On the homology of the space of knots

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Consider the space of 'long knots' in $\mathbb{R}^n$ $\mathcal{K}_{n,1}$. This is the space of knots as studied by V. Vassiliev. Based on previous work [5, 12], it follows that the rational homology of $\mathcal{K}_{3,1}$ is free Gerstenhaber-Poisson algebra. A partial description of a basis is given here. In addition, the mod-p homology of this space is a 'free, restricted Gerstenhaber-Poisson algebra'. Recursive application of this theorem allows us to deduce that there is p-torsion of all orders in the integral homology of $\mathcal{K}_{3,1}$.

This leads to some natural questions about the homotopy type of the space of long knots in $\mathbb{R}^n$ for $n > 3$, as well as consequences for the space of smooth embeddings of S^1 in S^3 and embeddings of S^1 in $\mathbb{R}^3$.

knots, embeddings, spaces, cubes, homology

1 Introduction

The purpose of this paper is to give homological properties of the classical spaces of smooth 'long' embeddings $\mathcal{K}_{3,1} = \mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$ and smooth embeddings $\mathrm{Emb}(S^1, S^3)$. Some results here also apply to the embedding spaces $\mathrm{Emb}(S^j, S^n)$, and 'long' embedding spaces $\mathcal{K}_{n,j} = \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$, with the main results focused on the 3-dimensional case $j = 1$, $n = 3$.

The approach here to these homological problems follows recent work of Hatcher [23, 24] and Budney [5, 7]. The homotopy type of the components of $\mathrm{Emb}(S^1, S^3)$ and $\mathcal{K}_{3,1}$ are understood completely in terms of configuration spaces in the plane, Stiefel manifolds, isometry groups of certain hyperbolic link complements and various natural iterated bundle operations. Many of the homological properties of both $\mathcal{K}_{3,1}$, and $\mathrm{Emb}(S^1, S^3)$ follow from combining this information together with earlier work of Cohen [12] on configuration spaces.

The space $\mathcal{K}_{n,1}$ admits the structure of an H-space induced by concatenation of 'long' embeddings. In addition, this H-space structure for $\mathcal{K}_{3,1}$ was shown to extend to a free $\mathcal{C}_2$-space in the sense of May, with "generating set" given by the space of prime long knots [5].

One consequence is that the homotopy type of the space of long knots is determined completely by the homotopy type of the prime long knots. Information concerning spaces of prime long knots is combined with bundle theoretic constructions to give a large contribution to the homology groups for spaces of long knots, as well as $\mathrm{Emb}(S^1, S^3)$.

The structure of a graded analogue of a Poisson algebra, a Poisson-Gerstenhaber algebra, arises in the work here. An introduction to Poisson algebras is given in [11], pages 177-182 while some applications are given in [12] pages 215-216 and [49]. A Poisson-Gerstenhaber algebra A is a graded commutative algebra over $\mathbb{Q}$ given by A_n in gradation n together with a graded skew symmetric bilinear map

$$\{-,-\} : A_s \otimes A_t \to A_{s+t+1}$$

which satisfies the following where $|a|$ denotes the degree of an element a in A:

(1) the Jacobi identity

$$\{a, \{b, c\}\} = (-1)^{|a|+|b|+|c|+1}\{\{a, b\}, c\} + (-1)^{|b||c|+|c|+1}\{\{a, c\}, b\}$$

where the signs will be typically omitted with the above written as

$$\{\{a, \{b, c\}\} = (\pm 1)\{\{a, b\}, c\} + (\pm 1)\{\{a, c\}, b\}.$$

(2) the Leibniz formula

$$\{a \cdot b, c\} = a \cdot \{b, c\} + (-1)^{|b||a|} b\{a, c\}.$$

A standard example of such a Gerstenhaber-Poisson algebra is given by the rational homology algebra of $\Omega^2(X)$ for X a bouquet of spheres of dimension at least 3 by [12] in which the precise axioms are recorded.

The mod-p homology of $\mathcal{K}_{3,1}$ has more detailed structure, and is, loosely speaking, a 'free restricted Gerstenhaber-Poisson algebra' with additional structure satisfied by free $\mathcal{C}_2$-spaces [12], freely generated on the mod-p homology of the subspace of prime long knots.

The main results of the current article are Theorem 1.3 on the structure of the homology of $\mathcal{K}_{3,1}$, Proposition 1.4 concerning implications for the space of smooth embeddings of S^1 in S^3, Proposition 1.5, a homological characterization of the unknot, as well as Theorem 9.1 on the minimal i such that $H_i(\mathcal{K}_{3,1};\mathbb{Z})$ contains p-torsion.

Definition 1.1 *$\mathcal{K}_{n,j} := \{f : \mathbb{R}^j \to \mathbb{R}^n : f$ is an embedding and $f(x_1, x_2, \cdots, x_j) = (x_1, x_2, \cdots, x_j, 0, \cdots, 0)$ for $|x| \geq 1\}$. $\mathcal{K}_{n,1}$ is traditionally called the space of long knots in $\mathbb{R}^n$, and $\mathcal{K}_{n,j}$ the space of long j-knots in $\mathbb{R}^n$. Given an element $f \in \mathcal{K}_{n,j}$ the connected-component of $\mathcal{K}_{n,j}$ containing f is denoted $\mathcal{K}_{n,j}(f)$. Two knots are considered equivalent if they are in the same connected component of $\mathcal{K}_{n,j}$ (That is the knots are isotopic).*

Let $X_{\mathcal{K}} = \{f \in \mathcal{K}_{3,1} : f \text{ is prime}\}$, where the word 'prime' is used in the traditional sense of Schubert [43], that is $X_{\mathcal{K}}$ is the union of the connected components of $\mathcal{K}_{3,1}$ which contain knots that are not connected sums of two or more non-trivial knots, nor are they allowed to contain the unknot.

Theorem 1.2 *[6] The space $\mathcal{K}_{3,1}$ is homotopy equivalent to*

$$C(\mathbb{R}^2, X_{\mathcal{K}} \amalg \{*\})$$

that is the labelled configuration space of points in the plane with labels in $X_{\mathcal{K}} \amalg \{*\}$. Furthermore, the following hold:

(1) *Each path-component of $\mathcal{K}_{3,1}$ is a $K(\pi, 1)$.*

(2) *The path-components of $\mathcal{C}_2(n) \times_{\Sigma_n} (X_{\mathcal{K}})^n$ for all n, and thus the path-components of $\mathcal{K}_{3,1}$ are given by*

$$\mathcal{C}_2(n) \times_{\Sigma_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i)$$

for certain choices of $f_1, \cdots, f_n \in X_{\mathcal{K}}$, and Young subgroups Σ_f.

The above theorem can be thought of as a generalization of Schubert's Theorem which states that $\pi_0\mathcal{K}_{3,1}$ is a free commutative monoid on countably-infinite many generators [43]. Schubert's theorem is about the monoid structure on $\pi_0\mathcal{K}_{3,1}$ induced by the cubes action, while the above theorem is space-level on $\mathcal{K}_{3,1}$.

In general, $\mathcal{K}_{n,1}$ is a homotopy-associative H-space with multiplication induced by concatenation. This multiplication gives a product operation

$$H_s(\mathcal{K}_{n,1}) \otimes H_t(\mathcal{K}_{n,1}) \to H_{s+t}(\mathcal{K}_{n,1})$$

Since $\mathcal{K}_{3,1}$ admits the action of the operad of little 2-cubes, there is an induced map

$$\theta : S^1 \times \mathcal{K}_{3,1} \times \mathcal{K}_{3,1} \to \mathcal{K}_{3,1}$$

together with an operation in homology with any coefficients

$$H_s(\mathcal{K}_{3,1}) \otimes H_t(\mathcal{K}_{3,1}) \to H_{1+s+t}(\mathcal{K}_{3,1})$$

which is denoted, up to sign, by

$$\{\alpha, \beta\} \equiv \lambda_1(\alpha, \beta) = \theta_*(\iota \otimes \alpha \otimes \beta)$$

for α in $H_s(\mathcal{K}_{3,1})$, β in $H_t(\mathcal{K}_{3,1})$ and $\iota \in H_1(S^1)$ the fundamental class. These operations satisfy the structure of a graded Poisson algebra for which the bracket operation $\lambda_1(\alpha, \beta)$ is called the Browder operation in [12].

The next result uses the product operation above as well as the bracket operation $\{\alpha, \beta\} = \lambda_1(\alpha, \beta)$, and follows by interweaving the results of 1.2, and [12]. We will use the notation $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ to denote the field with p elements, when p is a prime. To state these results, additional information given by three functors from graded vector spaces V over a field $\mathbb{F}$ are described next with complete details given in Section 5:

(1) If the characteristic of the field is 0 then the value of the functor on objects V is the symmetric algebra generated by an algebraically 'desuspended' free Lie algebra generated by the suspension of V, and denoted

$$S[\sigma^{-1}L[\sigma(V)]].$$

This last algebra is a free Gerstenhaber-Poisson algebra.

(2) For the field $\mathbb{F}_2$ the value of the functor on objects V is the symmetric algebra generated by an algebraically 'desuspended' free, mod-2 restricted Lie algebra generated by the suspension of V, and denoted

$$S[\sigma^{-1}L^{(2)}[\sigma(V)]].$$

This last algebra is a graded version of a free restricted Lie algebra.

(3) For the field $\mathbb{F}_p$ (p an odd prime), then the value of the functor on objects V is the symmetric algebra generated by an algebraically "desuspended" mod-p free restricted Lie algebra generated by the suspension of V plus an additional summand as described in Section 5, and denoted

$$S[\sigma^{-1}L^{(p)}[\sigma(V)] \oplus \sigma^{-2}W^p[\sigma(V)]].$$

Theorem 1.3 *The homology of $\mathcal{K}_{3,1}$ satisfies the following properties.*

(1) *The rational homology of $\mathcal{K}_{3,1}$ is a free Gerstenhaber-Poisson algebra generated by $V = H_*(X_{\mathcal{K}};\mathbb{Q})$.*

(2) *The homology of $\mathcal{K}_{3,1}$ with $\mathbb{F}_p$ coefficients is a free restricted Gerstenhaber-Poisson algebra generated by $V = H_*(X_{\mathcal{K}};\mathbb{F}_p)$ as described in [12].*

(3) *There are isomorphisms of Hopf algebras*

   (a) $S[\sigma^{-1}L[\sigma(V)]] \to H_*(\mathcal{K}_{3,1};\mathbb{Q})$ *for* $V = H_*(X_{\mathcal{K}};\mathbb{Q})$,

   (b) $S[\sigma^{-1}L^{(2)}[\sigma(V)]] \to H_*(\mathcal{K}_{3,1};\mathbb{F}_2)$ *for* $V = H_*(X_{\mathcal{K}};\mathbb{F}_2)$, *and*

   (c) $S[\sigma^{-1}L^{(p)}[\sigma(V)] \oplus \sigma^{-2}W^p[\sigma(V)]] \to H_*(\mathcal{K}_{3,1};\mathbb{F}_p)$ *for* $V = H_*(X_{\mathcal{K}};\mathbb{F}_p)$ *in case p is an odd prime.*

   These isomorphisms specialize to an identification of the homology of each path-component of $\mathcal{K}_{3,1}$ with the one ambiguity that the homology of the components of knots arising from hyperbolic satellite operations is not given in a closed form here. More information is described in Section 3.

(4) *The integer homology of $\mathcal{K}_{3,1}$ has p-torsion of arbitrarily large order (with examples listed in Section 5, and 6).*

A primary development in this paper is our recursive application of the above theorem. Let $K_c \subset \mathcal{K}_{3,1}$ denote the subspace of $\mathcal{K}_{3,1}$ consisting of all cable knots. There is a homotopy-equivalence $\mathbb{Z} \times S^1 \times \mathcal{K}_{3,1} \to K_c$. Let $K_s \subset \mathcal{K}_{3,1}$ denote the subspace of $\mathcal{K}_{3,1}$ which are connect-sums of any number of cable knots. Then there is a homotopy-equivalence $\mathcal{C}_2(K_c \sqcup \{*\}) \to K_s$. The composite of the two maps is a homotopy-equivalence $\mathcal{C}_2((\mathbb{Z} \times S^1 \times \mathcal{K}_{3,1}) \sqcup \{*\}) \to K_s$. Since K_s is a collection of path-components of $\mathcal{K}_{3,1}$, this map can be iterated, giving a the homology of $\mathcal{K}_{3,1}$, as a Gerstenhaber-Poisson algebra, a fractal-like structure. These statements will be justified in Sections 2 and 3, and explored more fully in Sections 7 and 10.

These results lead to some natural questions about the structure of the homology of the higher-dimensional embedding spaces $\mathcal{K}_{n,1}$ ($n \geq 4$) studied recently by Sinha [45], Volic [51], Lambrechts [32] as well as others [1, 10, 17, 30, 41]. Constructions related to these questions are also addressed here.

By Theorem 1.3, there is arbitrarily large p-torsion in the homology of $\mathcal{K}_{3,1}$. Examples of Theorem 1.3 for knots whose path-components have higher 2-torsion in their integer homology is given next. This higher torsion can be regarded as a coarse "measure" of the "complexity" of a knot's JSJ-decomposition.

(1) Let $\mathcal{K}_{3,1}(f)$ denote the path-component of a torus knot f. Thus $\mathcal{K}_{3,1}(f)$ has the homotopy type of a circle [24, 7].

(2) Given any space X, and a strictly positive integer q, define

$$E(q,X) = \mathrm{Conf}(\mathbb{R}^2, q) \times_{\Sigma_q} X^q.$$

Assume that $\mathcal{K}_{3,1}(f)$ has the homotopy type of S^1. $E(4, \mathcal{K}_{3,1}(f))$ has the property that

(a) $H_2(E(4, \mathcal{K}_{3,1}(f))) = \mathbb{Z}/2\mathbb{Z}$,

(b) $H_3(E(4, \mathcal{K}_{3,1}(f))) = 0$,

(c) $H_3(S^1 \times E(4, \mathcal{K}_{3,1}(f)))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(d) Furthermore, $E(2^s, S^1 \times E(4, \mathcal{K}_{3,1}(f)))$ has the homotopy type of a path-component $\mathcal{K}_{3,1}(g)$ for a long knot g as given in [5], and has torsion of order 2^{s+1} in its integer homology by Section 6, and 7. In this case, g is a connected-sum of 2^s copies of the same summand, and that summand is a p/q-cable of a connected sum of four copies of the same torus knot. In the language of [6],

$$g = \left(\left(T^{(p,q)}\Delta_4 \bowtie \mathcal{H}^4\right) \bowtie \mathcal{S}^{(p,q)}\right) \Delta_{2^s} \bowtie \mathcal{H}^{2^s}$$

The elements of this notation is described in detail in [6] and is summarized in Section 2.

(e) A second example is $E(2^s, \mathcal{K}_{3,1}(h))$ where $h = T^{(p,q)} \bowtie W$ and W is the Whitehead link. In this case,

$$\mathcal{K}_{3,1}(h) \simeq S^1 \times \left(S^1 \times_{\Sigma_2} S^1\right)$$

where $S^1 \times_{\Sigma_2} S^1$ is the Klein bottle [7]. $H_1(\mathcal{K}_{3,1}(h)) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^2$.

A more complete description of the homology of $\mathcal{K}_{3,1}$ is given in Sections 5, and 6. The homology of each path-component is given in terms of Theorem 1.3 as well as filtrations of the values of the functors listed in that theorem.

Consider the subspace $\mathcal{T}\mathcal{K}_{3,1}$ of $\mathcal{K}_{3,1}$ consisting of the union of the components of $\mathcal{K}_{3,1}$ corresponding to knots which can be obtained from torus knots via iterations of the operations given by cabling and connected-sum. These are the knots whose complements are 'graph manifolds' ie: a union of Seifert-fibered manifolds. The structure of the homology of $\mathcal{T}\mathcal{K}_{3,1}$ is described in Section 10. This is our primary source of p^n-torsion in $H_*(\mathcal{K}_{3,1}; \mathbb{Z})$.

A further consequence of Theorem 1.2 is the next result which follows directly, and is proven in Section 4. Let $\mathrm{Emb}_*(S^1, S^n)$ denote the space of smooth pointed embeddings.

Proposition 1.4 *The group $SO(n-1)$ acts naturally on $\mathcal{K}_{n,1}$ (rotations that fix the 'long axis'), and there are morphisms of bundles for which each vertical map is a homotopy equivalence:*

$$\begin{tikzcd} SO(n) \times_{SO(n-1)} \mathcal{K}_{n,1} \arrow[r, "i"] \arrow[d, "\theta_n"] & SO(n+1) \times_{SO(n-1)} \mathcal{K}_{n,1} \arrow[r, "p"] \arrow[d, "\theta_n"] & SO(n+1)/SO(n) \arrow[d, "1"] \\ Emb_*(S^1, S^n) \arrow[r, "j"] & \mathrm{Emb}(S^1, S^n) \arrow[r] & S^n \end{tikzcd}$$

Thus, there is a bundle

$$SO(n+1) \times_{SO(n-1)} \mathcal{K}_{n,1} \to SO(n+1)/SO(n-1)$$

with fibre $\mathcal{K}_{n,1}$. Furthermore, there is a homeomorphism

$$\mathrm{Emb}(S^1, S^3) \to S^3 \times \mathrm{Emb}_*(S^1, S^3)$$

for which $\mathrm{Emb}_*(S^1, S^3)$ denotes the space of smooth pointed embeddings, and the bundle

$$\mathcal{K}_{3,1} \to \mathrm{Emb}_*(S^1, S^3) \to S^2$$

is the induced bundle with fibre $\mathcal{K}_{3,1}$ from the bundle

$$SO(2) \to SO(3) \to S^2.$$

Section 4 gives precise relationships (such as the above proposition) between the homotopy-type of the embedding spaces $\mathcal{K}_{n,j}$, $\mathrm{Emb}(S^j, S^n)$ and $\mathrm{Emb}(S^j, \mathbb{R}^n)$.

