BOUNDARIES AND JSJ DECOMPOSITIONS OF $CAT(0)$-GROUPS

PANOS PAPASOGLU AND ERIC SWENSON

ABSTRACT. Let G be a one-ended group acting discretely and co-compactly on a CAT(0) space X. We show that ∂X has no cut points and that one can detect splittings of G over two-ended groups and recover its JSJ decomposition from ∂X.

We show that any discrete action of a group G on a CAT(0) space X satisfies a convergence type property. This is used in the proof of the results above but it is also of independent interest. In particular, if G acts co-compactly on X, then one obtains as a Corollary that if the Tits diameter of ∂X is bigger than $\frac{3\pi}{2}$ then it is infinite and G contains a free subgroup of rank 2.

1. Introduction

The purpose of this paper is to generalize results about boundaries and splittings of hyperbolic groups in the case of $CAT(0)$ groups. Before stating our results we summarize what is known in the hyperbolic case.

Bestvina and Mess ([5]) showed that if the boundary of a one-ended hyperbolic group does not have a cut point then it is locally connected. They asked whether the boundary of a one-ended hyperbolic group can contain a cut point. The negative answer to this is mainly due to Bowditch ([6], [7], [8]) with contributions by Swarup and Levitt ([27], [18]). Bowditch [6] further showed that the boundary of a one-ended hyperbolic group has local cut points if and only if either the group splits over a 2-ended group or it is a hyperbolic triangle group. He deduced from this a canonical JSJ decomposition for hyperbolic groups (compare [26]).

We remark that in the CAT(0) case, boundaries are not necessarily locally connected (e.g. if $G = F_2 \times \mathbb{Z}$, where F_2 is the free group of rank 2, the boundary is a suspension of a Cantor set, hence it is not locally connected). However the question whether the boundaries have

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cut points also makes sense in this case. Indeed the second author showed that boundaries have no cut points under the assumption that the group does not contain an infinite torsion subgroup ([28]).

To state our results we recall some terminology. An action of a group G on a space X is called *proper* if for every compact $K \subset X$, the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite. An action of a group G on a space X is called *co-compact* if the quotient space of X by the action of G, X/G is compact. An action of a group G on a metric space X is called *geometric* if G acts properly, co-compactly by isometries on X. It follows that G is quasi-isometric to X, that is they have the same coarse geometry. If Z is a compact connected metric space we say that c is a *cut point* of Z if $Z - c$ is not connected. We say that a pair of points $\{a, b\}$ is a *cut pair* of Z if $Z - \{a, b\}$ is not connected.

We show the following.

Theorem 1. *Let G be a one-ended group acting geometrically on the CAT(0) space X. Then ∂X has no cut points.*

We note that Croke-Kleiner ([14]) showed it is possible for a 1-ended group G to act geometrically on CAT(0) spaces X, Y such that ∂X, ∂Y are not homeomorphic. Thus one can not talk about 'the' boundary of a CAT(0) group as in the case of hyperbolic groups.

We remark that if a one ended hyperbolic group G splits over a two ended group C, then the limit points of C separate the boundary of G. This holds also for $CAT(0)$ groups.

We show that the converse also holds, so one can detect splittings of $CAT(0)$ groups from their boundaries:

Theorem 2. *Let G be a one ended group acting geometrically on a CAT(0) space X. If ∂X has a cut pair then either G splits over a 2-ended group or G is virtually a surface group.*

We remark that in case ∂X is locally connected the theorem above follows from [20]. Indeed in this case if ∂X has a cut pair, it can be shown that a quasi-line coarsely separates X.

We showed in [21] that if Z is a continuum without cut points, then we can associate to Z an $\mathbb{R}$-tree T (called the JSJ tree of Z) encoding all pairs of points that separate Z.

We show here that if G is a 1-ended CAT(0) group and Z is any CAT(0) boundary then the $\mathbb{R}$-tree T is simplicial and gives the JSJ decomposition of G. More precisely we have:

Theorem 3. *Let G be a one ended group acting geometrically on a CAT(0) space X. Then the JSJ-tree of ∂X is a simplicial tree T and* *the graph of groups T/G gives a canonical JSJ decomposition of G over 2-ended groups.*

We note that the JSJ decomposition in the previous Theorem is closely related to the JSJ decomposition constructed by Scott-Swarup ([25]).

Our strategy for obtaining these results is similar to the one of Bowditch in the case of hyperbolic groups. For example to show that ∂G has no cut point, Bowditch associates a tree T to ∂G and studies the action of G on T via Rips theory.

The main difficulty in the case of $CAT(0)$ groups is that G does not act on the boundary as a convergence group so it is not immediate, as it is in the hyperbolic case, that the action on the trees we construct has no global fixed point. To deal with this difficulty we show that in the CAT(0) case the action has a convergence type property:

Theorem 4. *Let X be a CAT(0) space and G a group acting properly on X. For any sequence of distinct group elements of G, there exists a subsequence (g_i) and points $n, p \in \partial X$ such that for any compact set $C \subset \partial X - \bar{B}_T(n, \theta)$, $g_n(C) \to \bar{B}_T(p, \pi - \theta)$ (in the sense that for any open $U \supset \bar{B}_T(p, \pi - \theta)$, $g_i(C) \subset U$ for all i sufficiently large).*

We denote by $\bar{B}_T(n, \theta)$ the closed Tits ball with center n and radius θ. See [28] and [17] for partial results of this type.

The above Theorem plays a crucial role in our proofs. We think it is also of independent interest as it is a useful tool to study actions of CAT(0) groups on their boundaries. For example if you quotient out by the Tits components, G acts on this quotient as a convergence group (see Corollary 21). More importantly, we obtain the following improvement of a result of Ballmann and Buyalo [2]:

Theorem 5. *If the Tits diameter of ∂X is bigger than $\frac{3\pi}{2}$ then G contains a rank 1 hyperbolic element. In particular: if G doesn't fix a point of ∂X and doesn't have rank 1, and I is a minimal closed invariant set for the action of G on ∂X, then for any $x \in \partial X$, $d_T(x, I) \leq \frac{\pi}{2}$.*

It follows from this Theorem that if the Tits diameter of ∂X is bigger than $\frac{3\pi}{2}$ then G contains a free subgroup of rank 2 ([3], theorem A). We remark that the Tits alternative is not known for CAT(0) groups. In fact Ballmann-Buyalo ([2]) conjecture that if G acts geometrically on X and the Tits diameter of ∂X is bigger than π then G contains a rank 1 element (so G contains a free subgroup of rank 2). In the remaining case of Tits diameter π they conjecture that X is either a symmetric space or a Euclidean building or reducible. We note that the Tits alternative is known in the case of symmetric spaces and Euclidean buildings, so it would be implied for all CAT(0) groups by an affirmative answer to the above conjectures. On the other hand if X is a cube complex then the Tits alternative holds as shown by Sageev and Wise ([24]).

1.1. **Plan of the paper.** In section 2 we give a summary of our earlier paper [21]. We explain how to produce trees from cut points and cut pairs. More precisely, we show that if Z is a compact connected metric space, then there is an $\mathbb{R}$-tree T 'encoding' all cut points of Z. In particular if Z has a cut point, then T is non trivial. The construction of T is canonical, so the homeomorphism group of Z acts on T. If Z has no cut points, we show that there is an $\mathbb{R}$-tree T 'encoding' all cut pairs of Z (the 'JSJ tree' of Z). The JSJ tree is also canonical.

In section 3 we recall background and terminology for CAT(0) groups and spaces. In section 4 we show that actions of CAT(0) groups on boundaries satisfy a convergence type property (π-convergence). Sections 5 and 6 are devoted to the proof that CAT(0) boundaries of one ended groups have no cut points. In section 5 we apply the machinery of section 2 to construct an $\mathbb{R}$-tree on which G acts. Using π-convergence we show that the action is non-trivial. In section 6 we show that the action is stable and we apply the Rips machine to show that T is in fact simplicial. From this we arrive at a contradiction. In section 7 we assume that ∂X has a cut pair. We apply the construction of section 2 and we obtain the JSJ $\mathbb{R}$-tree for ∂X. Using π-convergence again we show that the action is non-trivial, and applying the Rips machine we show that the tree is in fact simplicial. We deduce that G splits over a 2-ended group unless it is virtually a surface group. We show further that the action on the tree gives the JSJ decomposition of G.

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2. Trees and continua

2.1. **From cut points to trees.** Whyburn ([29]) was the first to study the cut point set of a locally connected metric space and to show that it is 'treelike' (a dendrite). Bowditch ([8]) showed how to construct $\mathbb{R}$-trees for boundaries of hyperbolic groups while Swenson ([28]) gave a more direct construction. In [21] we showed how to associate to a continuum Z an $\mathbb{R}$-tree T encoding the cut points of Z. We recall here the results and terminology of [21].

In the following Z will be a (metric) continuum.

Definition . If $a, b, c \in Z$ we say that $c \in (a, b)$ if a, b lie in distinct components of $Z - \{c\}$.

We call (a,b) an interval and this relation an interval relation. We define closed and half open intervals in the obvious way i.e. $[a,b) = \{a\} \cup (a,b)$, $[a,b] = \{a,b\} \cup (a,b)$ for $a \neq b$ and $[a,a) = \emptyset$, $[a,a] = \{a\}$.

We remark that if $c \in (a,b)$ for some a,b then c is a cut point. Clearly $(a,a) = \emptyset$ for all $a \in Z$.

Definition . We define an equivalence relation on Z. Each cut point is equivalent only to itself and if $a,b \in Z$ are not cut points we say that a is equivalent to b, $a \sim b$ if $(a,b) = \emptyset$.

Let's denote by $\mathcal{P}$ the set of equivalence classes for this relation. We can define an interval relation on $\mathcal{P}$ as follows:

Definition . For $x,y,z \in \mathcal{P}$ we say that $z \in (x,y)$ if for all $a \in x, b \in y, c \in z$ we have that

$$[a,c) \cap (c,b] = \emptyset$$

We call elements of $\mathcal{P}$ which are not cut points *maximal inseparable sets*.

If $x,y,z \in \mathcal{P}$ we say that z is between x,y if $z \in (x,y)$. It is shown in [21] that $\mathcal{P}$ with this betweeness relation is a pretree.

The $\mathbb{R}$ tree T is obtained from $\mathcal{P}$ by "connecting the dots" according to the pretree relation on $\mathcal{P}$. We proceed now to give a rigorous definition of T.

We have the following results about intervals in pretrees from [8]:

Lemma 6. *If $x,y,z \in \mathcal{P}$, with $y \in [x,z]$ then $[x,y] \subset [x,z]$.*

Lemma 7. *Let $[x,y]$ be an interval of $\mathcal{P}$. The interval structure induces two linear orderings on $[x,y]$, one being the opposite of the other, with the property that if $<$ is one of the orderings, then for any $z,w \in [x,y]$ with $z < w$, $(z,w) = \{u \in [x,y] : z < u < w\}$. In other words the interval structure defined by one of the orderings is the same as our original interval structure.*

Definition . If x,y are distinct points of $\mathcal{P}$ we say that x,y are *adjacent* if $(x,y) = \emptyset$. We say $x \in \mathcal{P}$ is *terminal* if there is no pair $y,z \in \mathcal{P}$ with $x \in (y,z)$.

We recall [28, Lemma 4].

Lemma 8. *If $x,y \in \mathcal{P}$, are adjacent then exactly one of them is a cut point and the other is a nonsingleton equivalence class whose closure contains this cut point.*

We have the Theorem [28, Theorem 6]:

Theorem 9. *A nested union of intervals of $\mathcal{P}$ is an interval of $\mathcal{P}$.*

Corollary 10. *Any interval of $\mathcal{P}$ has the supremum property with respect to either of the linear orderings derived from the interval structure.*

Proof. Let $[x, y]$ be an interval of $\mathcal{P}$ with the linear order $\leq$. Let $A \subset [x, y]$. The set $\{[x, a] : a \in A\}$ is a set of nested intervals so their union is an interval $[x, s]$ or $[x, s)$ and $s = sup\, A$. $\square$

Definition . A *big arc* is the homeomorphic image of a compact connected nonsingleton linearly ordered topological space. A separable big arc is called an *arc*. A *big tree* is a uniquely big-arcwise connected topological space. If all the big arcs of a big tree are arcs, then the big tree is called a *real tree*. A metrizable real tree is called an $\mathbb{R}$-tree. An example of a real tree which is not an $\mathbb{R}$-tree is the long line (see [16] sec.2.5, p.56).

Definition . A pretree $\mathcal{R}$ is *complete* if every closed interval is complete as a linearly ordered topological space (this is slightly weaker than the definition given in [8]). Recall that a linearly ordered topological space is complete if every bounded set has a supremum.

Let $\mathcal{R}$ be a pretree, an interval $I \subset \mathcal{R}$ is called *preseparable* if there is a countable set $Q \subset I$ such that for every nonsingleton closed interval $J \subset I$, $J \cap Q \neq \emptyset$. A pretree is *preseparable* if every interval in it is preseparable.

We recall now the construction in [21]. Let $\mathcal{R}$ be a complete pretree. Set

$$T = \mathcal{R} \cup \bigsqcup_{x,y \text{ } adjacent} I_{x,y}$$

where $I_{x,y}$ is a copy of the real open interval $(0, 1)$ glued in between x and y. We extend the interval relation of $\mathcal{R}$ to T in the obvious way (as in [28]), so that in T, $(x, y) = I_{x,y}$. It is clear that T is a complete pretree with no adjacent elements. When $\mathcal{R} = \mathcal{P}$, we call the T so constructed the cut point tree of Z.

Definition . For A finite subset of T and $s \in T$ we define

$$U(s, A) = \{t \in T : [s, t] \cap A = \emptyset\}$$

The following is what the proof of [28, Theorem 7] proves in this setting.

