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AN OBSTRUCTION TO A KNOT BEING DEFORM-SPUN VIA ALEXANDER POLYNOMIALS

RYAN BUDNEY AND ALEXANDRA MOZGOVA

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ABSTRACT. We show that if a co-dimension two knot is deform-spun from a lower-dimensional co-dimension 2 knot, there are constraints on the Alexander polynomials. In particular this shows, for all n, that not all co-dimension 2 knots in S^n are deform-spun from knots in S^{n-1}.

In co-dimension 2 knot theory [6], typically the term 'n-knot' denotes a manifold pair (S^{n+2}, K) where K is the image of a smooth embedding $f : S^n \to S^{n+2}$. An n-ball pair is a pair (D^{n+2}, J) where J is the image of a smooth embedding $f : D^n \to D^{n+2}$ such that $f^{-1}(\partial D^{n+2}) = \partial D^n$. Every n-knot K is isotopic to a union $(S^{n+2}, K) = (D^{n+2}, J) \cup_\partial (D^{n+2}, D^n)$ for some unique isotopy class of n-ball pair (D^{n+2}, J) provided we consider K to be oriented. Let $\mathrm{Diff}(D^{n+2}, J)$ denote the group of diffeomorphisms of an n-ball pair (D^{n+2}, J). That is, $f \in \mathrm{Diff}(D^{n+2}, J)$ means that f is a diffeomorphism of D^{n+2} which restricts to the identity on $\partial D^{n+2} = S^{n+1}$, is isotopic to the identity (rel boundary) as a diffeomorphism of D^{n+1}, and f preserves J, $f(J) = J$. We say an n-knot (S^{n+2}, K) is deform-spun from an $(n-1)$-knot $(S^{n+1}, K') = (D^{n+1}, J') \cup_\partial (D^{n+1}, D^{n-1})$ if there exists $g \in \mathrm{Diff}(D^{n+1}, J')$ such that the pair $\left((D^{n+1}, J') \times_g S^1\right) \cup_\partial \left((S^n, S^{n-1}) \times D^2\right)$ is diffeomorphic to the pair (S^{n+2}, K). Here $(D^{n+1}, J') \times_g S^1$ is the bundle over S^1 with fibre (D^{n+1}, J') and monodromy given by g, ie: $(D^{n+1}, J') \times_g S^1 = ((D^{n+1}, J') \times \mathbb{R})/\mathbb{Z}$ where $\mathbb{Z}$ acts diagonally, by g on (D^{n+1}, J') and as the group of universal covering transformations for $\mathbb{R} \to S^1$.

To picture a deform-spun knot, let g_t be a null-isotopy of g, ie: $g_0 = g$, $g_1 = Id_{D^{n+1}}$ and g_t is a diffeomorphism of D^{n+1} which restricts to the identity on ∂D^{n+2} for all $0 \le t \le 1$. Consider S^{n+2} to be the union of a great n-sphere S^n and a disjoint trivial vector bundle over S^1. Identify this trivial vector bundle over S^1 with $S^1 \times int(D^{n+1})$, and identify S^1 with $\mathbb{R}/\mathbb{Z}$. We assume that the inclusion $S^1 \times int(D^{n+1}) \to S^{n+2}$ extends to a map $S^1 \times D^{n+1} \to S^{n+2}$ such that the restriction $S^1 \times S^n \to S^{n+2}$ factors as projection onto the great sphere S^n followed by inclusion $S^n \to S^{n+2}$. Then the set $\{(t, x) \in S^1 \times int(D^{n+1}) : x = g_t(p), p \in$

