Y\big(w^{(n-1-p)}\big)\Big)-\sum_{p=0}^{n-1}X\cdot\tilde{R}\Big(S\big(w^{(p)}\big)*w^{(n-1-p)}\Big)
\end{aligned}$$

$$\begin{aligned}
&= R\big(\mathfrak{l}_1^{(n-1)}(X)\big)\cdot X + \sum_{p=0}^{n-1} X\cdot \tilde{R}\Big(S\big(w^{(p)}\big) * Y\big(w^{(n-1-p)}\big)\Big) - X\cdot\tilde{R}\big((S\star\mathrm{id})(w^{(n-1)})\big)\\
&= R\big(\mathfrak{l}_1^{(n-1)})\big)\cdot X + X\cdot\tilde{R}\big((S\star Y)(w^{(n-1)})\big)\\
&= R\big(\mathfrak{l}_1^{(n-1)}(X)\big)\cdot X + X\cdot\tilde{R}\big(D(w^{(n-1)})\big)\\
&= R\big(\mathfrak{l}_1^{(n-1)}(X)\big)\cdot X + X\cdot\tilde{R}\big(\mathfrak{l}_1^{(n-1)}(X)\big) = \mathfrak{l}_1^{(n-1)}(X)\rhd_1 X = \mathfrak{l}_1^{(n)}(X)
\end{aligned}$$

We used once again that the antipode S is the convolution inverse of the identity, implying that $-X\cdot\tilde{R}\big((S\star\mathrm{id})(w^{(n-1)})\big)=0$. All this immediately implies the following important theorem.

Theorem 6.1. *We have the following identity in the Spitzer algebra $\mathcal{S}$:*

$$(RX)^{[n]} = \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{\mathfrak{L}_1^{(i_1)}(X)\cdot\ \cdots\ \cdot\mathfrak{L}_1^{(i_k)}(X)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)} \tag{25}$$

The theorems follows readily from Theorem 2.1 by applying the formula for the inverse of the Dynkin operator in Proposition 6.1. We obtain the equivalent expansion in the double Spitzer algebra $\mathcal{C}$

Corollary 6.1. *We have, in the double Spitzer algebra $\mathcal{C}$:*

$$(RX)^{[n-1]}\cdot X = \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{\mathfrak{l}_1^{(i_1)}(X)*\cdots*\mathfrak{l}_1^{(i_k)}(X)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}.$$

The reader should have no problem to verify the following statements.

Corollary 6.2. *We have:*

$$\begin{aligned}
(RX)^{\{n\}} &= \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{\mathfrak{R}_1^{(i_k)}(X)\cdot\ \cdots\ \cdot\mathfrak{R}_1^{(i_1)}(X)}{i_1(i_1+i_2)\ldots(i_1+\cdots+i_k)}\\
X\cdot(RX)^{\{n-1\}} &= \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{\mathfrak{r}_1^{(i_k)}(X)*\cdots*\mathfrak{r}_1^{(i_1)}(X)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}.
\end{aligned}$$

with $\mathfrak{r}_1^{(i)}(X)$ and $\mathfrak{R}_1^{(i)}(X)$ defined in (21) and (23), respectively.

At this point we may assume the Rota–Baxter algebra to be of weight θ, i.e. we replace the left-to-right bracketed Rota–Baxter pre-Lie words $R(\mathfrak{l}_1^{(i_k)}(X))$ by $R(\mathfrak{l}_\theta^{(i_k)}(X))$. C.S. Lam discovered in [31], see also [37], the weight zero case of identity (25), that is, for $\mathfrak{L}_0^{(i_k)}$.

Let us continue to follow closely Rota–Smith's work [45] implying the natural extension to the free Rota–Baxter algebra $\mathcal{R}$ in n generators, i.e., sequences $X_1,\ldots,X_n$. Now, working in the power series ring $\mathcal{R}[t_1,\ldots,t_n]$ with n commuting parameters $t_1,\ldots,t_n$, and replacing X by $X_1t_1+\cdots+X_nt_n\in\mathcal{R}[t_1,\ldots,t_n]$ in identity (25) of Theorem 6.1 we obtain a noncommutative generalization of the classical Bohnenblust–Spitzer identity by comparing the coefficients of the monomial $t_1\ldots t_n$ on both sides. We arrive at the following identity for arbitrary RB algebras:

Theorem 6.2. *Let R be a Rota–Baxter operator on a Rota–Baxter algebra A and $x_1,\ldots,x_n\in A$. Then:*

$$\sum_{\sigma\in S_n} R\Big(R\big(\cdots R(x_{\sigma_1})x_{\sigma_2}\ldots\big)x_{\sigma_n}\Big) = \sum_{\pi\in\mathcal{OP}_n}\omega(\pi)\mathtt{L}_\theta(\pi_1)\cdot\ \cdots\ \cdot\mathtt{L}_\theta(\pi_k)$$