Theorem 11. *Let $\mathcal{R}$ be a complete pretree. The pretree T, defined above, with the topology defined by the basis $\{U(s, A)\}$ is a regular big tree. If in addition $\mathcal{R}$ is preseparable, then T is a real tree.*

We recall [21, Theorem 13]

Theorem 12. *The pretree $\mathcal{P}$ is preseparable, so the cut point tree T of Z is a real tree.*

The real tree T is not always metrizable. Take for example Z to be the cone on a Cantor set C (the so called Cantor fan). Then Z has only one cut point, the cone point p, and $\mathcal{P}$ has uncountable many other elements q_c, one for each point $c \in C$. As a pretree, T consists of uncountable many arcs $\{[p, q_c] : c \in C\}$ radiating from a single central point p. However, in the topology defined from the basis $\{U(s, A)\}$, every open set containing p contains the arc $[p, q_c]$ for all but finitely many $c \in C$. There can be no metric, d, giving this topology since $d(p, q_c)$ could only be non-zero for countably many $c \in C$. In [21] we showed that it is possible to equip T with a metric that preserves the pretree structure of T. This metric is 'canonical' in the sense that any homeomorphism of Z induces a homeomorphism of T. We recall briefly how this is done. The idea is to metrize T in two steps. In the first step one metrizes the subtree obtained by the span of cut points of $\mathcal{P}$. This can be written as a countable union of intervals and it is easy to equip with a metric.

T is obtained from this tree by gluing intervals to some points of T. In this step, one might glue uncountably many intervals, but the situation is similar to the Cantor fan described above. The new intervals are metrized in the obvious way, e.g. one can give all of them length one.

So we have the following:

Theorem 13. *There is a path metric d on T, which preserves the pretree structure of T, such that (T, d) is a metric $\mathbb{R}$-tree. The topology so defined on T is canonical (and may be different from the topology with basis $\{U(s, A)\}$). Any homeomorphism ϕ of Z induces a homeomorphism $\hat{\phi}$ of T equipped with this metric.*

Since T has the supremum property, all ends of T correspond to elements of $\mathcal{P}$, and these elements will be terminal in $\mathcal{P}$ and in T. We remove these terminal points to obtain a new tree which we still call T. Clearly the previous Theorem holds for this new tree too.

2.2. **JSJ trees for continua.** In [21] we showed how to associate to cut-pairs of a continuum an $\mathbb{R}$-tree which we call the JSJ-tree of the continuum. The construction is similar as in the case of cut points but for JSJ-trees a new type of vertices has to be introduced which corresponds to the hanging orbifold vertex groups in the group theoretic setting. We recall briefly now the results of [21].

Let Z be a metric continuum without cut points. The set $\{a, b\}$ is a cut pair of Z separating $p \in Z$ from $q \in Z$ if there are continua $P \ni p$ and $Q \ni q$ such that $P \cap Q = \{a, b\}$ and $P \cup Q = Z$.

Definition . Let Z be a continuum without cut points. A finite set S with $|S| > 1$ is called a *cyclic* subset if S is a cut pair, or if there is an ordering $S = \{x_1, \dots x_n\}$ and continua $M_1, \dots M_n$ with the following properties:

- $M_n \cap M_1 = \{x_1\}$, and for $i > 1$, $\{x_i\} = M_{i-1} \cap M_i$
- $M_i \cap M_j = \emptyset$ for $i - j \neq \pm 1 \mod n$
- $\bigcup M_i = Z$

The collection $M_1, \dots M_n$ is called the (a) *cyclic decomposition* of Z by $\{x_1, \dots x_n\}$. (For $n > 2$, this decomposition is unique.) If S is an infinite subset of Z and every finite subset $A \subset S$ with $|A| > 1$ is cyclic, then we say S is cyclic. Every cyclic set is contained in a maximal cyclic set. We call a maximal cyclic subset of Z with more than 2 points a *necklace*.

Let S be a cyclic subset of Z. There exists a continuous function $f : Z \to S^1$ with the following properties:

(1) The function f is one to one on S, and $f^{-1}(f(S)) = S$
(2) For $x, y \in Z$ and $a, b \in S$:
    (a) If $\{f(a), f(b)\}$ separates $f(x)$ from $f(y)$ then $\{a, b\}$ separates x from y.
    (b) If $x \in S$ and $\{a, b\}$ separates x from y, then $\{f(a), f(b)\}$ separates $f(x)$ from $f(y)$

Furthermore if $|S| > 2$ (if S is a necklace for example), this function can be chosen in a canonical fashion up to isotopy and orientation. This function will be called the circle function for S and denoted f_S.

Definition . Let Z be a metric space without cut points. A non-empty non-degenerate set $A \subset Z$ is called inseparable if no pair of points in A can be separated by a cut pair. Every inseparable set is contained in a maximal inseparable set.

We define $\mathcal{P}$ to be the collection of all necklaces of Z, all maximal inseparable subsets of Z, and all inseparable cut pairs of Z.

If two elements of $\mathcal{P}$ intersect then the intersection is either a single point, or an inseparable cut pair. There is a natural pre-tree structure on $\mathcal{P}$ (given by separation). The pre-tree $\mathcal{P}$ has the property that every monotone sequence in a closed interval of $\mathcal{P}$ converges in that interval.

Distinct points of $\mathcal{P}$ are adjacent if there there are no points of $\mathcal{P}$ between them.

Lemma 14. [21, Lemma 28] *If $A, B \in \mathcal{P}$, with $A \subset B$ (thus A is an inseparable pair and B is maximal cyclic or maximal inseparable) then A and B are adjacent. The only other way two points of P can be adjacent is if one of them, say A, is a necklace and the other, B, is maximal inseparable and $|\partial A \cap B| \geq 2$.*

The Warsaw circle gives an example of this.

Recall that a point of $\mathcal{P}$ is called terminal if it is not in any open interval of $\mathcal{P}$. Terminal points of $\mathcal{P}$ do arise in CAT(0) boundaries even when the action on $\mathcal{P}$ is nontrivial (see [14] for example). We first remove all terminal points from $\mathcal{P}$, and then glue open intervals between the remaining adjacent points of $\mathcal{P}$ to form a topological $\mathbb{R}$-tree T. We call T the **JSJ-Tree** of the continuum Z. Notice that in the case where Z is a circle, $\mathcal{P}$ has only one element, a necklace, and so T is empty. This and similar issues will be dealt with by reducing to the case where G doesn't fix an element of $\mathcal{P}$, in which case T will be non-trivial.

3. CAT(0) Groups and boundaries

We give a brief introduction to CAT(0) spaces (see [9], [3] for details). Let Y be a metric space, and I be an interval of $\mathbb{R}$. A path $\gamma : I \to X$ is called a *geodesic* if for any $[a, b] \subset I$,

$$\ell_a^b(\gamma) = b - a = d(\gamma(a), \gamma(b))$$

where $\ell_a^b(\gamma)$ is the length of γ from a to b, defined as

$$\ell_a^b(\gamma) = \sup \left\{ \sum_{i=1}^{n} d\left(\gamma(x_{i-1}), \gamma(x_i)\right) : \{x_0, \dots x_n\} \text{ is a partition of } [a, b] \right\}$$

If I is a ray or $\mathbb{R}$ then we refer to γ (or more precisely its image) as a geodesic ray or geodesic line respectively. If I is a closed interval, we refer to γ as a geodesic segment.

A metric space is called *proper* if every closed metric ball is compact. In a proper metric space, closed and bounded implies compact.

A CAT(0) space X is a proper geodesic metric space with the property that every geodesic triangle (the union of three geodesic segments meeting in the obvious way) in X is a least as thin as the comparison Euclidean triangle, the triangle in the Euclidean plane with the same edge lengths. Every CAT(0) space is contractible.

The (visual) boundary ∂X is the set of equivalence classes of unit speed geodesic rays, where $R, S : [0, \infty) \to X$ are equivalent if $d(R(t), S(t))$ is bounded. Given a ray R and a point $x \in X$ there is a ray S emanating from x with $R \sim S$. Fixing a base point $\mathbf{0} \in X$ we define a

Topology on $\bar{X} = X \cup \partial X$ by taking the basic open sets of $x \in X$ to be the open metric balls about x. For $y \in \partial X$, and R a ray from $\mathbf{0}$ representing y, we construct basic open sets $U(R, n, \epsilon)$ where $n, \epsilon > 0$. We say $z \in U(R, n, \epsilon)$ if the unit speed geodesic, $S : [0, d(\mathbf{0}, z)] \to \bar{X}$, from $\mathbf{0}$ to z satisfies $d(R(n), S(n)) < \epsilon$. These sets form a basis for a topology on $\bar{X}$ under which $\bar{X}$ and ∂X are compact metrizable. Isometries of X act by homeomorphisms on ∂X.

For three points $a, b, p \in X$ $a \neq p \neq b$ the *comparison angle* $\overline{\angle}_p(a, b)$ is the Euclidean angle at the point corresponding to p in the comparison triangle in $\mathbb{E}^2$. By CAT(0), for any $c \in (p, a)$ and $d \in (p, b)$, $\overline{\angle}_p(a, b) \geq \overline{\angle}_p(c, d)$. This monotonicity implies that we can take limits. Thus the *angle* is defined as $\angle_p(a, b) = \lim \overline{\angle}_p(a_n, b_n)$ where $(a_n) \subset (p, a)$ with $a_n \to p$ and $(b_n) \subset (p, b)$ with $b_n \to p$. By monotonicity this limit exists, and this definition works equally well if one or both of a, b are in ∂X. Similarly we can define the *comparison angle*, $\overline{\angle}_p(a, b)$ even if one or both of a, b are in ∂X (the limit will exist by monotonicity). By [9, II 1.7(4)], $\overline{\angle}_p(a, b) \geq \angle_p(a, b)$.

On the other hand for $a, b \in \partial X$, we define $\angle(a, b) = \sup\limits_{p \in X} \angle_p(a, b)$. It follows from [9, II 9.8(1)] that $\angle(a, b) = \overline{\angle}_p(a, b)$ for any $p \in X$. Notice that isometries of X preserve the angle between points of ∂X. The angle defines a path metric, d_T on the set ∂X, called the Tits metric, whose topology is finer than the given topology of ∂X. Also $\angle(a, b)$ and $d_T(a, b)$ are equal whenever either of them are less than π ([9, II 9.21(2)].

The set ∂X with the Tits metric is called the Tits boundary of X, denoted TX. Isometries of X extend to isometries of TX.

The identity function $TX \to \partial X$ is continuous ([9, II 9.7(1)]), but the identity function $\partial X \to TX$ is only lower semi-continuous([9, II 9.5(2)]). That is for any sequences $(y_n), (x_n) \subset \partial X$ with $x_n \to x$ and $y_n \to y$ in ∂X, then

$$\underline{\lim}\, d_T(x_n, y_n) \geq d_T(x, y)$$

Recall that the action of a group H on a space Y is called:

- *Proper* if for every compact $K \subset Y$, the set $\{h \in H : h(K) \cap K \neq \emptyset\}$ is finite.
- *Cocompact* if the quotient space of Y by the action of H, Y/H is compact.
- *Geometric* if H acts properly, cocompactly by isometries on Y.

It follows that H is quasi-isometric to Y, that is they have the same coarse geometry.

A *hyperbolic* isometry g of a CAT(0) space X is an isometry which acts by translation on a geodesic line $L \subset X$. The distance that g translates along L is called the *translation length* of g denoted $|g|$. Any such line L is called an *axis* of g and the endpoints of L (in ∂X) are denoted by g^+ and g^-, where g^+ is the endpoint in the direction of translation. If a group acts geometrically on a CAT(0) space, then the non-torsion elements are exactly the hyperbolic elements, and every group element g has a *minimum* set ($\operatorname{Min} g$) which is the set of points moved a minimum distance by g. When g is hyperbolic, the $\operatorname{Min} g$ is just the union of the axes of g. When g is torsion (called *elliptic*) then $\operatorname{Min} g$ is just the set of points of X fixed by g. In either case, $\operatorname{Min} g$ is convex and the centralizer Z_g acts geometrically on $\operatorname{Min} g$.

If a group G acts geometrically on a non-compact CAT(0) space X, then G has a hyperbolic element, and ∂X is finite dimensional ([28, Theorem 12]).

Definition . If X is a CAT(0) space and $A \subset X$ we define ΛA to be the set of limit points of A in ∂X. If H is a group acting properly by isometries on the CAT(0) space X we define $\Lambda H = \Lambda H x$ for some $x \in X$. Observe that ΛH does not depend on the choice of x.

For the remainder of the paper, G will be a group acting geometrically on the non-compact CAT(0) space X

Lemma 15. *If $h \in G$ is hyperbolic and the centralizer Z_{h^n} is virtually cyclic for all n, then for some n, $stab(h^{\pm}) = Z_{h^n}$*

Proof. By [28, Zipper Lemma] hyperbolic elements with the same endpoints have a common power. Thus we may assume that h has the minimal translation length of any hyperbolic element with endpoints $h^{\pm}$. Let $H = stab(h^{\pm})$. Let Z be the union of all lines from h^+ to h^-. By [9] $W = R \times Y$ where R is an axis of h and Y is a convex subset of X, and $stab(h^{\pm})$ acts on W preserving the product structure. Since h has minimal translation length, every element of H translates the R factor by a multiple of $|h|$.

By [28, Theorem 8] $stab(h^{\pm}) = \bigcup_n Z_{h^n}$. Since $Z_{h^n} \cup Z_{h^m} < Z_{h^{nm}}$, it suffices to uniformly bound the number of cosets of $\langle h \rangle$ in Z_{h^n}. We do this by finding a section of the projection function $Z_{h^n} \to Z_{h^n}/\langle h \rangle$ whose image generates a finite group. Since there are only finitely many conjugacy classes of finite subgroups of G, there is a uniform bound on the size of a finite subgroup of G, and this number also bounds the number of cosets of $\langle h \rangle$ in Z_{h^n} for all n.