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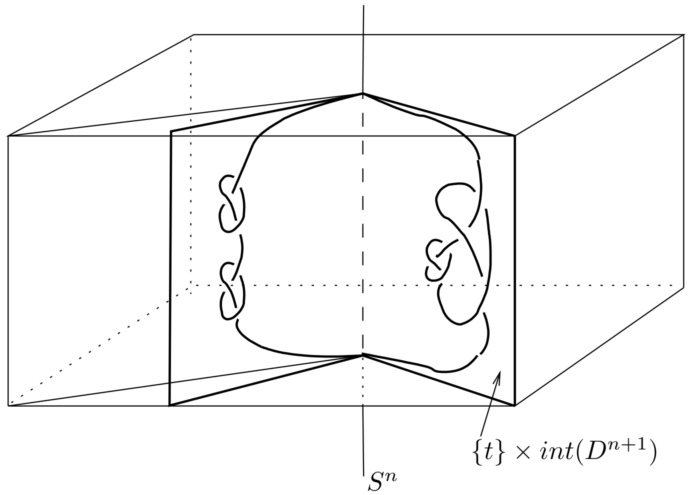

Figure 1

$int(J')\}$ is a subset of S^{n+2} whose closure is an n-knot. This is the deform-spun knot, see Figure 1.

The main observation of this paper is that if K is an n-knot, deform-spun from an $(n-1)$-knot K', then there is a relationship between the Alexander modules of K and K' which give rise to constraints on the Alexander polynomials $\Delta_1, \cdots, \Delta_n$ of K.

Theorem 0.1. *Let K be a n-knot which is deform-spun, then there exist polynomials $q_i \in \Lambda = \mathbb{Q}[t^{\pm 1}] = \mathbb{Q}[\mathbb{Z}]$ for $i = 0, 1, \cdots, n$ which satisfy $q_{i+1}q_i = \Delta_{i+1}$ ($q_0 = q_n = 1$) and $q_{n-i} = \overline{q_i}$ for all i, where we use the convention $\overline{q_i}(t) = q_i(t^{-1})$.*

An elementary consequence of this theorem is that for each $n \geq 2$, not every n-knot is deform-spun from an $(n-1)$-knot. This follows from the work of Levine [4] who gave a characterization of the Alexander modules of co-dimension 2 knots. In particular Levine shows that an n knot has Alexander polynomials $\Delta_1, \cdots, \Delta_n \in \Lambda$ which satisfy the relations $\Delta_i(1) \neq 0$, $\overline{\Delta_i} = \Delta_{n-i}$ for all i. Moreover, these relations are complete in the sense that given any n polynomials which satisfy these relations, there is an n-knot which has the specified Alexander polynomials. The case $n = 2$ has a particularly simple example. Theorem 0.1 states that if K is deform-spun, then $\overline{\Delta_1} = \Delta_1$, yet there are 2-knots such that Δ_1 is not symmetric. See example 10 of Fox's Quick Trip [2], which describes a 2-knot such that $\Delta_1(t) = 2t - 1$.

Litherland's deform-spinning construction has its origin in papers of Fox and Zeeman. Fox's 'Rolling' [3] paper gave a heuristic outline of the notion eventually called deform-spinning, as a graphing process from a 'relative 2-dimensional braid group' which nowadays is frequently called the fundamental group of the space of knots, or (in a slightly different setting) the mapping class group of the knot complement [1]. Zeeman proved that the complements of co-dimension two n-twist-spun knots fibre over S^1 provided $n \neq 0$ [8]. Litherland [7] went on to formulate a general situation where deform-spun knot complements fibre over S^1. Specifically, Litherland proved that if the diffeomorphism $g : (D^{n+1}, J') \to (D^{n+1}, J')$ preserves a Seifert surface for the knot (S^{n+1}, K') corresponding to the $(n-1)$-disc pair

(D^{n+1}, J'), then the deform-spun knot associated to the diffeomorphism $M \circ g : (D^{n+1}, J') \to (D^{n+1}, J')$ has a complement which fibres over S^1, provided $M : (D^{n+1}, J') \to (D^{n+1}, J')$ is a non-zero power of the meridional Dehn twist about J'.

