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Heisenberg-Type Families in $U_q(\widehat{sl_2})^\star$

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Abstract. Using the second Drinfeld formulation of the quantized universal enveloping algebra $U_q(\widehat{sl_2})$ we introduce a family of its Heisenberg-type elements which are endowed with a deformed commutator and satisfy properties similar to generators of a Heisenberg subalgebra. Explicit expressions for new family of generators are found.

Key words: quantized universal enveloping algebras; Heisenberg-type families

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1 Introduction

The purpose of this paper is to introduce a new family of elements in the quantized enveloping algebra $U_q(\widehat{sl_2})$ of an affine Lie algebra $\widehat{sl_2}$. This Heisenberg-type families possess properties similar to ordinary Heisenberg algebras. Heisenberg subalgebras of affine Lie algebras and of their q-deformed enveloping algebras are being actively used in various domains of mathematical physics. Most important applications of Heisenberg subalgebras can be found in the field of classical and quantum integrable models and field theories. Vertex operator constructions both for affine Lie algebras [5] and for q-deformations of their universal enveloping algebras [4, 6] are essentially based on Heisenberg subalgebras. Given a quantized universal enveloping algebra $U_q(\widehat{\mathcal{G}})$ of an affine Kac–Moody Lie algebra $\widehat{\mathcal{G}}$, it is rather important to be able to extract explicitly generators of a Heisenberg subalgebra associated to a chosen grading of $U_q(\widehat{\mathcal{G}})$ which is not always a trivial task. For instance, one can easily recognize elements of a Heisenberg subalgebra among generators in the homogeneous grading of the second Drinfeld realization of $U_q(\widehat{sl_2})$ [4] while it is not obvious how to extract a Heisenberg subalgebra in the principal grading. Ideally, one would expect to obtain a realization of the Heisenberg subalgebra associated to the principal graiding of $U_q(\widehat{sl_2})$ which would involve ordinary (rather then q-deformed) commutator in commutation relations with certain elements in the family giving central elements. This would lead to many direct applications both in quantum groups and quantum integrable theories in analogy with the homogeneous graiding case. In [2] the principal commuting subalgebra in the nilponent part of $U_q(\widehat{sl_2})$ was constructed. Its elements expressed in q-commuting coordinates commute with respect to the q-deformed bracket.

In this paper we introduce another possible version of a family elements in $U_q(\widehat{sl_2})$ which could play a role similar to ordinary Heisenberg subalgebra. Our general idea is to form certain sets of $U_q(\widehat{sl_2})$-elements containing linear combinations of generators $x_n^\pm$, $n \in \mathbb{Z}$, multiplied by various powers of K and the central element γ. Under certain conditions on corresponding powers we obtain commutation relations for a Heisenberg-type family with respect to an integral p-th power of $K \in U_q(\widehat{sl_2})$-deformed commutator. We consider this as some further generalization of various q-deformed commutator algebras (in particular, q-bracket Heisenberg subalgebras) which

*This paper is a contribution to the Proceedings of the XVIIth International Colloquium on Integrable Systems and Quantum Symmetries (June 19–22, 2008, Prague, Czech Republic). The full collection is available at http://www.emis.de/journals/SIGMA/ISQS2008.html find numerous examples in quantum algebras and applications in integrable models. Though the commutation relation we use in order to define a Heisenberg-type family look quite non-standard we believe that these families and their properties deserve a consideration as a new structure inside the quantized universal enveloping algebra of $\widehat{sl_2}$ even when $p \neq 0$.

The paper is organized as follows. In Section 2 we recall the definition of the quantized universal enveloping algebra $U_q(\widehat{sl_2})$ in the second Drinfeld realization. In Section 3 we find explicit expressions for elements of Heisenberg-type families. Then we prove their commutation relations. We conclude by making comments on possible generalizations and applications.

2 Second Drinfeld realization of $U_q(\widehat{sl_2})$

Let us recall the second Drinfeld realization [1, 3] of the quantized universal enveloping algebra $U_q(\widehat{sl_2})$. It is generated by the elements $\{x_k^\pm,\ k \in \mathbb{Z};\ a_n,\ n \in \{\mathbb{Z}\backslash 0\};\ \gamma^{\pm\frac{1}{2}},\ K\}$, subject to the commutation relations

$$
\begin{aligned}
&[K, a_k] = 0, \qquad K x_k^\pm K^{-1} = q^{\pm 2} x_k^\pm, \qquad [a_k, a_l] = \delta_{k,-l} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \\
&[a_n, x_k^\pm] = \pm \frac{[2n]}{n} \gamma^{\mp \frac{|n|}{2}} x_{n+k}^\pm, \qquad [x_n^+, x_k^-] = \frac{1}{q - q^{-1}} \left(\gamma^{\frac{1}{2}(n-k)} \psi_{n+k} - \gamma^{-\frac{1}{2}(n-k)} \phi_{n+k} \right), \\
&x_{k+1}^\pm x_l^\pm - q^{\pm 2} x_l^\pm x_{k+1}^\pm = q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm,
\end{aligned}
\tag{1}
$$

where $\gamma^{\pm\frac{1}{2}}$ belong to the center of $U_q(\widehat{sl_2})$, and

$$
[n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}.
$$

The elements ϕ_k and ψ_{-k}, for non-negative integers $k \in \mathbb{Z}_+$, are related to $a_{\pm k}$ by means of the expressions