Since Z_{h^n} is virtually cyclic, the number of cosets of $\langle h\rangle$ in Z_{h^n} is finite (say p). Choose coset representatives $g_1, \dots g_p \in Z_{h^n}$. Multiplying by a multiple of h, we may assume that each g_i fixes the R factor of W. Thus $K = \langle g_1, g_2, \dots g_p\rangle$ is a subgroup of the virtually cyclic Z_{h^n} which doesn't translate along the R direction. It follows that no power of h lies in K. If two subgroups of a virtually $\mathbb{Z}$ group intersect trivially then one of them is finite, and so K is finite. □

Lemma 16. *Let K be a group acting geometrically on a CAT(0) space U. If $H < K$ with ΛH finite then H is virtually cyclic.*

Proof. We assume that H is infinite. Since ΛH is finite, passing to a finite index subgroup, we may assume that H fixes ΛH pointwise.

We first show that H contains a hyperbolic element. Suppose not, then from [28, Theorem 17] there is a convex set $Y \subset U$ with $\dim \partial Y < \dim \partial U$ such that Y is invariant under the action of H and the action of H on Y can be extended to a geometric action on Y. Since $\dim \partial U < \infty$ ([28]), we get a contradiction by induction on boundary dimension (In the case where the boundary is zero dimensional, any group acting geometrically is virtually free.)

Thus there exists $h \in H$ hyperbolic. By [28] $stab(h^{\pm}) = \bigcup_n Z_{h^n}$, and so $H < \bigcup_n Z_{h^n}$. The union W of axis of h from h^- to h^+ decomposes as $W = R \times Y$ where R is an axis for h and Y is convex in U. Since H fixes $\{h^{\pm}\}$, then H acts on W preserving this product structure.

Suppose that $H/\langle h\rangle$ is infinite. Let ϵ be the translation length of h. Choose one element g_i from each coset with the translation distance of g_i on the R factor less than ϵ. Choose m larger than $|\{g \in K \mid g(B(u,\epsilon)) \cap B(u,\epsilon) \neq \emptyset\}|$ for any $u \in U$. Choose n such that $g_1, \dots g_m \in Z_{h^n}$. Notice that $\operatorname{Min} h^n = R \times Z$ where Z is a convex subset of Y. Now consider the images $\hat{g}_1, \dots \hat{g}_m$ of $g_1, \dots g_m$ in the quotient $Z_{h^n}/\langle h^n\rangle = J$ which acts geometrically on Z. Notice that $\hat{g}_1, \dots \hat{g}_m$ are distinct elements of J. By choice of m the subgroup $F = \langle \hat{g}_1, \dots \hat{g}_m\rangle$ doesn't fix a point of Z. It follows that F is infinite. Notice that $\Lambda F \subset \partial Z$ is finite (in fact $\Lambda F \subset \Lambda H$). It follows by the above that F contains a hyperbolic element $\hat{g}$. Let g be a preimage of $\hat{g}$ in H. Clearly g is hyperbolic and $\{g^{\pm}\} \cap \{h^{\pm}\} = \emptyset$. However $g \in Z_{h^j}$ for some j, so $\langle g, h^j\rangle \cong \mathbb{Z}^2$ and has limit set a circle. This contradicts the fact that ΛH is finite.

□

4. Convergence actions

4.1. **π-Convergence Action.** Notice that $\overline{\angle}_p(a,b)$ is a function of $d(a,b)$, $d(p,a)$ and $d(b,p)$. It follows that the function $\overline{\angle}$ is continuous on the subset of X^3 , $\{(a,b,p) \in X^3 : a \neq p,\ b \neq p\}$.

Lemma 17. *Let $p,q \in X$ $p \neq q$ and $a \in \partial X$. $\overline{\angle}_p(q,a) + \overline{\angle}_q(p,a) \leq \pi$.*

Proof. Suppose not, then by the definition of these angles as monotone limits, there exists $p' \in [p,a)$ and $q' \in [q,a)$ such that $\overline{\angle}_p(q,p') + \overline{\angle}_q(p,q') > \pi$. We may also assume that $d(p,p') = d(q,q')$. By convexity of the metric applied to the rays $[p,a)$ and $[q,a)$ we have that $d(p',q') \leq d(p,q)$.

By [9, II 1.1], for any quadrilateral (in our case $pp'q'q$) in X there is a comparison quadrilateral ($\bar{p}\bar{p}'\bar{q}'\bar{q}$) in $\mathbb{E}^2$ having the same edge lengths ($d(p,p') = d(\bar{p},\bar{p}')$, $d(p',q') = d(\bar{p}',\bar{q}')$,$d(q,q') = d(\bar{q},\bar{q}')$ and $d(q,p) = d(\bar{q},\bar{p})$) with diagonals no shorter than in the original quadrilateral ($d(p,q') \leq d(\bar{p},\bar{q}')$ and $d(q,p') \leq d(\bar{q},\bar{p}')$). It follows that $\overline{\angle}_q(p,q') \leq \angle_{\bar{q}}(\bar{p},\bar{q}')$ and that $\overline{\angle}_p(q,p') \leq \angle_{\bar{p}}(\bar{q},\bar{p}')$. Thus $\angle_{\bar{q}}(\bar{p},\bar{q}') + \angle_{\bar{p}}(\bar{q},\bar{p}') > \pi$. Using high school geometry we see that $d(\bar{p}',\bar{q}') > d(\bar{p},\bar{q})$ which contradicts the fact that $d(p',q') \leq d(p,q)$. ☐

Lemma 18. *Let X be a CAT(0) space, $\theta \in (0,\pi)$, $q \in \partial X$, and K a compact subset of $\partial X - \bar{B}_T(q,\theta)$. Then for any $x \in X$, there is a point $y \in [x,q)$ and $\epsilon > 0$ with $\angle_y(q,c) > \theta + \epsilon$ for all $c \in K$.*

Proof. We first find $y \in [x,q)$ such that $\angle_y(q,c) > \theta$ for all $c \in K$. Suppose there is not such y, then by [9, II 9.8 (2)], there is a sequence of points $(c_i) \subset K$ and a monotone sequence $(y_i) \subset [x,q)$ such that $y_i \to q$ with $\angle_{y_i}(q,c_i) \leq \theta$. By the monotonicity of (y_i) and [9, II 9.8 (2)]

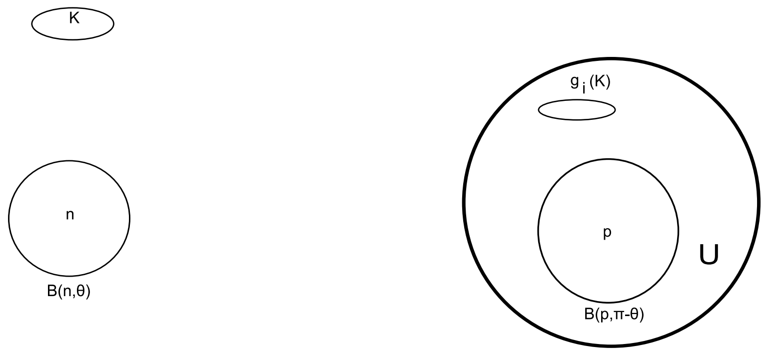
$$\angle_{y_i}(q,c_n) \leq \theta$$

for all $n \geq i$. By compactness of K, passing to a subsequence we may assume that $c_i \to c \in K$. For any fixed $i \in \mathbb{N}$, the sequence of rays $([y_i,c_n))$ converges to the ray $[y_i,c)$. Thus by [9, II 9.2(1)] $\angle_{y_i}(q,c_n) \to \angle_{y_i}(q,c) \leq \theta$ and this is true for each $i \in \mathbb{N}$. It follows from [9, II 9.8 (2)] that $\angle(q,c) \leq \theta$ which is a contradiction since $c \notin \overline{B}_T(q,\theta)$. Thus we have $y \in [x,q)$ such that $\angle_y(q,c) > \theta$ for all $c \in K$. Since the function $\angle_y$ is continuous on its domain, by compactness of K, there is a $d \in K$ with $\angle_y(q,d) \leq \angle_y(q,c)$ for all $c \in K$. We let $\epsilon < \theta - \angle_y(q,d)$ and the result follows. ☐

Lemma 19. *Let X be a CAT(0) space and G a group acting properly on X. Let $x \in X$, $\theta \in [0,\pi]$ and $(g_i) \subset G$ with the property that*

$g_i(x) \to p \in \partial X$ and $g_i^{-1}(x) \to n \in \partial X$. For any compact set $K \subset \partial X - \overline{B}_T(n,\theta)$, $g_n(K) \to \overline{B}_T(p,\pi-\theta)$ (in the sense that for any open $U \supset \overline{B}_T(p,\pi-\theta)$, $g_n(K) \subset U$ for all n sufficiently large).

The case where $\theta = \frac{\pi}{2}$ was done independently by Karlsson [17].



Proof. It suffices to consider the case $\theta \in (0,\pi)$.

Let $K \subset \partial X - \overline{B}_T(n,\theta)$ be compact, and U an open set in ∂X containing the closed ball $\overline{B}(p,\pi-\theta)$.

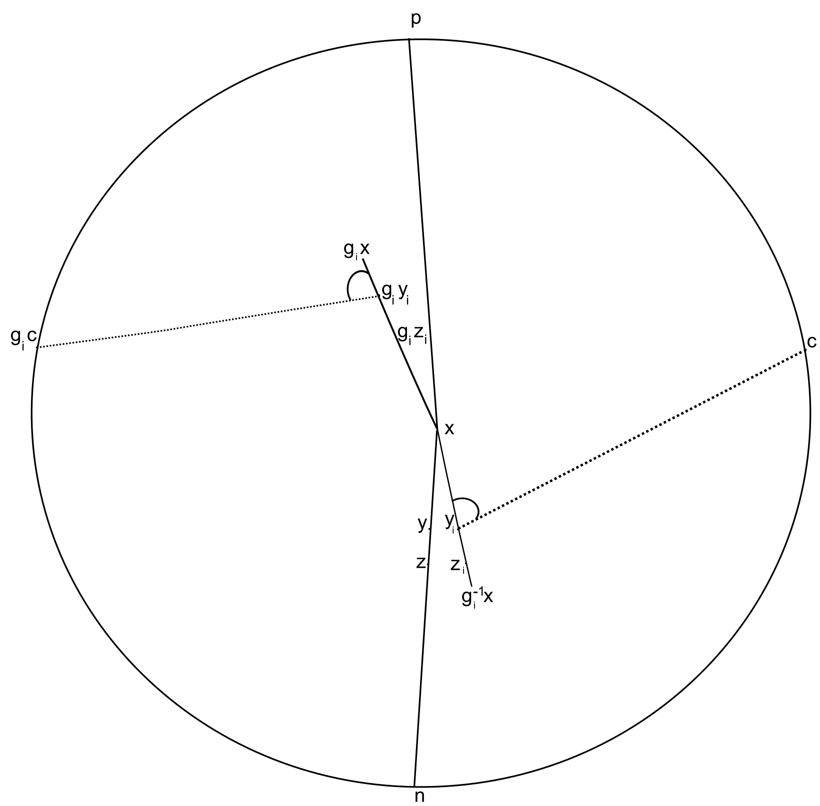
Fix $x \in X$. By Lemma 18, there is a $y \in [x,n)$ and $\epsilon > 0$ with $\angle_y(n,c) > \theta + \epsilon$ for all $c \in K$.

Notice that the sequence segments $\big([x, g_i^{-1}(x)]\big)$ converges to the ray $[x,n)$. For each $i \in \mathbb{N}$ let y_i be the projection of y onto $[x, g_i^{-1}(x)]$. Passing to a subsequence, we may assume that $d(y_i, g_i^{-1}(x)) \geq 1$. Let $z_i \in [y_i, g_i^{-1}(x)]$ such that $d(y_i, z_i) = 1$. It follows that $z_i \to z \in [y,n)$ where $d(y,z) = 1$.

Now comparison angles are continuous in all three variables for points on X and lower semi-continuous for points on ∂X. That is if $D = \{(u,v,w) \in \overline{X}^2 \times X \,|\, u \neq w \neq v\}$ then the comparison function $\overline{\angle} : D \to [0,\pi]$ is continuous on the third variable, it is continuous on $D' = \{(u,v,w) \in X^3 \,|\, u \neq w \neq v\}$ and it is lower semicontinuous on the first 2 variables. This is not true for the function $\angle : D \to [0,\pi]$, as it is only upper semicontinuous. However it is always that case that $\overline{\angle}_w(u,v) \geq \angle_w(u,v)$. Thus for any $c \in K$,

$$\overline{\angle}_{y_i}(z_i,c) \to \overline{\angle}_y(z,c) \geq \angle_y(z,c) = \angle_y(n,c) > \theta + \epsilon$$

It follows that for $i \gg 0$, $\overline{\angle}_{y_i}(z_i,c) > \theta + \epsilon$ for all $c \in K$.



Now consider the segments $g_i([x, g_i^{-1}(x)]) = [g_i(x), x]$ and note that $[g_i(x), x] \to (p, x]$ the ray. Notice that for any $c \in K$ and $i \gg 0$, $\overline{\angle}_{g_i(y_i)}(g_i(z_i), g_i(c)) = \overline{\angle}_{y_i}(z_i, c) > \theta + \epsilon$.

Let $u \in (p, x]$, for each i, let u_i be the projection of u to $[g_i(x), x]$. For $i \gg 0$, $u_i \in [g_i(z_i), x]$ and $d(u_i, g_i(z_i)) > 1$. For $i \gg 0$ choose $v_i \in [g_i(z_i), u_i]$ with $d(v_i, u_i) = 1$. By Lemma 17 and the monotonicity of the comparison angle,

$$\overline{\angle}_{g_i(y_i)}(g_i(z_i), g_i(c)) + \overline{\angle}_{u_i}(v_i, g_i(c)) \leq \pi$$

for all $c \in K$ and $i \gg 0$. Thus for $i \gg 0$

$$\overline{\angle}_{u_i}(v_i, g_i(c)) < \pi - \theta - \epsilon$$

for all $c \in K$.