Notice that the homological properties of each path component thus give knot invariants. This is illustrated by the following proposition.

Proposition 1.5 (1) *A knot $f : S^1 \to S^3$ is the unknot if and only if its component in $\mathrm{Emb}(S^1, S^3)$ contains no 2-torsion in its 1st homology group.*

(2) *A long knot $f : \mathbb{R}^1 \to \mathbb{R}^3$ in $\mathcal{K}_{3,1}$ is the 'long unknot' if and only if its component has trivial first homology group.*

(3) *An embedding of S^1 in $\mathbb{R}^3$ is the unknot if and only if its component in $\mathrm{Emb}(S^1, \mathbb{R}^3)$ has torsion first homology group. It is also true if and only if its 2nd homology group is trivial.*

Theorem 1.6 *Let $\mathcal{K}_{3,1}(f)$ denote a path-component of $\mathcal{K}_{3,1}$. Then*

$$H_1(\mathcal{K}_{3,1}(f); \mathbb{Z})$$

is a finite direct-sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

In addition, a characterization of the components of $\mathcal{K}_{3,1}$ such that $H_1(\mathcal{K}_{3,1}(f); \mathbb{Z})$ contains 2-torsion is given in Section 8. A precise identification of those knots f such that $H_1(\mathcal{K}_{3,1}(f); \mathbb{Z})$ contains a $\mathbb{Z}/2\mathbb{Z}$ summand is also given. In Section 9 the least degree in which odd p torsion in $H_*(\mathcal{K}_{3,1}; \mathbb{Z})$ occurs is as follow.

Theorem 1.7 *Let f denote a long knot and p an odd prime. If $H_i(\mathcal{K}_{3,1}(f);\mathbb{Z})$ contains $\mathbb{Z}/p\mathbb{Z}$, then $i \geq 2p-2$.*

Much recent progress has been made on the structure of spaces of embeddings via finite-dimensional model spaces and approximations. Some of this was first given by Vassiliev [50] and has been the subject of further study via the Goodwillie Calculus of Embeddings by Sinha [45], Volic [51] Lambrechts [32], or cohomological techniques such as Bott-Taubes integrals [4, 10].

The Gerstenhaber-Poisson algebra above was first considered on the E^2-level of the Vassiliev spectral sequence by Tourtchine [49]. Related progress is given in work of Altshuler-Freidel [1], Bar-Natan [2], Cattaneo, Cotta-Ramusino-Longoni [10], Kohno [30], Kontsevich [31], Lescop [33], Polyak-Viro [40], Sakai [41], Watanabe [52] as well as others.

This paper takes the direction of using Gramain and Hatcher's techniques for understanding the homotopy type of $\mathcal{K}_{3,1}$, one component at a time [18, 24]. The central construction of Hatcher [24] is to consider the components of the knot space as the classifying space of the mapping class group of the knot complement. One then studies how such a mapping class group acts on the JSJ-tree of the knot complement as in [5, 7], using Hatcher's results on the homotopy type of diffeomorphism groups of Haken manifolds [21] to assemble an answer. Thus most of the results here are complementary to the results of the authors mentioned in the previous two paragraphs.

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2 Notation, labelling components

Whitney [56] showed that the embedding space $\mathcal{K}_{n,j}$ is connected for $n > 2j+1$. By work of Wu [53], $\mathcal{K}_{n,j}$ is also connected provided both $n > 2j$ and $j > 1$. That $\mathcal{K}_{n,1}$ is also connected for $n = 1$ is elementary. The fact that $\mathcal{K}_{2,1}$ is connected is equivalent to the smooth Alexander/Schoenflies theorem in dimension 2. In co-dimension 3 and higher. Haefliger [20] vastly generalized Whitney's result, proving that $\mathcal{K}_{n,j}$ is connected provided $2n > 3j+3$, and $\pi_0\mathcal{K}_{n,j}$ is non-trivial for $2n = 3j+3$. This work has recently been extended by the first author to a computation of the first non-trivial homotopy group of $\mathcal{K}_{n,j}$ provided $2n-3j-3 \geq 0$ [8].

When $2n \leq 3j+3$ the space $\mathcal{K}_{n,j}$ could potentially have many connected components. $\pi_0\mathcal{K}_{n,j}$ was shown to be a group by Haefliger [20] provided $n-j > 2$, whereas it is only a monoid for $n-j \leq 2$. A fundamental example is the space $\mathcal{K}_{3,1}$ which has countably infinite many components, and no inverses in the monoid $\pi_0\mathcal{K}_{3,1}$ [44]. Given $f \in \mathcal{K}_{n,j}$, let $\mathcal{K}_{n,j}(f)$ denote the path-component of $\mathcal{K}_{n,j}$ containing f.

We will use the notation $\mathrm{EC}(1, D^{n-1})$ as defined in [5] for the space of framed long knots in $\mathbb{R}^n$. Given a compact manifold M, define

$$\mathrm{EC}(k,M) = \{f \in Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M), supp(f) \subset \mathsf{I}^k \times M\}.$$

Here the support of f, $supp(f)$ is defined by $supp(f) = \{x \in \mathbb{R}^k \times M : f(x) \neq x\}$ and $\mathsf{I} = [-1,1]$. $\mathrm{EC}(1, D^{n-1})$ is not homotopy equivalent to $\mathcal{K}_{n,1}$ in general, but as described in [5] there is a fibration

$$\Omega SO(n-1) \to \mathrm{EC}(1, D^{n-1}) \to \mathcal{K}_{n,1}$$

which splits at the fibre (via a 2-cubes map) for $n \in \{1,2,3\}$, allowing us to think of $\mathcal{K}_{3,1}$ as a sub 2-cubes object of $\mathrm{EC}(1, D^2)$.

This section collects information on the indexing of the components of $\mathcal{K}_{3,1}$ which is given in terms of the 'companionship tree' classification of knots, an application of the Jaco-Shalen-Johannson (JSJ) decomposition of knot complements. The indexing that we will use is described in detail in [6]. Aspects of this indexing have been partially described before in the works of Budney [5], Eisenbud and Neumann [14], Schubert [44], and the unpublished work of Bonahon and Siebenmann [3], as well as the survey work of Kawauchi [28]. Indeed, the results in [6] should be thought of as a uniqueness statement for Schubert's satellite operations that he describes in [44]. In the book by Eisenbud-Neumann [14] this method of indexing is called the splice decomposition of links, but is specialized to the case of links in homology spheres whose complements are graph manifolds. A terse statement of the results in [6] given next suffice for the applications here. More complete as well as more specific information is given in [6].

Definition 2.1 *An n-component link in S^3 is a compact, connected, oriented, 1-dimensional submanifold of S^3 consisting of n path components labelled with distinct numbers from the set $\{0,1,2,\cdots,n-1\}$. Thus the notation $L = (L_0, L_1, \cdots, L_{n-1})$ is used frequently for n-component links. A knot K (in S^3) is a 1-component link.*

An n-component link L is the unlink if there exists n disjointly embedded 2-discs $D = (D_0, D_1, \cdots, D_{n-1})$ in S^3 whose boundary is L, $\partial D = (\partial D_0, \partial D_1, \cdots, \partial D_{n-1}) = (L_0, L_1, \cdots, L_{n-1}) = L$.

For $n \geq 0$ an $(n+1)$-component link $L = (L_0, L_1, \cdots, L_n)$ is said to be a KGL (knot generating link) if the sublink $(L_1, L_2, \cdots, L_n)$ is the unlink.

Given an $(n+1)$-component KGL L and n knots $J = (J_1, J_2, \cdots, J_n)$ in S^3 there is an operation called splicing defined in [6] which produces a knot $J \bowtie L$ in S^3. Here is a rough statement of the splicing construction. Fix $D = (D_1, \cdots, D_n)$ n disjointly embedded discs in S^3 such that $\partial D = (L_1, L_2, \cdots, L_n)$. Let $\nu_D : [-\infty, \infty] \times D^2 \to S^3$ be a closed tubular neighbourhood of D. Let $C_{L'}$ be the complement of an open tubular neighbourhood of $(L_1, \cdots, L_n)$ in S^3, and define $R : C_{L'} \to S^3$ to be unique continuous function which is the identity outside img(νD), and on the image of ν_D define it to be the conjugate $\nu_D \circ (\tilde{J}_1 \sqcup \cdots \tilde{J}_n) \circ \nu_D^{-1}$, where $\tilde{J}_i \in \mathrm{EC}(1, D^2)$ is the framed long knot in the homotopy-fibre of the map $\mathrm{EC}(1, D^2) \to \Omega SO(2)$ corresponding to J_i under the map $\mathrm{EC}(1, D^2) \to \mathcal{K}_{3,1} \to \mathrm{Emb}(S^1, S^3)$. $J \bowtie L$ is defined to be the image of L_0 under the embedding R. See [6] for details.

Example 2.2 *Let W denote the Whitehead link and F_8 the figure-8 knot.*

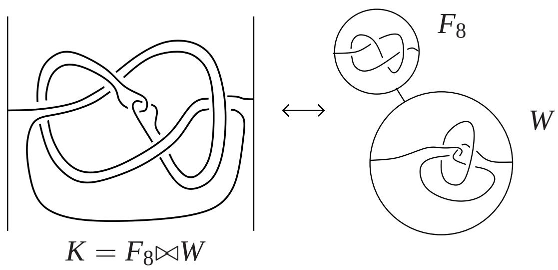


Figure 1: Whitehead double and companionship tree

The role of splicing is that it is an operation that takes knots and KGL's as input and produces a knot of greater ‘complexity’ in the sense that the companionship trees of the input data is spliced together to produce the companionship tree of $J \bowtie L$. We proceed to make these ideas more precise.

Definition 2.3 *The Hopf link $\mathcal{H}^1$ is the 2-component link in S^3 given by*

$$\{(z_1, 0) \in \mathbb{C}^2 : z_1 \in \mathbb{C}, |z_1| = 1\} \cup \{(0, z_2) : z_2 \in \mathbb{C}, |z_2| = 1\} \subset S^3$$

where S^3 is regarded the unit sphere in $\mathbb{C}^2$.

If one takes a connected-sum of p copies of the Hopf link along a common component, one obtains the $(p+1)$-component link, which we will call the $(p+1)$-component *keychain link* $\mathcal{H}^p$ *(see Figure 2).*

$$\{(z_1,0)\in\mathbb{C}^2 : |z_1|=1\}\cup\bigcup_{k=1}^{p}\{\frac{1}{\sqrt{2}}(e^{\frac{2\pi ik}{p}},z_2) : |z_2|=1\}\subset S^3$$

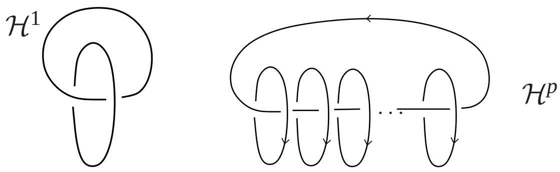


Figure 2: Hopf link and keychain link

For any $(p,q)\in\mathbb{Z}\times\mathbb{N}$*, the* (p,q)*-Seifert link* $\mathcal{S}^{(p,q)}$ *is defined to be*

$$\{(z_1,0)\in\mathbb{C}^2 : |z_1|=1\}\cup\{(z_1,z_2)\in\mathbb{C}^2 : |z_1|=|z_2|=\frac{1}{\sqrt{2}}, z_1^q=z_2^p\}\subset S^3$$

The (p,q)*-Seifert link has* $GCD(p,q)+1$ *components (see Figure 3).*

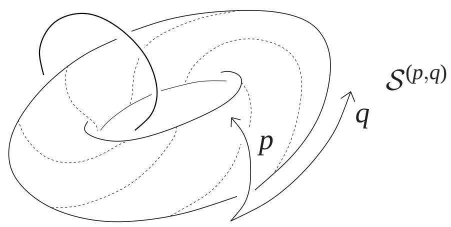


Figure 3: Seifert link

For any $(p,q)\in\mathbb{Z}\times\mathbb{N}$*,* $GCD(p,q)=1$*, the* (p,q)*-torus knot* $T^{(p,q)}$ *is*

$$\{(z_1,z_2)\in\mathbb{C}^2 : |z_1|=|z_2|=\frac{1}{\sqrt{2}}, z_1^q=z_2^p\}\subset S^3$$

Theorem 2.4 *[6] Given a knot* K *in* S^3 *there is a finite, labelled, rooted tree-valued invariant of* K *denoted* $\mathbb{G}_K$ *having the following properties:*

(1) *Each vertex of the tree is labelled by a link and any link from the following list is admissible:*

   (a) *Torus knots* $T^{(p,q)}$ *for* $p/q\in\mathbb{Q}$*,* $GCD(p,q)=1$*,* $q\geq 2$*.*

   (b) *Seifert links* $\mathcal{S}^{(p,q)}$ *for* $GCD(p,q)=1$*,* $q\geq 1$*.*

   (c) *Keychain links* $\mathcal{H}^p$ *for* $p\geq 2$*.*

   (d) *Hyperbolic KGLs.*

   (e) *The unknot.*

(2) *Given any vertex in* $\mathbb{G}_K$*, the number of children of the vertex is one less than the number of components of the link that decorates the vertex.*

(3) *If any vertex is decorated by a keychain link* $\mathcal{H}^p$*, none of its children are allowed to be decorated by keychain links.*

(4) *A vertex of the tree $\mathbb{G}_K$ can be decorated by the unknot if and only if the tree $\mathbb{G}_K$ consists of only one vertex.*

(5) *If one changes all the labels on the tree $\mathbb{G}_K$ by substituting for each vertex label L its complement C_L one obtains G_K, the JSJ-tree of K [6]. This is the tree whose vertex set is the set of path components of the knot complement C_K split along its JSJ-tori, and the edges are the JSJ-tori of C_K.*

(6) *If $\mathbb{G}_K$ consists of more than one vertex, then $K = J \bowtie L$ where the root of $\mathbb{G}_K$ is labelled by L and $\mathbb{G}_{J_i}$ are the subtrees rooted at the children of L in $\mathbb{G}_K$.*

(7) *The number of vertices of $\mathbb{G}_K$ is one more than the number of tori in the JSJ-decomposition of the complement of K in S^3. Thus for example, $\mathbb{G}_K$ is a one-vertex tree if and only if K is either hyperbolic, a torus knot, or the unknot.*

The above properties 1 through 4 are complete, in the sense that any tree satisfying properties 1 through 4 is realizable as $\mathbb{G}_K$ for some knot K. $\mathbb{G}_K$ is known as the companionship tree of K.

Given a vertex v of $\mathbb{G}_K$, there is a maximal subtree of $\mathbb{G}_K$ rooted at v, and this subtree is the companionship tree of a unique knot in S^3, K_v. K_v is called a companion knot to K.

Item (6) implies that if one writes down the 'postorder' (reverse Polish) listing of $\mathbb{G}_K$, one is simply writing K as an iterated splice knot where all the KGL's used in splicing come from the list (1). Thus $\mathbb{G}_K$ could simply be considered a precise way to specify K as a splice of atoroidal KGLs.

Unlike links with Seifert-fibred complements, hyperbolic KGLs have no known canonical enumeration.

Some elementary examples of hyperbolic KGLs are:

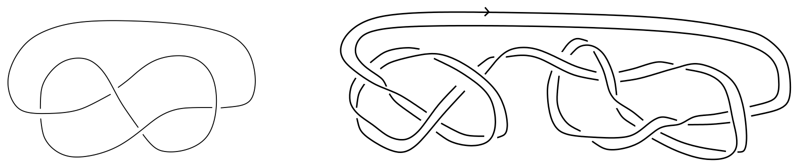
- the figure-8 knot
- the Whitehead link
- the Borromean rings

Hyperbolic KGLs of arbitrarily many components are known to exist by the work of Kanenobu [27]. For details on the hyperbolic structures, see for example the textbook of Thurston [48].

A trefoil knot $K = T^{(-3,2)}$
$\mathbb{G}_K = K$

$T^{(-3,2)} \bowtie S^{(17,2)}$

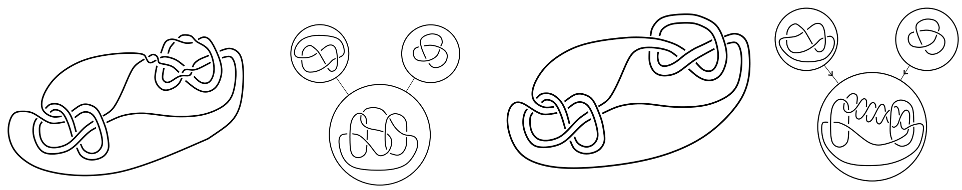
Figure 4: Trefoil and (17,2)-cable



F_8 $((T^{(3,2)}, F_8)\bowtie\mathcal{H}^2)\bowtie\mathcal{S}^{(-17,2)}$

Figure 5: Cable of connect sum of trefoil and figure-8 knot

Figures 4 and 5 give examples of knots K and the associated tree $\mathbb{G}_K$, and the corresponding splice notation, where F_8 denotes the figure-8 knot. Let $B = (B_0, B_1, B_2)$ denote the Borromean rings, and let $B_{i,j}$ be the 3-component link in S^3 obtained from B by doing i Dehn twists about the spanning disc of B_1 and j Dehn twists about the spanning disc for B_2.



$(F_8, T^{(3,2)})\bowtie B$ $\mathbb{G}_K$ $(F_8, T^{(3,2)})\bowtie B_{0,3}$ $\mathbb{G}_K$

Figure 6: Various Borromean splices

The spaces $\mathrm{EC}(1, D^{n-1})$ admit an action of the operad of little 2-cubes [5]. Using the connectedness of $\mathcal{K}_{n,1}$ for $n \geq 4$ together with the cubes action one can prove that $\mathrm{EC}(1, D^{n-1})$ has the homotopy type of a 2-fold loop space for $n \geq 4$. At present it is not known what the 2-fold de-looping of $\mathrm{EC}(1, D^{n-1})$ is. Recently, P. Salvatore [42] has constructed an action of the operad of 2-cubes on $\mathcal{K}_{n,1}$ for all $n \geq 4$.