The sum on the left hand side is over all permutations in S_n. The sum on the right hand side is over all *ordered* partitions, $\pi=[\pi_1]\ldots[\pi_k]$, that is, sequences of its disjoint subsets whose union is $[n]$. We denote by $\mathcal{OP}_n=\sum_{i=1}^n\mathcal{OP}_n^k$ the set of all ordered partitions and by $\mathcal{OP}_n^k$ the set of ordered partitions of $[n]$ with k blocks. We denote by $m_i:=|\pi_i|$ the number of elements in the block π_i of partition π. The coefficient function $\omega(\pi)$ is simply defined to be

$$\omega(\pi) := \big(m_1(m_1+m_2)\cdots(m_1+\cdots+m_k)\big)^{-1}.$$

Finally, we define $\mathtt{L}_\theta(\pi_i)$, $[\pi_i] = [j_1 \dots j_{m_i}]$, using the left-to-right bracketed RB pre-Lie words of weight θ (20) in $(\mathcal{R}, R)$ by:

$$\mathtt{L}_\theta(\pi_i) := \sum_{\sigma \in S_{m_i}} R\big(\mathfrak{l}_\theta^{(m_i)}(X_{j_{\sigma_1}}, \dots, X_{j_{\sigma_{m_i}}})\big). \tag{26}$$

We recover identity (25), that is, for $X = X_1 = \dots = X_n$, from the fact that the number of ordered partitions of type $m_1 + \dots + m_k = n$ is given by the multinomial coefficient $n!(m_1! \dots m_k!)^{-1}$ and from the fact that in that case we have

$$\mathtt{L}_\theta(\pi_i) = m_i! \mathfrak{L}_\theta^{(m_i)}(X).$$

Let us turn now to the classical, commutative case and show how the classical Spitzer identity can be recovered from the noncommutative one. Recall first that, in the commutative case, $a \rhd_\theta b = -\theta ab$, so that, for π_i as above, we have

$$\mathtt{L}_\theta(\pi_i) = m_i!(-\theta)^{m_i - 1} R\big(\prod_{j \in \pi_i} X_j\big)$$

and we get

$$\sum_{\sigma \in S_n} R(R(\dots R(X_{\sigma_1})X_{\sigma_2} \dots)X_{\sigma_n}) = \sum_{\pi \in \mathcal{OP}_n} \frac{(m_1)! \cdots (m_k)!}{m_1(m_1+m_2)\cdots(m_1+\cdots+m_k)} (-\theta)^{n-k} \prod_{i=1}^k R\big(\prod_{j \in \pi_i} X_j\big).$$

Lemma 6.1. *We have, for any sequence $(m_1, \dots, m_k)$:*

$$\sum_{\sigma \in S_k} \frac{1}{m_{\sigma_1}(m_{\sigma_1} + m_{\sigma_2}) \cdots (m_{\sigma_1} + \cdots + m_{\sigma_k})} = \prod_{i=1}^k \frac{1}{m_i}$$

Indeed, let us consider the integral expression

$$\int_0^1 x_k^{m_k - 1} dx_k \dots \int_0^1 x_1^{m_1 - 1} dx_1 = \prod_{i=1}^k \frac{1}{m_i}$$

Recall that

$$\int_0^1 x^p dx \int_0^1 y^q dy = \int_0^1 x^p \int_0^x y^q dy dx + \int_0^1 y^q \int_0^y x^p dx dy.$$

The formula (a weight zero RB relation for the Riemann integral map) follows from the geometric decomposition of the square into two triangles and generalizes to higher products of integrals. In the general case, the hypercube in dimension n is divided into $n!$ simplices. We get:

$$\begin{aligned}
\int_0^1 x_k^{m_k-1} dx_k \dots \int_0^1 x_1^{m_1-1} dx_1 &= \sum_{\sigma \in S_k} \int_0^1 x_{\sigma_1}^{m_{\sigma_1}-1} \int_0^{x_{\sigma_1}} x_{\sigma_2}^{m_{\sigma_2}-1} \dots \int_0^{x_{\sigma_{k-1}}} x_{\sigma_k}^{m_{\sigma_k}-1} dx_{\sigma_{k-1}} \dots dx_{\sigma_1} \\
&= \sum_{\sigma \in S_k} \frac{1}{m_{\sigma_1}(m_{\sigma_1} + m_{\sigma_2}) \cdots (m_{\sigma_1} + \cdots + m_{\sigma_k})}
\end{aligned}$$

which gives the expected formula.

This leads to the classical Bohnenblust–Spitzer formula [45] of weight θ

$$\sum_{\sigma \in S_n} R\Big(R\big(\cdots R(X_{\sigma_1})X_{\sigma_2} \cdots\big)X_{\sigma_n}\Big) = \sum_{\pi \in \mathcal{P}_n} (-\theta)^{n-|\pi|} \prod_{\pi_i \in \pi} (m_i - 1)!\, R\Big(\prod_{j \in \pi_i} X_j\Big), \tag{27}$$

Here π now runs through all unordered set partitions $\mathcal{P}_n$ of $[n]$; by $|\pi|$ we denote the number of blocks in π; and m_i was the size of the particular block π_i. In the commutative case with weight $\theta = 0$ we get the generalized integration by parts formula

$$\sum_{\sigma \in S_n} R\Big(R\big(\cdots R(X_{\sigma_1})X_{\sigma_2} \cdots\big)X_{\sigma_n}\Big) = \prod_{j=1}^n R(X_j). \tag{28}$$

Also, for $n > 0$ and $X_1 = \cdots = X_n = X$ we find

$$R\Big(R\big(\cdots R(X)X\cdots\big)X\Big) = \frac{1}{n!}\sum_{\pi\in\mathcal{P}_n}(-\theta)^{n-|\pi|}\prod_{\pi_i\in\pi}(m_i-1)!\,R\big(X^{m_i}\big). \tag{29}$$

7. A new identity for Rota–Baxter algebras

In this section we provide a detailed proof of a Theorem announced in [5, 22].

