Kobayashi-Hitchin correspondence for tame harmonic bundles II

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Abstract

Let X be a smooth projective complex variety with an ample line bundle L, and let D be a simple normal crossing divisor. We establish the Kobayashi-Hitchin correspondence between tame harmonic bundles on $X-D$ and μ_L-stable parabolic λ-flat bundles with trivial characteristic numbers on (X,D). Especially, we obtain the quasiprojective version of the Corlette-Simpson correspondence between flat bundles and Higgs bundles.

Keywords: harmonic bundle, λ-connection, Kobayashi-Hitchin correspondence
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Contents

1 Introduction

1.1 Main results

We explain the main results in this paper. We do not recall history or background about the study of Kobayashi-Hitchin correspondence and harmonic bundles, for which we refer to the introductions of [38], [24] or [31], for example. The notion of regular filtered λ-flat bundles and parabolic λ-flat bundles are explained in Subsection 2.1. (See also Subsections 3.1–3.2 of [31]. But, we also use a slightly different notation and terminology, as is explained in Subsection 2.1.7.) They are equivalent, and we will not care about the distinction of them. The notion of filtered local systems is explained in Section 6.

1.1.1 Kobayashi-Hitchin Correspondence

Let X be a smooth complex projective variety with an ample line bundle L. Let D be a normal crossing divisor of X. Our main purpose is to show the following theorem.

Theorem 1.1 (Theorem 5.15, Proposition 2.52, Proposition 2.53) *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat bundle on (X, D). We put $E := \boldsymbol{E}_{|X-D}$. Then, the following conditions are equivalent.*

- *It is μ_L-polystable with the trivial characteristic numbers* $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$.
- *There exists a pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)$ adapted to the parabolic structure.*

Such a metric is unique up to obvious ambiguity. ∎

Remark 1.2 *The claims of Theorem* 1.1 *in the case $\lambda = 0$ has already been proved in our previous paper* [31]. *Hence, we restrict ourselves to the case $\lambda \neq 0$ in this paper.* ∎

Corollary 1.3 (Corollary 5.17) *Let $\mathcal{C}_\lambda^{poly}$ denote the category of μ_L-polystable regular filtered λ-flat bundles on (X, D) with trivial characteristic numbers. Then, we have the natural equivalence of the categories $\mathcal{C}_{\lambda_1}^{poly} \simeq \mathcal{C}_{\lambda_2}^{poly}$ for any $\lambda_i \in \boldsymbol{C}$ $(i = 1, 2)$. The equivalence preserves the tensor products, direct sums and duals.* ∎

Remark 1.4 *Let $\lambda_i \in \boldsymbol{C}^*$ $(i = 1, 2)$. A λ_2-connection $\mathbb{D}^{\lambda_2} = d'' + (\lambda_2/\lambda_1) \cdot d'$ is induced from a λ_1-connection $\mathbb{D}^{\lambda_1} = d'' + d'$. Hence we have the obvious functor* $\text{Obv} : \mathcal{C}_{\lambda_1}^{poly} \longrightarrow \mathcal{C}_{\lambda_2}^{poly}$. *But this is not the same as the above functor $\Xi_{\lambda_1,\lambda_2}$.* ∎

Especially, we obtain a generalization of the Corlette-Simpson correspondence between flat bundles and Higgs bundles in the so-called non-abelian Hodge theory.

Corollary 1.5 *We have the equivalences of the following two categories:*

- *The category of μ_L-polystable regular filtered Higgs bundles on (X, D) with trivial characteristic numbers.*
- *The category of μ_L-polystable regular filtered flat bundles on (X, D) with trivial characteristic numbers.* ∎

1.1.2 Bogomolov-Gieseker inequality and some formula for the characteristic numbers

Let X, L and D be as above.