This paper was largely motivated by a result in 'high' co-dimension knot theory. In the paper [1] the first author gave a new proof of Haefliger's theorem, that the monoid of isotopy classes of smooth embeddings of S^j in S^n is a group, provided $n-j > 2$. The heart of the proof is showing that if $n-j > 2$ then every knot (S^n, K) (where $K \simeq S^j$) is deform-spun from a lower-dimensional knot (S^{n-1}, K'), where $K' \simeq S^{j-1}$. Moreover, all knots (S^n, K) are i-fold deform-spun for $i = 2(n-j)-4$, in the sense that one obtains (S^n, K) be iterating the deform-spinning process i times. So in a sense this paper represents an investigation of the extreme case $n-j = 2$. A second motivation is the observation that frequently the groups $\pi_0 \mathrm{Diff}(D^3, J')$ ((D^3, J') a 1-ball pair) are quite large [1], in the sense that their classifying spaces all have the homotopy-type of finite-dimensional manifolds, but the dimension of these manifolds can be arbitrarily large. So there are many ways to construct 2-knots by deform-spinning a 1-knot. As far as the authors know, this paper represents the first known obstructions to knots being deform-spun.

1. Asymmetry obstruction

Given a co-dimension 2 knot K in S^{n+2}, the complement of the knot, C_K is a homology S^1. Let $\tilde{C}_K$ denote the universal abelian cover of C_K, ie: the cover corresponding to the kernel of the abelianization map $\pi_1 C_K \to \mathbb{Z}$, and consider $H_i(\tilde{C}_K; \mathbb{Q})$ to be a module over the group-ring of covering transformations $\Lambda = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$, this is called the i-th Alexander module of K. $H_i(\tilde{C}_K; \mathbb{Q})$ is a finitely-generated torsion Λ-module [4] for each i, so $H_i(\tilde{C}_K; \mathbb{Q}) \simeq \bigoplus_j \Lambda/p_j$ for some collection of polynomials p_j. The product of these polynomials $\prod_j p_j$ is called the i-th Alexander polynomial of K, or the order ideal of the i-th Alexander module $H_i(\tilde{C}_K; \mathbb{Q})$, denoted Δ_i. In general, the order ideal of a finitely generated torsion Λ-module M will be denoted Δ_M. A theorem of Levine's [4] is that Poincaré Duality combined with the Universal Coefficient Theorem induces an isomorphism $\overline{H_i(\tilde{C}_K; \mathbb{Q})} \simeq Ext_\Lambda(H_{n+1-i}(\tilde{C}_K; \mathbb{Q}), \Lambda)$. Here, if M is a Λ-module, $\overline{M}$ denotes the conjugate Λ-module. This is a module whose underlying $\mathbb{Q}$-vector space is M, but where action of the generator t on $\overline{M}$ is defined as the action of t^{-1} on M. Thus, the only Alexander polynomials of K which can be non-trivial are $\Delta_1, \cdots, \Delta_n$, and they satisfy the relation $\overline{\Delta_i} = \Delta_{n+1-i}$ for all i.

We collect some elementary results about Λ-modules that will be of use in the proof of Theorem 0.1. To state the lemma, let $\mathbb{Q}(\Lambda)$ denote the field of fractions of Λ, ie: the field which consists of rational Laurent polynomials.

Lemma 1.1. (a) *(see [6] 7.2.7) Given a short exact sequence of finitely generated torsion Λ-modules*

$$0 \to H_1 \to H \to H_2 \to 0$$

the order ideals satisfy $\Delta_{H_1}\Delta_{H_2} = \Delta_H$.