First, recall that in a RB algebra (A, R) we find for the RB double product (6) $R(a *_\theta b) = R(a)R(b)$ which is just a reformulation of the Rota–Baxter relation. Next, we introduce some notation.

Let (A, R) be a RB algebra and $a_1, \ldots, a_n$ be a collection of elements in A. For any permutation $\sigma \in S_n$ we define the element $T_\sigma(a_1, \ldots, a_n)$ as follows: define first the subset $E_\sigma \subset \{1, \ldots, n\}$ by $k \in E_\sigma$ if and only if $\sigma_{k+1} > \sigma_j$ for any $j \leq k$. We write E_σ in the increasing order $1 \leq k_1 < \cdots < k_p \leq n-1$. Then we set:

$$T_\sigma(a_1,\ldots,a_n) := \Big(\cdots\big((a_{\sigma_1}\rhd_\theta a_{\sigma_2})\rhd_\theta\cdots\big)\rhd_\theta a_{\sigma_{k_1}}\Big) *_\theta\cdots *_\theta \Big(\cdots\big((a_{\sigma_{k_p+1}}\rhd_\theta a_{\sigma_{k_p+2}})\rhd_\theta\cdots\big)\rhd_\theta a_{\sigma_n}\Big). \tag{30}$$

There are $p+1$ packets separated by p double RB products on the right-hand side of the expression (30) above, and the parentheses are set to the left inside each packet. Quite symmetrically we define the element $U_\sigma(a_1, \ldots, a_n)$ by considering first the subset $F_\sigma \subset \{1, \ldots, n\}$ defined by $l \in F_\sigma$ if and only if $\sigma_l < \sigma_j$ for any $j \geq l+1$. We write F_σ in the increasing order: $1 \leq l_1 < \cdots < l_q \leq n-1$. Then we set:

$$U_\sigma(a_1,\ldots,a_n) := \Big(a_{\sigma_1}\lhd_\theta\big(\cdots(a_{\sigma_{l_1-1}}\lhd_\theta a_{\sigma_{l_1}})\big)\cdots\Big) *_\theta\cdots *_\theta \Big(a_{\sigma_{l_q+1}}\lhd_\theta\big(\cdots(a_{\sigma_{n-1}}\lhd_\theta a_{\sigma_n})\big)\cdots\Big). \tag{31}$$

There are $q+1$ packets separated by q double RB products on the right-hand side of the expression (31) above, and the parentheses are set to the right inside each packet. The pre-Lie operations $\rhd_\theta$ and $\lhd_\theta$ involved in the right-hand side of equality (30) respectively (31) are given by (9) respectively (10).

Following [31] it is convenient to write a permutation by putting a vertical bar after each element of E_σ or F_σ according to the case. For example, for the permutation $\sigma = (3261457)$ inside S_7 we have $E_\sigma = \{2, 6\}$ and $F_\sigma = \{4, 5, 6\}$. Putting the vertical bars:

$$\sigma = (32|6145|7), \qquad\qquad \sigma = (3261|4|5|7) \tag{32}$$

we see that the corresponding elements in A will then be:

$$T_\sigma(a_1,\ldots,a_7) \;=\; (a_3\rhd_\theta a_2) *_\theta \Big(\big((a_6\rhd_\theta a_1)\rhd_\theta a_4\big)\rhd_\theta a_5\Big) *_\theta a_7, \tag{33}$$

$$U_\sigma(a_1,\ldots,a_7) \;=\; \Big(a_3\lhd_\theta\big(a_2\lhd_\theta(a_6\lhd_\theta a_1)\big)\Big) *_\theta a_4 *_\theta a_5 *_\theta a_7. \tag{34}$$

Theorem 7.1. *(New noncommutative Spitzer formula) We have:*

$$\sum_{\sigma\in S_n} R\Big(\cdots R\big(R(X_{\sigma_1})X_{\sigma_2}\big)\cdots X_{\sigma_n}\Big) \;=\; \sum_{\sigma\in S_n} R\big(T_\sigma(X_1,\ldots,X_n)\big), \tag{35}$$

$$\sum_{\sigma\in S_n} R\Big(X_{\sigma_1}\cdots R\big(X_{\sigma_{n-1}}R(X_{\sigma_n})\big)\ldots\Big) \;=\; \sum_{\sigma\in S_n} R\big(U_\sigma(X_1,\ldots,X_n)\big), \tag{36}$$

In the weight $\theta = 0$ case, the pre-Lie operations involved on the right-hand side of the above identities reduce to $a \rhd_0 b = [R(a), b] = -b \lhd_0 a$. This case, in the form (36), has been handled by C.S. Lam in [31], in the concrete situation when A is a function space on the real line, and when $R(f)$ is the primitive of f which vanishes at a fixed $T \in \mathbb{R}$. In the case of a commutative RB algebra both identities agree since both the left and right RB pre-Lie products (9), (10), respectively, agree. See [23] for analogous statements in the context of dendriform algebras.