Theorem 1.6 (Corollary 3.20) *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a μ_L-stable regular filtered λ-flat bundle on (X,D) in codimension two. Then, we have the following inequality holds for the parabolic characteristic numbers for $\boldsymbol{E}_*$:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) \le \frac{\int_X \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2\,\mathrm{rank}\,E}. \tag{1}$$

It is a generalization of the so-called Bogomolov-Gieseker inequality. ∎

In the case $\lambda \neq 0$, we also have some formulas about the parabolic Chern characteristic numbers, which are valid for any parabolic λ-flat bundles in codimension two. One of the formulas can be stated simply, after we see the correspondence of regular filtered λ-flat sheaves and filtered local systems. Let$(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf on (X,D). As is explained in Remark 1.4, we have the obvious correspondence of flat λ-connection $\mathbb{D}^\lambda = d'' + d'$ ($\lambda \neq 0$) and flat connection $\mathbb{D}^{\lambda\, f} = d'' + \lambda^{-1} d'$. In particular, we obtain the local system $\mathcal{L}$ on $X - D$ from the flat bundle $(\boldsymbol{E}_*, \mathbb{D}^{\lambda,f})_{|X-D}$. Moreover, the parabolic structure of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ induces the filtered structure of $\mathcal{L}$, and we have the more refined claims as in the following proposition.

Proposition 1.7 (Corollary 6.5 and Corollary 6.7) *Let $\widetilde{\mathcal{C}}(X,D)$ denote the category of filtered local system on (X,D), and let $\mathcal{C}_\lambda^{sat}(X,D)$ denote the category of saturated regular filtered λ-flat sheaves on (X,D) for $\lambda \neq 0$. Then, we have the equivalent functor $\Phi_\lambda : \widetilde{\mathcal{C}}(X,D) \longrightarrow \mathcal{C}_\lambda^{sat}(X,D)$ such that $\text{par-c}_1(\mathcal{L}_*) = \text{par-c}_1\big(\Phi_\lambda(\mathcal{L}_*)\big)$ and $\int_X \text{par-ch}_{2,L}(\mathcal{L}_*) = \int_X \text{par-ch}_{2,L}\big(\Phi_\lambda(\mathcal{L}_*)\big)$. The functor Φ_λ preserves the μ_L-stability.* ∎

Remark 1.8 *From Theorem* 1.6 *and Proposition* 1.7*, we obtain the Bogomolov-Gieseker inequality for μ_L-stable filtered local systems (Corollary* 6.8*). Such a kind of the inequality is discussed in* [41]*.* ∎

Remark 1.9 *Let us describe the formula $\int_X \text{par-ch}_{2,L}(\mathcal{L}_*) = \int_X \text{par-ch}_{2,L}(\Phi(\mathcal{L}_*))$ in terms of the $\boldsymbol{c}$-truncation $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)$ of saturated regular filtered λ-flat bundle $\Phi_\lambda(\mathcal{L}_*)$. For simplicity, we assume* $\dim X = 2$.

$$\begin{aligned}\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*) &= \frac{1}{2}\sum_{i\in S}\sum_{u\in\mathcal{KMS}({}_{\boldsymbol{c}}E_*,i)} \big(\mathrm{Re}(\lambda^{-1}\alpha)+a\big)^2\cdot r(i,u)\cdot(D_i,D_i)\\ &+\frac{1}{2}\sum_{i\in S}\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)\in\mathcal{KMS}({}_{\boldsymbol{c}}E_*,P)}\big(\mathrm{Re}(\lambda^{-1}\alpha_i)+a_i\big)\big(\mathrm{Re}(\lambda^{-1}\alpha_j)+a_j\big)\cdot r(P,u_i,u_j).\end{aligned} \tag{2}$$

Here, $u=(a,\alpha)$, $u_i=(a_i,\alpha_i)$ and $u_j=(a_j,\alpha_j)$ denote the KMS-spectra of $({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda)$, which are elements of $\boldsymbol{R}\times\boldsymbol{C}$. We put $r(i,u) := \mathrm{rank}\,{}^i\,\mathrm{Gr}_u^{F,\mathbb{E}}({}_{\boldsymbol{c}}E)$ for $u\in\mathcal{KMS}({}_{\boldsymbol{c}}E_*,i)$, and $r(P,u_i,u_j) := \mathrm{rank}\,{}^P\,\mathrm{Gr}_{(u_i,u_j)}^{F,\mathbb{E}}({}_{\boldsymbol{c}}E_{|P})$ for $(u_i,u_j)\in\mathcal{KMS}({}_{\boldsymbol{c}}E,P)$ and $P\in D_i\cap D_j$. And (D_i,D_j) and $\big(D_i,c_1(L)\big)$ denote the intersection numbers.