The previously described fibre bundle

$$\mathrm{Emb}(S^1, S^3) \to SO(4)/SO(2)$$

whose fibre inclusion $i : \mathcal{K}_{3,1} \to \mathrm{Emb}(S^1, S^3)$ is induced by the one-point compactification is explored more deeply in Section 4. For the purpose of this section and the study of the components of $\mathcal{K}_{3,1}$ and $\mathrm{Emb}(S^1, S^3)$ respectively, we note that the inclusion $\mathcal{K}_{3,1} \to \mathrm{Emb}(S^1, S^3)$ induces bijection on path-components. Thus, our indexing of $\pi_0\mathrm{Emb}(S^1, S^3)$ above by companionship trees $\mathbb{G}_K$ is also an indexing of $\pi_0\mathcal{K}_{3,1}$.

3 The homotopy type of $\mathcal{K}_{3,1}$

A detailed description of the homotopy type of $\mathcal{K}_{3,1}$ is given in this section. This description is given in terms of the splicing operations as described in Section 2. A good general reference for these results is the work [7].

(1) If f is the unknot then $\mathcal{K}_{3,1}(f)$ is contractible by work of Hatcher [24].

(2) If f is a p/q-cabling of g then by work of Hatcher [24], there is a homotopy equivalence

$$S^1 \times \mathcal{K}_{3,1}(g) \to \mathcal{K}_{3,1}(f).$$

We consider a torus knot to be a cable of the unknot, so we are claiming all non-trivial torus knots f satisfy $\mathcal{K}_{3,1}(f) \simeq S^1$.

(3) If $f = (f_1, f_2, \cdots, f_n) \bowtie \mathcal{H}^n$ where $\{f_i : i \in \{1, 2, \cdots, n\}\}$ are the prime summands of f and $n \geq 2$, then there is a homotopy equivalence

$$\mathcal{C}_2(n) \times_{\Sigma_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \to \mathcal{K}_{3,1}(f)$$

where $\Sigma_f \subset \Sigma_n$ is the Young subgroup corresponding to the partition $\sim$ of $\{1, 2, \cdots, n\}$ given by $i \sim j \Leftrightarrow \mathcal{K}_{3,1}(f_i) = \mathcal{K}_{3,1}(f_j)$. This result originally appears in [5].

(4) If a knot $f = (f_1, f_2, \cdots, f_n) \bowtie L$ where L is a hyperbolic KGL then there is a homotopy-equivalence:

$$S^1 \times \left(SO(2) \times_{A_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right) \to \mathcal{K}_{3,1}(f).$$

we define A_f as a subgroup of B_L. B_L is the subgroup of the group of hyperbolic isometries of the complement of L in S^3 which:

- extend to diffeomorphisms of S^3.
- the extensions preserve L_0 and its orientation (ie: they act on the cusp corresponding to L_0 by translations).
- put together, the above two properties imply there is an embedding $B_L \to \mathrm{Diff}(S^3; L, L_0)$.

It is a non-trivial fact [7] that the composite is an embedding of groups $B_L \to \mathrm{Diff}(S^3; L, L_0) \to \mathrm{Diff}^+(L_0)$ where $\mathrm{Diff}^+(L_0)$ is the group of orientation-preserving diffeomorphisms of L_0. Regard B_L as a finite subgroup of $SO(2)$. There is a representation of B_L given by the composite: $B_L \to \mathrm{Diff}(S^3, L, L_0) \to \mathrm{Diff}(\sqcup_{i=1}^n L_i) \to \pi_0 \mathrm{Diff}(\sqcup_{i=1}^n L_i) \equiv \Sigma_n^+$ where we identify $\pi_0 \mathrm{Diff}(\sqcup_{i=1}^n L_i)$ with Σ_n^+, the signed symmetric group on $\{1, 2, \cdots, n\}$. Σ_n acts on $\mathcal{K}_{3,1}^n$ by permutation of factors. Σ_2 acts on $\mathcal{K}_{3,1}$ by knot inversion – fix an axis perpendicular to the long axis, and rotate a knot by π about this axis, this is knot inversion. Stated another way, the group of rotations which preserve the long axis, $O(2) \subset SO(3)$, acts on $\mathcal{K}_{3,1}$ by conjugation. Fix an element $\varpi \in O(2) \setminus SO(2)$, then ϖ acts as an involution on $\mathcal{K}_{3,1}$, thus defining an action of Σ_2 on $\mathcal{K}_{3,1}$. These two actions extend to an action of Σ_n^+ on $\mathcal{K}_{3,1}^n$. A_f is the subgroup of B_L that preserves the path-component $\prod_{i=1}^n \mathcal{K}_{3,1}(f_i)$ of $\mathcal{K}_{3,1}^n$.

As mentioned in part (4) above, $\mathcal{K}_{3,1}$ is naturally an $O(2)$-space. Parts (1), (2) and (3) above all are $O(2)$-equivariant homotopy equivalences, as shown in [7]. Case (4) is only an $SO(2)$-equivariant homotopy-equivalence, although the homotopy-class of ϖ acting on $\mathcal{K}_{3,1}$ is computed.

4 Relations among various spaces

The goal of this section is to compare the homotopy types of the spaces:

- $\mathcal{K}_{n,1}$
- $\mathrm{Emb}(S^1, S^n)$
- $\mathrm{Emb}(S^1, \mathbb{R}^n)$

The space of pointed, smooth embeddings $\mathrm{Emb}_*(S^1, S^n)$ will be a useful auxiliary space. Relationships between the embedding spaces $\mathcal{K}_{n,j}$, $\mathrm{Emb}(S^j, S^n)$ and $\mathrm{Emb}(S^j, \mathbb{R}^n)$ will also be listed.

Proposition 4.1 *For all $n \geq 1$ there are morphisms of fibrations for which each vertical map is a homotopy equivalence:*

$$\begin{array}{ccccc}
SO(n) \times_{SO(n-1)} \mathcal{K}_{n,1} & \xrightarrow{i} & SO(n+1) \times_{SO(n-1)} \mathcal{K}_{n,1} & \xrightarrow{p} & SO(n+1)/SO(n) \\
\downarrow \Theta_n & & \downarrow \Theta_n & & \downarrow 1 \\
Emb_*(S^1, S^n) & \xrightarrow{j} & \mathrm{Emb}(S^1, S^n) & \longrightarrow & S^n
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{K}_{n,1} & \xrightarrow{i} & SO(n) \times_{SO(n-1)} \mathcal{K}_{n,1} & \xrightarrow{p} & SO(n)/SO(n-1) \\
\downarrow 1 & & \downarrow \Theta_n & & \downarrow 1 \\
\mathcal{K}_{n,1} & \xrightarrow{j} & \mathrm{Emb}_*(S^1, S^n) & \longrightarrow & S^{n-1}.
\end{array}$$

Proof Consider the maps $\Theta_n : SO(n+1) \times \mathcal{K}_{n,1} \to \mathrm{Emb}(S^1, S^n)$ obtained from the natural $SO(n+1)$-action on $\mathrm{Emb}(S^1, S^n)$ together with the natural inclusion $\mathcal{K}_{n,1} \to \mathrm{Emb}(S^1, S^n)$. Notice that the map Θ_n is $SO(n-1)$-equivariant and thus there is an induced map

$$\Theta_n : SO(n+1) \times_{SO(n-1)} \mathcal{K}_{n,1} \to \mathrm{Emb}(S^1, S^n).$$

Consider the natural fibrations

$$\mathrm{Emb}_*(S^1, S^n) \to \mathrm{Emb}(S^1, S^n) \to S^n,$$

and

$$\mathcal{K}_{n,1} \to \mathrm{Emb}_*(S^1, S^n) \to S^{n-1}.$$

The first map is a fibration by the isotopy extension theorem. Indeed, Palais proved that in general 'restriction maps' are locally trivial fibre bundles [39]. The map $\mathrm{Emb}_*(S^1, S^n) \to S^{n-1}$ is the composition of the restriction map $\mathrm{Emb}_*(S^1, S^n) \to \mathrm{Emb}_*(U, S^n)$ with the homotopy equivalence $\mathrm{Emb}_*(U, S^n) \to S^{n-1}$ given by the derivative at $*$ where U is some closed interval neighbourhood of $*$ in S^1. Thus there is a map of fibrations:

$$\begin{array}{ccccc} SO(n)\times_{SO(n-1)}\mathcal{K}_{n,1} & \xrightarrow{i} & SO(n+1)\times_{SO(n-1)}\mathcal{K}_{n,1} & \xrightarrow{p} & SO(n+1)/SO(n) \\ \downarrow{\scriptstyle\Theta_n} & & \downarrow{\scriptstyle\Theta_n} & & \downarrow{\scriptstyle 1} \\ Emb_*(S^1,S^n) & \xrightarrow{j} & \mathrm{Emb}(S^1,S^n) & \longrightarrow & S^n \end{array}$$

as well as

$$\begin{array}{ccccc} \mathcal{K}_{n,1} & \xrightarrow{i} & SO(n)\times_{SO(n-1)}\mathcal{K}_{n,1} & \xrightarrow{p} & SO(n)/SO(n-1) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle\Theta_n} & & \downarrow{\scriptstyle 1} \\ \mathcal{K}_{n,1} & \xrightarrow{j} & \mathrm{Emb}_*(S^1,S^n) & \longrightarrow & S^{n-1}. \end{array}$$

The map $\Theta_n : SO(n)\times_{SO(n-1)}\mathcal{K}_{n,1} \to \mathrm{Emb}_*(S^1,S^n)$ is thus a homotopy equivalence. Hence the map

$$\Theta_n : SO(n+1)\times_{SO(n-1)}\mathcal{K}_{n,1} \to \mathrm{Emb}(S^1,S^n)$$

is also a homotopy equivalence. □

Restrict attention to the special case given by $n = 3$.

Corollary 4.2 *There is a homeomorphism*

$$S^3\times \mathrm{Emb}_*(S^1,S^3)\to \mathrm{Emb}(S^1,S^3).$$

Furthermore, the bundle

$$\mathcal{K}_{3,1}\to \mathrm{Emb}_*(S^1,S^3)\to S^2$$

is the induced bundle with fibre $\mathcal{K}_{3,1}$ from the bundle

$$SO(2)\to SO(3)\to S^2$$

where $SO(2)$ acts on $\mathcal{K}_{3,1}$ by rotation about the long axis, as previously described. Thus, up to a homotopy-equivalence $\mathrm{Emb}_*(S^1,S^3)$ is the union of two copies of $D^2\times\mathcal{K}_{3,1}$ along their common boundary, where the gluing map $S^1\times\mathcal{K}_{3,1}\to S^1\times\mathcal{K}_{3,1}$ is given by $(z,f)\longmapsto(z,z^2.f)\in S^1\times\mathcal{K}_{3,1}$, where we identify $S^1\equiv SO(2)$ and its action by rotation about the long axis.

Observe that Proposition 4.1 generalizes to a proposition about the embedding spaces $\mathrm{Emb}(S^k,S^n)$. We skip the proof as it is essentially the same as Proposition 4.1.

Proposition 4.3 *Provided $n-j\geq 1$, there is a homotopy-equivalence $SO(n+1)\times_{SO(n-j)}\mathcal{K}_{n,j}\to\mathrm{Emb}(S^j,S^n)$.*

Given $f \in \mathcal{K}_{n,j}$, let $\dot{f} \in \mathrm{Emb}(S^j, S^n)$ be the one-point compactification of f. Consider S^n to be the one-point compactification of $\mathbb{R}^n$. The inclusion $\mathbb{R}^n \to S^n$ induces an inclusion $\mathrm{Emb}(S^j, \mathbb{R}^n) \to \mathrm{Emb}(S^j, S^n)$. This inclusion induces a bijection $\pi_0\mathrm{Emb}(S^j, \mathbb{R}^n) \to \pi_0\mathrm{Emb}(S^j, S^n)$ provided $n - j \geq 2$. Given $f \in \mathcal{K}_{n,j}$ let $\overline{f} \in \mathrm{Emb}(S^j, \mathbb{R}^n)$ be such that $\dot{f}$ is isotopic (in S^n) to $\overline{f}$. These conventions give us a one-to-one correspondence between $\pi_0\mathcal{K}_{n,j}$, $\pi_0\mathrm{Emb}(S^j, S^n)$ and $\pi_0\mathrm{Emb}(S^j, \mathbb{R}^n)$ for $n - j \geq 2$.

If $f \in \mathcal{K}_{n,j}$ is a long knot, let X_f denote the component of $\overline{f}$ in $\mathrm{Emb}(S^j, D^n)$ and let C_f denote the complement of an open tubular neighbourhood of $\dot{f}$ in S^n. Given $f \in \mathcal{K}_{n,j}$ define $C_f \rtimes \mathcal{K}_{n,j}(f) = \{(p, g) : g \in \mathcal{K}_{n,j}, p \in C_g,$ where g isotopic to $f\}$, and define $C \rtimes \mathcal{K}_{n,j}$ to be the union of the spaces $C_f \rtimes \mathcal{K}_{n,j}(f)$ for all $f \in \mathcal{K}_{n,j}$

Proposition 4.4 *Provided $n - j > 0$, $\mathrm{Emb}(S^j, \mathbb{R}^n)$ is homotopy-equivalent to the space $SO(n) \times_{SO(n-j)} \left(C \rtimes \mathcal{K}_{n,j}\right)$. In particular the components X_f of $\mathrm{Emb}(S^j, \mathbb{R}^n)$ have the homotopy-type of $SO(n) \times_{SO(n-j)} \left(C_f \rtimes \mathcal{K}_{n,j}(f)\right)$. $SO(n-j) \subset SO(n)$ acts on $SO(n)$ as the subgroup fixing a j-dimensional subspace, and $SO(n-j)$ acts on $C_f \rtimes \mathcal{K}_{n,j}$ diagonally.*

Proof See Proposition 2.2 of [8]. □

5 On the homology of $\mathcal{C}_2(X \amalg \{*\})$

The purpose of this section is to recall the homology of

$$\mathcal{C}_2(X \amalg \{*\})$$

for X not necessarily path-connected. These results will then be combined with Theorem 1.2 to obtain Theorem 1.3. The space X is assumed to be compactly generated and weak Hausdorff as a topological space [12]; the base-point $\{*\}$ is non-degenerate by construction.

Formal constructions are given next for which $\mathbb{F}$ is a field and all modules are assumed to be vector spaces over $\mathbb{F}$. Let V denote a graded vector space which splits as a direct sum

$$V = V_+ \oplus V_-$$

for which V_+ consists of the elements concentrated in even degrees and V_- consists of the elements concentrated in odd degrees. Let $\sigma(V)$ denote the “algebraic suspension" of V. That is $\sigma(V)$ is the module V with all degrees raised by one. In addition, define $\sigma^n(V) = \sigma(\sigma^{n-1}(V))$. The “algebraic desuspension" of V denoted $\sigma^{-1}(V)$ is defined by requiring $\sigma(\sigma^{-1}(V)) = V$.

Next consider the free Lie algebra

$$L[\sigma(V)]$$

and, if $\mathbb{F} = \mathbb{F}_p$, the free restricted Lie algebra over $\mathbb{F}_p$ denoted $L^{(p)}[\sigma(V)]$. In this last case, consider the natural inclusion

$$j : L[\sigma(V)] \to L^{(p)}[\sigma(V)]$$

with co-kernel denoted $W^p[\sigma(V)]$ (for which $L[\sigma(V)]$ is the free Lie algebra defined over the field $\mathbb{F}_p$). The definition of a restricted Lie algebra is given in Jacobson's book "Lie Algebras" [26] with graded restricted Lie algebras treated in [38]. Graded restricted Lie algebras may be regarded as the module of primitive elements in the tensor algebra $T[\sigma(V)]$ defined over the field $\mathbb{F}_p$. Notice that $\sigma^{-1}(\sigma V) = V$, but that $\sigma^{-1}L[\sigma(V)]$ is not isomorphic to $L[V]$ in general.

Let $E[V_-]$ denote the exterior algebra generated by V_- and let $\mathbb{F}[V_+]$ denote the polynomial algebra generated by V_+. Consider the symmetric algebra $S[V]$ defined as follows:

(1) For the field $\mathbb{Q}$, $S[V] = E[V_-] \otimes \mathbb{F}[V_+]$.

(2) For the field $\mathbb{F}_2$, $S[V] = \mathbb{F}[V]$, the polynomial algebra generated by V.

(3) For the field $\mathbb{F}_p$ with p an odd prime, $S[V] = E[V_-] \otimes \mathbb{F}[V_+]$.

We describe the homology of $\mathcal{C}_2(X \amalg \{*\})$ with coefficients in the field (1) $\mathbb{Q}$, (2) $\mathbb{F}_2$ and (3) $\mathbb{F}_p$ for p and odd prime.

Consider case (1). Let $V = H_*(X, \mathbb{Q})$, and form the symmetric algebra $S[\sigma^{-1}L[\sigma(H_*(X;\mathbb{Q})]]$. By [12], there is an isomorphism of Hopf algebras

$$S[\sigma^{-1}L[\sigma(H_*(X;\mathbb{Q}))]] \to H_*(\mathcal{C}_2(X \amalg \{*\});\mathbb{Q})$$

with co-product determined by that of $H_*(X;\mathbb{Q})$.