Proof. The proof of (35) proceeds by induction on the number n of arguments, and (36) follows easily by analogy. The case $n = 2$ reduces to the identity:

$$R\big(R(X_1)X_2\big) + R\big(R(X_2)X_1\big) = R(X_1)R(X_2) + R(X_2\rhd_\theta X_1), \tag{37}$$

which immediately follows from the definitions. The case $n = 3$ is already non obvious and relies on considering the six permutations in S_3:

$$(1|2|3), \qquad (1|32), \qquad (2|31), \qquad (21|3), \qquad (321), \qquad (312),$$

so that

$$\sum_{\sigma\in S_3} R\Big(R\big(R(X_{\sigma_1})\,X_{\sigma_2}\big)X_{\sigma_3}\Big) \;=\; R(X_1)\,R(X_2)\,R(X_3)+R(X_1)\,R(X_3\rhd_\theta X_2)+R(X_2)\,R(X_3\rhd_\theta X_1)$$
$$+R(X_2\rhd_\theta X_1)\,R(X_3)+R\big((X_3\rhd_\theta X_2)\rhd_\theta X_1\big)+R\big((X_3\rhd_\theta X_1)\rhd_\theta X_2\big),$$

To prove the identity, we consider the following partition of the group S_n:

$$S_n = S_n^n \amalg \coprod_{j,k=1}^{n-1} S_n^{j,k}, \tag{38}$$

where S_n^n is the stabilizer of n in S_n, and where $S_n^{j,k}$ is the subset of those $\sigma\in S_n$ such that $\sigma_j=n$ and $\sigma_{j+1}=k$. For $k\in\{1,\ldots,n-1\}$ we set:

$$S_n^k := \coprod_{j=1}^{n-1} S_n^{j,k}. \tag{39}$$

This is the subset of permutations in S_n in which the two-terms subsequence (n,k) appears in some place. We have:

$$S_n = \coprod_{k=1}^{n} S_n^k. \tag{40}$$

Each S_n^k is in bijective correspondence with S_{n-1}, in an obvious way for $k=n$, and by considering the two-term subsequence (n,k) as a single letter for $k\neq n$. Precisely, in that case, in the expansion of $\sigma\in S_n$ as a sequence $(\sigma_1,\ldots,\sigma_n)$, we replace the pair (n,k) by $n-1$ and any j, $k<j<n$ by $j-1$, so that, for example, $(2,1,5,3,4)\in S_5^{3,3}$ is sent to $(2,1,4,3)$ by the bijection. For each $\sigma\in S_n^k$ we denote by $\widetilde{\sigma}$ its counterpart in S_{n-1}. Notice that for any $k\neq n$ and for any $j\in\{1,\ldots,n-1\}$, the correspondence $\sigma\mapsto\widetilde{\sigma}$ sends $S_n^{j,k}$ onto the subset of S_{n-1} formed by the permutations τ such that $\tau_j=n-1$. The following lemma is almost immediate:

Lemma 7.1. *For $\sigma\in S_n^n$ we have:*

$$T_\sigma(a_1,\ldots,a_n)=T_{\widetilde{\sigma}}(a_1,\ldots,a_{n-1})*_\theta a_n, \tag{41}$$

and for $\sigma\in S_n^k, k<n$ we have:

$$T_\sigma(a_1,\ldots,a_n)=T_{\widetilde{\sigma}}(a_1,\ldots,\widehat{a_k},\ldots,a_{n-1},a_n\rhd_\theta a_k), \tag{42}$$

where a_k under the hat has been omitted.

We rewrite the $n-1$-term sequence $(a_1,\ldots,\widehat{a_k},\ldots,a_{n-1},a_n\rhd_\theta a_k)$ as $(c_1^k,\ldots,c_{n-1}^k)$. We are now ready to compute, using the last lemma and the induction hypothesis:

$$\sum_{\sigma\in S_n} R\big(T_\sigma(a_1,\ldots,a_n)\big)=\sum_{k=1}^{n}\sum_{\sigma\in S_n^k} R\big(T_\sigma(a_1,\ldots,a_n)\big)$$
$$= \sum_{\tau\in S_{n-1}} R\Big(\big(R(\cdots R(R(a_{\tau_1})a_{\tau_2})\cdots)a_{\tau_{n-1}}\big)*_\theta a_n\Big)+\sum_{k=1}^{n-1}\sum_{\tau\in S_{n-1}} R\Big(R\big(\ldots R(R(c^k_{\tau_1})c^k_{\tau_2})\cdots\big)c^k_{\tau_{n-1}}\Big)$$
$$= \sum_{\tau\in S_{n-1}} R\Big(R\big(R(\cdots R(R(a_{\tau_1})a_{\tau_2})\cdots)a_{\tau_{n-1}}\big)a_n\Big)-\sum_{\tau\in S_{n-1}} R\Big(R\big(\cdots R(R(a_{\tau_1})a_{\tau_2})\cdots\big)a_{\tau_{n-1}}\tilde R(a_n)\Big)$$
$$+\sum_{k=1}^{n-1}\sum_{\tau\in S_{n-1}} R\Big(R\big(\ldots R(R(c^k_{\tau_1})c^k_{\tau_2})\cdots(a_n\rhd_\theta a_k)\cdots\big)c^k_{\tau_{n-1}}\Big)$$

where $a_n\rhd_\theta a_k=R(a_n)a_k+a_k\tilde R(a_n)=c^k_{\tau_j}=c^k_{n-1}$ lies in position j. Recall that $x*_\theta y=R(x)y-x\tilde R(y)$. Using the definition of the pre-Lie operation $\rhd_\theta$ and the RB relation we get:

$$\sum_{\sigma\in S_n} R\big(T_\sigma(a_1,\ldots,a_n)\big)$$

$$
\begin{aligned}
= & \sum_{\tau\in S_{n-1}} R\Big(R\big(R(\cdots R(R(a_{\tau_1})a_{\tau_2})\cdots)a_{\tau_{n-1}}\big)a_n\Big) - \sum_{\tau\in S_{n-1}} R\Big(R\big(\cdots R(R(a_{\tau_1})a_{\tau_2})\cdots\big)a_{\tau_{n-1}}\tilde{R}(a_n)\Big) \\
& + \sum_{k=1}^{n-1}\sum_{\substack{\tau\in S_{n-1}\\ \tau_1=n-1}} R\Big(R\big(\cdots R\big(R(R(a_n)a_k)c^k_{\tau_2}\big)\cdots\big)c^k_{\tau_{n-1}}\Big) + \sum_{k=1}^{n-1}\sum_{\substack{\tau\in S_{n-1}\\ \tau_1=n-1}} R\Big(R\big(\cdots R\big(R(a_k\tilde{R}(a_n))c^k_{\tau_2}\big)\cdots\big)c^k_{\tau_{n-1}}\Big) \\
& + \sum_{k=1}^{n}\sum_{j=2}^{n-1}\sum_{\substack{\tau\in S_{n-1}\\ \tau_j=n-1}} R\bigg(R\Big(\cdots R\Big(R\Big(R\big(\cdots R(c^k_{\tau_1})c^k_{\tau_2}\big)\cdots\Big)a_n\Big)a_k\cdots\Big)c^k_{\tau_{n-1}}\bigg) \\
& - \sum_{k=1}^{n}\sum_{j=2}^{n-1}\sum_{\substack{\tau\in S_{n-1}\\ \tau_j=n-1}} R\bigg(R\Big(\cdots R\Big(R\big(\cdots R(c^k_{\tau_1})c^k_{\tau_2}\big)\cdots\tilde{R}(a_n)\Big)a_k\cdots\Big)c^k_{\tau_{n-1}}\bigg) \\
& + \sum_{k=1}^{n-1}\sum_{j=2}^{n-1}\sum_{\substack{\tau\in S_{n-1}\\ \tau_j=n-1}} R\bigg(R\Big(\cdots R\Big(R\Big(R\big(\cdots R(c^k_{\tau_1})c^k_{\tau_2}\big)\cdots\Big)a_k\tilde{R}(a_n)\Big)\cdots\Big)c^k_{\tau_{n-1}}\bigg)
\end{aligned}
$$

where a_n lies in position j (resp. $j+1$) in lines 4 and 5 (resp. in the last line) in the above computation, and where a_k lies in position $j+1$ (resp. j) in lines 4 and 5 (resp. in the last line). We can rewrite this going back to the permutation group S_n and using the partition (38):

$$
\begin{aligned}
& \sum_{\sigma\in S_n} R\big(T_\sigma(a_1,\ldots,a_n)\big) \\
& \quad = \sum_{\sigma\in S_n^n} R\Big(R\big(R(\cdots R(R(a_{\sigma_1})a_{\sigma_2})\cdots)a_{\sigma_{n-1}}\big)a_{\sigma_n}\Big) \\
& \qquad - \sum_{\sigma\in S_n^n} R\Big(R\big(\cdots R(R(a_{\sigma_1})a_{\sigma_2})\cdots\big)a_{\sigma_{n-1}}\tilde{R}(a_{\sigma_n})\Big) \\
& \qquad + \sum_{k=1}^{n-1}\sum_{\sigma\in S_n^{1,k}} R\Big(R\big(R(\cdots R(R(a_{\sigma_1})a_{\sigma_2})\cdots)a_{\sigma_{n-1}}\big)a_{\sigma_n}\Big) \\
& \qquad + \sum_{k=1}^{n-1}\sum_{\sigma\in S_n^{1,k}} R\Big(\big(R(\cdots R(a_{\sigma_2}\tilde{R}(a_{\sigma_1}))\cdots\big)a_{\sigma_{n-1}}\Big) \\
& \qquad + \sum_{k=1}^{n}\sum_{j=2}^{n-1}\sum_{\sigma\in S_n^{j,k}} R\bigg(R\Big(\cdots R\Big(R\Big(R\big(\cdots R(a_{\sigma_1})a_{\sigma_2}\cdots\big)a_{\sigma_j}\Big)a_{\sigma_{j+1}}\Big)\cdots\Big)a_{\sigma_n}\bigg) \\
& \qquad - \sum_{k=1}^{n}\sum_{j=2}^{n-1}\sum_{\sigma\in S_n^{j,k}} R\bigg(R\Big(\cdots R\Big(R\Big(\cdots R(a_{\sigma_1})a_{\sigma_2}\cdots\tilde{R}(a_{\sigma_j})\Big)a_{\sigma_{j+1}}\Big)\cdots\Big)a_{\sigma_n}\bigg) \\
& \qquad + \sum_{k=1}^{n-1}\sum_{j=2}^{n-1}\sum_{\sigma\in S_n^{j,k}} R\bigg(R\Big(\cdots R\Big(R\Big(\cdots R(a_{\sigma_1})a_{\sigma_2}\cdots\big(a_{\sigma_{j+1}}\tilde{R}(a_{\sigma_j})\big)\Big)\cdots\Big)a_{\sigma_n}\bigg).
\end{aligned}
$$

Lines 2, 4 and 6 together give the left-hand side of (35) whereas lines 3, 5, 7 and 8 cancel. More precisely line 3 cancels with the partial sum corresponding to $j = n-1$ in line 8, line 5 cancels with the partial sum corresponding to $j = 2$ in line 7, and (for $n \geq 4$), the partial sum corresponding to some fixed $j \in \{3,\ldots,n-1\}$ in line 7 cancels with the partial sum corresponding to $j-1$ in line 8. This proves equality (35). □