We also have some other formulas for $\int_X \text{par-ch}_2\big({}_{\boldsymbol{c}}E_*\big)$ (Proposition 3.22*) or some vanishings for the data of $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)$ at D (Corollary* 3.20 *and Proposition* 3.22*).* ∎

1.1.3 Vanishing of the characteristic numbers and existence of the Corlette-Jost-Zuo metric

Due to Proposition 1.7, we obtain the vanishings $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$, when $(\boldsymbol{E}_*, \nabla)$ corresponds to the filtered local system whose parabolic structure is trivial, in other words, $\mathrm{Re}(\alpha)+a=0$ is satisfied for any KMS-spectrum $u=(a,\alpha)\in\mathcal{KMS}(i)$ and for any $i\in S$. We can apply such a consideration to the canonical prolongation of a flat bundle due to P. Deligne [4]. Let (E,∇) be a flat bundle on $X-D$. Then, it is shown that there exists the holomorphic vector bundle $\widetilde{E}$ on X satisfying (i) $\widetilde{E}_{|X-D}=E$, (ii) $\nabla\widetilde{E}\subset\widetilde{E}\otimes\Omega_X^{1,0}(\log D)$, (iii) the real parts of the eigenvalues of $\mathrm{Res}_i(\nabla)$ are contained in $[0,1[$. In that case, we have the naturally defined parabolic structure $\boldsymbol{F}$ for which $\mathrm{Re}(\alpha)+a=0$ are satisfied for any KMS-spectrum (a,α). Hence, we obtain the vanishing $\text{par-deg}_L(\widetilde{E},\boldsymbol{F}) = \int_X \text{par-ch}_{2,L}(\widetilde{E},\boldsymbol{F}) = 0$.

This vanishing is significant to understand the existence theorem of the Corlette-Jost-Zuo metric from the view point of Kobayashi-Hitchin correspondence. When (E,∇) is semisimple, we know the existence of a tame pure imaginary pluri-harmonic metric, which we call the Corlette-Jost-Zuo metric. (See [3] for the case $D=\emptyset$ and [16] for the general case. See also [30].) Since semisimplicity obviously implies the μ_L-polystability of $(\widetilde{E},\boldsymbol{F},\nabla)$ ([35], for example), we can derive the existence of the Corlette-Jost-Zuo metric from Theorem 1.1 due to the vanishing of the characteristic numbers.

1.2 Methods and difficulty

1.2.1 Perturbation of parabolic structure

Let X be a smooth projective surface, and let D be a simple normal crossing divisor of X. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a parabolic λ-flat bundle on (X, D). For any small $\epsilon > 0$, we take an ϵ-perturbation $\boldsymbol{F}^{(\epsilon)}$ of the parabolic structure, and then $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ is *graded semisimple* (Subsection 2.1.6). It can be shown that the pseudo curvature of ordinary metrics for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ $(\epsilon > 0)$ satisfy the appropriate finiteness (Section 3). By using the theorem of Simpson, we can take a Hermitian-Einstein metric $h_{HE}^{(\epsilon)}$ of $(E_{|X-D}, \mathbb{D}^\lambda)$ which is adapted to $\boldsymbol{F}^{(\epsilon)}$ $(\epsilon > 0)$. Then, we can easily derive the Bogomolov-Gieseker inequality (Theorem 1.6). We also obtain the formulas by calculating the integrals of the characteristic numbers for pseudo curvatures, for example (2).

Let us consider the existence of a pluri-harmonic metric (Theorem 1.1). Ideally, the limit $\lim_{\epsilon \to 0} h_{HE}^{(\epsilon)}$ should give the desired pluri-harmonic metric for the given flat parabolic bundle $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$. However, it is not easy to show such a convergence. It is the main problem which we have to overcome in this paper.