(b) *(see [4] Proposition 4.1) Let H be a finitely-generated torsion Λ-module. There is a natural isomorphism of Λ-modules*

$$Ext_\Lambda(H, \Lambda) \simeq Hom_\Lambda(H, \mathbb{Q}(\Lambda)/\Lambda).$$

(c) *With the same setup as (b), there is a natural isomorphism of $\mathbb{Q}$-vector spaces*

$$Hom_\Lambda(H, \mathbb{Q}(\Lambda)/\Lambda) \simeq Hom_\mathbb{Q}(H, \mathbb{Q})$$

where we interpret $\Lambda \subset \mathbb{Q}(\Lambda)$ as the rational Laurent polynomials with denominator 1*.*

(d) *Let $g : H \to H$ be a Λ-linear map, where H is a finitely-generated torsion Λ-module. Let $g^* : Ext_\Lambda(H, \Lambda) \to Ext_\Lambda(H, \Lambda)$ the Ext-dual of g. Then $ker(g)$ and $ker(g^*)$ have the same order ideals.*

Proof. (of item (c)) Consider a rational polynomial $\frac{p}{q} \in \mathbb{Q}(\Lambda)$. The division algorithm allows us to write $p = sq + r$ for Laurent polynomials $s, r \in \Lambda$ where $r \in \mathbb{Q}[t]$ and $deg(r) < deg(q)$. To ensure that r is unique, we demand that $GCD(p, q) = 1$, $q \in \mathbb{Q}[t]$ and the constant coefficient of q is 1. Define a function $\mathbb{Q}(\Lambda)/\Lambda \to \mathbb{Q}$ by sending $\frac{p}{q}$ to the constant coefficient of r. Composition with this map is a $\mathbb{Q}$-linear homomorphism $Hom_\Lambda(H, \mathbb{Q}(\Lambda)/\Lambda) \to Hom_\mathbb{Q}(H, \mathbb{Q})$ which is natural and respects connect-sum decompositions of the domain H. Thus to verify that it is an isomorphism, we need to only check it on a torsion Λ-module with one generator.

$$Hom_\Lambda(\Lambda/p, \mathbb{Q}(\Lambda)/\Lambda) \to Hom_\mathbb{Q}(\Lambda/p, \mathbb{Q})$$

In this case the target space has dimension $deg(p)$; the basis given by the dual basis to the polynomials t^i for $0 \leq i < deg(p)$. The domain also has dimension $deg(p)$, with basis given by homomorphisms that send 1 to t^i/p where $0 \leq i < deg(p)$. Hence the map is a bijection between these basis vectors.

To prove item (d), consider the 'prime factorization' of H. Let $P \subset \Lambda$ be the prime factors of the order ideal Δ_H. Given $p \in P$ let $H_p \subset H$ be the sub-module of elements of H killed by a power of p, thus $\bigoplus_{p \in P} H_p \simeq H$. g must respect the splitting, so we have maps g_p such that:

$$g = \bigoplus_{p \in P} g_p : H_p \to H_p.$$

Thus,

$$\Delta_{ker(g)} = \prod_{p \in P} \Delta_{ker(g_p)}.$$

Let $d_p \in \mathbb{Z}$ be defined so that $\Delta_{ker(g_p)} = p^{d_p}$. By part (c), g and g^* can be thought of as the $Hom_\mathbb{Q}(\cdot, \mathbb{Q})$-duals of each other, thus $ker(g)$ and $ker(g^*)$ have the same dimension as $\mathbb{Q}$-vector spaces, and so $dim_\mathbb{Q}(ker(g_p)) = deg(p)d_p$, and $\Delta_{ker(g_p)}$ is determined by the rank of $ker(g_p)$ as a $\mathbb{Q}$-vector space. Hence $ker(g)$ and $ker(g^*)$ have the same order ideals. □

Remark. Although they have the same order ideals, in general the two kernels are not isomorphic as Λ-modules. An example is given by $g : \Lambda/p \oplus \Lambda/p^2 \to \Lambda/p \oplus \Lambda/p^2$ defined by $g(a, b) = (0, pa)$. In this case, $ker(g) \simeq \Lambda/p^2$, while $ker(g^*) \simeq \bigoplus_2 \Lambda/p$.