8. On the Magnus and Atkinson's recursions

Now we return to Atkinson's recursions in Theorem 3.1. We will focus only on the first equation in (15). Recall that the Spitzer algebra is naturally a graded Hopf algebra. Moreover, we will assume the —free— Rota–Baxter algebra to be of weight θ. For computational convenience, we consider the embedding of the Spitzer algebra $\mathcal{S}$ into $\mathcal{S}[[t]]$ defined on homogeneous elements z of degree n in $\mathcal{S}$ by $z \longmapsto t^n \cdot z$. We agree to identify $\mathcal{S}$ with its image in $\mathcal{S}[[t]]$, so that this image is naturally provided with a graded Hopf algebra structure and that the grading operation Y now is naturally given by $t\partial_t$. It is then obvious that the equation (a generalized integral equation if we view the Rota–Baxter operation R as a generalized integral operator)

$$F = 1 + R(F \cdot Xt) \tag{43}$$

is solved by the series $F = F(t) := \sum_{n\geq 0} t^n (RX)^{[n]}$ which is a group-like element in the Hopf algebra. The operation of the Dynkin map on F is given by

$$D(F(t)) = F(t)^{-1} \cdot t\partial_t F(t) = \mathfrak{L}(t) := \sum_{n>0} t^n \mathfrak{L}_\theta^{(n)}(X),$$

which, of course, implies the linear differential equation $t\partial_t F(t) = F(t) \cdot \mathfrak{L}(t)$ and hence, by comparing coefficients on both sides, the recursion:

$$n(RX)^{[n]} = \sum_{k=0}^{n-1} (RX)^{[k]} \cdot \mathfrak{L}_\theta^{(n-k)}(X),$$

which is a way to relate our Theorem 6.1 to the classical problem of finding explicit solutions to the first order linear differential equations $\partial_t X = XA$. Now, the linear differential equation $t\partial_t F(t) = F(t)\mathfrak{L}(t)$ is a classical differential equation in noncommutative variables with associated integral operator P (the weight zero RB operator $P = \int_0^t$) so that the equation can be solved with the usual techniques for solving matricial or functional first order differential equations. Actually, it is well-known that in the noncommutative setting the differential equation $\partial_t F(t) = F(t) \cdot \frac{1}{t}\mathfrak{L}(t)$ respectively the integral equation:

$$F(t) = 1 + \int_0^t F(t') \cdot \frac{1}{t'} \mathfrak{L}(t') dt'$$

can be solved via the exponential function. Notice that this equation is a particular case of Atkinson's recursion with $F = 1 + P(F \cdot \hat{\mathfrak{L}})$, with $\hat{\mathfrak{L}}(t) = \frac{\mathfrak{L}(t)}{t}$. Recall Magnus' seminal work [32]. He proposed the exponential Ansatz

$$F(t) = \exp\big(\Omega[\hat{\mathfrak{L}}](t)\big), \tag{44}$$

where $\Omega[\hat{\mathfrak{L}}](0) = 0$. Following [25], the series for $\Omega[\hat{\mathfrak{L}}]$

$$\Omega[\hat{\mathfrak{L}}](t) = \sum_{n>0} \Omega_n t^n, \tag{45}$$

can be expressed in terms of multiple integrals of nested commutators. Magnus provided a differential equation which in turn can be easily solved recursively for the terms Ω_n

$$\frac{d}{dt}\Omega[\hat{\mathfrak{L}}](t) = \frac{-\mathrm{ad}\,\Omega[\hat{\mathfrak{L}}]}{\mathrm{e}^{-\mathrm{ad}\,\Omega[\hat{\mathfrak{L}}]} - 1}(\hat{\mathfrak{L}})(t).$$

which leads to the Magnus recursion

$$\Omega[\hat{\mathfrak{L}}](t) = P\bigg(\hat{\mathfrak{L}} + \sum_{n>0} (-1)^n b_n \Big[\mathrm{ad}\big(\Omega[\hat{\mathfrak{L}}]\big)\Big]^n (\hat{\mathfrak{L}})\bigg)(t). \tag{46}$$

The coefficients are $b_n := B_n/n!$ with B_n the Bernoulli numbers. For $n = 1, 2, 4$ we find $b_1 = -1/2, b_2 = 1/12$ and $b_4 = -1/720$. We have $b_3 = b_5 = \cdots = 0$.

Strichartz succeeded in giving a closed solution to Magnus' expansion [49], see also [35, 25]. He found

$$\Omega[\hat{\mathfrak{L}}](t) = \sum_{n>0} \sum_{\sigma \in S_n} \frac{(-1)^{d(\sigma)}}{n^2 \binom{n-1}{d(\sigma)}} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \big[[\ldots[\hat{\mathfrak{L}}(t_{\sigma_n}), \hat{\mathfrak{L}}(t_{\sigma_{n-1}})] \ldots], \hat{\mathfrak{L}}(t_{\sigma_1})\big]\, dt_n \ldots dt_2\, dt_1. \tag{47}$$

Here $d(\sigma)$ denotes the number of *descents* in the permutation $\sigma \in S_n$, that is, $d(\sigma) = |\{i < n,\ \sigma(i) > \sigma(i+1)\}|$. In fact, more detail can be provided. In [25] we find the following theorem for $\Omega(t) := \Omega[\hat{\mathfrak{L}}](t)$.