Notice that $v_i \to v \in (p, u]$ where $d(v, u) = 1$. Using the continuity of $\overline{\angle}$, we can show that for $i \gg 0$, $\pi - \theta > \overline{\angle}_u(v, g_i(c)) \geq \angle_u(p, g_i(c))$. So for any $u \in (p, x]$ there is $I \in \mathbb{N}$ such that for all $i > I$ and for all $c \in K$, $\angle_u(p, g_i(c)) < \pi - \theta$.

Claim: $g_i(K) \subset U$ for all $i \gg 0$. Suppose not, then passing to a subsequence, for each i, there exists $c_i \in K$, such that $g_i(c_i) \notin U$.

Passing to a subsequence we may assume that $g_i(c_i) \to \hat{c} \in \partial X - B_T(p, \pi - \theta)$. Since $\angle(\hat{c}, p) > \pi - \theta$, there exists $w \in (p, x]$ such that $\angle_w(\hat{c}, p) > \pi - \theta$. However this contradicts the fact that for $i \gg 0$, $\angle_w(p, g_i(c_i)) < \pi - \theta$. This proves the claim and the Lemma. ☐

Theorem 20. *Let X be a CAT(0) space and G a group acting properly by isometries on X. For any sequence of distinct group elements of G, there exists a subsequence (g_i) and points $n, p \in \partial X$ such that for any $\theta \in [0, \pi]$ and for any compact set $K \subset \partial X - \overline{B}_T(n, \theta)$, $g_n(K) \to \overline{B}_T(p, \pi - \theta)$ (in the sense that for any open $U \supset \overline{B}_T(p, \pi - \theta)$, $g_i(K) \subset U$ for all i sufficiently large).*

Proof. For any sequence of distinct elements of G, using the compactness of $\overline{X} = X \cup \partial X$, there exists a subsequence (g_i) and points $n, p \in \partial X$ such that $g_i(x) \to p$, $g_i^{-1}(x) \to n$ for some $x \in X$, and so Lemma 19 applies. ☐

We will refer to the results of Lemma 19 and Theorem 20 collectively as *π-convergence.*

Corollary 21. *Let G act properly by isometries on X, a CAT(0) space. If $\tau : \partial X \to Z$ is a G-quotient map, with the property that for any $a, b \in \partial X$ with $d_T(a, b) \leq \frac{\pi}{2}$, $\tau(a) = \tau(b)$, then the induced action of G on Z is a convergence action.*

Proof. Let (g_i) be a sequence of distinct elements of G. By π-convergence, passing to a subsequence there exists $n, p \in \partial X$ such that for any compact $K \subset \partial X - B_T(n, \frac{\pi}{2})$ and any open $U \supset B_T(p, \frac{\pi}{2})$, $g_i(K) \subset U$ for all $i \gg 0$. Now let $\hat{p} = \tau(p)$ and $\hat{n} = \tau(n)$. For any compact $\hat{K} \subset Z$ with $\hat{n} \notin \hat{K}$, and any open $\hat{U} \ni \hat{p}$, $\tau^{-1}(\hat{U}) = U$ is an open set containing $B_T(p, \frac{\pi}{2})$ and $\tau^{-1}(\hat{K}) = K$ is a compact set missing $B_T(n, \frac{\pi}{2})$. Thus for $i \gg 0$, $g_i(K) \subset U$ and so $g_i(\hat{K}) \subset \hat{U}$ as required.

☐

In the later sections we will be using the following result of Ballmann-Buyalo ([2, Proposition 1.10])

Theorem 22. *If the Tits diameter of ∂X is bigger than 2π then G contains a rank 1 hyperbolic element. In particular: If G doesn't have rank 1, and I is a minimal closed invariant set for the action of G on ∂X, then for any $x \in \partial X$ and $m \in I$, $d_T(x, m) \leq \pi$.*

Definition . We say a hyperbolic element g has *rank 1*, if no axis of g bounds a half flat (isometric to $[0, \infty) \times \mathbb{R}$). In particular if $d_T(g^+, g^-) > \pi$ then the hyperbolic element g is rank 1. If G contains a rank 1 hyperbolic element, then we say G has *rank 1*. The endpoints of a rank 1 hyperbolic element have infinite Tits distance from any other point in ∂X [1]. (In the case where $d_T(g^+, g^-) > \pi$, this follows trivially from π-convergence)

Now if G has rank 1, then for any open U, V in ∂X, there is a rank 1 hyperbolic element g with one endpoint in U and the other in V satisfying $g(\partial X - U) \subset V$ and $g^{-1}(\partial X - V) \subset U$ [3, Theorem A] (Notice that U and V need not be disjoint.) If G has rank 1, then the only nonempty closed invariant subset of ∂X is ∂X.

Definition . Let Z be a metric space and $A \subset Z$. The radius of A,

$$radius(A) = \inf\{r : A \subset \bar{B}(z, r) \text{ for some } z \in Z\}$$

We say that $c \in Z$ is a *centroid* of $A \subset Z$ if $A \subset \bar{B}(c, radius A)$. When using the Tits metric on ∂X we signal this to the reader using the subscript $_T$.

Using π-convergence, we obtain the following strengthening of the Theorem above.

Theorem 23. *If the Tits diameter of ∂X is bigger than $\frac{3\pi}{2}$ then G contains a rank 1 hyperbolic element. In particular: if G doesn't fix a point of ∂X and doesn't have rank 1, and I is a minimal closed invariant set for the action of G on ∂X, then for any $x \in \partial X$, $d_T(x, I) \leq \frac{\pi}{2}$.*

Proof. We assume that G doesn't have rank 1. Let I be a minimal closed invariant set for the action of G on ∂X. If $radius_T(I) < \frac{\pi}{2}$, then by [9, II 2.7], I has a unique centroid $b \in \partial X$. Since I is invariant, G fixes b, and it follows by a result of Ruane [23] (see also Lemma 26 of the present paper) that G virtually splits as $H \times \mathbb{Z}$. But then $diam_T(\partial X) = \pi$. Thus we may assume that for any $a \in \partial X$ and any $\delta > 0$ there is a point of $y \in I$ with $d_T(a, y) > \frac{\pi}{2} - \delta$.

Let $x \in \partial X$. Assume that $d_T(x, I) = \frac{\pi}{2} + \epsilon$ for some $x \in \partial X$ and some $\epsilon > 0$. Let $g_n \in G$ such that $g_n t \to x$ for some (hence for all) $t \in X$. We may assume by passing to a subsequence that $g_n^{-1} t \to n \in \partial X$. There is some $y \in I$ such that $d_T(y, n) > \frac{\pi}{2} - \frac{\epsilon}{2}$.

We apply now π-convergence to the sequence g_n and the point y. Clearly y does not lie in the Tits ball of radius $\frac{\pi}{2} - \frac{\epsilon}{2}$ and center n. Passing to a subsequence, we have that $g_n y \to z \in \bar{B}_T(x, \frac{\pi}{2} + \frac{\epsilon}{2})$. Since $z \in I$ we have that $d_T(x, I) \leq \frac{\pi}{2} + \frac{\epsilon}{2}$, a contradiction.

Thus we have shown that $d_T(x, I) \leq \frac{\pi}{2}$. Choose $m \in I$ with $d_T(x, m) \leq \frac{\pi}{2}$. Since G doesn't have rank 1, by Theorem 22, $\partial X \subset \bar{B}_T(m, \pi)$. By the triangle inequality, $\partial X \subset \bar{B}_T(x, \frac{3\pi}{2})$ as required.

□

Corollary 24. *If the Tits diameter of ∂X is bigger than $\frac{3\pi}{2}$, then G contains a free subgroup of rank 2.*

5. The action on the cut-point tree is non trivial

For the next two sections, we assume that ∂X has a cut point. We apply the construction of Section 1 to $Z = \partial X$ to obtain a non-trivial $\mathbb{R}$-tree T encoding the cut-points of ∂X. Since the construction of T is canonical, G acts on T by homeomorphisms.

By the construction of T if G fixes a point of T then G fixes a point of $\mathcal{P}$. Points of $\mathcal{P}$ are of one of the following two types: i) cut points and ii)maximal inseparable sets of ∂X.

Lemma 25. *The action of G on the $\mathbb{R}$-tree T is non-nesting. That is for any closed arc I of T, there is no $g \in G$ with $g(I) \subsetneq I$.*

Proof. Suppose not, then by the Brouwer fixed point Theorem, there is an $A \in T$ with $g(A) = A$.

Replacing g with g^2 if need be, we may assume that $I = [A, B]$ with $g(B) \in (A, B)$. We may also assume that $A, B \in \mathcal{P}$. Similarly we may assume that $|(g(B), B)| > 2$, so $d_T(g(B), B) > 0$. Since G acts by isometries on the Tits boundary, it follows that $d_T(B, A) = \infty$, moreover all limit points of the sequence of sets $(g^i B)$ are at infinite Tits distance from all limit points of the sequence of sets $(g^{-i}B)$. So by π-convergence g is a rank 1 hyperbolic element.

Now by [1] every point of ∂X is at infinite Tits distance from both of the endpoints, $g^{\pm} \in \partial X$, of g. By π-convergence $g^+ \in A$. We consider $g^{-i}B$ and we note that as $i \to \infty$ it converges either to a point C of T or to an end of T corresponding to an element C of $\mathcal{P}$. We remark that g^- lies in C and that $B \in (A, C)$.

Since A is not terminal in $\mathcal{P}$, there is a $D \in \mathcal{P}$ with $A \in (D, C)$. Thus by π-convergence $g^{-i}D$ (for some $i > 0$) lies in the component of $T - A$ containing C. This is however impossible since A is fixed by g. $\square$

We deal now with the first type, i.e. we assume that G fixes a cut point. We need the following unpublished result of Ruane. We provide a proof for completeness

Lemma 26. *If G virtually stabilizes a finite subset A of ∂X, then G virtually has $\mathbb{Z}$ as a direct factor.*

Proof. Clearly there is a finite index subgroup $H < G$ which fixes A pointwise. Let $\{h_1, \dots h_n\}$ be a finite generating set of H. By [23], the centralizer Z_{h_i} acts geometrically on the convex subset $Min(h_i) \subset Y$.

By [28] $A \subset \Lambda Min(h_i) = \Lambda Z_{h_i}$ for all i. Since Z_{h_i} is convex, we have by [28] that the centralizer of H, $Z_H = \cap Z_{h_i}$ is convex and

$$\bigcap \Lambda Z_{h_i} = \Lambda \left[\bigcap Z_{h_i} \right] \supset A$$

Thus $A \subset \Lambda Z_H$, so Z_H is an infinite CAT(0) group, and by [28] Z_H has an element of infinite order. Thus H contains a central $\mathbb{Z}$ subgroup. By [9, II 6.12], H virtually has $\mathbb{Z}$ as a direct factor. $\square$

From this Lemma it follows that if G fixes a point of ∂X then G is virtually $K \times \mathbb{Z}$. Thus ∂X is a suspension and $|\Lambda K| \neq 1$ ([28], Cor. p.345) so ∂X has no cut points, a contradiction.

We now prove the following:

Theorem 27. *G fixes no maximal inseparable subset.*

Proof. It follows from Theorem 23 [3, Theorem A] that if G fixes a maximal inseparable set then the Tits diameter of ∂X is at most $\frac{3}{2}\pi$.

Let's assume that G fixes the maximal inseparable set B. We remark that if $x \in B$ then Gx is contained in B so the closure of Gx is contained in $\bar{B}$. Thus $\bar{B}$ contains a minimal invariant set that we denote by I. By Theorem 23, $\partial X \subset \mathrm{Nbh}_T(\bar{B}, \frac{\pi}{2})$, the closed $\frac{\pi}{2}$ Tits neighborhood of $\bar{B}$.

If c is a cut point of ∂X we have $\partial X - \{c\} = U \cup V$ with U, V open and B is contained, say, in U. With this notation we have the following:

Lemma 28. *There is a 2π geodesic circle contained in $\bar{V}$.*

Proof. Let α be a geodesic arc of length smaller than π contained in V. Let m be the midpoint of α and let a, b be its endpoints. Fix $t \in X$ and consider the geodesic ray γ from t to m. Construct an increasing sequence $(n_i) \subset \mathbb{N}$ and $(g_i) \in G$ such that $g_i(\gamma(n_i)) \to y \in X$, $g_i(m) \to m' \in \partial X$, $g_i(a) \to a' \in \partial X$ and $g_i(b) \to b'$. By [9] $\angle_y(a', b') = d_T(a, b) \geq d_T(a', b') \geq \angle_y(a', b')$. Thus $\angle_y(a', b') = d_T(a', b')$ which implies by the flat sector Theorem ([9], p.283, cor.9.9) that the sector bounded by the rays $[y, a')$ and $[y, b')$ is flat. Let's denote the limit set of this sector by α'.

By passing to a subsequence if necessary we have that $g_i(y)$ converges to some $p \in \partial X$. Notice that by construction $g_i^{-1}(y) \to m$. We remark that $d_T(p, m') \geq \pi$, so by π-convergence $g_i^{-1}(\alpha') \cap V \neq \emptyset$ for $i >> 0$. So there is a flat sector Q (based at $v \in X$) whose limit set is contained in V.

To simplify notation we denote the limit set of this sector again by α (and a, b, m its endpoints and midpoint respectively).

Repeating this process (by pulling back along the ray $[v, m))$ we obtain a sequence $(h_i) \subset G$ and a flat F with $h_i(Q) \to F$ (uniformly on compact subsets of F), and so a geodesic 2π-circle $\gamma = \Lambda F$ with $h_i(\alpha)$ converging to an arc of γ. As before, using π-convergence, we show that for $i >> 0$, $h_i^{-1}(\gamma) \cap V \neq \emptyset$ and since γ is a **simple** closed curve, $h_i^{-1}(\gamma) \subset \bar{V}$. ☐

Since no geodesic circle can lie in a Tits ball of radius less than $\frac{\pi}{2}$, and $X \subset \mathrm{Nbh}_T(B, \frac{\pi}{2})$, it follows from Lemma 28 that:

- $c \in \bar{B}$
- $\exists p \in V$ such that $d_T(p, \bar{B}) = \pi/2$.