Proof. (*of Theorem 0.1*) Let C_K be the complement of an open tubular neighbourhood of $K \subset S^{n+2}$, and $C_{K'}$ the complement of an open tubular neighbourhood of $K' \subset S^{n+1}$. As in the introduction, let $g : (D^{n+1}, J') \to (D^{n+1}, J')$ be the diffeomorphism for the deform-spinning construction of K from K', so we can isotope g so that it preserves a regular neighbourhood of $J' \cup S^n$, therefore g restricts to a diffeomorphism of $C_{K'}$ (which we can think of as the complement of an open regular neighbourhood of $S^n \cup J'$ in D^{n+1}), giving a diffeomorphism

$$C_K \simeq (C_{K'} \times_g S^1) \cup_{\nu S^1 \times S^1} ((\nu S^1) \times D^2).$$

where νS^1 is a trivial D^{n-1}-bundle over S^1 (a meridian of $\partial C_{K'}$). The decomposition lifts to the universal abelian covering space, giving the isomorphism $H_1(\tilde{C}_K;\mathbb{Q}) \simeq coker(I - g_{1*})$ and short exact sequences

$$0 \to coker(g_{i*} - I) \to H_i(\tilde{C}_K;\mathbb{Q}) \to ker(g_{(i-1)*} - I) \to 0, \quad i > 1$$

with $g_{i*} : H_i(\tilde{C}_{K'};\mathbb{Q}) \to H_i(\tilde{C}_{K'};\mathbb{Q})$ the induced map coming from $\tilde{g} : \tilde{C}_{K'} \to \tilde{C}_{K'}$. Let q_i be the order ideal of $coker(g_{i*} - I)$.

The map $g_{i*} - I : H_i(\tilde{C}_{K'};\mathbb{Q}) \to H_i(\tilde{C}_{K'};\mathbb{Q})$ give rise to a canonical short exact sequence

$$0 \to ker(g_{i*} - I) \to H_i(\tilde{C}_{K'};\mathbb{Q}) \to img(g_{i*} - I) \to 0$$

and the inclusion $img(g_{i*} - I) \to H_i(\tilde{C}_{K'};\mathbb{Q})$ to another

$$0 \to img(g_{i*} - I) \to H_i(\tilde{C}_{K'};\mathbb{Q}) \to coker(g_{i*} - I) \to 0.$$

Lemma 1.1 (a) applied to our short exact sequences tells us that $\Delta_i = q_i q_{i-1}$.

We now reconsider the proof of the symmetry of the Alexander polynomial of a knot in S^3 [5, 6], or more precisely, the isomorphism $\overline{H_i(\tilde{C}_{K'};\mathbb{Q})} \simeq H_{n-i}(\tilde{C}_{K'};\mathbb{Q})$ derived from Poincaré Duality [4], paying special attention to naturality with respect to diffeomorphisms $g \in \mathrm{Diff}(C_{K'})$, with an eye towards proving the symmetry conditions $\overline{q_{n-i}} = q_i$.

1. $H_i(\tilde{C}_{K'};\mathbb{Q}) \simeq H_i(\tilde{C}_{K'}, \partial;\mathbb{Q})$: this is a natural isomorphism coming from the long exact sequence of a pair.
2. $H_i(\tilde{C}_{K'}, \partial;\mathbb{Q}) \simeq \overline{H^{n+1-i}(\tilde{C}_{K'};\mathbb{Q})}$: this is the Poincaré duality isomorphism; it is also natural, although it reverses arrows [4].
3. $H^{n+1-i}(\tilde{C}_{K'};\mathbb{Q}) \simeq Ext_\Lambda(H_{n-i}(\tilde{C}_{K'};\mathbb{Q}), \Lambda)$: this is a natural isomorphism coming from the universal coefficient theorem [4].
4. $Ext_\Lambda(H_{n-i}(\tilde{C}_{K'};\mathbb{Q}), \Lambda) \simeq H_{n-i}(\tilde{C}_{K'};\mathbb{Q})$. This last result uses that both modules have a square presentation matrix, with one being the transpose of the other. Since Λ is a principal ideal domain, the presentation matrices are equivalent to the same diagonal matrices. This isomorphism is not natural.