The analogous theorem for $\mathbb{F}_2$ is given as follows. Let $V = H_*(X;\mathbb{F}_2)$, and form the symmetric algebra

$$S[\sigma^{-1}L^{(2)}[\sigma(H_*(X;\mathbb{F}_2))]].$$

By [12], there is an isomorphism of Hopf algebras

$$S[\sigma^{-1}L^{(2)}[\sigma(H_*(X;\mathbb{F}_2))]] \to H_*(\mathcal{C}_2(X \amalg \{*\});\mathbb{F}_2)$$

with co-product determined by that of $H_*(X;\mathbb{F}_2)$. Remark: The role of the restriction in a restricted Lie algebra over $\mathbb{F}_2$ is to create the Araki-Kudo-Dyer-Lashof operation, the operation which sends an element $\sigma(v)$ to $Q_1(\sigma(v))$.

The result for odd primes p with $\mathbb{F}_p$ is given as follows. Let $V = H_*(X;\mathbb{F}_p)$, and form the symmetric algebra

$$S[\sigma^{-1}L^{(p)}[\sigma(V)] \oplus \sigma^{-2}W^p[\sigma(V)]].$$

By [12], there is an isomorphism of Hopf algebras

$$S[\sigma^{-1}L^{(p)}[\sigma(V)] \oplus \sigma^{-2}W^p[\sigma(V)]] \to H_*(\mathcal{C}_2(X \amalg \{*\});\mathbb{F}_p)$$

with coproduct determined by that of $H_*(X;\mathbb{F}_p)$.

6 On the homology of $\mathcal{K}_{3,1}$

Recall the homotopy equivalence of Theorem 1.2,

$$C(\mathbb{R}^2, X_{\mathcal{K}} \amalg \{*\}) \to \mathcal{K}_{3,1}$$

Thus there are isomorphisms of Hopf algebras

(1) $S[\sigma^{-1}L[\sigma(V)]] \to H_*(\mathcal{K}_{3,1};\mathbb{Q})$ for $V = H_*(X_{\mathcal{K}};\mathbb{Q})$,

(2) $S[\sigma^{-1}L^2[\sigma(V)]] \to H_*(\mathcal{K}_{3,1};\mathbb{F}_2)$ for $V = H_*(X_{\mathcal{K}};\mathbb{F}_2)$ and

(3) $S[\sigma^{-1}L^{(p)}[\sigma(V)] \oplus \sigma^{-2}W^p[\sigma(V_-)]]$ for $V = H_*(X_{\mathcal{K}};\mathbb{F}_p)$ for odd primes p.

Further information concerning the the homology of $X_{\mathcal{K}}$ is given in Section 10.

Thus the above isomorphisms give the homology of $\mathcal{K}_{3,1}$ with field coefficients. The first and second parts of Theorem 1.3 follow. The proof of Theorem 1.3 will be concluded in Section 7 in which higher torsion is constructed.

Notice that the space $\mathcal{C}_2(X \amalg \{*\})$ is naturally a disjoint union of X with another space. Thus, there is a natural direct sum decomposition of graded vectors spaces

$$\bar{H}_*(X;\mathbb{F}) \oplus \Gamma(X;\mathbb{F}) \to H_*(\mathcal{C}_2(X \amalg \{*\});\mathbb{F}))$$

for a choice of graded vector space

$$\Gamma(X;\mathbb{F})$$

which is functor of $H_*(X;\mathbb{F})$.

The construction $\Gamma(X;\mathbb{F})$ is used in Section 10 to describe the homology of the subspace of $\mathcal{K}_{3,1}$ generated by torus knots, as well as the operations of connected sums, cablings and the action of the little two-cubes.

7 Higher p-torsion in the integer homology of $\mathcal{K}_{3,1}$

One way in which higher order p-torsion in the homology of $\mathcal{K}_{3,1}$ arises is summarized next. The way in which little cubes $\mathcal{C}_2(n)$ are related to configuration spaces $\mathrm{Conf}(\mathbb{R}^2, n)$ is as follows. There are maps $\mathcal{C}_2(n) \to \mathrm{Conf}(\mathbb{R}^2, n)$ which are both homotopy equivalences and equivariant with respect to the action of the symmetric group Σ_n [37]. Thus it suffices to exhibit higher torsion in the integer homology of $\mathrm{Conf}(\mathbb{R}^2, n) \times_{\Sigma_n} X^n$ for certain choices of spaces $X = \mathcal{K}_{3,1}(f)$. Since the construction $\mathrm{Conf}(\mathbb{R}^2, n) \times_{\Sigma_n} X^n$ occurs numerous times below, it is convenient to define

$$E_n(X) = \mathrm{Conf}(\mathbb{R}^2, n) \times_{\Sigma_n} X^n$$

as given in the Introduction.

Given a prime long knot f, consider the path-component $\mathcal{K}_{3,1}(g)$ where

$$g = \#_n f \equiv f\Delta_n \bowtie \mathcal{H}^n$$

Here $\#_n f$ denotes the connected-sum of n copies of the same knot f, with the splice notation from Section 2 given. There are homotopy equivalences

$$\mathcal{K}_{3,1}(g) \to \mathrm{Conf}(\mathbb{R}^2, n) \times_{\Sigma_n} \mathcal{K}_{3,1}(f)^n = E_L(\mathcal{K}_{3,1}(f)).$$

First consider p-torsion of order exactly p obtained from the equivariant cohomology of $\mathrm{Conf}(\mathbb{R}^2, p)$ as constructed in [12].

Proposition 7.1 *Let Y denote any connected CW-complex.*

(1) *If $H_{2t-1}(Y;\mathbb{F}_p)$ is non-zero, then $\mathbb{F}_p$ is a direct summand of*

$$H_{2pt-2}(\mathrm{Conf}(\mathbb{R}^2, p) \times_{\Sigma_p} Y^p; \mathbb{Z}).$$

(2) *If $\mathbb{Z}/p^s\mathbb{Z}$ is a direct summand of $H_{2t-1}(Y;\mathbb{Z})$, then*

$$H_{2tp^r-1}(E_{p^r}(Y);\mathbb{Z}) = H_{2tp^r-1}(\mathrm{Conf}(\mathbb{R}^2, p^r) \times_{\Sigma_{p^r}} Y^{p^r}; \mathbb{Z})$$

has a $\mathbb{Z}/p^{s+q}\mathbb{Z}$-summand.

(3) *There is a homotopy equivalence*

$$E_n(\mathcal{K}_{3,1}(f)) \to \mathcal{K}_{3,1}(\#_n f).$$

(4) *Thus if $\mathcal{K}_{3,1}(f)$ has any non-trivial mod-p homology in degree $2t-1$, then $\mathbb{F}_p$ is a direct summand of*

$$H_{2pt-2}(E_p(\mathcal{K}_{3,1}(f));\mathbb{Z}).$$

Hence

$$H_{2pt-2}(\mathcal{K}_{3,1}(\#_p f);\mathbb{Z}) = \mathbb{F}_p \oplus A$$

for some abelian group A.

(5) *Furthermore, if $\mathbb{Z}/p^s\mathbb{Z}$ is a direct summand of $H_{2t-1}(\mathcal{K}_{3,1}(f);\mathbb{Z})$, then*

$$H_{2tp^r-1}(E_{p^r}(\mathcal{K}_{3,1}(f));\mathbb{Z}) = H_{2tp^r-1}(\mathcal{K}_{3,1}(\#_{p^r} f);\mathbb{Z}) = \mathbb{Z}/p^{s+r}\mathbb{Z} \oplus A$$

for some abelian group A.

Assume that

$$H_j(\mathcal{K}_{3,1}(f);\mathbb{Z}) = \mathbb{Z}/p^s\mathbb{Z} \oplus A$$

for some abelian group A. Label this $\mathbb{Z}/p^s\mathbb{Z}$-summand (ambiguously) by $< f, j, \mathbb{Z}/p^s\mathbb{Z} >$.

Example 7.2 By [24] or [7] if f is a non-trivial torus knot, then $\mathcal{K}_{3,1}(f)$ has the homotopy type of a circle. A direct application of Proposition 7.1 gives that

$$H_{2p-2}(\mathrm{Conf}(\mathbb{R}^2, p) \times_{\Sigma_p} (\mathcal{K}_{3,1}(f))^p; \mathbb{Z}) = H_{2p-2}(\mathcal{K}_{3,1}(\#_p f);\mathbb{Z}) = < \#_p f, 2p-2, \mathbb{F}_p > \oplus A$$

for some abelian group A.

Next, consider the prime knot h given by a cabling of f.

Example 7.3 The examples here arise by an iterated cabling construction as follows. Let $\alpha/\beta \in \mathbb{Q}$ satisfy $\beta \geq 1$ with $GCD(\alpha, \beta) = 1$. $h = f \bowtie \mathcal{S}^{(\alpha,\beta)}$ is the α/β-cabling of f.

There is a homotopy equivalence

$$\mathcal{K}_{3,1}(f) \times S^1 \to \mathcal{K}_{3,1}(h) = \mathcal{K}_{3,1}(f \bowtie \mathcal{S}^{(\alpha,\beta)})$$

as described in Section 3.

Next, consider an m-fold iterated cabling h_m of f defined by

- $h_1 = f \bowtie \mathcal{S}^{(\alpha,\beta)}$,
- $h_{i+1} = h_i \bowtie \mathcal{S}^{(\alpha,\beta)}$ for $i \geq 1$, defined recursively.

Then there are homotopy equivalences

$$\mathcal{K}_{3,1}(h_m) \to \mathcal{K}_{3,1}(f) \times (S^1)^m.$$

Assume that

$$H_{2t-1}(\mathcal{K}_{3,1}(f)); \mathbb{F}_p) =< f, 2t-1, \mathbb{F}_p > \oplus A$$

for some abelian group A. Then the integer homology of $\mathcal{K}_{3,1}(h_m) = \mathcal{K}_{3,1}(f) \times (S^1)^m$ has a summand denoted (ambiguously) by

$$< f, 2t-1, \mathbb{F}_p > \otimes H_*((S^1)^m; \mathbb{Z}).$$

Thus if $m \geq 1$, there are elements of order p in both odd as well as even degrees in the integer homology of $\mathcal{K}_{3,1}(h_m))$. If m is “large", then there are many elements of order exactly p which are of both odd and even degree. This fussiness concerning parity of degrees has consequences for higher torsion in homology.

The above remarks give examples of long knots with torsion of order exactly p concentrated in odd degrees for the integer homology of their path-components. One choice of f is a torus knot. The next proposition shows that p-torsion of order exactly p in the homology of $\mathcal{K}_{3,1}(g)$ gives rise higher p^s-torsion in the homology of other components related components as follows.

Recall that Example 7.3 provides instances of prime knots f such that

$$H_{2t-1}(\mathcal{K}_{3,1}(f); \mathbb{Z}) =< f, 2t-1, \mathbb{F}_p > \oplus A$$

for some abelian group A. Consider the long knot g given by the p^s-fold connected sum

$$g = f\Delta_{p^s} \bowtie \mathcal{H}^{p^s} \equiv \#_{p^s} f$$

as used in the next Proposition.

Proposition 7.4 *Let f denote a prime long knot such that*

$$H_{2t-1}(\mathcal{K}_{3,1}(f);\mathbb{Z}) =< f, 2t-1, \mathbb{F}_p > \oplus A$$

for some abelian group A. Let

$$g = f\#\cdots\#f = \#_{p^s}f.$$

Then

$$H_{2tp^s-1}(\mathcal{K}_{3,1}(g);\mathbb{Z}) =< \#_{p^s}f, 2tp^s-1, \mathbb{Z}/p^{s+1}\mathbb{Z} > \oplus A$$

for some abelian group A.

Proof Assume that the integer homology of $\mathcal{K}_{3,1}(f)$ has a non-trivial $\mathbb{F}_p$ summand in degree $2t-1$ as guaranteed by example 7.3. Thus, in the mod-p reduction of the integer homology of $\mathcal{K}_{3,1}(f)$, there are classes x of degree $2t-1$, the mod-p reduction of a class of order p, as well as a class y in degree $2t$ which corresponds to the contribution forced by x in the “Tor" term in the classical universal coefficient Theorem.

Since $H_{2t-1}(\mathcal{K}_{3,1}(f);\mathbb{Z})$ has a $\mathbb{F}_p$-summand, there are elements in the mod-p homology of $\mathcal{K}_{3,1}(f)$ with

(1) x in $H_{2t-1}(\mathcal{K}_{3,1}(f);\mathbb{F}_p)$,

(2) y in $H_{2t-1}(\mathcal{K}_{3,1}(f);\mathbb{F}_p)$ and

(3) the first Bockstein of y is x,

$$\beta_1(y) = x.$$

A classical computation of the Bockstein spectral sequence gives that the $(s+1)$-st Bockstein is defined with

$$\beta_{s+1}(y^{p^s}) = x \cdot y^{-1+p^s} + I$$

in case $s \geq 1$ for which I denotes the indeterminacy of this operation. The Proposition follows as these classes survive in the Bockstein spectral sequence for $C(\mathbb{R}^2, X_{\mathcal{K}} \amalg \{*\})$. □

Two concrete examples are listed next.

Example 7.5 (1) Let f denote a non-trivial torus knot with

$$H_1\mathcal{K}_{3,1}(f) = \mathbb{Z}.$$

Then the long knot

$$\#_pf = f\#\cdots\#f$$

satisfies the property that

$$\mathcal{K}_{3,1}(\#_pf) = E_p(\mathcal{K}_{3,1}(f))$$

with

$$H_{2p-2}(\mathcal{K}_{3,1}(\#_pf);\mathbb{Z}) =< \#_pf, 2p-2, \mathbb{Z}/p^1\mathbb{Z} > \oplus A$$

for some abelian group A.

(2) Denote a cable of $\#_p f$ by $(\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}$. There are homotopy equivalences

$$\mathcal{K}_{3,1}((\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}) \to \mathcal{K}_{3,1}(\#_p f) \times S^1$$

with the property that

$$H_{2p-1}(\mathcal{K}_{3,1}((\#_p f)\bowtie\mathcal{S}^{\alpha,\beta});\mathbb{Z}) =< (\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}, 2p-1, \mathbb{F}_p > \oplus A$$

for some abelian group A.

(3) The p^s-fold connected sum of $(\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}$,

$$\#_{p^s}\left((\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}\right)$$

has the property that

$$H_{2p^{s+1}-1}\left(\mathcal{K}_{3,1}\left(\#_{p^s}\left((\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}\right)\right);\mathbb{Z}\right) =< \#_{p^s}\left((\#_p f)\bowtie\mathcal{S}^{\alpha,\beta}\right), 2p^{s+1}-1, \mathbb{Z}/p^{s+1}\mathbb{Z} > \oplus A$$

for some abelian group A.

The third part of Theorem 1.3 follows, thus concluding the proof.

8 On $H_1\mathcal{K}_{3,1}$

$H_*\mathcal{K}_{3,1}$ has torsion of all orders, it is natural to ask for the lowest dimension $i_{(p,n)}$ so that $H_{i_{(p,n)}}\mathcal{K}_{3,1}$ contains torsion of order p^n. This question is answered in this section for the special case $(p,n) = (2,1)$. This section contains a proof of Theorem 1.6.

The idea of the proof is to describe $H_1\mathcal{K}_{3,1}$ inductively, component-by-component. The most complicated case from the point of view of torsion is the hyperbolic satellite case, since there is currently insufficient control of the representations $B_L \to \Sigma_n^+$. In addition, better control over the class of the inversion-map $H_1\mathcal{K}_{3,1} \to H_1\mathcal{K}_{3,1}$ is required.

First, the base-case: knots whose JSJ-trees have one vertex.

- If f is a torus knot $H_1\mathcal{K}_{3,1}(f) \simeq \mathbb{Z}$ and f is invertible and the Σ_2 action (inversion action on $H_1\mathcal{K}_{3,1}(f)$) is given by multiplication by (-1). This is a direct corollary of [7].
- If f is a hyperbolic knot $H_1\mathcal{K}_{3,1}(f) \simeq \mathbb{Z}^2$. In the case that f is invertible, the (inversion) action of Σ_2 on $\mathbb{Z}^2$ is multiplication by (-1). This also follows immediately from [7].

The next proposition gives $H_1(\mathcal{K}_{3,1}(f);\mathbb{Z})$ inductively, via the JSJ-tree of f. Given a group G acting on an abelian group A, let A_G denote the module of co-invariants, the quotient group of A modulo the subgroup generated by $\{a - g\cdot a : g \in G, a \in A\}$. The following lemma follows from the Leray-Serre spectral sequence for any fibration with a section.

Lemma 8.1 *Given a fibration $F \to E \to B$ with a section, with both the base and the fibre path-connected, then $H_1 E \simeq H_1 B \oplus (H_1 F)_{\pi_1 B}$.*

In principle, one can deduce the following result from the presentation of the groups $\pi_1(\mathrm{Conf}(\mathbb{R}^2, n)/Y; \mathbb{Z})$ given by Manfredini [36], alternatively using some elementary facts about the abelianization of the braid group or from the description in [12]. The proof is omitted.

Lemma 8.2 *Let Y be a Young subgroup of Σ_n. We think of Σ_n as the group of bijections of the set $\{1, 2, \cdots, n\}$. Let k be the number of distinct orbits of Y, and let f be the number of fixed-points of Y acting on $\{1, 2, \cdots, n\}$. Let $l = k - f$. Then $H_1(\mathrm{Conf}(\mathbb{R}^2, n)/Y; \mathbb{Z})$ is a free-abelian group of rank $l + \binom{k}{2}$.*

Proposition 8.3 *Given any component $\mathcal{K}_{3,1}(f)$ of $\mathcal{K}_{3,1}$, $H_1(\mathcal{K}_{3,1}(f); \mathbb{Z})$ is finitely-generated and a direct-sum of groups of the form: $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Moreover, if f is an invertible knot, the involution of $H_1(\mathcal{K}_{3,1}(f); \mathbb{Z})$ preserves a splitting of $H_1(\mathcal{K}_{3,1}(f); \mathbb{Z})$ into a direct sum $H_1(\mathcal{K}_{3,1}(f); \mathbb{Z}) = V_1 \oplus V_2$ where the involution acts on V_1 as the identity and acts on V_2 by multiplication by (-1).*

Proof The proof is by induction on the height of the JSJ-tree of f. The height one case was dealt with at the start of this section. The inductive step is as follows.