Notice that we have shown that every cut point of X is adjacent to B (in the pre-tree structure) and so every cut point is contained in $\bar{B}$.

Fix $t \in X$ and consider a sequence $(g_i) \in G$ such that $g_i(t) \to p \in \partial X$. By passing to a subsequence we may assume that $g_i^{-1}(t) \to n \in \partial X$. Since G does not virtually fix c, there are distinct translates $h_1(c), h_2(c) \neq n$ which do not separate B from n. Thus $d_T(h_1(p), n) > \frac{\pi}{2}$ and $d_T(h_2(p), n) > \frac{\pi}{2}$. By π-convergence $g_i(h_1(p)), g_i(h_2(p))$ lie in V for sufficiently large i. Since all cut points are adjacent to E, we have $g_i(h_1(c)) = g_i(h_2(c)) = c$ for sufficiently big i. This is impossible as h_1c, h_2c are distinct, and the proof of Theorem 27 is complete. ☐

Corollary 29. *In our setting, the action of G on X is rank 1, so ∂X has infinite Tits diameter.*

Proof. The action of G on the $\mathbb{R}$-tree T is non-nesting and without global fixed points. It follows from [21, Proposition 35] that G has an element h which acts by translation on a line of T. Thus there is a cut point c such that c separates $h^{-1}(c)$ from $h(c)$. It follows by π-convergence that $d_T(h^+, h^-) = \infty$. ☐

6. Cat(0) groups have no cut points

We study now further the action of G on T.

Lemma 30. *If an interval $I \subset \mathcal{P}$ is infinite then the stablizer of I in G is finite.*

Proof. Assume that I is fixed (pointwise) by an infinite subgroup of G, say H. Let $(h_i) \subset H$ be an infinite sequence of distinct elements of H. By passing to a subsequence we may assume that $h_i(t) \to p \in \partial X$, $h_i^{-1}(t) \to n \in \partial X$ for any $t \in X$.

The interval I contains infinitely many cut points, so by Corollary 29 we can find $x \in \partial X$, $d_T(n, x) = \infty$, with the property that x is separated from p by two cut points $c, d \in I$. We may assume that d

separates c from p. By π-convergence $h_i(x) \to p$. There exist subcontinua $B \ni p$ and $A \ni c, x$ with $X = A \cup B$ and $A \cap B = d$. Furthermore, since (h_i) fixes c which separates x from d, it follows that c separates $h_i(x)$ from d. Thus $\{h_i(x)\} \subset A$, contradicting $h_i(x) \to p \in B - \{d\}$.

□

Lemma 31. *The action of G on the $\mathbb{R}$-tree T is stable.*

Proof. We recall that a non-degenerate arc I is called stable if there is a (non degenerate) arc $J \subset I$ such that for any non degenerate arc $K \subset J$, $stab(K) = stab(J)$.

An action is called stable if any closed arc I of T is stable.

We remark that by construction, arcs of T that correspond to adjacent elements of $\mathcal{P}$ are stable. Further if an arc contains a stable arc it is itself stable, so every unstable arc contains infinitely many elements of $\mathcal{P}$.

Suppose now that an arc I is not stable. Then there is a properly decreasing sequence $I = I_1 \supset I_2 \supset ...$ so that $stab(I_1) \subset stab(I_2) \subset ...$ where all inclusions are proper. Notice that $I_i \cap \mathcal{P}$ is infinite for each i, and so $stab(I_i)$ is finite by Lemma 30. On the other hand, there is a uniform bound on the size of a finite subgroup of G, which is a contradiction.

□

Lemma 32. *T is discrete.*

Proof. Suppose that T is not discrete. Then by [18] there is an $\mathbb{R}$-tree S and a G-invariant quotient map $f : T \to S$ such that G acts non-trivially on S by isometries. Furthermore for each non-singleton arc $\alpha \subset S$, $f^{-1}(\alpha) \cap \mathcal{P}$ is an infinite interval of $\mathcal{P}$, and the stabilizer of an arc of S is the stabilizer of an arc of T. It follows that the arc stabilizers of S are finite (Lemma 30) and that the action of G on S is stable (Lemma 31).

Since G is one-ended, by the Rips machine applied to S, there is an element $h \in G$ which acts by translation on a line in S such that G virtually splits over h. This implies that $\{h^{\pm}\}$ is a cut pair of ∂X, however since h acts by translation on a line in S, it acts by translation on a line in T, and so $h^{\pm}$ will correspond to the ends of this line (by π-convergence). Thus $\{h^{\pm}\}$ lie in terminal elements of $\mathcal{P}$. This contradicts the fact that $\{h^{\pm}\}$ separates. Thus T is discrete.

□

The following is easy and well known, but we provide a proof for completeness

Lemma 33. *If $H < G$ then the centralizer Z_H fixes ΛH in ∂X.*

Proof. Let $\alpha \in \Lambda H \subset \partial X$. By definition there exists a sequence of elements $(h_n) \subset H$ with $h_n(x) \to \alpha$ for any $x \in X$. Since G acts by homeomorphisms on $\bar{X} = X \cup \partial X$, $g(h_n(x)) \to g(\alpha)$ for any $g \in Z_H$. Notice however that $g(h_n(x)) = h_n(g(x))$, and $h_n(g(x)) \to \alpha$. It follows that $\alpha = g(\alpha)$. □

For a finitely generated $H < G$, the centralizer Z_H will be a *convex* subgroup of G, that is Z_H will act geometrically on a convex subset of X.

Lemma 34. *Let $H < G$ with H fixing the adjacent pair $\{c, B\}$ of $\mathcal{P}$ (so c is a cut point, $c \in \bar{B}$, $h(c) = c$ and $h(B) = B$ for each $h \in H$). Then $\Lambda H \subset \bar{B}$.*

Proof. Let $(h_i) \subset H$ be a sequence with $h_i(x) \to p \in \Lambda H - \bar{B}$ for any $x \in X$. Passing to a subsequence we have $h_i^{-1}(x) \to n \in \Lambda H$.

The adjacent pair $\{c, B\}$ gives a separation of $\partial X - \bar{B}$ in the obvious way namely: Let $U = \{\beta \in \partial X - \bar{B} : c \in ([\beta], B)\}$ and $V = \{\beta \in \partial X - \bar{B} : B \in ([\beta], c)\}$. Using the definition of $\mathcal{P}$ one can show that both U and V are open in ∂X (being the union of open sets). Using the pretree axioms we see that $U \cap V = \emptyset$ and $U \cup V = \partial X - \bar{B}$. Since H leaves B and c invariant, $h_i(U) = U$ and $h_i(V) = V$ for all i.

Without loss of generality let $p \in U$. Since G is rank 1 in its action on X, there is a point $\alpha \in V$ with $d_T(\alpha, n) > \pi$. It follows by π-convergence that $h_i(\alpha) \to p$. This contradicts the fact that $p \in U$ and $h_i(\alpha) \in V$ (in particular $h_i(\alpha) \notin U$) for all i. Thus $\Lambda H \subset \bar{B}$. □

Lemma 35. *Suppose that we have adjacent pairs $\{B, c\}$ and $\{\hat{B}, c\}$ in $\mathcal{P}$ with $B \neq \hat{B}$ with the stabilizer of $\{B, c\}$ infinite and the stabilizer of $\{\hat{B}, c\}$ infinite. Then there is a hyperbolic $g \in G$ with $g(B) = B$, $g(\hat{B}) = \hat{B}$, and $g(c) = c$.*

Proof. Since there are only finitely many conjugacy classes of finite subgroups in G, there are infinite finitely generated subgroups $H = \langle h_1 \dots h_n \rangle$ and $\hat{H} = \langle \hat{h}_1 \dots \hat{h}_m \rangle$ stabilizing $\{B, c\}$ and $\{\hat{B}, c\}$ respectively. Since $h_i(c) = c = \hat{h}_j(c)$ for all i, j it follows from the [28] that $c \in \Lambda Z_{h_i}$ and $c \in \Lambda Z_{\hat{h}_j}$ for all i, j. Since Z_{h_i} and $Z_{\hat{h}_j}$ are convex, it follows from [28, Theorem 16] that $Z_H = \bigcap_i Z_{h_i}$ and $Z_{\hat{H}} = \bigcap_i Z_{h_i}$ are convex and that $c \in \Lambda Z_H$ and $c \in \Lambda Z_{\hat{H}}$. Thus by [28, Theorem 16] $Z = Z_H \cap Z_{\hat{H}}$ is convex and $c \in \Lambda Z = \Lambda Z_H \cap \Lambda Z_{\hat{H}}$.

Since Z is convex, it is a CAT(0) group, and by [28, Theorem 11] there is an element $g \in Z$ of infinite order. For any $K < G$, ΛK is not a single point [28, Corollary to Theorem 17]. Since H is infinite, ΛH is non-empty with more than one point. By Lemma 34 $\Lambda H \subset \bar{B}$. By Lemma 33, g fixes ΛH. It follows that $g(B) = B$. Similarly $g(\hat{B}) = \hat{B}$. It follows that $g(c) = c$ as required. □

Theorem 36. *Let G be a one-ended group acting geometrically on the CAT(0) space X. Then ∂X has no cut points.*

Proof. Assume that ∂X has a cut point. Let T be the cut point tree of ∂X. By Lemma 32 T is a simplicial tree. Since G is one-ended, all edge stabilizers are infinite. By Lemma 35 there is a hyperbolic element $g \in G$ stabilizing two adjacent edges of $\{B, c\}$ and $\{c, D\}$ of T. Thus by Lemma 34, $\{g^{\pm}\} \subset \bar{B}$ and $\{g^{\pm}\} \subset \bar{D}$. This is absurd since the intersection of $\bar{B}$ and $\bar{D}$ is a single point (namely c). □

7. The action on the JSJ-tree is non-trivial

Setting . For the remainder of the paper G will be a one-ended group acting geometrically on the CAT(0) space X. As we showed in the previous section, ∂X has no cut points. We will further assume that ∂X contains a cut pair.

Let T be the JSJ-tree of $Z = \partial X$ constructed in section 1. Since the construction of T is canonical, G acts on T.

Lemma 37. *The action of G on the $\mathbb{R}$-tree T is non-nesting. That is for any closed arc I of T, there is no $g \in G$ with $g(I) \subsetneq I$.*

Proof. The proof is basically the same as that of Lemma 25. We sketch it here. Suppose not, then we may assume that:

- $I = [A, B]$
- $A, B \in \mathcal{P}$
- $g(A) = A$
- $g(B) \in (A, B)$
- $|(g(B), B)| > 2$, so $d_T(g(B), B) > 0$

Using π-convergence we see that :

- $d_T(B, A) = \infty$.
- g is a rank 1 hyperbolic element.
- $g^+ \in A$.
- B separates A from g^-

No element of $\mathcal{P}$ is a singleton, so let $a \in A - \{g^+\}$. Notice that g is rank 1 so $d_T(a, g^+) = \infty$. Thus by π-convergence $g^{-i}(a) \to g^-$. Since A is fixed by g, then $g^- \in \bar{A}$. This contradicts the fact that B separates A from g^- □

By the construction of T if G fixes a point of T then G fixes a point of $\mathcal{P}$. Points of $\mathcal{P}$ are of one of the following types:

- inseparable cut pairs of ∂X
- maximal cyclic subsets of ∂X which we call *necklaces*
- maximal inseparable subsets of ∂X.

We deal now with the first type, i.e. we assume that G leaves an inseparable cut pair invariant, so in fact G virtually fixes each element of the cut pair. By Lemma 26 if G virtually fixes a point of ∂X then G is virtually $H \times \mathbb{Z}$ for some finitely presented group. Since ∂X has a cut pair, H will have more than one end. In this case either G is virtually $\mathbb{Z}^2$ or ∂X has only one cut pair and all the other elements of $\mathcal{P}$ are maximal inseparable sets. Having classified these possibilities, we now ignore them.

Setting . For the remainder of the paper, we assume that G doesn't virtually have $\mathbb{Z}$ as a direct factor, in particular G doesn't leave a cut pair (or any other finite subset) invariant.

7.1. **Maximal inseparable subsets of ∂X.**

Lemma 38. *If the Tits diameter of ∂X is more than $\frac{3\pi}{2}$, then G leaves no maximal inseparable set invariant.*

Proof. If A is a cut pair, then there are nonempty disjoint open subsets U and V such that $U \cup V = \partial X - A$. Choose a rank 1 hyperbolic element $g \in G$ with $g^- \in U$ and $g^+ \in V$. By π-convergence, it follows that $g(A) \subset V$ and $g(A)$ separates A from g^+. By π-convergence every maximal inseparable set will lie "between" $g^i(A)$ and $g^{i+1}(A)$ for some $i \in Z$, and so will be moved off itself by g. □

Let's assume now that G leaves invariant a maximal inseparable subset of ∂X, say D, so the Tits diameter is at most $\frac{3\pi}{2}$. Let a, b be a pair of points separating ∂X.

Let B be the closure of the component of $\partial X - \{a, b\}$ which contains D, and I be a minimal non-empty closed invariant set for the action of G of necessity contained in D. By [9, Proposition II 2.7] if the Tits radius of I were less than $\frac{\pi}{2}$, then I would have a unique centroid (in ∂X) which would be fixed by G. Since G fixes no point of ∂X, it follows that the Tits radius of I is at least $\frac{\pi}{2}$. Similarly I must be infinite.