Theorem 8.1. [25] *The expansion of $\Omega(t)$ in terms of the $\hat{\mathfrak{L}}_\theta^{(i)}$s writes*

$$\Omega[\hat{\mathfrak{L}}](t) = \sum_{n>0} \sum_{\sigma \in S_n} \frac{(-1)^{d(\sigma)}}{n \binom{n-1}{d(\sigma)}} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \hat{\mathfrak{L}}(t_{\sigma_n}) \cdots \hat{\mathfrak{L}}(t_{\sigma_2}) \hat{\mathfrak{L}}(t_{\sigma_1})\, dt_n \ldots dt_2\, dt_1.$$

The coefficient of the term $\mathfrak{L}_\theta^{(i_1)} \cdots \mathfrak{L}_\theta^{(i_m)}$ in the above expansion of Ω_k was also given in [25].

$$k \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m \sum_{\sigma \in S_m} \frac{(-1)^{d(\sigma)}}{m \binom{m-1}{d(\sigma)}} t_{\sigma_m}^{i_m - 1} \ldots t_{\sigma_1}^{i_1 - 1}. \tag{48}$$

In other terms, the Magnus expansion solves the Atkinson recursion. We summarize our results in the following Theorem.

Theorem 8.2. *For an arbitrary weight θ RB algebra A, the Atkinson recursion $F = 1 + R(F \cdot Xt)$, $X \in A$ is solved by the Strichartz' expansion:*

$$F(t) = \exp\left(\sum_{m>0} \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\ldots,i_n>0}} m \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \sum_{\sigma \in S_n} \frac{(-1)^{d(\sigma)}}{n \binom{n-1}{d(\sigma)}} t_{\sigma_n}^{i_n-1} \ldots t_{\sigma_1}^{i_1-1} \mathfrak{L}_\theta^{(i_1)}(X) \cdots \mathfrak{L}_\theta^{(i_n)}(X) \right)$$

9. Solving Bogoliubov's counterterm recursion

Let us return to the Bogoliubov recursion (13), as described in Section 3. As already noticed, its original setting is perturbative quantum field theory (pQFT). Connes and Kreimer associated to a renormalizable quantum field theory the Hopf algebra $H = \bigoplus_{n=0}^\infty H_n$ of ultraviolet (UV) superficially divergent one-particle irreducible (1PI) Feynman graphs [11, 12], see [24, 19, 34] for reviews. This Hopf algebra is polynomially generated by UV superficially divergent 1PI Feynman graphs, graded by the number of loops and non-cocommutative. It is connected with the base field being, say, the complex numbers $\mathbb{C}$. We exclude from considerations theories with gauge symmetries, for which the Hopf algebra is still commutative but, in general, is not polynomially generated [28, 50, 51] anymore.

The relevant quantities for the theory such as the Green functions can be deduced in principle from Feynman rules –a prescription associating to each (1PI) Feynman graph an integral. If the integrals were convergent, Feynman rules would be imbedded into the group $G(\mathbb{C}) \subset \mathrm{Lin}(H, \mathbb{C})$ of algebra maps from H to $\mathbb{C}$ with the Hopf algebra counit ε as the group unit. Here, $\mathrm{Lin}(H, \mathbb{C})$ denotes as usual the associative algebra of linear maps from H to $\mathbb{C}$ equipped with the usual convolution product, $f \star g := m_{\mathbb{C}}(f \otimes g)\Delta$. Then, there would be no obvious need for a renormalization process. However, these integrals are most often divergent, hence can not be interpreted as elements of $G(\mathbb{C})$, and require to be renormalized to make sense.

The process of regularization, one of the most common ways to proceed, is encoded in the change of the target space from $\mathbb{C}$ to a commutative algebra A which is supposed to be equipped with an idempotent operator (called the renormalization scheme operator) denoted by $R \in \mathrm{End}(A)$. Feynman rules have, after regularization, a rigorous meaning as elements of $G(A)$, see e.g. [9]. We denote the projections $R(A) := A_-$ and $\tilde{R}(A) := (1 - R)(A) = A_+$ corresponding to the vector space splitting of $A = R(A) \oplus \tilde{R}(A)$. In the dimensional regularization and minimal substraction scheme, for example, $A = \mathbb{C}[\epsilon^{-1}, \epsilon]]$, $A_- = \epsilon^{-1}\mathbb{C}[\epsilon^{-1}]$, $A_+ = \mathbb{C}[[\epsilon]]$. As already pointed out, one can show that for (A, R) an idempotent Rota–Baxter algebra such as $\mathbb{C}[\epsilon^{-1}, \epsilon]]$, $\mathrm{Lin}(H, A)$ with the idempotent operator $\mathcal{R}$ defined by $\mathcal{R}(f) = R \circ f$, for any $f \in \mathrm{Lin}(H, A)$, is a noncommutative complete filtered unital Rota–Baxter algebra.

The Bogoliubov/Atkinson recursion allows then to decompose $G(A)$ as the set-theoretic product of its subgroups $G_-(A)$ and $G_+(A)$:

$$\forall \gamma \in G(A),\ \exists! \gamma_- \in G_-(A),\ \gamma_+ \in G_+(A),\ \text{such that } \gamma = \gamma_-^{-1} * \gamma_+,$$

where we use the notations of Section 3. Moreover γ_+ is a multiplicative map from the Hopf algebra of Feynman diagrams H to $\mathbb{C}[[\epsilon]]$, and $\gamma^{ren} := \lim_{\epsilon \to 0} \gamma_+$ is therefore a well-defined element of $G(\mathbb{C})$, the "renormalized Feynman rule" one was looking for to compute the relevant properties of the theory.