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \psi_m z^{-m} = K \exp\left((q - q^{-1}) \sum_{k=1}^{+\infty} a_k z^{-k} \right), \\
&\sum_{m=0}^{\infty} \phi_{-m} z^{m} = K^{-1} \exp\left(-(q - q^{-1}) \sum_{k=1}^{+\infty} a_{-k} z^{k} \right),
\end{aligned}
\tag{2}
$$

i.e., $\psi_m = 0$, $m < 0$, $\phi_m = 0$, $m > 0$.

3 Heisenberg-type families

In this section we introduce a family of $U_q(\widehat{sl_2})$-elements which have properties similar to a ordinary Heisenberg subalgebra of an affine Kac–Moody Lie algebra [5]. We consider families of linear combinations of $x_n^\pm$-generators of $U_q(\widehat{sl_2})$ multiplied by powers of K and central element γ.

Let us introduce for m, l, η, $\theta \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, the following elements:

$$
E_n^\pm(m, \eta) = \gamma^{\pm(n+\frac{1}{2})} x_n^+ K^m + x_{n+1}^- K^\eta, \tag{3}
$$

$$
E_{-n-1}^\pm(l, \theta) = x_{-n-1}^+ K^l + \gamma^{\pm(n+\frac{1}{2})} x_{-n}^- K^\theta. \tag{4}
$$

Denote also for some $p \in \mathbb{Z}$, a deformed commutator

$$
[A, B]_{K^p} = A K^p B - B K^p A. \tag{5}
$$

We then formulate

Proposition. *Let p, $m \in \mathbb{Z}$, $l = m$, $\theta = \eta = -m - 2p$. Then the family of elements*

$$\{E^{\pm}_{p,n}(m),\ E^{\pm}_{p,-n-1}(m),\ n \in \mathbb{Z}_+\},\tag{6}$$

where $E^{\pm}_{p,n}(m) \equiv E^{\pm}_{n}(m, -m-2p)$, $E^{\pm}_{p,-n-1}(m) \equiv E^{\pm}_{-n-1}(m, -m-2p)$, we have denoted in (3), (4), are subject to the commutation relations with $k \in \mathbb{Z}_+$,

$$\left[E^{+}_{p,n}(m), E^{+}_{p,-k-1}(m)\right]_{K^p} = 0, \qquad \text{for all } n < k,\tag{7}$$

$$\left[E^{-}_{p,n}(m), E^{-}_{p,-k-1}(m)\right]_{K^p} = 0, \qquad \text{for all } n > k,\tag{8}$$

$$\left[E^{\pm}_{\pm 1,n}(m), E^{\pm}_{\pm 1,-n-1}(m)\right]_{K^{\mp 1}} = c^{\pm}_n(m),\tag{9}$$

where

$$c^{+}_n(m) = \frac{q^{-2(m-1)}}{q-q^{-1}}\gamma^{2n+1}(\gamma^{n} - \gamma^{-n-1}), \qquad c^{-}_n(m) = \frac{q^{-2(m+1)}}{q-q^{-1}}\gamma^{-n-1}(\gamma^{2n+2} - \gamma^{-n}),$$

belong to the center $\mathcal{Z}(U_q(\widehat{sl_2}))$ of $U_q(\widehat{sl_2})$.

We call a subset (6) of $U_q(\widehat{sl_2})$-elements with all appropriate m, $p \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, such that it satisfies the commutation relations (7)–(9) the *Heisenberg-type* family. In particular, when $p = 0$, (7)–(8) reduce to ordinary commutativity conditions. Note that if we formally substitute $n \mapsto -n-1$, then,

$$E^{\pm}_{p,n}(m,\eta) \mapsto \gamma^{\mp(n+1/2)} E^{\mp}_{p,-n-1}(m,\eta).$$

Under the action of an automorphism ω of $U_q(\widehat{sl_2})$ which maps $K \mapsto K^{-1}$, $\gamma \mapsto \gamma^{-1}$, $x^{\pm}_n \mapsto x^{\mp}_{-n}$, $a_n \mapsto a_{-n}$, one has $\omega(E^{\pm}_{p,n}(m)) = E^{\mp}_{p,-n-1}(m)\, K^{2p}$.