7.1.1. *C-geodesics.* In order to show that G leaves no maximal inseparable set invariant, we must understand a special type of local Tits geodesic in ∂X. Let $\{c, d\} \in \partial X$ be a cut pair and $C \subset \partial X$ the closure of a component of $\partial X - \{c, d\}$. Consider the path metric of C using the Tits metric $d_T^C(x,y) = \inf\{\ell_T(\alpha) : \alpha$ is a path in C from x to $y\}$. For $e, f \in C$, let $\epsilon > 0$, consider the sets $F_\epsilon = \cup\{$ paths $\alpha \subset C$ with $f \in \alpha$ and with Tits length $\ell_T(\alpha) \leq \epsilon\}$ and $E_\epsilon = \cup\{$ paths $\alpha \subset C$ with $e \in \alpha$ and with Tits length $\ell_T(\alpha) \leq \epsilon\}$. Using lower semi-continuity of the (identity) function from the Tits boundary TX to the boundary ∂X, we see that F_ϵ and E_ϵ are closed connected subsets of the continua C. The Tits diameter of ∂X is at most $\frac{3\pi}{2}$. It follows that for some ϵ, $C = E_\epsilon \cup F_\epsilon$, so there exists a path $\alpha \subset C$ of finite Tits length from f to e. Using lower semi-continuity of the identity map (from the Tits boundary to the regular boundary) and a limiting argument, we can show that there is a shortest path in C from f to e which we call a C-geodesic from f to e.

Local geodesics of length $\leq \pi$ are geodesics, [9], so if $d_T^C(d,e) < \pi$, then the C-geodesic from d to e is the Tits geodesic from d to e and therefore unique.

Theorem 39. *G doesn't leave a maximal inseparable set invariant.*

Proof. Let's assume that G leaves invariant a maximal inseparable subset of ∂X, say D. Let I be a minimal invariant set for the action of G on ∂X. Observe that I is contained in the closure of D, $\bar{D}$. It follows that ∂X has finite Tits diameter.

D is a point of T. Let R be a component of $T - D$. We can view R as a subset of ∂X in the appropriate way. With this viewpoint, the closure of R in the Tits topology (i.e. the topology defined by d_T) intersects D at either one or two points.

We distinguish two cases:

(1) There is no element of $\mathcal{P}$ contained in R adjacent to D.
(2) There is an element of $\mathcal{P}$ contained in R adjacent to D.

In the first case there is a non trivial loop contained in R separated from D by infinitely many elements of $\mathcal{P}$. We homotope this loop to a geodesic circle w separated from D by infinitely many elements of $\mathcal{P}$. Thus we have $\{\hat{a}, \hat{b}\}$, a cut pair disjoint from w and D, separating w from D.

Then we claim that for some $e \in w$, $d_T(e, \{\hat{a}, \hat{b}\}) \geq \frac{\pi}{2}$. Indeed if every point on w is at Tits distance less than $\frac{\pi}{2}$ from $\hat{a}, \hat{b}$ then there are two antipodal points on w, both at Tits distance less than $\frac{\pi}{2}$ from one of $\hat{a}, \hat{b}$. This contradicts the fact that w is geodesic. It follows now that $d_T(e, I) > \frac{\pi}{2}$ which is a contradiction.

We deal now with the second case. Since the Tits diameter of ∂X is at most $\frac{3\pi}{2}$, it follows that the element of $\mathcal{P}$ adjacent to D in R is a cut pair, $\{a, b\}$. All translates (there are infinitely many of them) of

$\{a,b\}$ are adjacent to D. Let p be a point of ∂X separated from D by $\{a,b\}$, and let $g_i \in G$ such that for some (hence for all) $x \in X$, $g_i(x)$ converges to p. By passing to a subsequence we may assume that $g_i^{-1}x$ converges to some $n \in \partial X$.

By [19], X has almost extendable geodesics, and this implies that there is $q \in \partial X$ with $d_T(n,q) = \pi$, and by π-convergence $g_i(q) \to p$. It follows that there are translates a_1, b_1 of a, b respectively so that q is separated from D by $\{a_1, b_1\}$. Further we claim that $g_i\{a_1, b_1\} = \{a,b\}$ for all i big enough. Indeed since $\{a,b\}$ is a cut pair there is a neighborhood U of p with $\partial U = \{a,b\}$. On the other hand if $g_i\{a_1,b_1\} \neq \{a,b\}$, $g_i(q) \notin U$. We have in fact shown that every point of ∂X at Tits distance $\geq \pi$ from n is separated from D by $\{a_1, b_1\}$.

Let B_1 be the component of $\partial X - \{a_1, b_1\}$ containing D and C_1 the component of $\partial X - \{a_1, b_1\}$ containing q. We claim that $n \in B_1$. Indeed suppose that n lies in another component, say B_2 of $\partial X - \{a_1, b_1\}$. By Theorem 23, every point is at Tits distance less than $\frac{\pi}{2}$ from I, so there is a B_2-geodesic γ_1 of length less than π from a_1 to b_1. It follows that if γ_2 is a B_1 geodesic from a_1 to b_1 the length of γ_2 is at least π. This is because $\gamma_1 \cup \gamma_2$ is a non-contractible loop. It follows that there is some point on γ_2 at distance more than π from n. This contradicts our earlier observation that all such points are separated from D by $\{a_1, b_1\}$.

Now consider the loop S consisting of the B_1-segments $[n, a_1]$, $[n, b_1]$ and a C_1-segment $[a_1, b_1]$. We may assume that $q \in [a_1, b_1]$ because S is non-contractible, so some point of S is at Tits distance $\geq \pi$ from n. The B_1-segments $[n, a_1]$ and $[n, b_1]$, having length less than π, will be Tits geodesics. Furthermore, since every point of ∂X is within $\frac{\pi}{2}$ of $I \subset D$, the C_1-segment $[a_1, b_1]$ must also have length less than π and is therefore a Tits geodesic.

Since every essential loops of ∂X has Tits length at least 2π, one can show that only finitely many translates of $\{a,b\}$ separate a subarc of S from D. Let $\{a_2, b_2\}$, C_2, B_2 be translates of $\{a_1, b_1\}$, C_1, B_1 respectively so that $\{a_2, b_2\}$ doesn't separate a subarc of S from D. As before let S' be the union of the B_2-segments $[n, a_2]$, $[n, b_2]$ and a C_2-segment $[a_2, b_2]$, and as before these three segments will be Tits geodesics. Notice that since S' is not contractible, there is a point of S' at distance $\geq \pi$ from n, and so some point of S' is separated from D by $\{a_1, b_1\}$

Notice by construction that for every $s \in S$, S contains a Tits geodesic from s to n and similarly for S'. Then S, S' both contain a_1, b_1, have length at least 2π. Let S_1, S_1' be respectively the subarcs of S, S'

from n to a_1 contained in B_1 and similarly let S_2, S_2' be respectively the subarcs of S, S' from n to b_1 contained in B_1. Then as we note before all have length smaller than π. On the other hand at least one of $S_1 \cup S_1'$, $S_2 \cup S_2'$ runs through C_2 once, and so is a non contractible loop. This is a contradiction.

□

7.2. **Stabilizers of necklaces of ∂X.**

Theorem 40. *Let H be a group of homeomorphisms of the circle S^1, and let A be a nonempty closed H-invariant subset of S^1. Either A is countable and H virtually fixes a point of A, or there is an H-equivariant cellular quotient map $q : S^1 \to S^1$ such that $q(A) = S^1$.*

Proof. Recall that a compact Hausdorff space E is called perfect if each point of E is a limit point of E. We note that every perfect space is uncountable.

Let

$$B = \{a \in A : \exists \text{ an open } U \ni a \text{ with } U \cap A \text{ countable } \}$$

and let $C = A - B$. Since A is Lindeloef, and B is a subset of A, B is countable. Since B is an open subset of A, C is compact, and it follows that C is perfect since every neighborhood of a point of C contains uncountably many points of A, and therefore a point of C (since B is countable). Thus A is the union of a countable set B, and a perfect set C.

First consider the case where A is countable, ($C = \emptyset$). We remark that if $A \supset A_1 \supset A_2 \supset ...$ and A_i are closed H-invariant sets, then $\cap A_i$ is non-empty and H-invariant. By Zorn's Lemma there is a minimal, closed, non-empty, H-invariant subset of A. Let's call this set A'. If $a \in A'$ then A' is the closure of the orbit of a under H, $\overline{Ha}$. If A' is infinite then there is some limit point b in A'. Since A' is minimal closed H-invariant set, and $\overline{Hb} \subset A'$ we have that $\overline{Hb} = A'$. It follows that a is a limit point of A'. Since $A' = \overline{Ha}$, every point of A' is a limit point of A', so A' is perfect, a contradiction. Therefore A' is finite and H virtually fixes a point of A.

Now consider the case where A is uncountable, so $C \neq \emptyset$.

Let I be a closed interval of $\mathbb{R}$ and let D be a perfect subset of I. For $x, y \in I$ define $x \sim y$ if there is no point of d strictly between them. This is an equivalence relation (transitivity follows from the perfection of D). The linear order of I extends to a linear order on the quotient $I/\sim$. It follows that $I/\sim$ is a compact connected separable linearly ordered topological space, also known as a closed interval. Since the image of D in $I/\sim$ is compact and dense, it follows that the image is the entire quotient $I/\sim$.

Of course S^1 is just the quotient space of a closed interval which identifies exactly the endpoints. We now say that $x, y \in S^1$ are equivalent if there is an open arc in $S^1 - C$ from x to y. It follows from the previous paragraph that this is an equivalence relation on S^1 and that $S^1/\sim = S^1$. Let $q : S^1 \to S^1$ be the quotient map. Since C is H-invariant, it follows that q is H-equivariant, and by the previous paragraph $q(C) = S^1$. By constuction, the inverse image of a point is a closed arc, so q is cellular. □

Lemma 41. *If G leaves a necklace N of ∂X invariant, then the action of G on X is rank 1.*

Proof. Suppose not. Since the Tits diameter of ∂X is at most $\frac{3}{2}\pi$, N is contained in a geodesic circle α of length at most 3π. If any arc of $\alpha - N$ has length at least π, then the endpoints of that arc are virtually stabilized by G which contradicts our hypothesis. Thus α is the unique geodesic circle containing N, and so G stabilizes α.

Since G acts on the Tits boundary by isometries, there is a homomorphism $\rho : G \to Isom(\alpha)$ (which is virtually the Lie group S^1 and so virtually abelian). Since G doesn't virtually fix any point of ∂X, $\rho(G)$ is infinite. Since G is finitely generated, the virtually abelian group $\rho(G)$ contains an element of infinite order, $\rho(g)$. That is g(or g^2) rotates α by an irrational multiple of $\ell_T(\alpha)$. It follows by π-convergence that $d_T(s, g^+) = d_T(s, g^-) = \frac{\pi}{2}$ for all $s \in \alpha$. Thus every point of N is joined to g^+ by a Tits geodesic of length exactly $\frac{\pi}{2}$. However since N is a necklace and $g^+ \notin N$, there must be $a, b, c \in N$ such that $\{a, b\}$ separates c from g^+, and so $d_T(c, g^+) > \min\{d_T(a, g^+), d_T(b, g^+)\}$, a contradiction. □

Corollary 42. *In our setting (∂X has a cut pair and G doesn't virtually have $\mathbb{Z}$ as a direct factor) the action of G on X is rank 1.*

Proof. By Lemma 41 we may assume that G doesn't leave a necklace of ∂X invariant. Thus G acts on $\mathcal{P}$ without fixing a point, so $T \neq \emptyset$ and G acts on T without fixed points. By [21] that there is $h \in G$ which acts as a hyperbolic element on T, that is h acts by translation on a line $L \subset T$. Let $A \in L \cap \mathcal{P}$. For some $n > 0$, $d_T(h^n(A), A) > 0$. It follows by π-convergence that $d_T(h^+, h^-) = \infty$, and by definition h is rank 1 in its action on X. □

Definition . Let S be a necklace of X. We say $y, z \in X - S$ are S equivalent, denoted $y \sim_S z$ if for every cyclic decomposition $M_1, \ldots M_n$

of X by $\{x_1, \dots x_n\} \subset S$, both $y, z \in M_i$ for some $1 \leq i \leq n$. The relation $\sim_S$ is an equivalence relation on $X - S$. The closure (in X) of a $\sim_S$-equivalence class of $X - S$ is called a *gap* of S. Notice that every gap is a nested intersection of continua, and so is a continuum. Every inseparable cut pair in S defines a unique gap. The converse is true if X is locally connected, but false in the non-locally connected case.

Theorem 43. *Let B be a necklace of ∂X, and let H be the stabilizer of B, so $H = \{g \in G : g(B) = B\}$. Then one of the following is true:*

1. *H is virtually cyclic and there is at most one gap J of B with the stabilizer of J in H infinite .*
2. *H acts properly by isometries on $\mathbb{H}^2$ with limit set the circle at infinity. Furthermore for any gap J of B, the stabilizer of J in H is finite or it is a peripheral subgroup of H (as a Fuchsian group). Distinct gaps correspond to distinct peripheral subgroups.*

Proof. Let $f_B : \partial X \to S^1$ be the circle function for B (see Theorem 22 of [21]). Thus the action of H on B extends to an action of H on S^1. By Theorem 40 either H virtually fixes a point in $\overline{f_B(B)}$, or by composing f_B with an H-invariant cellular quotient map q, we have an H invariant $\pi : \partial X \to S^1$. We may assume that H is infinite.

Case I. We first deal with the case where H virtually fixes a point $r \in \overline{f_B(B)}$. Passing to a finite index subgroup, we assume H fixes r.