- Consider the cases that f is a cable of g, then $H_1 \mathcal{K}_{3,1}(f) = \mathbb{Z} \oplus H_1 \mathcal{K}_{3,1}(g)$ [7]. In the case that f is invertible, the homotopy-equivalence is $\mathbb{F}_2$-equivariant with $\mathbb{F}_2$-action on $S^1 \times \mathcal{K}_g$ being complex conjugation on S^1 and the inversion involution on $\mathcal{K}_g$. Thus the $\mathbb{F}_2$-action on $H_1 S^1 \times \mathcal{K}_{3,1}(g) = \mathbb{Z} \oplus H_1 \mathcal{K}_{3,1}(g)$ is simply the direct sum of $\mathbb{F}_2$-modules $\mathbb{Z}$ (with the non-trivial involution) and $H_1 \mathcal{K}_{3,1}(g)$ with its own inversion involution, completing this part of the inductive step.
- Consider the case that f is a connected sum of prime knots $f_1, f_2, \cdots, f_n$ with $n \geq 2$. Then by Lemma 8.1,
$$H_1(\mathcal{K}_{3,1}(f); \mathbb{Z}) = H_1(\mathrm{Conf}(\mathbb{R}^2, n)/Y; \mathbb{Z}) \oplus \left(\bigoplus_{i=1}^{n} H_1(\mathcal{K}_{3,1}(f_i); \mathbb{Z}) \right) /Y$$
where Y is the Young subgroup of Σ_n given by the equivalence relation $i \sim j \Leftrightarrow \mathcal{K}_{3,1}(f_i) = \mathcal{K}_{3,1}(f_j)$. $H_1(\mathrm{Conf}(\mathbb{R}^2, n)/Y; \mathbb{Z}) \simeq \mathbb{Z}^{l+\binom{k}{2}}$ where l is the number of orbits of Y with more than 1 element, and k is the number of orbits of Y by Lemma 8.2. If f is invertible, the involution action on $\mathcal{K}_{3,1}(f) \simeq \mathrm{Conf}(\mathbb{R}^2, n) \times_Y \prod_{i=1}^n \mathcal{K}_{3,1}(f_i)$ was described in [7] as a map that preserved the above product structure, acting by mirror reflection along a line in $\mathbb{R}^2$ on $\mathrm{Conf}(\mathbb{R}^2, n)$ and by permutation of the factors of $\mathcal{K}_{3,1}(f_i)$ combined with the inversion involution on $\mathcal{K}_{3,1}(f_i)$ for each $i \in \{1, 2, \cdots, n\}$. Since the abelianisation of $\pi_1(\mathrm{Conf}(\mathbb{R}^2, n)/Y)$ was computed entirely in terms of linking numbers, mirror reflection along a line induces multiplication by (-1) on $H_1(\mathrm{Conf}(\mathbb{R}^2, n)/Y; \mathbb{Z})$. This completes this step of the inductive argument.

- Consider the case of a hyperbolic satellite operation. In this case, $H_1\mathcal{K}_f = \mathbb{Z}^2 \oplus \left(\oplus H_1\mathcal{K}_{3,1}(f_i)\right)/A_f$ [7]. Thus $H_1\mathcal{K}_f$ consists of $\mathbb{Z}^2$ direct sum various groups, one for each orbit of A_f acting on $\{1,2,\cdots,n\}$. Denote the orbits by $\{1,2,\cdots,n\} = Y_1 \cup Y_2 \cup \cdots \cup Y_k$. The summand corresponding to orbit Y_i is either $H_1\mathcal{K}_{3,1}(f_j)$ for $j \in Y_i$ or $(H_1\mathcal{K}_{3,1}(f_j))/\Sigma_2$ depending on whether or not A_f has an element that reverses the orientation of L_j or not. $(H_1\mathcal{K}_{3,1}(f_j))/\Sigma_2$ is also a direct sum of groups of the form $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ by the inductive hypothesis. Now consider the case that f is invertible. By [7] the Σ_2-action on $\mathcal{K}_f \simeq M \times (SO(2) \times_{A_f} \prod_{i=1}^n \mathcal{K}_{3,1}(f_i))$ respects the bundle structure, thus on $\mathbb{Z}^2 \oplus \left(\oplus H_1\mathcal{K}_{3,1}(f_i)\right)/A_f$ it acts by multiplication by (-1) on the $\mathbb{Z}^2$-factor. On the remaining factors it either acts trivially on the i-th summand if the inversion symmetry of L does not reverse the orientation of L_i or it acts by inversion on that summand.

Thus the result follows. □

Corollary 8.4 *$H_1(\mathcal{K}_{3,1}(f);\mathbb{Z})$ contains 2-torsion if and only if there is a hyperbolic link L so that one of the vertices of $\mathbb{G}_f$ is decorated by L, and if we let g be the knot whose JSJ-tree is the subtree rooted at L, then A_g must contain an isometry that reverses the orientation of some L_i.*

9 The first occurrence of odd torsion

Theorem 9.1 *Let f denote a long knot and p an odd prime. If $H_i(\mathcal{K}_{3,1}(f);\mathbb{Z})$ contains $\mathbb{F}_p$, then $i \geq 2p-2$.*

Notice that the Theorem does not assert that there is torsion in $H_*\mathcal{K}_{3,1}(f)$, but rather the least dimension in which p-torsion can possibly occur. There are long knots f such that $H_*\mathcal{K}_{3,1}(f)$ is torsion free. Furthermore, there are long knots g such that $H_{2p-2}(\mathcal{K}_{3,1}(g);\mathbb{Z})$ contains copies of $\mathbb{F}_p$ by Example 7.2. This theorem follows a classical pattern which is exhibited for both $K(B_n,1)$ as well as $\Omega^n S^{n+1}$.

Proof It suffices to prove that $H_i(\mathcal{K}_{3,1};\mathbb{Z})$ contains no p-torsion for $i < 2p-2$.

Since the homology groups $H_*\mathcal{K}_{3,1}(f)$ are torsion free for the unknot, torus knots and hyperbolic knots, it suffices to check that that p-torsion cannot occur in dimensions less than $2p-2$ in the following three cases.

(1) The knot f is a cable of g in which case $\mathcal{K}_{3,1}(f) \simeq S^1 \times \mathcal{K}_{3,1}(g)$.

(2) The knot f is hyperbolically spliced.

(3) The knot f is a connected-sum of knots g_i such that the homology of $\mathcal{K}_{3,1}(g_i)$ is p-torsion free in dimensions less than $2p-2$.

Case 1 follows directly from the classical Künneth theorem. Cases 2 and 3 follow inductively by the next three lemmas. □

Consider the k-fold product X^k with the natural (left) action of Σ_k on X^k. The free $\mathcal{C}_n$-space generated by $X \amalg +$ is denoted $\mathcal{C}_n(X \amalg +)$. Recall that this space is homotopy-equivalent to the disjoint union of $\mathrm{Conf}(\mathbb{R}^n, k) \times_{\Sigma_k} X^k$ for all $k \geq 0$.

Lemma 9.2 *Assume that X is a topological space of the homotopy type of a CW-complex (alternatively, one can substitute compactly generated, weak Hausdorff for having the homotopy-type of a CW-complex in this lemma) without p-torsion in homology of dimensions less than $2p-2$ for p an odd prime. Then the homology of $\mathcal{C}_n(X \amalg +)$ and $\Omega^n\Sigma^n(X \amalg +)$ also do not have p-torsion in homology of dimensions less than $2p-2$. Thus the homology of $\mathrm{Conf}(\mathbb{R}^n, k) \times_{\Sigma_k} X^k$ does not have p-torsion in homology of dimensions less than $2p-2$.*

The proof follows directly from the computations in [12] or can be done classically by chain level arguments (in the spirit of Nakaoka and Steenrod).

Lemma 9.3 *Let $A = \mathbb{Z}/p^r\mathbb{Z}$ act on the p^r-fold product of a path-connected CW-complex X^{p^r} by a cyclic permutation of order p^r and on S^1 freely via a rotation of order p^r. If $H_i(X;\mathbb{Z})$ is p-torsion free and finitely generated for all $i < q$, then*

$$H_j(S^1 \times_A X^{p^r}; \mathbb{Z})$$

is p-torsion free for all $j < q$.

Proof Consider the space

$$S^1 \times_A X^{p^r}.$$

Classically, there are chain equivalences

$$\mathbb{B}_* \otimes_{\mathbb{Z}[A]} C_*(X)^{\otimes p^r} \xrightarrow{D \otimes 1^{p^r}} \mathbb{B}_* \otimes_{\mathbb{Z}[A]} C_*(X^{p^r}) \longrightarrow C_*(S^1 \times_A X^{p^r})$$

where

(1) $A = \mathbb{Z}/p^r\mathbb{Z}$,

(2) $\mathbb{B}_*$ denotes the chain complex of right $\mathbb{Z}[A]$-modules (chain equivalent to the total singular chain complex of a circle)

$$\cdots \longrightarrow \{0\} \longrightarrow \mathbb{Z}[A] \xrightarrow{D} \mathbb{Z}[A]$$

for which $D(1) = 1 - \tau$ where τ is a generator for A,

(3) $C_*(X)$ denotes the total singular chain complex of X for which $C_i(X)$ denotes the singular chains in degree i and

(4) $C_*(X)^{\otimes p^r}$ and $C_*(X^{p^r})$ is given the natural structure of left $\mathbb{Z}[A]$-modules.

Since X is assumed to be of finite type,

$$H_i(X) = F_i \oplus T_i$$

where F_i is a finite direct sum of copies of $\mathbb{Z}$ and T_i is a finite direct sum of finite cyclic groups. If $i < q$, it may be assumed that T_i is of order prime to p and thus this summand will not contribute p-torsion to the homology of the chain complex $\mathbb{B}_* \otimes_{\mathbb{Z}[A]} C_*(X)^{\otimes p^r}$ (Details are omitted).

Furthermore, there is a map

$$\rho_i : F_i \to C_i(X)$$

which

(1) induces a map of chains complexes (with trivial differential for the source) and
(2) induces a homology isomorphism in degrees $i < q$ with coefficients in $\mathbb{Z}_{(p)}$ (the integers localized at p meaning those rational numbers with denominators prime to p).

Thus it suffices to check that the homology of the chain complex

$$\mathbb{B}_* \otimes_{\mathbb{Z}[A]} (\oplus_{i<q} F_i)^{\otimes p^r}$$

is p-torsion free homology in dimensions less than q for p an odd prime. Since F_i is free abelian, notice that $(\oplus_{i<q} F_i)^{\otimes p^r}$ is a sum of permutation representations (over the integral group ring of A) each of which are cyclic $\mathbb{Z}[A]$-modules which have the following generators.

(1) $v^{\otimes p^r}$ where v is an element in F_i of even degree.
(2) $v^{\otimes p^r}$ where v is an element in F_i of odd degree.
(3) $v_1 \otimes v_2 \otimes \cdots \otimes v_{p^r}$ where the v_i run over a basis for the $\oplus_{i<q} F_i$ with at least two distinct basis elements appearing.

Thus it suffices to work out the torsion in the chain complex

$$\mathbb{B}_* \otimes_{\mathbb{Z}[A]} M$$

where M denotes the free abelian group which is a cyclic $\mathbb{Z}[A]$-module with one of the elements in (1-3) as generators. These are considered next.

(1) Let M denote the cyclic $\mathbb{Z}[A]$-module generated by $v^{\otimes p^r}$ where v is an element in F_i of either odd or even degree. Since p is odd, the associated permutation representation is trivial and thus the chain complex

$$\mathbb{B}_* \otimes_{\mathbb{Z}[A]} M$$

is isomorphic to $(\mathbb{B}_* \otimes_{\mathbb{Z}[A]} \mathbb{Z}) \otimes_{\mathbb{Z}} M$. The homology of this chain complex is isomorphic to $H_*(S^1) \otimes_{\mathbb{Z}} M$ as a graded abelian group and is thus torsion free.

(2) Let M denote the cyclic $\mathbb{Z}[A]$-module generated by $v_1 \otimes v_2 \otimes \cdots \otimes v_{p^r}$ where the v_i run over a basis for the $\oplus_{i<q} F_i$ with at least two distinct basis elements appearing among the v_i. The action of $A = \mathbb{Z}/p^r\mathbb{Z}$ has isotropy subgroup given by $\mathbb{Z}/p^s\mathbb{Z}$ for some $0 \leq s < r$. Thus the module M is isomorphic to to

$$\mathbb{Z}[A] \otimes_{\mathbb{Z}[\mathbb{Z}/p^s\mathbb{Z}]} \mathbb{Z}$$

as a left $\mathbb{Z}[A]$-module and there is an induced isomorphism of chain complexes

$$\mathbb{B}_* \otimes_{\mathbb{Z}[A]} (\mathbb{Z}[A] \otimes_{\mathbb{Z}[\mathbb{Z}/p^s\mathbb{Z}]} \mathbb{Z}) \to \mathbb{B}_* \otimes_{\mathbb{Z}[A]} M.$$

Since the chain complex $\mathbb{B}_* \otimes_{\mathbb{Z}[A]} (\mathbb{Z}[A] \otimes_{\mathbb{Z}[\mathbb{Z}/p^s\mathbb{Z}]} \mathbb{Z})$ is isomorphic to

$$\mathbb{B}_* \otimes_{\mathbb{Z}[\mathbb{Z}/p^s\mathbb{Z}]} \mathbb{Z},$$

the chain complex has torsion free homology.

□

Lemma 9.4 *Let $g = (f_1, \cdots, f_n) \bowtie L$ where $n \geq 1$, and L a hyperbolic KGL. If for all $j \in \{1, 2, \cdots, n\}$, $H_i\mathcal{K}_{3,1}(f_j)$ contains no elements of order p for all $i < 2p-2$, then $H_i\mathcal{K}_{3,1}(g)$ contains no elements of order p for $i < 2p-2$.*

Proof In this case, there is a homotopy equivalence

$$\mathcal{K}_{3,1}(g) \simeq S^1 \times \left(SO(2) \times_{A_g} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right)$$

where A_g is a cyclic group acting via permutations on the factors in $\prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i)$.

To determine whether there is p-torsion in the homology of $\mathcal{K}_{3,1}(g)$, it suffices to determine the p-torsion in case A_g is replaced by the p-Sylow subgroup of A_g given by $H = \mathbb{Z}/p^n\mathbb{Z}$ as the induced map

$$S^1 \times \left(SO(2) \times_H \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right) \to S^1 \times \left(SO(2) \times_{A_g} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right)$$

induces a split epimorphism on the $p-$*torsion* subgroup by a classical transfer argument ala' Cartan-Eilenberg.

Consider the covering map

$$\left(SO(2) \times_H \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right) \to \left(SO(2) \times_{A_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right)$$

for which the group of covering translations is abelian with group of covering transformations A_f/H.

Homologically, this map is onto the p-torsion elements of $H_*\left(SO(2) \times_{A_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i)\right)$ since the composite of the transfer map with the covering map:

$$H_*\left(SO(2) \times_{A_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right) \to H_*\left(SO(2) \times_{A_f} \prod_{i=1}^{n} \mathcal{K}_{3,1}(f_i) \right)$$

is multiplication by $|\frac{A_f}{H}|$ which is coprime to p.

The Lemma follows at once from Lemma 9.3. □

10 On the subspace generated by cabling and summation

The purpose of this section is to describe the subspace $\mathcal{TK}_{3,1}$ of $\mathcal{K}_{3,1}$, consisting of the path components of $\mathcal{K}_{3,1}$ containing the unknot and all knots generated from the unknot by iterating the cabling and connected-sum operations. An alternate description of the space $\mathcal{TK}_{3,1}$ is that it consists of precisely those long knots whose complements have JSJ-decompositions containing only Seifert-fibred manifolds.

First define the space

$$\mathcal{T} = \amalg_{1<p<q,(p,q)=1}\mathcal{K}_{3,1}(f(p,q))$$

where $f(p,q)$ denotes a (p,q)-torus knot. Thus $\mathcal{K}_{3,1}(f(p,q))$ has the homotopy type of S^1. Consider the James construction

$$J(\mathcal{T} \amalg \{*\}) = \amalg_{0\leq n}\mathcal{T}^n$$

with $\mathcal{T}^0 = \{*\}$, the base-point. Write

$$J_{\mathcal{K}} = \amalg_{1\leq n}\mathcal{T}^n$$

with

$$J(\mathcal{T} \amalg \{*\}) = J_{\mathcal{K}} \amalg \{*\}.$$

Spaces Y_n are specified inductively in terms of $J_{\mathcal{K}}$ as follows.

* $Y_0 = \mathcal{C}_2(J_{\mathcal{K}} \amalg \{*\})$ and
* $Y_{n+1} = \{([\mathcal{C}_2(Y_n)] - Y_n) \times J_{\mathcal{K}})\} \amalg Y_n$.

Notice that Y_n is naturally a subspace of Y_{n+1} and that all of these may be regarded as subspaces of $\mathcal{K}_{3,1}$ in the following way: There are induced maps

$$\mathcal{K}_{3,1} \times J(\mathcal{T} \amalg \{*\}) \to \mathcal{K}_{3,1}$$

induced by cabling.