Thus we have a non-natural isomorphism $H_i(\tilde{C}_K;\mathbb{Q}) \simeq \overline{H_{n-i}(\tilde{C}_K;\mathbb{Q})}$. The natural part of the isomorphism can be expressed by the commutative diagram

$$\begin{tikzcd}
\overline{H_i(\tilde{C}_K)} \arrow[r] \arrow[d, "g_*"] & \overline{H_i(\tilde{C}_K, \partial)} \arrow[r, "PD"] \arrow[d, "g_*"] & H^{n+1-i}(\tilde{C}_K) & Ext_\Lambda\left(H_{n-i}(\tilde{C}_K), \Lambda\right) \arrow[l, "UCT"'] \\
\overline{H_i(\tilde{C}_K)} \arrow[r] & \overline{H_i(\tilde{C}_K, \partial)} \arrow[r, "PD"] & H^{n+1-i}(\tilde{C}_K) \arrow[u, "g^*"] & Ext_\Lambda\left(H_{n-i}(\tilde{C}_K), \Lambda\right) \arrow[l, "UCT"'] \arrow[u, "(g_*)^*"]
\end{tikzcd}$$

This gives us an isomorphism of Λ-modules $\overline{ker(I - g_{i*})} \simeq ker(I - (g^{-1}_{(n-i)*})^*)$, so

$$\overline{ker(I - g_{i*})} \simeq ker(I - (g^{-1}_{(n-i)*})^*) = ker(I - (g_{(n-i)*})^*).$$

Lemma 1.1 (d), tells us that $ker(I - (g_{(n-i)*})^*)$ and $ker(I - g_{(n-i)*})$ have the same order ideals. Thus, $\overline{q_i} = q_{n-i}$. $\square$

2. Comments and questions

Levine [4] has a complete characterization of the Alexander modules of co-dimension two knots. A natural question would be, could one derive further other obstructions to deform-spinning from the Alexander modules of knots? The primary aspect of Levine's work that we've neglected is the $\mathbb{Z}$-torsion submodule of $H_i(\tilde{C}_K;\mathbb{Z})$. Simple experiments show that when $K \subset S^{n+2}$ is deform-spun from a knot $K' \subset S^{n+1}$, the Alexander modules of K can have $\mathbb{Z}$-torsion, even when the Alexander modules of K' do not. Moreover, twist-spinning sufficies to produce many such examples. So any torsion obstructions to deform-spinning, if they exist, would likely be fairly subtle.

In co-dimension larger than two, deform-spinning is the boundary map in the pseudo-isotopy long exact sequence for embedding spaces and diffeomorphism groups [1]. Moreover, Cerf's Pseudoisotopy Theorem states that, in the case of diffeomorphism groups of discs, this map is onto, provided the dimension of the disc is 6 or larger. So one might expect an analogy.

Question 2.1. *Is there a simple characterization of deform-spun co-dimension two knots $K \subset S^{n+2}$ (provided n is large)?*

One would certainly expect more obstructions to deform-spinning than the ones in this paper. For example, let K_1 and K_2 be two otherwise unrelated 2-knots such that $\Delta_{K_1}(t) = 2 - t$ and $\Delta_{K_2}(t) = 2t - 1$. Their connect sum has Alexander polynomial $\Delta_{K_1\#K_2}(t) = -2t^2 + 3t - 2$ which is symmetric, but we have no reason to expect $K_1\#K_2$ is deform-spun.

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Mathematics and Statistics, University of Victoria, PO BOX 3045 STN CSC, Victoria, B.C., Canada V8W 3P4
E-mail address: `rybu@uvic.ca,`

ACRI, 260 route du Pin Montard, BP 234, F-06904 sophia-Antipolis Cedex - France
E-mail address: `sasha.mozgova@gmail.com`