First consider the case where an infinite subgroup $K < H$ fixes three points $a, b, c \in B$. We have the unique cyclic decomposition U, V, W, continua with $U \cup V \cup W = \partial X$ and $U \cap V = a$, $V \cap W = b$, and $W \cap U = c$. With no loss of generality, there is $p \in \Lambda K - U$. Choose $(k_i) \subset K$ with $k_i(x) \to p$ and $k_i^{-1}(x) \to n$ for some $n \in \partial X$ and for all $x \in X$. Since G is rank one, there exists $q \in U$ with $d_T(q, n) = \infty$. Thus $k_i(q) \to p \notin U$, but since K fixes a, b, c, by uniqueness of the cyclic decomposition K leaves U invariant, a contradiction. *Thus no infinite subgroup of H fixes three points of B; in particular no hyperbolic element of H fixes more than two points of B.*

We remark now that $\overline{f_B(B)} - p$ is linearly ordered so a finite order element of H either fixes all points of $\overline{f_B(B)}$ or its square fixes all points of $\overline{f_B(B)}$. If H is torsion, then passing to an index two subgroup, we may assume that H fixes $\overline{f_B(B)}$, so H fixes B. As noted above, this is not possible, so H contains a hyperbolic element.

Since H fixes a point in S^1, the action of H on S^1 comes from an action on a closed interval. Thus there exists $b \in B$ and hyperbolic $h \in H$ such that $\{b, h^2(b)\}$ separates $h^{-1}(b)$ from $h(b)$.

Since $d_T(b, g(b)) > 0$, for each i, $h^i(b)$ is at infinite Tits distance from any fixed point of h in ∂X, in particular from $h^\pm$. It follows by π-convergence that $h^i(b) \to h^+$ and $h^{-i}(b) \to h^-$. For $m \in \mathbb{Z}$ let M_m be equal to the closure of the component of $\partial X - \{h^{m-1}(b), h^m(b)\}$ which doesn't contain $h^\pm$. Notice that $h(M_m) = M_{m+1}$, and that $d_T(M_m, h^\pm) = \infty$. From π-convergence, the closure of $M = \cup M_m$ is $M \cup \{h^\pm\}$. It follows by [21, Lemma 15] that $B \cup \{h^\pm\}$ is cyclic subset of ∂X, and by maximality, $\{h^\pm\} \subset B$. Since h fixes at most 2 points of B, $h^\pm$ are the only points of B fixed by h, and $r \in f_B(h^\pm)$. Since $f_B^{-1}(f_B(h^\pm)) = \{h^\pm\}$, it follows that H fixes one of $h^\pm$. From [9], H fixes both $h^\pm$.

We now show that $\Lambda H = \{h^\pm\}$. Suppose not, then there is $(g_i) \subset H$ with $g_i(x) \to p \notin \{h^\pm\}$ and $g_i^{-1}(x) \to n$ for all $x \in X$. Notice that (g_i) leaves M invariant. Since (g_i) fixes $h^\pm$, by π-convergence, $n, p \in B_T(h^\pm, \pi)$. It follows that $d_T(M_m, n) = \infty$ for all $m \in \mathbb{Z}$. Thus $g_i(M_0) \to p$. However $g_i(M_0) \subset M$ for all i and the closure of M is $M \cup \{h^\pm\}$ a contradiction. We have shown that $\Lambda H = \{h^\pm\}$ and by Lemma 16, H is virtually $\langle h \rangle$. If there is a gap of B with sides $h^\pm$ then that gap will be stabilized by $\langle h \rangle$ (there can be at most one such gap), but no other gaps of B will be stabilized by a positive power of h, and so will have finite stabilizer in H.

Case II. By Theorem 40 we are left with the case where there is an H invariant function $\rho : \partial X \to S^1$ with $\overline{\rho(B)} = S^1$, where $\rho = q \circ f_B$, and q is the quotient map of Theorem 40. Notice that ρ is a quotient map, using the equivalence relation defined by $x \not\sim y$ if there exists uncountable many disjoint cut pairs $\{s, t\} \subset f_B(\bar{B})$ separating $f_B(x)$ from $f_B(y)$. Since the map f_B is canonical up to isotopy, this is an equivalence relation, and so ρ is a H quotient.

We first show that the action of H on this quotient S^1 is a convergence action. Let (g_i) be a sequence of distinct elements of G. Passing to a subsequence we find $n, p \in \partial X$ such that for any $x \in \partial X$, $g_i(x) \to p$ and $g_i^{-1}(x) \to n$. Since G is rank 1, every non-empty open set of ∂X contains points at infinite Tits distance from n. Every arc of S^1 contains the image of a non-empty open subset of ∂X. Thus every open arc of S^1 contains the image of a point at infinite Tits distance from n.

Let $\hat{p} = \rho(p)$ and $\hat{n} = \rho(n)$. For $v, u \in S^1$ we define the Tits distance $d_T(u, v) = d_T(\rho^{-1}(u), \rho^{-1}(v))$. Since the action of G preserves Tits distance on ∂X, it also preserves it on S^1. Let U, open arc about of $\hat{p}$. There exists $\epsilon > 0$ and, such that $d_T(\hat{p}, J) > \epsilon$ where $J = S^1 - U$.

Let K be a closed arc of S^1 with endpoints a and b and $\hat{n} \notin K$. It suffices to show that $g_n(K) \subset U$ for all $i >> 0$. If $\rho^{-1}(K) \cap B_T(n, \pi) = \emptyset$

then by π-convergence, $g_i(\rho^{-1}(K)) \to \bar{B}_T(p,\epsilon)$ so $g_i(K) \subset U$ for all $i >> 0$. Thus we may assume that $K \subset \rho(\bar{B}_T(n,\pi))$. In fact we can now reduce to the case where $K = \rho(\alpha)$ where α is a Tits arc of length less that $\frac{\epsilon}{2}$. The arc α contains the image q of a point at infinite Tits distance from n. Thus $g_i(q) \to p$. By the lower semi-continuity of the Tits metric on ∂X, for $i >> 0$, $d_T(g_i(q), J) > \frac{\epsilon}{2}$, and it follows that $g_i(K) \subset U$.

Thus H acts as a convergence group on S^1, so H acts properly on $\mathbb{H}^2$ with the action of H on S^1 coming the induced action of H on $\partial\mathbb{H}^2 = S^1$. Clearly for any gap J of B, $\pi(J)$ is a single point. Thus $stab(J) \cap H < stab(\pi(J))$. Since the stabilizer of single boundary point in a Fuchsian group is either finite or peripheral, the proof is complete. □

Corollary 44. *If I is an interval of $\mathcal{P}$ which contains a necklace N, then $stab(I)$ is virtually cyclic. If N is interior to I then $stab(I)$ is finite.*

Proof. In any case the $stab(I)$ will stabilize N and a gap of N, and thus be virtually cyclic. If N is interior to I, then the $stab(I)$ stabilizes N and two distinct gaps of N, and so $stab(I)$ is finite. □

Corollary 45. *If G leaves a necklace invariant, then G is virtually a closed hyperbolic surface group.*

Proof. By Theorem 43, G is virtually a Fuchsian group. Since G has one end, G is virtually a closed hyperbolic surface group. □

For the remainder of the paper, we assume (in addition) that G is not virtually a closed surface group, so G acts on T without global fixed point.

7.3. **T is discrete.** The tree T we have constructed from the boundary of X is a-priori a possibly non discrete $\mathbb{R}-$tree. We recall that a G-tree is termed minimal if it does not contain a G-invariant proper subtree.

Lemma 46. *The tree T is minimal.*

Proof. Suppose not, then we pass to the minimal subtree contained in T, which we denote by T_m. Let $\mathcal{P}_m = \mathcal{P} \cap T_m$ (i.e. $\mathcal{P}_m$ are the elements of $\mathcal{P}$ that correspond to points of T_m). Let A be a necklace or an inseparable cut pair. Then there are open sets U and V of ∂X separated by a cut pair contained in A with $U \cap A = \emptyset$. Since the action of G on X is rank 1, there is a rank 1 hyperbolic element h with $h^- \in U$ and $h^+ \in V$. It can be shown using π-convergence that h acts by translation on a line $L \subset T$ with $A \in L$. Thus $A \in \mathcal{P}_m$ and so every cut pair and necklace is in $\mathcal{P}_m$.

Recall by definition that we removed the terminal points of $\mathcal{P}$ before we glued in intervals to form T. Notice that any non-terminal maximal inseparable set is between two elements which are either necklaces or inseparable cut pairs. Since all necklaces and inseparable cut pairs are in $\mathcal{P}_m$, all non-terminal elements of $\mathcal{P}$ are in $\mathcal{P}_m \subset T_m$. It follows that $T_m = T$. ☐

Theorem 47. *T is discrete.*

Proof. We first show that T is stable. Let I be an interval of $\mathcal{P}$ with infinitely many elements. We will show that $stab(I)$ is a finite group. If I contains a necklace in its interior then $stab(I)$ is finite by Corollary 44. Otherwise there are infinitely many inseparable cut pairs in I.

Assume that $stab(I)$ is infinite and let $g_n \in stab(I)$ be an infinite sequence of distinct elements. By passing to a subsequence we may assume that $g_n(x) \to p$, and $g_n^{-1}(x) \to n$ for some (all) $x \in X$. There are inseparable cut pairs $A, B \in I$ with A separating B from $p \in \partial X$. Choose continua Y, Z with $Y \cup Z = \partial X$ and $Y \cap Z = B$. We may assume that $A \subset Y$, which forces $p \in Y$.

Since G is rank 1, there exists $z \in Z$ with $d_T(z, n) = \infty$. By π-convergence $g_n(z) \to p$. Consider $V = \bigcup_{n \geq 0} h^n(Z)$. The set V is connected since each set in the union contains B. Also $A \notin V$. But $p \in \bar{V}$, the closure of V. This contradicts the fact that A separates B from p. Thus $stab(I)$ is finite.

Since there is a bound on the size of a finite subgroup in G, the action of G on T is stable.

Suppose that T is not discrete. Then by [18] there is an $\mathbb{R}$-tree S and a G-invariant quotient map $f : T \to S$ such that G acts non-trivially on S by isometries. Furthermore for each non-singleton arc $\alpha \subset S$, $f^{-1}(\alpha) \cap \mathcal{P}$ is an infinite interval of $\mathcal{P}$, and the stabilizer of an arc of S is the stabilizer of an arc of T. It follows that the arc stabilizers of S are finite and that the action of G on S is stable.

Since G is one-ended, by the Rips machine applied to S, there is a element $h \in G$ which acts by translation on a line in S such that G virtually splits over h. This implies that $\{h^{\pm}\}$ is a cut pair, however since h acts by translation on a line in S, it acts by translation on a line in T, and so $h^{\pm}$ will correspond to the ends of this line (by π-convergence). Thus $\{h^{\pm}\}$ is not an element of $\mathcal{P}$, nor is it a subset of a necklace of $\mathcal{P}$. This contradicts the fact that $\{h^{\pm}\}$ is a cut pair. Thus T is discrete. ☐

Lemma 48. *Let $A \in \mathcal{P}$ be an inseparable cut pair and $C \in \mathcal{P} \cap T$ adjacent to A. Then A doesn't separate C from a point of $\Lambda stab[C, A]$. Furthermore $\Lambda stab[C, A]$ is contained in C.*

Proof. Suppose that $p \in \Lambda stab[C, A]$ is separated from C by A. Choose a $(g_i) \subset stab[C, A]$ with $g_i(x) \to p$ and $g_i^{-1}(x) \to n$ for some $n \in \partial X$. Since $C \in \mathcal{P} \cap T$, there is a necklace or cut pair $E \in \mathcal{P}$ with $C \in (E, A)$. Since G is rank one, there is a point q at infinite Tits distance from n separated from C by E. It follows that $g_i(q) \to p$ but this contradicts A separating C from p.

To show that $\Lambda stab[C, A]$ is contained in C we argue similarly. Suppose $p \in \Lambda stab[C, A]$ does not lie in C. Choose a $(g_i) \subset stab[C, A]$ with $g_i(x) \to p$ and $g_i^{-1}(x) \to n$ for some $n \in \partial X$. Since A is not a terminal point of T, there is a point q at infinite Tits distance from n separated from p by A. It follows that $g_i(q) \to p$. We note however that p is separated from C by a cut pair c, d. So we can write $\partial X = E \cup F$ with E, F continua such that $E \cap F = \{c, d\}$. Let's say $A \subset F$, $p \in E$. Now since g_i fixes A, $g_i(q) \in F$ for all i. So $g_i(q)$ does not converge to p, a contradiction. ∎

Theorem 49. *If A is an inseparable cut pair then there is a hyperbolic g with $A = \{g^{\pm}\}$.*

Proof. Suppose not. Let C, D be elements of $\mathcal{P} \cap T$ adjacent to A. Since G does not split over a finite group, there are infinite finitely generated subgroups $H = \langle h_1 \dots h_n \rangle$ and $\hat{H} = \langle \hat{h}_1 \dots \hat{h}_m \rangle$ stabilizing $\{C, A\}$ and $\{A, D\}$ respectively.

By passing to subgroups of index 2, if necessary, we may assume that H and $\hat{H}$ fix A pointwise.

It follows from [28] that $A \subset \Lambda Z_{h_i}$ and $A \subset \Lambda Z_{\hat{h}_j}$ for all i, j. Since Z_{h_i} and $Z_{\hat{h}_j}$ are convex, it follows from [28, Theorem 16] that $Z_H = \bigcap_i Z_{h_i}$ and $Z_{\hat{H}} = \bigcap_i Z_{h_i}$ are convex and that $A \subset \Lambda Z_H$ and $A \subset \Lambda Z_{\hat{H}}$. Thus by [28, Theorem 16] $Z = Z_H \cap Z_{\hat{H}}$ is convex and $A \subset \Lambda Z = \Lambda Z_H \cap \Lambda Z_{\hat{H}}$.

Since Z is convex, it is a CAT(0) group, and by [28, Theorem 11] there is an element $g \in Z$ of infinite order [28, Corollary to Theorem 17]. Since $H, \hat{H}$ are infinite, $\Lambda H, \Lambda \hat{H}$ are non-empty.

By Lemma 48, ΛH is not separated from C by A and similarly $\Lambda \hat{H}$ is not separated from D by A. By Lemma 33, g fixes ΛH and $\Lambda \hat{H}$. If either $\Lambda H \subset A$ or $\Lambda H \subset A$, we are done by Lemma 16. Either $\Lambda H \subset C$, in which case $g(C) = C$, or points of ΛH are separated from D by C, and similarly for $\Lambda \hat{H}$ and D.