Define

$$\mathcal{TK}_{3,1} = \cup_{n\geq 0} Y_n.$$

Notice that $\mathcal{TK}_{3,1}$ is the subspace of $\mathcal{K}_{3,1}$ which contains the path-components of p/q-torus knots and which is closed under the operations of cabling and sums. That is, there are induced maps $\mathcal{K}_{3,1} \times J(\mathcal{T} \amalg \{*\}) \to \mathcal{K}_{3,1}$ induced by cabling. There is an induced inclusion $\mathcal{TK}_{3,1} \to \mathcal{K}_{3,1}$.

The remainder of this section gives features of the homology of $\mathcal{TK}_{3,1}$. First notice that homology commutes with inductive co-limits and so there are isomorphisms

$$\varinjlim H_*(Y_n) \to H_*(\varinjlim Y_n) \to H_*(\mathcal{TK}_{3,1}).$$

Recall the construction $\Gamma(X)$ as given in section 5.

To describe the homology of Y_n, restrict to field coefficients. $\mathbb{F}$, Recall the natural splitting of graded vectors spaces

$$\bar{H}_*(X;\mathbb{F}) \oplus \Gamma(X;\mathbb{F}) \to H_*(\mathcal{C}_2(X);\mathbb{F}))$$

for a choice of graded vector space

$$\Gamma(X;\mathbb{F})$$

which is functor of $H_*(X;\mathbb{F})$ as listed in section 5. Thus $H_*(Y_{n+1})$ is given in terms of the construction $\Gamma(X;\mathbb{F})$ in case $X = Y_n$.

Proposition 10.1 *The natural map $H_*(Y_n) \to H_*(Y_{n+1})$ is a split monomorphism.*

Notice that the homology of $\mathcal{TK}_{3,1}$ exhibits a fractal-like behaviour reflecting the geometry in Budney's theorem [5] and iterations of the constructions $\Gamma(X)$ as given in section 5. Namely, this homological behaviour arises by first considering $Y_1 = \mathcal{C}_2(J_{\mathcal{K}})$ together with the homology $H_*\left(\mathcal{C}_2(X \amalg \{*\});\mathbb{F}\right)$ as given in section 5 as follows where

$$V = H_*(X;\mathbb{F}).$$

(1) $S[\sigma^{-1}L[\sigma(V]]$ if the characteristic of $\mathbb{F}$ is 0,

(2) $S[\sigma^{-1}L^{(2)}[\sigma(V)]]$ if $\mathbb{F} = \mathbb{F}_2$ and

(3) $S[\sigma^{-1}L^{(p)}[\sigma(V)] \oplus \sigma^{-2}W^p[\sigma(V)]]$ if $\mathbb{F} = \mathbb{F}_p$ for odd primes p.

On a simpler note, let $\aleph\mathcal{K}_{3,1}$ denote the subspace of $\mathcal{K}_{3,1}$ consisting of: all unknots, torus knots, and all connect-sums of torus knots. Thus, $\aleph\mathcal{K}_{3,1}$ is a 2-cubes subspace of $\mathcal{K}_{3,1}$ and

$$\aleph\mathcal{K}_{3,1} \simeq \mathcal{C}_2\left(\{*\} \sqcup \bigsqcup_{\mathbb{Z}} S^1\right).$$

By May [37],

$$B(\aleph\mathcal{K}_{3,1}) \simeq \Omega^2\Sigma^2\left(\{*\} \sqcup \bigsqcup_{\mathbb{Z}} S^1\right)$$

which has the homotopy-type of

$$\Omega^2\left(\bigvee_{\mathbb{Z}} (S^2 \vee S^3)\right)$$

where the union and wedge index set $\mathbb{Z}$ corresponds to the isotopy classes of torus knots. Thus, by the Hilton-Milnor theorem the homotopy groups of $B(\aleph\mathcal{K}_{3,1})$ contain the homotopy groups of all spheres (of dimension ≥ 2) in profusion.

11 Closed Knots and Homology

The purpose of this section is to use results of the earlier sections to give information about the space $\mathrm{Emb}(S^1,S^3)$. Recall the homeomorphism

$$S^3 \times \mathrm{Emb}_*(S^1,S^3) \to \mathrm{Emb}(S^1,S^3)$$

for which $Emb_*(S^1,S^3)$ denotes the pointed embeddings. Thus there are isomorphisms

$$H_*(\mathrm{Emb}(S^1,S^3)) \to H_*(S^3) \otimes H_*(\mathrm{Emb}_*(S^1,S^3))$$

by Proposition 4.2.

Information giving the structure of the bundle $\mathcal{K}_{3,1} \to \mathrm{Emb}_*(S^1, S^3) \to S^2$ was worked out earlier. That structure is used to provide information concerning $H_*(\mathrm{Emb}_*(S^1, S^3))$ by a Mayer-Vietoris argument.

Let D_1 and D_2 be two discs in S^2 whose union is S^2 and whose intersection is S^1. Let A_1 and A_2 be the preimages of D_1 and D_2 under the projection map $\mathrm{Emb}_*(S^1, S^3) \to S^2$, then both A_1 and A_2 are homeomorphic to $\mathcal{K}_{3,1} \times D^2$. Consider the Mayer-Vietoris sequence for $\mathrm{Emb}_*(S^1, S^3) = A_1 \cup A_2$ where $A_1 \cap A_2$ is homeomorphic to $S^1 \times \mathcal{K}_{3,1}$.

Identify $A_1 \equiv D^2 \times \mathcal{K}_{3,1}$ and $A_2 \equiv D^2 \times \mathcal{K}_{3,1}$ then the gluing map from $\partial A_1 = S^1 \times \mathcal{K}_{3,1} \to \partial A_2 = S^1 \times \mathcal{K}_{3,1}$ is the map $S^1 \times \mathcal{K}_{3,1} \ni (t, x) \longmapsto (t, t^2.x) \in S^1 \times \mathcal{K}_{3,1}$. Thus the Meyer-Vietoris sequence has the form:

$$\cdots \to H_*(S^1 \times \mathcal{K}_{3,1}) \to H_*(D^2 \times \mathcal{K}_{3,1}) \oplus H_*(D^2 \times \mathcal{K}_{3,1}) \to H_*(\mathrm{Emb}_*(S^1, S^3)) \to \cdots$$

where $H_*(S^1 \times \mathcal{K}_{3,1})$ is identified with $H_*\mathcal{K}_{3,1} \oplus H_{*-1}\mathcal{K}_{3,1}$ and $H_*(D^2 \times \mathcal{K}_{3,1})$ is identified with $H_*(\mathcal{K}_{3,1})$. The map $H_n\mathcal{K}_{3,1} \oplus H_{n-1}\mathcal{K}_{3,1} \to H_n(\mathcal{K}_{3,1}) \oplus H_n(\mathcal{K}_{3,1})$ is given by the 2×2 matrix $\begin{pmatrix} I & 0 \\ I & 2\kappa_n \end{pmatrix}$ where $\mu : SO(2) \times \mathcal{K}_{3,1} \to \mathcal{K}_{3,1}$ is the $SO(2)$-action on $\mathcal{K}_{3,1}$ and $\kappa_n : H_{n-1}\mathcal{K}_{3,1} \to H_n\mathcal{K}_{3,1}$ satisfies $\kappa_n(x) = \mu_*(SO(2) \times x)$

Corollary 11.1 *There is a short exact sequence*

$$0 \to coker(2\kappa_n) \to H_n\mathrm{Emb}_*(S^1, S^3) \to ker(2\kappa_{n-1}) \to 0.$$

Corollary 11.2 *A knot $f : S^1 \to S^3$ is the unknot if and only if its component in* $\mathrm{Emb}(S^1, S^3)$ *contains no torsion in its homology. Moreover, the component of a non-trivial knot in* $\mathrm{Emb}(S^1, S^3)$ *always has* 2*-torsion in its integral homology.*

Proof If f is the unknot, the component of f has the homotopy type of $V_{4,2} = S^3 \times S^2$ which has no torsion in its homology.

If f is non-trivial, then first consider its long knot component, $\mathcal{K}_{3,1}(f)$. This has a $\mathbb{Z}$ embedded in its its fundamental group, embedded as the 2π rotation around the long axis [18]. We call the embedding $\mathbb{Z} \to \pi_1\mathcal{K}_{3,1}(f)$ the Gramain map. In [7] it's shown that there is a map $\mathcal{K}_{3,1}(f) \to S^1$ which when composed with the Gramain map is not null-homotopic. It follows that $H_1\mathcal{K}_{3,1}(f)$ contains a copy of the integers, generated by the Gramain element. Thus since the image of $2\kappa_1$ is generated by twice the Gramain element, $coker(2\kappa_1)$ must contain 2-torsion. □

The short exact sequence in Corollary 11.1 is not ideal because it leaves us with extension problems. We show how the extension problems can be solved using the techniques of Section 3.

Observe that, if $f \in \mathcal{K}_{3,1}$ is a prime knot, then there is an $SO(2)$-equivariant homotopy-equivalence $\mathcal{K}_{3,1}(f) \simeq SO(2) \times X(f_i)$ where the $SO(2)$-action on $SO(2) \times X(f)$ is a product action, given by left-multiplication on $SO(2)$ and the trivial action on $X(f)$ (here $X(f)$ is just $\mathcal{K}_{3,1}(f)/SO(2)$). So prime knot components of $Emb(S^1,S^3)$ have the homotopy type of $S^3 \times SO(3) \times X(f)$. As mentioned earlier, the unknot component has the homotopy-type of $S^3 \times S^2$.

We now investigate the case of a connected-sum of $n \geq 2$ prime knots, $f = f_1\#\cdots\#f_n$. By the above argument, we can assume $\mathcal{K}_{3,1}(f_i) \simeq SO(2)\times X(f_i)$ for $X(f_i) = \mathcal{K}_{3,1}(f_i)/SO(2)$. Thus, the component corresponding to f in $\mathrm{Emb}_*(S^1,S^3)$ has the homotopy-type of $\mathcal{C}_2(n) \times_{\Sigma_f} \left(\left(SO(3) \times_{SO(2)} SO(2)^n\right) \times \prod_{i=1}^n X(f_i)\right)$

We determine the homotopy-type of $SO(3) \times_{SO(2)} SO(2)^n$ as a Σ_n-space. Consider $SO(2)^n$ to be $\mathbb{R}^n/\mathbb{Z}^n$. Let $D \subset \mathbb{R}^n$ be the diagonal $D = \{(t,t,\cdots,t) : t \in \mathbb{R}\}$. Let $P \subset \mathbb{R}^n$ be the perp of D, ie: $P = \{(x_1,x_2,\cdots,x_n) : \sum_{i=1}^n x_i = 0\}$. Thus $P/(P\cap\mathbb{Z}^n)$ is an $(n-1)$-dimensional torus, which we will denote $P(n)$. We also define a subgroup $Z(n) \subset P(n)$. The integer lattice $\mathbb{Z}^n$ projects (orthogonally) onto a subgroup of P, we further take the image of this subgroup under the quotient map $P \to P(n)$ and denote this image $Z(n)$. There is a naturally defined homomorphism $Z(n) \to SO(2)$ given by considering the embedding $P(n) \to SO(2)^n \equiv \mathbb{R}^n/\mathbb{Z}^n$. For every element $z \in Z(n)$ there is a unique element $t \in SO(2)$ so that $t.z = 0 \in SO(2)^n$.

Proposition 11.3 *Provided $n \geq 1$,*

$$SO(3) \times_{SO(2)} SO(2)^n \simeq SO(3) \times_{Z(n)} P(n)$$

where the action of $Z(n)$ on $SO(3)$ is given by the homomorphism $Z(n) \to SO(2)$. Moreover, this is an Σ_n-equivariant homeomorphism where the action of Σ_n on $SO(3) \times_{Z(n)} P(n)$ is a product action, trivial on $SO(3)$ and the natural action on $P(n) \subset SO(2)^n$. Since the homomorphism $Z(n) \to SO(2)$ is null-homotopic (as a continuous function), the above bundle is homeomorphic to a product $SO(3) \times_{Z(n)} P(n) \simeq SO(3) \times (P(n)/Z(n))$.

Corollary 11.4 *If f is a connected-sum of n prime knots $n \geq 2$, then*

$$SO(3) \times_{SO(2)} \mathcal{K}_{3,1}(f) \simeq SO(3) \times \mathcal{C}_2(n) \times_{\Sigma_f} \left(P(n)/Z(n) \times \prod_{i=1}^n X(f_i)\right)$$

where $\mathcal{K}_{3,1}(f_i) = SO(2) \times X(f_i)$, so the fibration $SO(3) \times_{SO(2)} \mathcal{K}_{3,1}(f) \to S^2$ is just projection onto $SO(3)$ then onto S^2.

We now perform the analogous computations for $\mathrm{Emb}(S^1,\mathbb{R}^3)$. Proposition 4.4 gives us the analogous bundle $C \rtimes \mathcal{K}_{3,1} \to \mathrm{Emb}(S^1,\mathbb{R}^3) \to S^2$. Decomposing S^2 as the union of two discs, one gets a Meyer-Vietoris sequence

$$\cdots \to H_*(S^1 \times (C \rtimes \mathcal{K}_{3,1})) \to H_*(C \rtimes \mathcal{K}_{3,1}) \oplus H_*(C \rtimes \mathcal{K}_{3,1}) \to H_*(\mathrm{Emb}(S^1,\mathbb{R}^3)) \to \cdots$$

which splits into short exact sequences as in Corollary 11.1:

$$0 \to coker(2\kappa'_n) \to H_n\mathrm{Emb}(S^1,\mathbb{R}^3) \to ker(2\kappa'_{n-1}) \to 0$$

where $\kappa'_n : H_{n-1}(C \rtimes \mathcal{K}_{3,1}) \to H_n(C \rtimes \mathcal{K}_{3,1})$ is given by $\mu_*(SO(2) \times \cdot)$ for the $SO(2)$ action μ on $C \rtimes \mathcal{K}_{3,1}$. The bundle $C_f \to C_f \rtimes \mathcal{K}_{3,1}(f) \to \mathcal{K}_{3,1}$ is split, and the monodromy acts trivially on H_*C_f since C_f is a homology S^1 with $H_1(C_f)$ generated by a meridional curve. Thus, $H_*(C \rtimes \mathcal{K}_{3,1}) \simeq H_*S^1 \otimes H_*\mathcal{K}_{3,1}$ and $H_n(C \rtimes \mathcal{K}_{3,1}) = (H_0S^1 \otimes H_n\mathcal{K}_{3,1}) \oplus (H_1S^1 \otimes H_{n-1}\mathcal{K}_{3,1})$.

κ'_n has a description in terms of κ_n and κ_{n-1}. Let $\alpha_i \in H_iS^1$ represent the standard generators of H_iS^1 for $i = 0,1$. Then $\kappa'_n(\alpha_0 \otimes x) = \alpha_1 \otimes x + \alpha_0 \otimes \kappa_n(x)$ and $\kappa'_n(\alpha_1 \otimes x) = -\alpha_1 \otimes \kappa_{n-1}(x)$. Thus, κ'_n can be thought of as a map $\kappa'_n : H_{n-2}\mathcal{K}_{3,1} \oplus H_{n-1}\mathcal{K}_{3,1} \to H_{n-1}\mathcal{K}_{3,1} \oplus H_n\mathcal{K}_{3,1}$ given by $\kappa'_n(x,y) = (-\kappa_{n-1}(x) + y, \kappa_n(y))$. Since $\kappa_n \circ \kappa_{n-1} = 0$, $ker(2\kappa'_n)$ is given by the solutions to the equation $-2\kappa_{n-1}(x) + 2y = 0$ for $(x,y) \in H_{n-2}\mathcal{K}_{3,1} \oplus H_{n-1}\mathcal{K}_{3,1}$. Thus,

$$ker(2\kappa'_n) \simeq H_{n-2}\mathcal{K}_{3,1} \oplus \tau_2 H_{n-1}\mathcal{K}_{3,1}$$

where if A is an abelian group and p an integer, τ_pA is the subgroup of A killed by multiplication by p. Similarly,

$$coker(2\kappa'_n) \simeq H_{n-1}\mathcal{K}_{3,1}/2H_{n-1}\mathcal{K}_{3,1} \oplus H_n\mathcal{K}_{3,1}$$

Proposition 11.5 *There is a short exact sequence*

$$0 \to H_{n-1}\mathcal{K}_{3,1}/2H_{n-1}\mathcal{K}_{3,1} \oplus H_n\mathcal{K}_{3,1} \to H_n\mathrm{Emb}(S^1,\mathbb{R}^3) \to H_{n-3}\mathcal{K}_{3,1} \oplus \tau_2 H_{n-2}\mathcal{K}_{3,1} \to 0$$

Thus, the component of the unknot in $\mathrm{Emb}(S^1,\mathbb{R}^3)$ *is the unique component such that its first homology group is torsion. It is also the unique component so that its 2nd homology group is trivial.*

Proof The short exact sequence follows from the above observations.

That H_1 of a non-trivial component is non-torsion follows from Proposition 6.1 of [7] and the above short exact sequence. That H_1 of the unknot component is torsion follows from Proposition 4.4 and the results in Section 3. Thus, the component of the unknot in $\mathrm{Emb}(S^1,\mathbb{R}^3)$ has the homotopy-type of $SO(3)$, and $H_1SO(3) \simeq \mathbb{Z}_2$.

The statement about H_2 follows from Proposition 6.1 of [7] and the above short exact sequence. □

Proposition 1.5 gives a criteria for testing whether a long knot f is the 'long unknot' which can be modified to to compare two arbitrary long knots. That procedure involves forming a 'difference' to be defined precisely. Namely, it is not the case that an arbitrary long knot admits an inverse. It is necessary to pass to a setting for which the inverses exists in order to 'take differences'. That 'world' is the group completion of $\mathcal{K}_{3,1}$,

$$\Omega(B\mathcal{K}_{3,1}).$$

This last space admits inverses up to homotopy. Given two long knots f and g, consider their classes in $\Omega(B\mathcal{K}_{3,1})$ denoted $[f]$, and $[g]$ respectively. Next, consider the product

$$[f]\cdot[g]^{-1}\in\Omega(B\mathcal{K}_{3,1}).$$

The path-component of $[f]\cdot[g]^{-1}\in\Omega(B\mathcal{K}_{3,1})$ has vanishing first homology group if and only if f and g are in the same path-component of $\mathcal{K}_{3,1}$.