Proof. The proof is the direct calculation of the commutation relations (7)–(9). Indeed, consider K^p-deformed commutator (5) of $E^{\pm}_n(m,\eta)$ and $E^{\pm}_{-k-1}(l,\theta)$ with some $m, \eta, l, \theta \in \mathbb{Z}$, $n, k \in \mathbb{Z}_+$. Using the commutation relations (1) we obtain

$$\begin{aligned}\left[E^{\pm}_n(m,\eta), E^{\pm}_{-k-1}(l,\theta)\right]_{K^p} &= \gamma^{\pm(n+k+1)}\big(q^{-2m-2p}x^{+}_n x^{-}_{-k} - q^{2\theta+2p}x^{-}_{-k}x^{+}_n\big)K^{m+\theta+p}\\ &+ \big(q^{2\eta+2p}x^{-}_{n+1}x^{+}_{-k-1} - q^{-2l-2p}x^{+}_{-k-1}x^{-}_{n+1}\big)K^{\eta+l+p}\\ &+ \gamma^{\pm(n+1/2)}\big(q^{2m+2p}x^{+}_n x^{+}_{-k-1} - q^{2l+2p}x^{+}_{-k-1}x^{+}_n\big)K^{\eta+\theta+p}\\ &+ \gamma^{\pm(k+1/2)}\big(q^{-2\eta-2p}x^{-}_{n+1}x^{-}_{-k} - q^{-2\theta-2p}x^{-}_{-k}x^{-}_{n+1}\big)K^{\eta+\theta+p}.\end{aligned}$$

Then for $m = l$, $\theta = \eta = -m-2p$, from (1) it follows

$$\begin{aligned}\left[E^{\pm}_{p,n}(m), E^{\pm}_{p,-k-1}(m)\right]_{K^p} &= \frac{q^{-2(m+p)}}{q-q^{-1}}\big[\gamma^{\pm(n+k+1)}\big(\gamma^{1/2(n+k)}\psi_{n-k} - \gamma^{-1/2(n+k)}\phi_{n-k}\big)\\ &- \big(\gamma^{-1/2(n+k+2)}\psi_{n-k} - \gamma^{1/2(n+k+2)}\phi_{n-k}\big)\big]K^{p}.\end{aligned}$$

Since for $n < k$, $\psi_{n-k} = 0$, and the reaming terms containing ϕ_{n-k} cancels, we obtain (7). Similarly, for $n > k$, $\phi_{n-k} = 0$, the terms containing ψ_{n-k} cancels, and (8) follows.

From (2) we see that $\psi_0 = K$, $\phi_0 = K^{-1}$. Taking $K^{\mp 1}$-deformed commutators (5) of $E^{\pm}_{1,n}(m)$, $E^{\pm}_{1,-k-1}(m)$, we then have

$$\left[E^{+}_{1,n}(m), E^{+}_{1,-n-1}(m)\right]_{K^{-1}} = \frac{q^{-2m+2}}{q-q^{-1}}\big[\gamma^{2n+1}(\gamma^{n}K)\,K^{-1} - (\gamma^{-n-1}K)K^{-1}\big] = c^{+}_n(m),$$

$$\left[E^{-}_{-1,n}(m), E^{-}_{-1,-n-1}(m)\right]_{K} = \frac{q^{-2m-2}}{q-q^{-1}}\big[\gamma^{-2n-1}K(-\gamma^{-n}K^{-1}) + K(\gamma^{n+1}K^{-1})\big] = c^{-}_n(m),$$

for $n = k$. ∎

4 Conclusions

In the second Drinfeld realization of $U_q(\widehat{sl_2})$ we have defined a subset of elements that constitutes a Heisenberg-type family, explicitly constructed their elements, and proved corresponding commutation relations. Properties of a Heisenberg-type family are similar to ordinary Heisenberg subalgebra properties. These families might be very useful in construction of special types of vertex operators in $U_q(\widehat{sl_2})$, and, in particular, might have their further applications in the soliton theory of non-linear integrable partial differential equations [6]. One of our aims to introduce Heisenberg-type families is the development of corresponding vertex operator representation which plays the main role in the theory of quantum soliton operators in exactly solvable field models associated to the infinite-dimensional Lie algebra $\widehat{sl_2}$ [6].

Finally, we would like also to make some comments comparing present work to [2] where the quantum principal commutative subalgebra in $U_q(\widehat{sl_2})$ associated to the principal grading of $\widehat{sl_2}$ was found. Here we introduce Heisenberg-type families of $U_q(\widehat{sl_2})$ in the principal grading of $U_q(\widehat{sl_2})$ [6]. Although we use K^p-deformed commutators (which for $p = 1$ can be seen quite similar to q-deformed commutators in [2]) these two approaches are quite different. We prefer to work with the explicit set of $U_q(\widehat{sl_2})$ generators (q-commutative coordinates) in its second Drinfeld realization [1], and introduce elements of our Heisenberg-type families not involving lattice constructions or trace invariants as in [2]. A generalization of our results to an arbitrary $\widehat{\mathcal{G}}$ case does not face any serious technical problems. We assume that Heisenberg-type families introduced which exhibit properties similar to a Heisenberg subalgebra in $U_q(\widehat{sl_2})$ are not the most general ones. At the same time the construction described in this paper allows further generalization to cases of arbitrary affine Lie algebras [7]. Using formulae from [2] we see that even in the q-commutator case there exist more complicated q-commutative elements in $U_q(\widehat{sl_2})$. Thus one would expect the same phenomena for K^p-deformed algebras.

More advanced examples of Heisenberg-type families associated to various gradings of $U_q(\widehat{\mathcal{G}})$ in the Drinfeld–Jimbo and second Drinfeld realizations as well as corresponding vertex operators will be discussed in a forthcoming paper [7].

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