It could a priori happen that g acts by translation on a line of T which contains A, C, D. In that case however, g is rank 1 (see proof of Corollary 42). By Lemma 16 $|\Lambda H|, |\Lambda \hat{H}| \geq 2$ Thus $\Lambda H \cup \Lambda \hat{H} \not\subset \{g^{\pm}\}$ contradicting the fact that g fixes both and is rank 1.

The only remaining possibility is that $g(C) = C$ and $g(D) = D$, so $g(A) = A$. It follows by Lemma 48 that $g^{\pm} \subset C$ and $g^{\pm} \subset D$. Therefore $g^{\pm} \in C \cap D = A$ and so $\{g^{\pm}\} = A$. $\square$

Corollary 50. *Let $A, B, C \in \mathcal{P} \cap T$ with $B \in (A, C)$ and B inseparable cut pair. If A and C are both adjacent to B then the stabilizer of the interval $[A, C]$ is finite or virtually $\langle g \rangle$ where g is hyperbolic with $\{g^{\pm}\} = B$.*

Proof. Let $H = stab[A, C]$, so $H = stab[A, B] \cap stab[B, C]$. By Lemma 48, $\Lambda H \subset A$ and $\Lambda H \subset C$, so $\Lambda H \subset A \cap C = B$. If H is not finite, it follows that H is virtually $\langle g \rangle$ where g is hyperbolic with $\{g^{\pm}\} = B$. $\square$

Let $I \subset \mathcal{P} \cap T$ be an interval of $\mathcal{P}$. Notice that if I contains three elements of $\mathcal{P}$ which are not inseparable cut pairs, then $stab(I)$ is finite.

7.4. G **splits over a 2-ended group.** We recall here some terminology for group actions on trees and graphs of groups.

If Γ is a graph of groups and V is a vertex group of Γ then we say that we can refine Γ at V if the following holds: There is a graph of groups decomposition of V such that all edge groups of Γ adjacent to V are contained in vertex groups of the graph of groups decomposition of V. We obtain the refinement of Γ by substituting V by its graph decomposition and joining the edges of Γ adjacent to V to the vertex groups of the graph containing the corresponding edge groups. A vertex v of a graph of groups labeled by a group V is called non-reduced if it is adjacent to at most 2 edges, none of which is a loop, and all edges adjacent to v are labeled by V. A graph of groups which contains no such vertices is called reduced. We say that a subgroup $H < G$ is elliptic in Γ if it is contained in a conjugate of a vertex group of Γ.

We proceed now to show that G splits over a 2-ended group. We will show this considering the action of G on T. In the next section we show that this action gives also the JSJ decomposition of G. We remark that T has at least one vertex that corresponds either to an inseparable cut pair or to a necklace. We will show in each case that one can get a splitting of G over a 2-ended group.

We consider the graph of groups Γ obtained from the action of G on T.

By Theorem 43 if a vertex v of Γ corresponds to a necklace then the vertex is labeled either by virtually a subgroup of $\mathbb{Z}^2$ or it is a fuchsian group. In the second case, Γ has an edge labeled by a peripheral subgroup of the fuchsian group. In the first case, if the vertex is labeled by $\mathbb{Z}^2$, then the limit set of the vertex stabilizer is a circle. Since this limit set is a necklace, we have that the boundary of G is a circle so G is virtually $\mathbb{Z}^2$. Otherwise the vertex is labeled by virtually $\mathbb{Z}$. If the vertex corresponds to a branch point, we see that all edges adjacent to it are labeled by 2-ended groups and the graph is reduced at this vertex. Otherwise a vertex u adjacent to it is labeled by a group containing the stabilizer of a maximal inseparable set which is a branch point of T. The edge with endpoints $[u, v]$ is labeled by a 2-ended group and gives a splitting of G.

Let's now consider a vertex v of T corresponding to an inseparable pair $\{a, b\}$. We consider first the case that the stabilizer of v is a two-ended group. If v corresponds to a branch point of T, then, since the edges adjacent to v are labeled by subgroups of the stabilizer of v, all these edges give reduced splittings of G over two ended groups. If v does not correspond to a branch point, then an edge adjacent to v corresponds to a branch point of T. This edge is labeled by a two ended group and gives a reduced splitting of G over a two ended group.

Let V be the stabilizer of the vertex v corresponding to a, b. By theorem 49 there is a hyperbolic element g in V with $\{g^\infty, g^{-\infty}\} = \{a, b\}$. By the algebraic torus theorem [11] to show that G splits over a 2-ended group it suffices to show that $X/ < g >$ has more than one end.

Let $\bar{X} = X \cup \partial X$. Let L be an axis for g and let D be the translation length of g. Let U, V be small neighborhoods around a, b respectively. To fix ideas we define U, V as follows: we take a base point x on L and we consider the ray from x to a. Let's call this ray c_a. Let's call c_y the (possibly finite) ray from x to a point $y \in \bar{X}$. We define U as follows:

$$U = \{y \in \partial X : d(c_y(10D), c_a(10D)) < 1\}$$

V is defined similarly. We claim that there is a K such that a K-neighborhood of L union $U \cup V$ separates any two components of $\partial X - a, b$. Suppose that there are components C_1, C_2 of $\partial X - \{a, b\}$ such that for any n there is a component T_n of $\bar{X}$ minus $U \cup V \cup N_n(L)$ such that T_n intersects both C_1, C_2. Without loss of generality we can assume that the $T_n's$ are nested. Then if $T = \bigcup T_n$, $T \subset \partial X$ is closed, connected and intersects both C_1, C_2. This is a contradiction since T does not contain a or b.

We recall the fact (see [19]) that geodesics in X are 'almost extendable' i.e. there is an $A > 0$ such that if $[a, b]$ is a finite geodesic in X then there is an infinite ray $[a, c]$ ($c \in \partial X$) such that $d(b, [a, c]) \leq A$.

Let $p : X \to L$ be the projection map from X to L.

We claim that there is an $M > 0$ such that the M-neighborhood of L separates X in at least two components Y_1, Y_2 such that Y_i is not contained in any finite neighborhood of L ($i = 1, 2$). This claim implies that G either splits over a 2-ended group or is virtually a surface group by [11].

We now prove the claim. Let X_1, X_2 be two distinct components of $\partial X - \{a, b\}$. We claim that there are two infinite rays r_1, r_2 from x to X_1, X_2 respectively which are perpendicular to L. Indeed we consider rays from x to ,say, points on $\bar{X}_1$. The angle that these rays form with L varies continuously and takes the values $\{0, \pi\}$ at $\{a, b\}$. Therefore some such ray r_1 is perpendicular to L. We argue similarly for r_2.

Given $n > A, n \in \mathbb{N}$ let $R_1 = r_1(n + 1), R_2 = r_2(n + 1)$. If R_1, R_2 are not contained in the same component of $X - N_n(L)$ then the claim is proven. Otherwise there is a path p in $X - N_n(L)$ joining R_1 to R_2. For every $y \in X$ we consider $p(y) \in L$ and we pick an infinite ray r_y from $p(y)$ such that $d(r_y, y) \leq A$. For R_1, R_2 we choose the corresponding rays to be r_1, r_2 respectively. Clearly there are $y_1, y_2 \in p$ such that $d(y_1, y_2) < 1$ and the corresponding rays r_{y_1}, r_{y_2} define points in distinct components of $\partial X - \{a, b\}$. Since $< g >$ acts cocompactly on L we may translate $p(y_1), p(y_2)$ close to x, say

$$d(g^k(p(y_1)), x) < 2D, d(g^k(p(y_2)), x) < 2D$$

Then $g^k(r_{y_1}), g^k(r_{y_2})$ define points in distinct components of $\partial X - \{a, b\}$ and if $n > K$ these two points are not separated by the K-neighborhood of L union $U \cup V$. This is a contradiction.

We have shown the following:

Theorem 51. *Let G be a one ended group acting geometrically on a CAT(0) space X. If a pair of points $\{a, b\}$ separates ∂X then either G splits over a 2-ended group or G is virtually a surface group.*

7.5. **JSJ decompositions.**

Definition . Let G be a group acting on a tree T and let Γ be the quotient graph of groups. We say that this action (or the decomposition Γ) is canonical if for every automorphism α of G there is an automorphism C_α of T such that $\alpha(g)C_\alpha = C_\alpha g$ for every $g \in G$.

Bowditch ([6]) constructed a canonical JSJ-decomposition for hyperbolic groups. Swarup-Scott ([25]) constructed a canonical JSJ-decomposition for finitely presented groups in general. We show here how to deduce a canonical JSJ-decomposition for CAT(0) groups using their CAT(0) boundary. Our JSJ-decomposition is similar to the one in [25].

To describe our decomposition we use the notation of the previous sections. So G is a CAT(0) group acting on a CAT(0) space X. From the pairs of cut points of the boundary of X, we construct an $\mathbb{R}$-tree T on which G acts non-trivially (unless G has no splittings over two-ended groups or G is virtually of the form $H \times \mathbb{Z}$, in these cases our JSJ decomposition is trivial). As we showed in the previous section, T is discrete. Then the graph of groups decomposition Γ given by the quotient graph of the action of G on T is the JSJ decomposition of G.

Vertex groups of this graph are either fuchsian or groups that have no splitting over 2-ended groups in which the adjacent edge groups are elliptic or groups that stabilize inseparable cut pairs. Edges incident to fuchsian groups are labeled by 2-ended groups by Theorem 43. The only other type of edges are edges joining an element C of $\mathcal{P}$ corresponding to an inseparable subset of ∂X and an inseparable pair $\{a, b\}$ of $\mathcal{P}$. These edges are not labeled necessarily by 2-ended groups. So our decomposition might differ from the one obtained by Rips-Sela [22] or Dunwoody-Sageev [12]. We give an example to illustrate this point. Let

$$A = \mathbb{F}_4 \times \mathbb{Z} =< a, b, c, d > \times < x >$$
$$B = \mathbb{F}_2 \times \mathbb{F}_2 =< a, b > \times < x, y >$$
$$C = \mathbb{F}_2 \times \mathbb{F}_2 =< c, d > \times < x, z >$$

and consider the group given by the graph of groups

$$G = B *_E A *_D C$$

where $E =< a, b > \times < x >= \mathbb{F}_2 \times \mathbb{Z}$, $D =< c, d > \times < x >= \mathbb{F}_2 \times \mathbb{Z}$. Then the JSJ decomposition that we obtain is the graph of groups $B *_E A *_D C$. However we remark that this decomposition can be refined to give the ordinary JSJ decomposition by splitting A over x, i.e. we have

$$G = B *_E A *_D C = B *_E *(E *_{<x>} D) *_D C = A *_{<x>} C$$

where the latter decomposition is the ordinary JSJ decomposition.

We claim that one can always refine our decomposition to a JSJ decomposition over 2-ended groups given in [12]. Let T_J be the tree corresponding to the JSJ decomposition in [12]. To show that Γ can be refined, it is enough to show that edge groups of Γ act elliptically on T_J. This is clearly true for the 2-ended vertex groups. Now let V be the stabilizer of an inseparable cut pair $\{a, b\}$ and let e be an edge joining $\{a, b\}$ to an element C of $\mathcal{P}$ corresponding to an inseparable subset of ∂X. If E is the edge stabilizer of e by lemma 48 ΛE is contained in

C. If E acts hyperbolically on T_J, then there is a hyperbolic element $h \in E$ and the 2 points of $\Lambda\langle h\rangle$ are separated by any pair of points of ∂X corresponding to limit points of an edge on the axis of h on T_J. This is a contradiction since $\Lambda\langle h\rangle \subset C$. It follows that any edge group of Γ fixes a point of T_J. We remark that the only vertex groups of Γ that might act hyperbolically on T_J are vertex groups that stabilize inseparable cut pairs. To refine Γ we let all such groups act on T_J and we replace them in Γ by the decompositions we obtain in this way. Doing this refinement and collapsing we obtain a JSJ-decomposition of G over 2-ended groups in the sense of Dunwoody-Sageev.

In order to show that our JSJ decomposition is canonical, we observe that if G splits over a 2-ended subgroup Z, then the two limit points of Z form a cut pair of ∂X. So if α is an automorphism of G the limit points of $\alpha(Z)$ is also a separating pair for ∂X. This observation suffices to show that our JSJ decomposition is canonical. We now explain this in detail.

Let α be an automorphism of G. If H is a fuchsian hanging subgroup of Γ, then the limit set of H in ∂X is a necklace and H fixes the vertex of T corresponding to this necklace. $\alpha(H)$ is also a hanging fuchsian group whose limit set is a necklace and $\alpha(H)$ fixes the corresponding vertex. A vertex V of Γ which is virtually of the form $H \times \mathbb{Z}$ is the stabilizer of an inseparable pair $\{a, b\}$. Clearly $\alpha(V)$ is also a stabilizer of an inseparable pair so it fixes also a vertex of T. Finally we remark that π convergence implies that the stabilizer V of a maximal inseparable set has a limit set which is contained in the maximal inseparable set. So $\alpha(V)$ fixes also a vertex of T.

We observe that adjacent vertex groups are mapped to adjacent vertex groups by α. This is because splittings $A *_C B$ are mapped to splittings $\alpha(A) *_{\alpha(C)} \alpha(B)$ by α.

We have shown the following:

Theorem 52. *Let G be a one ended group acting geometrically on a CAT(0) space X. Then the JSJ-tree of ∂X is a simplicial tree T and the graph of groups T/G gives a canonical JSJ decomposition of G over 2-ended groups.*

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E-mail address, Panos Papasoglu: `panos@math.uoa.gr`
E-mail address, Eric Swenson: `eric@math.byu.edu`

(Panos Papasoglu) Mathematics Department, University of Athens, Athens 157 84, Greece

(Eric Swenson) Mathematics Department, Brigham Young University, Provo UT 84602