12 Problems

The purpose of this section is to list problems which arise naturally from the work above.

(1): Interpret the rational cohomology of $\mathcal{K}_{3,1}$ in terms of iterated integrals in the sense of Kohno-Kontsevich-Chen.

(2): Compare the Vassiliev invariants of braids as studied by T. Kohno [29] and the Lie algebra obtained from the descending central series for the fundamental groups of the spaces $\mathcal{C}_2(n)\times\mathcal{K}_{3,1}(f_1)\times\cdots\times\mathcal{K}_{3,1}(f_n)$ as well as the induced invariants for $\mathcal{C}_2(n)\times_{\Sigma_f}\left(\mathcal{K}_{3,1}(f_1)\times\cdots\times\mathcal{K}_{3,1}(f_n)\right)$.

(3): A natural connection between the space of long knots and the mod-2 Steenrod algebra arises from the "group completion" of $\mathcal{K}_{3,1}$ given $\Omega B(\mathcal{K}_{3,1})$ [37]. Notice that the collapse map $X_{\mathcal{K}}\to\{*\}$ induces a map $p: C(\mathbb{R}^2, X_{\mathcal{K}}\amalg\{*\})\to C(\mathbb{R}^2,S^0)$ and there is an induced map $p: C(\mathbb{R}^2, X_{\mathcal{K}}\amalg\{*\})\to\mathbb{Z}\times BO$ induced by the regular representation bundle. Thus there are maps

$$\Omega B(\mathcal{K}_{3,1})\to\Omega_0^2S^2\to BO$$

with composite denoted $\phi:\Omega B(\mathcal{K}_{3,1})\to\mathbb{Z}\times BO$. The Thom spectrum of ϕ is a wedge of Eilenberg-Mac Lane spectra $H\mathbb{F}_2$ and thus the mod-2 co-homology of the Thom spectrum $M\Omega B(\mathcal{K}_{3,1})$ is free over the mod-2 Steenrod algebra. Interpret the Steenrod operations in terms of knots.

(4): The Goodwillie Calculus mapping space models $AM_j(\mathsf{I}^n)$ for $\mathcal{K}_{n,1}$ constructed by Sinha [45] have a natural homotopy-associative pairing. This pairing makes $\pi_0 AM_3(\mathsf{I}^3)$ into a group, isomorphic to the integers. As an invariant of knots, $\pi_0\mathcal{K}_{3,1}\to\pi_0 AM_3(\mathsf{I}^3)\simeq\mathbb{Z}$ is the essentially unique type-2 finite-type invariant of knots [9]. This raises the question, do the maps $\mathcal{K}_{n,1}\to AM_j(\mathsf{I}^n)$ factor through the group completion, $\mathcal{K}_{n,1}\to\Omega B\mathcal{K}_{n,1}\to AM_j(\mathsf{I}^n)$?

(5): Combine the structures here with that of Khovanov homology.

References

[1] D. Altshuler and L. Freidel, *On universal Vassiliev invariants*, Comm. Math. Phys., **170** (1995), 41–62.

[2] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology, **34**(1995), 423–472.

[3] F. Bonahon, L. Siebenmann, *Geometric splittings of knots and Conway's algebraic knots,* unpublished preprint (1987).

[4] R. Bott and C. Taubes, *On the self-linking of knots*, J. Math. Phys., **35**(1994), 5247-5287.

[5] R. Budney, *Little cubes and long knots*, Topology **46** (2007), 1–27. arXiv *[math.GT/0309427]*

[6] R. Budney, *JSJ-decompositions of knot complements in S^3,* l'Enseignement Mathématique (2) **52** (2006) 319–259. arXiv *[math.GT/0506523]*

[7] R. Budney, *The topology of knotspaces in dimension* 3, arXiv *[math.GT/0506523]*

[8] R. Budney, *A family of embedding spaces,* Geometry and Topology Monographs **13** (2007). arXiv *[math.AT/0605069]*

[9] R. Budney, J. Conant, K. Scannell, D. Sinha, *New perspectives on self-linking,* Advances in Mathematics **191** (2005) 78–113.

[10] A. Cattaneo, P. Cotta-Ramusino and R. Longoni, *Configuration spaces and Vassiliev invariants in any dimension*, Algebraic, and Geometric Topology, **2**(2002), 949–1000.

[11] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[12] F. R. Cohen, T. J. Lada and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Math., vol. **533**. Springer-Verlag (1976).

[13] C. Day, *On link invariants of Vassiliev-type*, Ph.D. thesis, University of North Carolina, 1993.

[14] D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. Math. Stud. **110**.

[15] M. Falk, R. Randell, *The lower central series of a fiber-type arrangement*, Invent. Math., **82** (1985), 77–88.

[16] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand., **10** (1962), 111-118.

[17] T. Goodwillie, and M. Weiss, *Embeddings from the point of view of immersion theory II*, Geometric Topology, **3** (1999), 103-118.

[18] A. Gramain, *Sur le groupe fondamental de l'espace des noeuds*, Ann. Inst. Fourier, Grenoble. **27,3** (1977), 29–44.

[19] A. Haefliger, M. Hirsch, *On the existence and classification of differentiable embeddings*, Topology **2** (1963), 129–135.

[20] A. Haefliger, *Differentiable embeddings of S^n in S^{n+q} for $q > 2$*, Ann. of Math., 2nd Ser. Vol **83** No. 3 (1966). 402–436.

[21] A. Hatcher, *Homeomorphisms of sufficiently-large P^2-irreducible 3-manifolds*, Topology, **15** (1976)

[22] A. Hatcher, *A proof of the Smale conjecture*, Ann. of Math. **177** (1983)

[23] A. Hatcher, *Topological Moduli Spaces of Knots*, Oct, 2002. *Personal webpage at [http://www.math.cornell.edu/]*

[24] A. Hatcher, *Spaces of Knots, arXiv [math.GT/9909095]*.

[25] M. W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, 33. Springer-Verlag, New York, 1976.

[26] N. Jacobson, *Lie Algebras*, Interscience Tracts in Pure and Applied Mathematics, No. 10. Interscience Publishers (John Wiley & Sons), New York-London. 1962

[27] T. Kanenobu, *Hyperbolic links with Brunnian properties*, J. Math. Soc. Japan. Vol. **38** No. 2, (1986) 295–308.

[28] A. Kawauchi, *A Survey of Knot Theory*, Birkhauser Verlag. (1996)

[29] T. Kohno, *Vassiliev invariants and de Rham complex on the space of knots*, Contemp. Math. **179**(1994), 123-138.

[30] T. Kohno, *Loop spaces of configuration spaces and finite-type invariants*, Geometry and Topology Monographs, Vol. 4. (2001) 143–160.

[31] M. Kontsevich, *Feynman diagrams and low diemensional topology*, Progress in Mathematics, vol. 2, 1990, 120.

[32] P. Lambrechts, V. Turchin, I. Volic, *The rational homotopy type of spaces of knots in codimension* > 2, preprint.

[33] C. Lescop, *On configuration space integrals for links*, Invariants of knots and 3-manifolds (Kyoto, 2001), 183–199 (electronic), Geom. Topol. Monogr., **4**, Geom. Topol. Publ., Coventry, 2002.

[34] G. I. Lehrer, *The cohomology of a regular semisimple variety*, J. Alg. 199(1998), 666-689.

[35] G. I. Lehrer, G. B. Segal, *Homology stability for classical semisimple varieties*, Math. Zeit., **236** (2001) no. 2, 251–290.

[36] S. Manfredini, *Some subgroups of Artin's braid group*, Special issue on braid groups and related topics (Jerusalem, 1995). Topology Appl. **78** (1997), no. 1-2, 123-142.

[37] J. P. May, *The Geometry of Iterated Loop Spaces*, Lect. Notes in Math., **268**, Springer-Verlag, Berlin, 1972.

[38] J. Milnor, J. Moore, *On the structure of Hopf algebras*, Ann. of Math., (2) **81** 1965 211–264.

[39] R. Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **34** (1960) 305–312.

[40] M. Polyak, O. Viro, *On the Casson knot invariant*, Knots in Hellas '98, vol. 3(Delphi). J. Knot Theory Ramifications, **10** (2001), no. 5, 711–738.

[41] K. Sakai, *Non-trivalent graph cocycle and cohomology of the long knot space*, Algebraic & Geometric Topology **8**(2008) 1499.1522.

[42] P. Salvatore, *Knots, operads and double loop spaces.* IMRN (2006) arXiv *[math.AT/0608490]*

[43] H. Schubert, *Die eindeutige Zerlegbarkeit eines Knoten in Primknoten*, Sitzungsber. Akad. Wiss. Heidelberg, math.-nat. KI., **3:57-167**. (1949)

[44] H. Schubert, *Knoten und vollringe*, Acta Mat. **90**, 131–286 (1953)

[45] D. Sinha, *The topology of spaces of knots*, arXiv *[math.AT/0202287]*

[46] D. Sinha D. Sinha, *Operads and knot spaces*, J. Amer. Math. Soc. **19** (2006), no. 2, 461–486

[47] D. Tamaki, *A dual Rothenberg-Steenrod spectral sequence*, Topology, **33**(4)(1994), 631-662.

[48] W. Thurston, *Three-dimensional geometry and topology.* Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp. ISBN: 0-691-08304-5

[49] V. Tourtchine *On the other side of the bialgebra of chord diagrams*, J. Knot Thry. Ram. **16** (5) May 2007 575–629. arXiv.QA/0411436.

[50] V. Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, Translations of Mathematical Monographs, **98**, American Mathematical Society, Providence, RI, 1992.

[51] I. Volic, *Finite-type knot invariants and calculus of functors*, Compositio Mathematica **142** (2006) 222–250.

[52] T. Watanabe, *Configuration space integral for long n-knots, the Alexander polynomial and knot space cohomology,* Algebraic and Geometric Topology **7** (2007) 47–92. arXiv *[math.GT/0609742]*

[53] W. Wu, *On the isotopy of C^r-manifolds of dimension n in euclidean $(2n+1)$-space.* Sci. Record (N.S.) **2** (1958), 271–275.

[54] H. Whitney, *Topological properties of differentiable manifolds*, Bull. Amer. Math. Soc., **43** (1937), 785–805.

[55] H. Whitney, *The self-intersections of a smooth n-manifold in $2n$-space*, Ann. of Math. (2) **45**, 220–246 (1944).

[56] H. Whitney, *Differentiable Manifolds*, Ann. of Math. (2) **37** (1936), 645–680.

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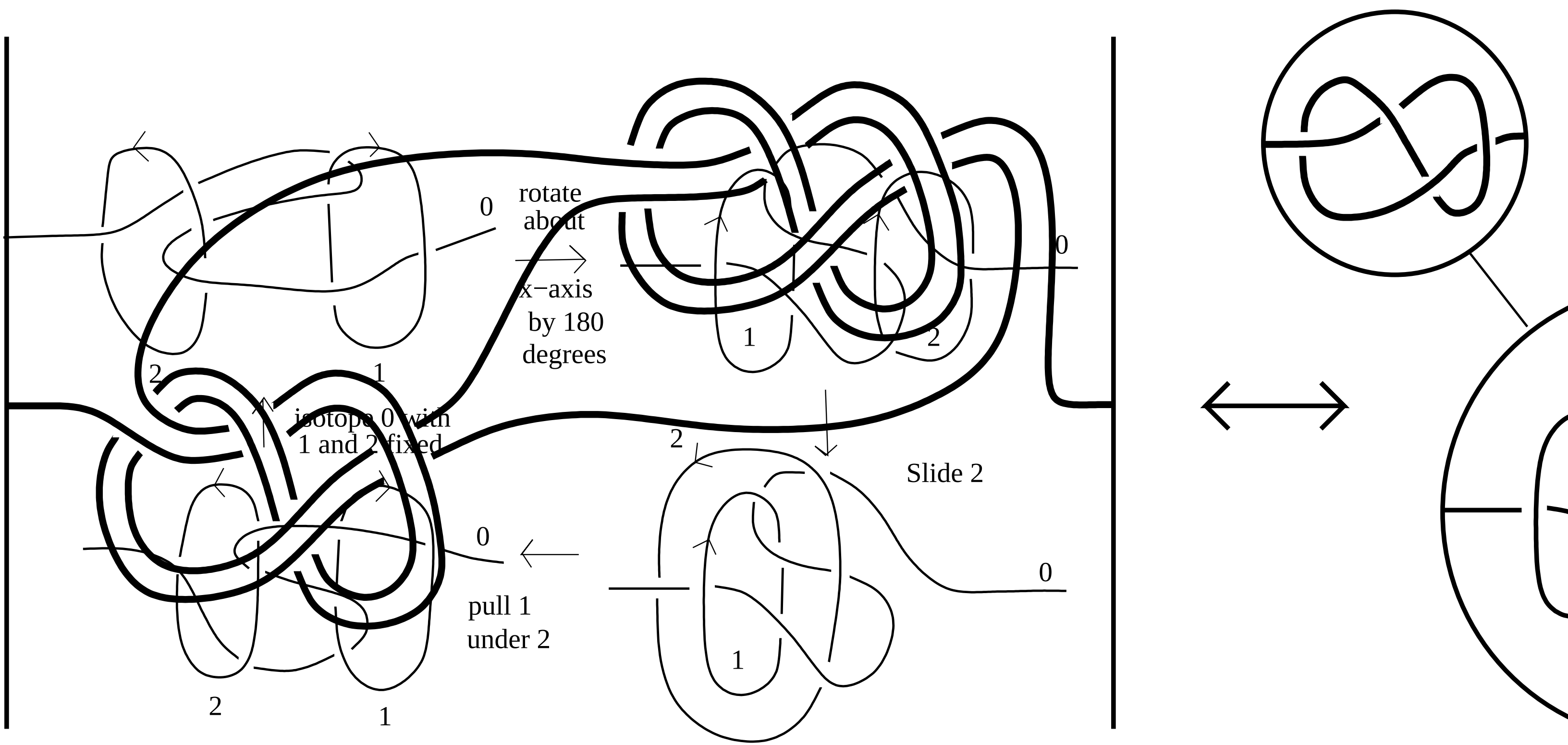

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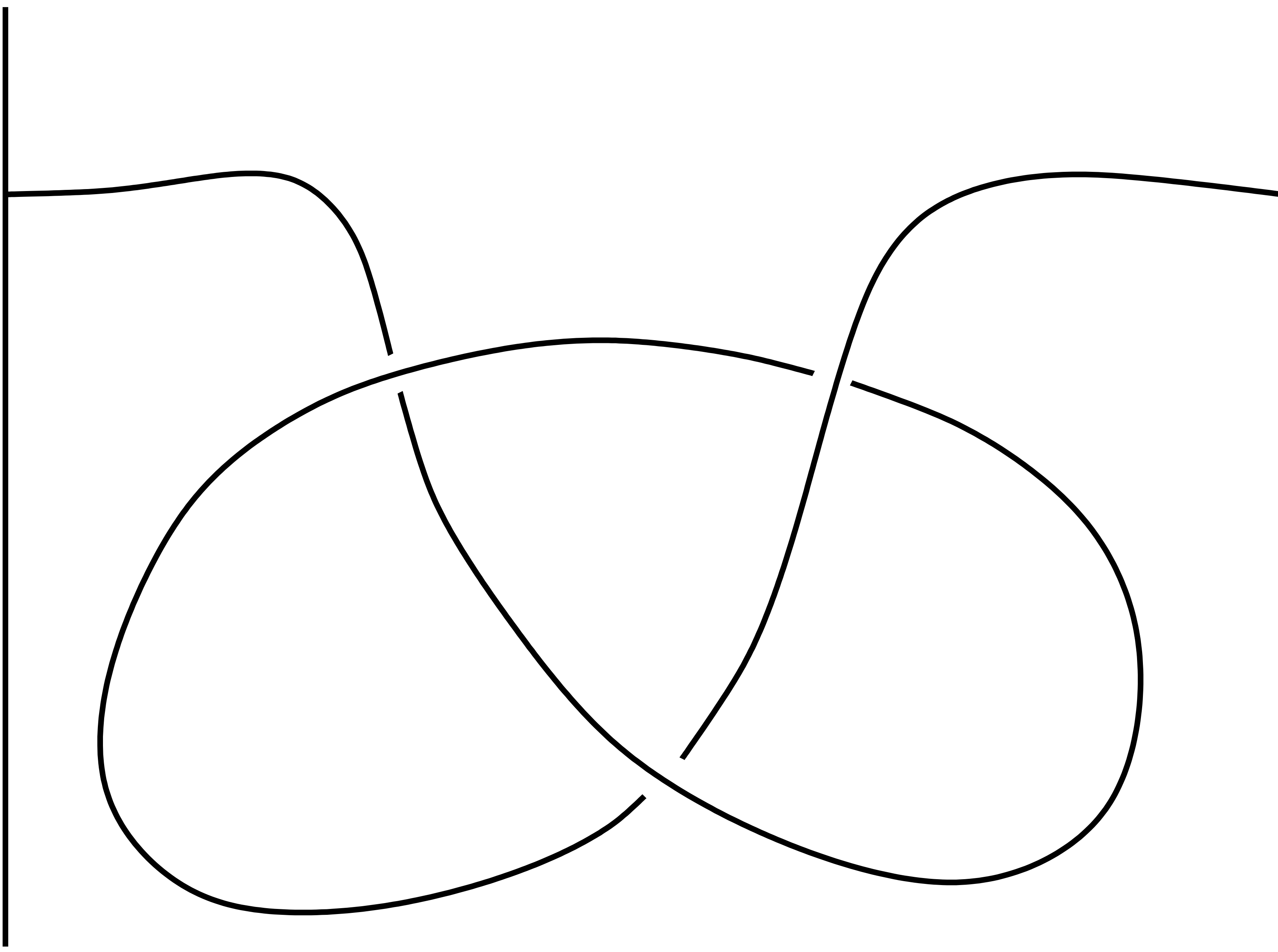