1.2.2 Difficulty

In [31], we gave an argument to deal with such a convergence problem for the case $\lambda = 0$. The argument doesn't work in the case $\lambda \neq 0$. Let us explain what is the difference heuristically and imprecisely in the case $\lambda = 1$. Since we have $\text{par-deg}_L(E, \boldsymbol{F}^{(\epsilon)}) = 0$, the metrics $h_{HE}^{(\epsilon)}$ give the harmonic metrics in this case. Recall that a harmonic metric can be regarded as a harmonic map, at least locally, and that we know a well established argument for the convergence of a sequence of harmonic maps when the energies are dominated ([8]). In our case, the energies of $h_{HE}^{(\epsilon)}$ over $X - D$ are not finite, in general. Even if we consider the energies over a compact subset $Z \subset X - D$, it is not clear how to derive a uniform estimate which is independent of ϵ. On the other hand, the Higgs field is fixed for such a convergence problem in the case $\lambda = 0$. In particular, the eigenvalues of the Higgs fields are fixed. Then, we can derive the estimate of the local L^2-norm of the Higgs fields independently from ϵ. Since such L^2-norms play the role of the energies, the local convergence can be easily shown in the Higgs case, although we need some technical argument for global convergence. On the contrary, even the local convergence is not easy to show in the case $\lambda \neq 0$.

1.2.3 Convergences

To attack the problem, we discuss similar convergence problems in the curve case where the Kobayashi-Hitchin correspondence was established and well understood by the work of C. Simpson [37]. Let C be a smooth projective curve, and let D be a divisor of C. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a λ-flat stable parabolic bundle on (C, D), and let $\boldsymbol{F}^{(\epsilon)}$ be ϵ-perturbations. Note we have $\det(E, \boldsymbol{F}, \mathbb{D}^\lambda) = \det(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$. We can take a sequence of harmonic metrics $h^{(\epsilon)}$ for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ $(\epsilon \geq 0)$ such that $\det h^{(\epsilon)} = \det h^{(0)}$, due to the result of Simpson.

First, we will show that the sequence $\{h^{(\epsilon)} \,|\, \epsilon > 0\}$ converges to $h^{(0)}$. Namely, let $h_{in}^{(\epsilon)}$ $(\epsilon > 0)$ be initial metrics for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$, and let $s^{(\epsilon)}$ be the endomorphism determined by $h^{(\epsilon)} = h_{in}^{(\epsilon)} \cdot s^{(\epsilon)}$. Then, we can show the following relations:

$$M(h_{in}^{(\epsilon)}, h^{(\epsilon)}) \leq 0, \quad \big|\log s^{(\epsilon)}\big|_{h_{in}^{(\epsilon)}} \leq C_{1,\epsilon} + C_{2,\epsilon} \cdot M(h_{in}^{(\epsilon)}, h^{(\epsilon)}), \quad \big\|\mathbb{D}^\lambda s^{(\epsilon)}\big\|^2_{L^2, h_{in}^{(\epsilon)}, \omega_\epsilon} \leq \int \big|\mathrm{tr}\big(s^{(\epsilon)} \cdot G(h_{in}^{(\epsilon)})\big)\big| \,\mathrm{dvol}_{\omega_\epsilon}. \tag{3}$$

Here, $M(h_{in}^{(\epsilon)}, h^{(\epsilon)})$ denote the Donaldson functionals, and ω_ϵ denote appropriate metrics of $C - D$. Hence, if we show that $C_{i,\epsilon}$ can be taken independently from ϵ for some ω_ϵ, and if we can construct appropriate family of initial metrics $h_{in}^{(\epsilon)}$ such that $G(h_{in}^{(\epsilon)})$ are uniformly bounded with respect to ω_ϵ and $h_{in}^{(\epsilon)}$, then we obtain the L_1^2-boundedness of the family $\{s^{(\epsilon)}\}$. Then, by using a standard bootstrapping argument, we can show that the sequence $\{s^{(\epsilon)}\}$ is convergent to the identity in the C^∞-sense, i.e., $\{h^{(\epsilon)}\}$ is convergent to $h^{(0)}$ (Section 4).