Now, our results allow to give closed form expansions for γ_-, γ_+ and γ^{ren}. Recall indeed from Section 3 that γ_- solves Atkinson's recursion:

$$\gamma_- \quad = \quad e_A + \sum_{n>0} (\mathcal{R}a)^{[n]}$$

for $a := e_A - \gamma$, $e_A := \eta_A \circ \varepsilon$, and analogously for γ_+ in terms of $\tilde{\mathcal{R}}$. In conclusion, we get, in the weight $\theta := -1$ RB algebra $(Lin(H, A), \mathcal{R})$ the following theorems.

Theorem 9.1. *We have for $a = e_A - \gamma$*

$$\begin{aligned}\gamma_- &= e_A + \sum_{n>0} (\mathcal{R}a)^{[n]} \\ &= e_A + \sum_{n>0} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \frac{\mathfrak{L}_1^{(i_1)}(a) \star \cdots \star \mathfrak{L}_1^{(i_k)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)} \end{aligned} \tag{49}$$

where $\mathfrak{l}_1^{(m)}(a) = (\mathfrak{l}_1^{(m-1)}(a) \triangleright_1 a)$ and $\mathfrak{L}_1^{(n+1)}(a) := R\big(\mathfrak{l}_1^{(n+1)}(a)\big)$ was defined in (23). It is important to underline that the expression given here applies in principle, besides the minimal substraction and dimensional regularization scheme, to any renormalization procedure which can be formulated in terms of a Rota–Baxter structure.

The reader should notice the formal similarity of this solution for γ_- (the "counterterm character") with Connes–Marcolli's formula for the universal singular frame, see [13, 14, 15], see also [21, 34]. In fact, in the context of dimensional regularization together with the minimal subtraction scheme, there exists a linear map from the Connes–Kreimer Hopf algebra of Feynman graphs to the complex numbers $\beta = \sum_{n>0} \beta_n$, naturally associated to the counterterm γ_- (see [21] for details and a Lie theoretic construction of β) and such that:

$$\gamma_- = e_A + \sum_{n>0} \sum_{\substack{k_1+\cdots+k_m=n \\ k_1,\ldots,k_m>0}} \frac{\beta_{k_1} \star \cdots \star \beta_{k_m}}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_m)} \frac{1}{\epsilon^n} \tag{50}$$

However, in spite of the similarity of formulas (49) and (50) together with the techniques to obtain them, the $\frac{\beta_i}{\epsilon^i}$'s do not coincide with the $\mathfrak{L}_1^{(i)}(a)$'s. In fact, both are obtained from the action of the Dynkin operator $D = S \star Y$. They are the homogeneous components of $D(\gamma_-)$ but, with respect to two different graded Hopf algebra structures: $\frac{\beta_i}{\epsilon^i}$ is simply the homogeneous component of $D(\gamma_-)$ in the completed Hopf algebra $\mathrm{Lin}(H, A)$, whereas $\mathfrak{L}_1^{(i)}(a)$ is the image in $\mathrm{Lin}(H, A)$ of the homogeneous component of $D(\gamma_-)$ in the completed Spitzer algebra $\mathcal{S}$ built on one generator (still written abusively a), with its associated Hopf structure as described in Section 5.
From Theorem 8.2 we conclude immediately:

Theorem 9.2. *We have*

$$\gamma_- = \exp\Bigg(\sum_{\substack{m>0 \\ i_1+\cdots+i_n=m \\ i_1,\ldots,i_n>0}} \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \sum_{\sigma \in S_n} \frac{m(-1)^{d(\sigma)}}{n \binom{n-1}{d(\sigma)}} \; t_{\sigma_n}^{i_n-1} \ldots t_{\sigma_1}^{i_1-1} \; \mathfrak{L}_1^{(i_1)}(a) \star \cdots \star \mathfrak{L}_1^{(i_n)}(a) \Bigg)$$

Returning to [25] we may give a more combinatorial expression for the formula in Theorem 9.2, omitting the dummy integrations. Recall the notion of a composition I of an integer m, i.e., a vector of positive integers, its parts, $I := (i_1, \ldots, i_k)$, of length $\ell(I) := k$ and weight $|I| := \sum_{j=1}^k i_j = m$. For instance, all compositions of weight 3 are $C_3 := \{(111), (21), (12), (3)\}$. The set of all compositions $C := \bigcup_{m \geq 0} C_m$ is partially ordered by reversed refinement, that is, $I \preceq J$ iff each part of I is a sum of parts of J. We call J finer than I. For instance, $(1234) \preceq (11112211)$. Recall

$$\omega(I) := \big(i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)\big)^{-1}.$$

Let J be a composition finer than I. Define $\tilde{J} := (J_1, \ldots, J_m)$ to be the unique decomposition of the composition J, such that $|J_k| = i_k$ for $k = 1, \ldots, m$. Now define

$$\omega(J, I) := \prod_{k=1}^m \omega(J_k).$$

We then deduce from [25] (paragraphs 4.2 and 4.3) the following formula for the exponent $\Omega[\mathfrak{L}] := \sum_{i>0} \Omega_i[\mathfrak{L}]$ in the formula in Theorem 9.2:

$$\Omega_n = \sum_{|J|=n} n \frac{(-1)^{\ell(J)-1}}{\ell(J)} \sum_{J \preceq K = \{(k_1,\ldots,k_p)\}} \omega(K, J) \; \mathfrak{L}_1^{(k_1)}(a) \star \cdots \star \mathfrak{L}_1^{(k_p)}(a).$$

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