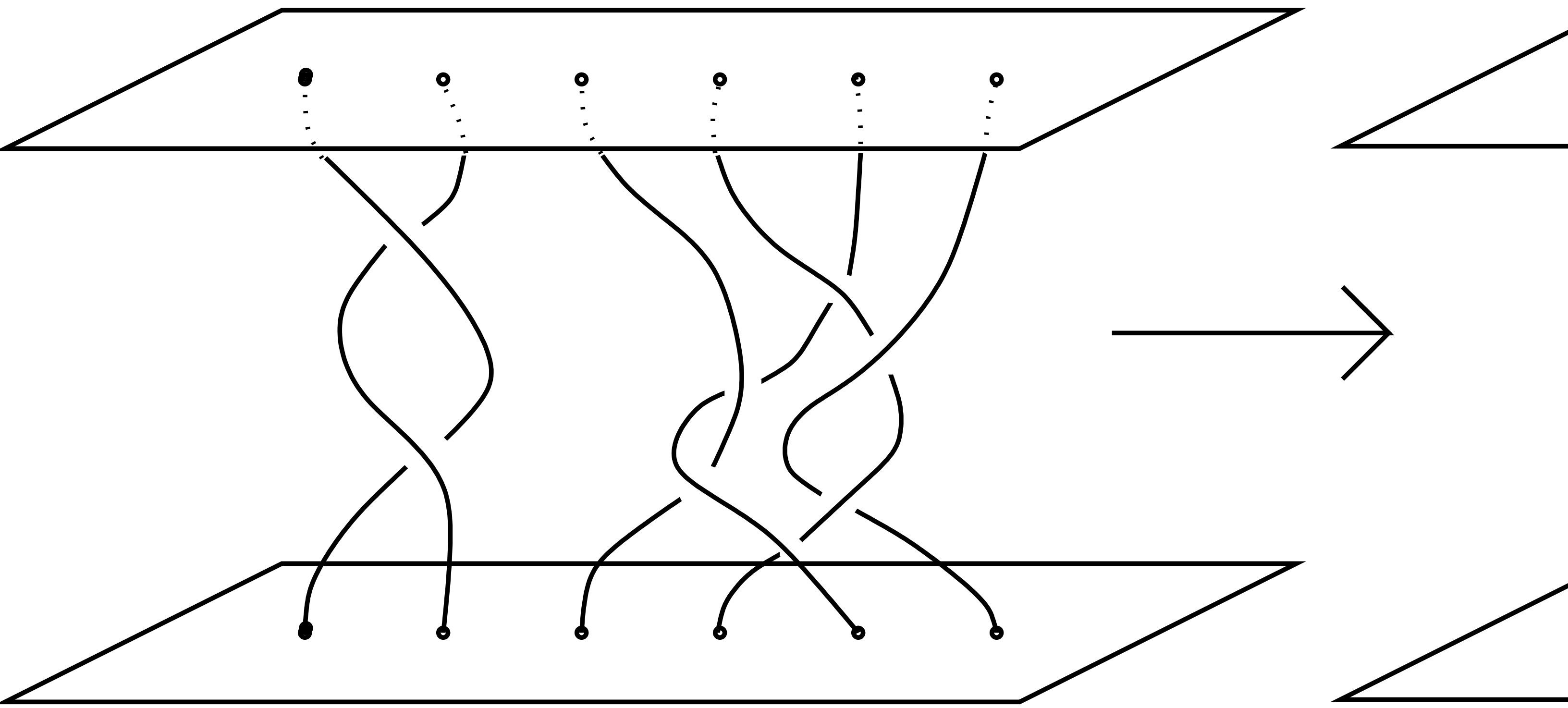

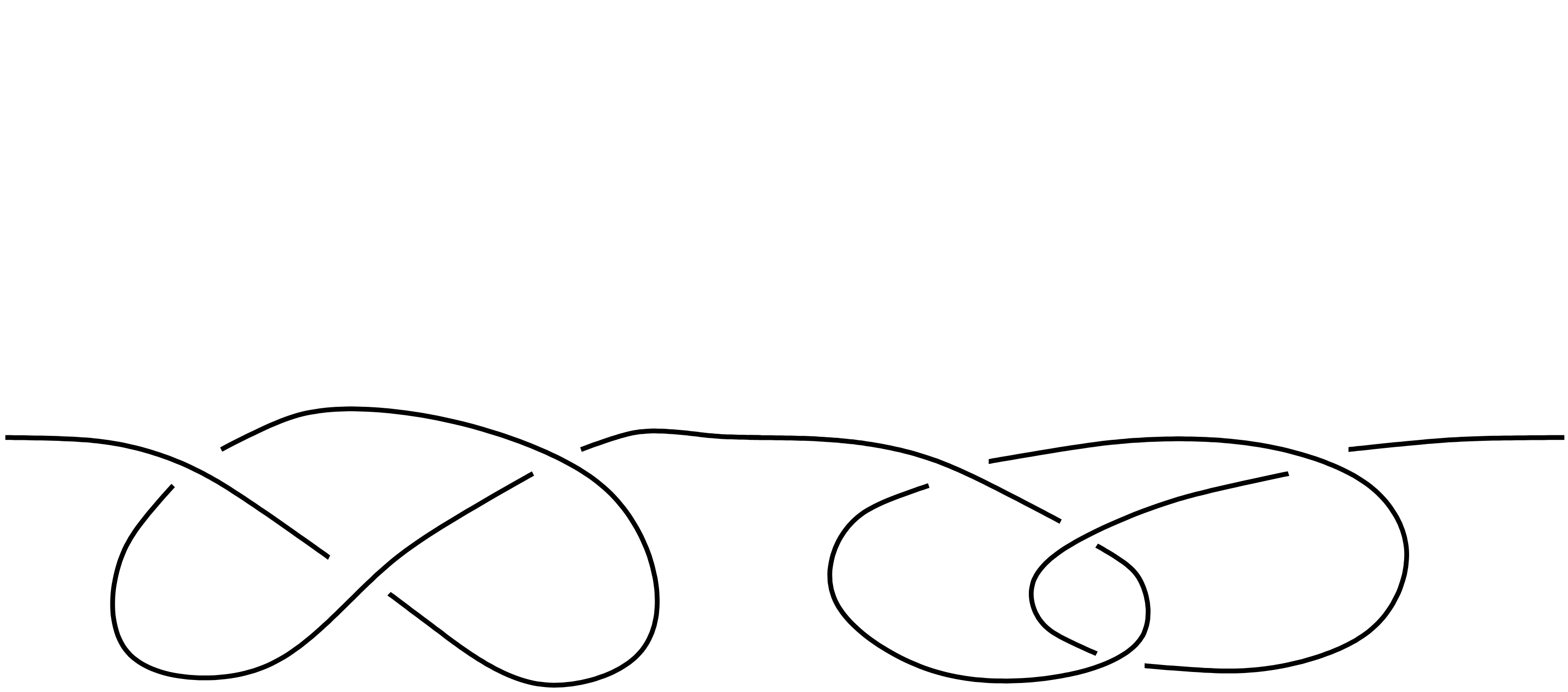

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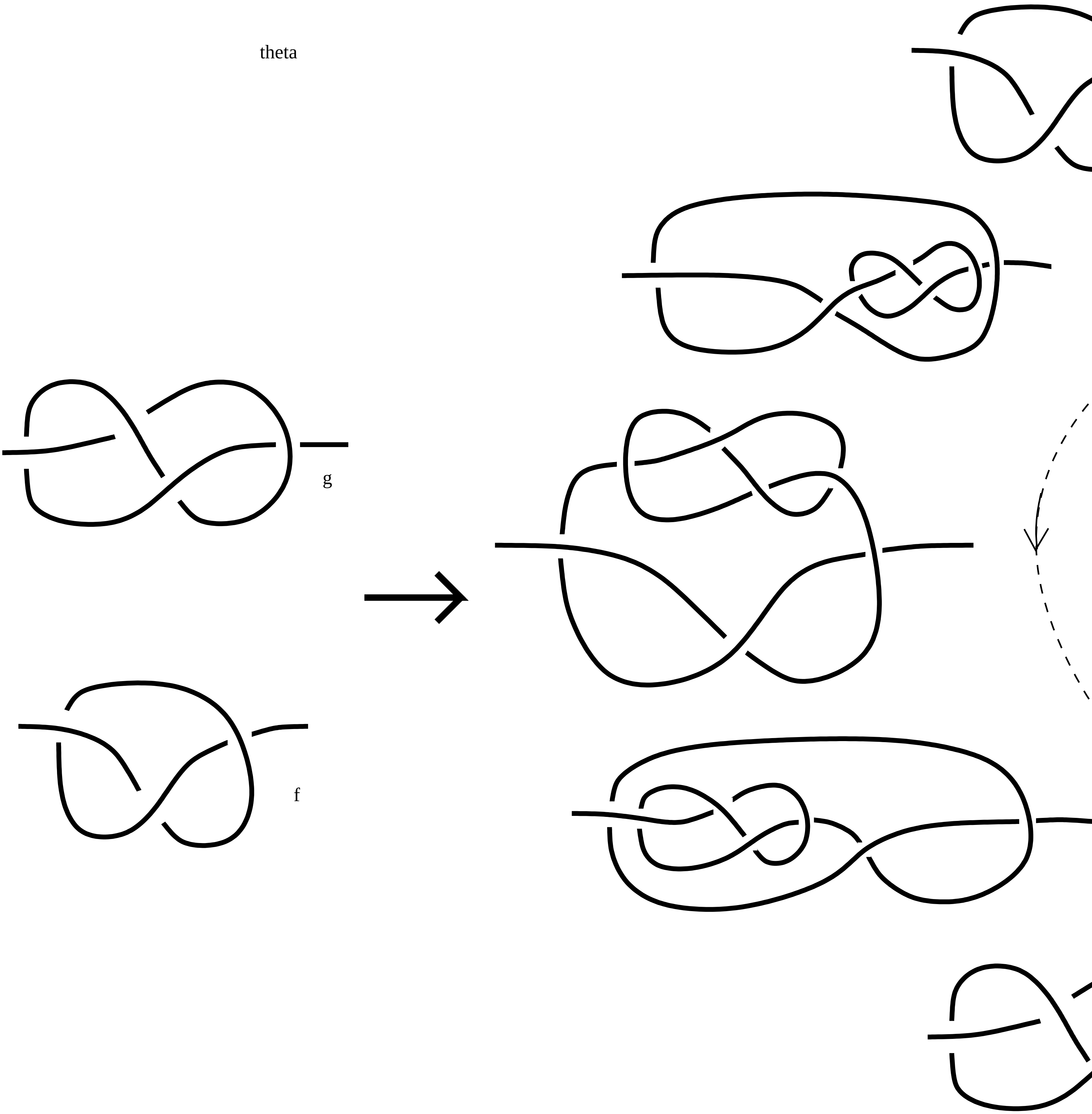
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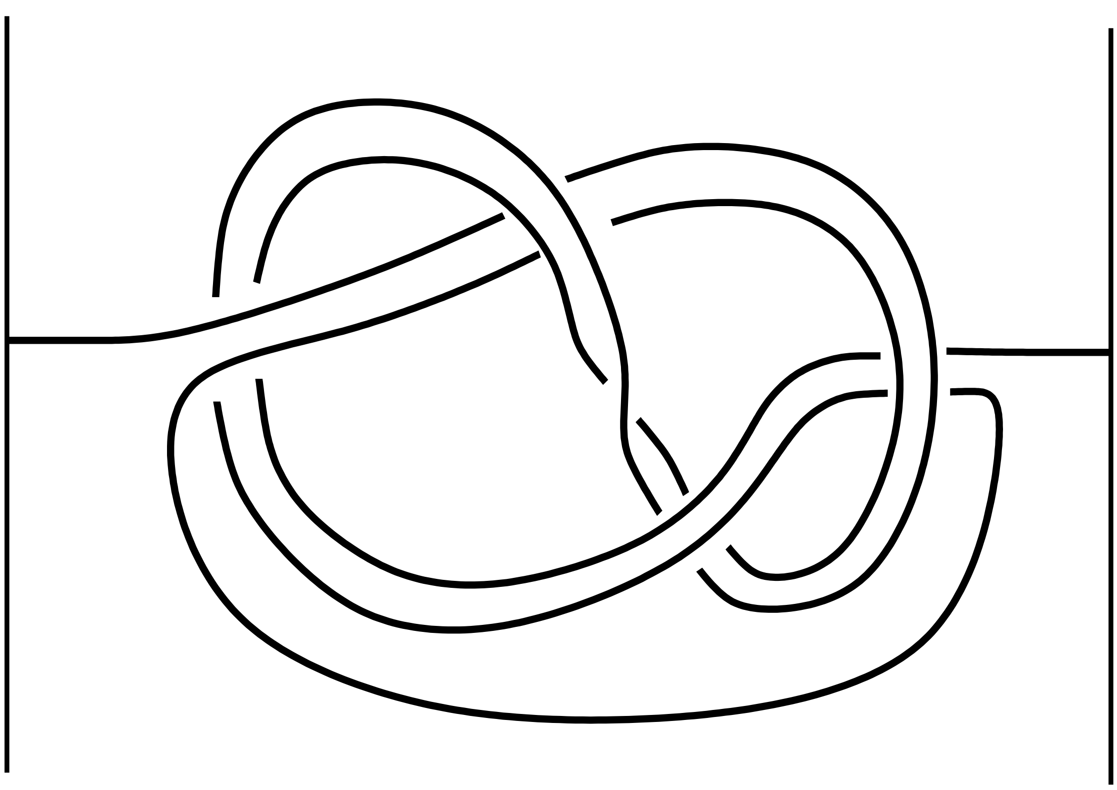

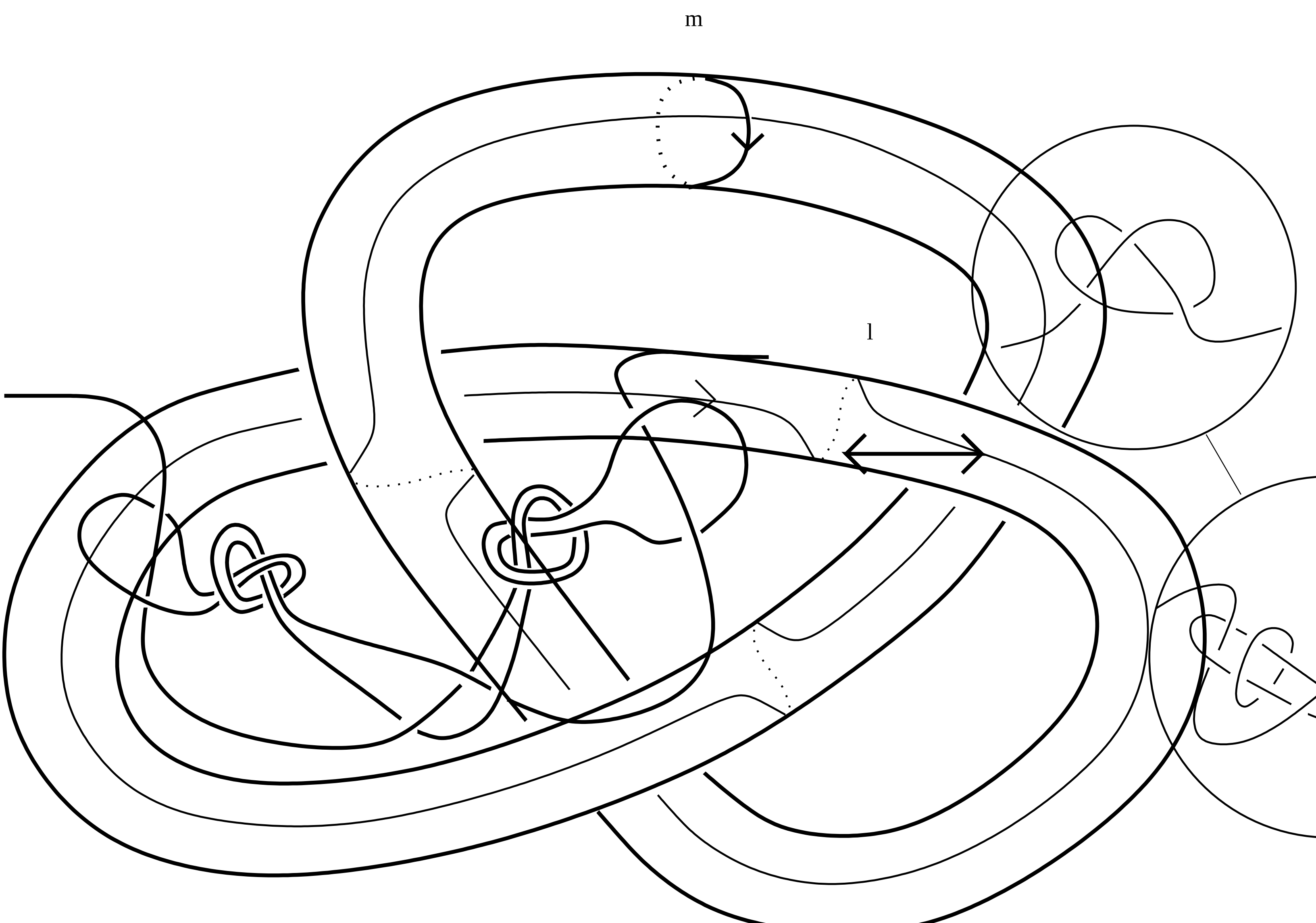

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