Next, suppose that we are given hermitian metrics $\widetilde{h}^{(\epsilon)} := h^{(\epsilon)} \cdot \widetilde{s}^{(\epsilon)}$ for $\epsilon > 0$, with the following properties:

- $\det \widetilde{h}^{(\epsilon)} = \det h^{(\epsilon)}$.

- $\int |G(\widetilde{h}^{(\epsilon)})|^2 \longrightarrow 0$.
- $\big\|\mathbb{D}^{\lambda} s^{(\epsilon)}\big\|^2 < \infty$. (We do not need uniform bound.)

Then, we can show that $\{\widetilde{h}^{(\epsilon)}\}$ is convergent to $h^{(0)}$. (See Subsection 5.1 for more precise claims.)

We apply the above results to our convergence problem explained in Subsection 1.2.1. Due to the standard Mehta-Ramanathan type theorem (Proposition 2.9), the restriction $(E, \boldsymbol{F}, \mathbb{D}^{\lambda})_{|C}$ is also stable for almost every very ample $C \subset X$. Let h_C be a harmonic bundle of $(E, \boldsymbol{F}, \mathbb{D}^{\lambda})_{|C}$. Then, we can show that $\{h^{(\epsilon)}_{HE\,|\,C}\}$ is convergent to h_C almost everywhere on C for almost every very ample $C \subset X$, by using the above result. Therefore, we obtain a metric $h_{\mathcal{V}}$ defined almost everywhere on $X - D$ such that $h_{\mathcal{V}\,|\,C} = h_C$ almost everywhere on C for almost every curve $C \subset X$. With some more additional argument, we can show that $h_{\mathcal{V}}$ gives the desired pluri-harmonic metric, indeed (Subsection 5.2).

Remark 1.10 *Perhaps, the argument of this paper can be used in the Higgs case, to show the existence of a pluri-harmonic metric. However, we remark that the argument for a convergence given in* [31] *can be applied in a wider range. In fact, we used it to discuss the convergence of a family of harmonic bundles induced by the constant multiplication of Higgs fields.* ∎

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2 Preliminary

2.1 Generality of regular filtered λ-flat sheaf in complex geometry

The notion of a parabolic bundle, filtered bundle and their characteristic numbers are explained in Sections 3.1–3.2 of [31]. We use the notation there.

2.1.1 λ-connection

Let Y be a complex manifold, and let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Recall that a λ-connection of $\mathcal{E}$ is defined to be a map $\mathbb{D}^{\lambda} : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{1,0}_Y$ satisfying the twisted Leibniz rule $\mathbb{D}^{\lambda}(f \cdot s) = f \cdot \mathbb{D}^{\lambda}(s) + \lambda \cdot d_Y(f) \cdot s$, where f and s denote holomorphic sections of $\mathcal{O}_Y$ and $\mathcal{E}$ respectively. The maps $\mathbb{D}^{\lambda} : \mathcal{E} \otimes \Omega^{p,0} \longrightarrow \mathcal{E} \otimes \Omega^{p+1,0}$ are induced. When $\mathbb{D}^{\lambda} \circ \mathbb{D}^{\lambda} = 0$ is satisfied, it is called flat.

Let X be a complex manifold, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let $\mathcal{E}_* = \big(\mathcal{E}, \{{}^i\mathcal{F} \,\big|\, i \in S\}\big)$ be a $\boldsymbol{c}$-parabolic sheaf on (X, D) for some $\boldsymbol{c} \in \boldsymbol{R}^S$. A flat logarithmic λ-connection $\mathbb{D}^{\lambda}$ of $\mathcal{E}_*$ is defined to be a map $\mathbb{D}^{\lambda} : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{1,0}(\log D)$ satisfying the same twisted Leibniz rule as above, the flatness $\mathbb{D}^{\lambda} \circ \mathbb{D}^{\lambda} = 0$ and $\mathbb{D}^{\lambda}({}^i\mathcal{F}_a) \subset {}^i\mathcal{F}_a \otimes \Omega^{1,0}(\log D)$. Such a tuple $(\mathcal{E}_*, \mathbb{D}^{\lambda})$ will be called a regular parabolic λ-flat sheaf. When the underlying $\boldsymbol{c}$-parabolic sheaf $\mathcal{E}_*$ is a $\boldsymbol{c}$-parabolic bundle in codimension k, it is called a regular λ-flat $\boldsymbol{c}$-parabolic bundle in codimension k.

Remark 2.1 *We often omit to state "regular" in this paper, because we always assume regularity. Non-regular case is discussed in* [32]. ∎

Let $\boldsymbol{E}_* = \big(\boldsymbol{E}, \{{}_{\boldsymbol{c}}E\} \,\big|\, \boldsymbol{c} \in \boldsymbol{R}^S\big)$ be a filtered sheaf on (X, D). A regular λ-connection of $\boldsymbol{E}_*$ is a λ-connection $\mathbb{D}^{\lambda}$ of $\boldsymbol{E}$ satisfying $\mathbb{D}^{\lambda}\big({}_{\boldsymbol{c}}E\big) \subset {}_{\boldsymbol{c}}E \otimes \Omega^{1,0}_X(\log D)$. A tuple $(\boldsymbol{E}_*, \mathbb{D}^{\lambda})$ is called a regular filtered λ-flat sheaf. When the underlying filtered sheaf is a filtered bundle in codimension k, it is called a regular filtered λ-flat bundle in codimension k.

Lemma 2.2 *A regular filtered λ-flat sheaf on (X, D) is a regular filtered λ-flat bundle in codimension one.*

Proof We have only to check that there exists a subset $W \subset D$ with $\operatorname{codim}_X(W) \geq 2$, such that ${}_{\boldsymbol{c}}E_{*\,|\,X\setminus W}$ is a $\boldsymbol{c}$-parabolic bundle on $(X \setminus W, D \setminus W)$ for some $\boldsymbol{c}$. We can take W as $\bigcup_{i\neq j} D_i \cap D_j \subset W$, and hence we may assume D is smooth. Since $E = \boldsymbol{E}_{|X-D}$ is locally free and ${}_{\boldsymbol{c}}E$ is torsion-free, we can take $W' \subset D$ with $\operatorname{codim}_X(W') \geq 2$ such that ${}_{\boldsymbol{c}}E_{|X-W'}$ is locally free. We may also take a subset $W'' \subset D \setminus W'$ with $\operatorname{codim}_X(W'') \geq 2$ such that the parabolic filtration of ${}_{\boldsymbol{c}}E_{|D\setminus(W'\cup W'')}$ is filtration in the category of vector bundles. Then, $W = W' \cup W''$ gives the desired subset. ∎

When X is an n-dimensional projective variety with an ample line bundle L, we can define the μ-stability, μ-semistability, and μ-polystability of regular filtered λ-flat sheaves with respect to L, in the standard manner. "μ-stability with respect to L" will be called μ_L-stability, in this paper.

2.1.2 KMS-structure

Let X be a complex manifold, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat bundle in codimension one over (X, D). For simplicity, we consider only the case $\lambda \neq 0$. Let us take any element $\boldsymbol{c} \in \boldsymbol{R}^S$, and the $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E_*$ of $\boldsymbol{E}_*$. We would like to recall the KMS-structure at D_i, or more precisely, at the generic point of D_i. We may assume that $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)$ is a $\boldsymbol{c}$-parabolic bundle. We have the induced filtration ${}^iF$ on ${}_{\boldsymbol{c}}E_{|D_i}$. We put ${}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E) := {}^iF_a\big({}_{\boldsymbol{c}}E\big)\big/{}^iF_{<a}\big({}_{\boldsymbol{c}}E\big)$. Recall that we use the notation:

$$\mathcal{P}ar({}_{\boldsymbol{c}}E_*, i) := \big\{a \,\big|\, c_i - 1 < a \leq c_i,\ {}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E) \neq 0\big\}, \quad \mathcal{P}ar(\boldsymbol{E}_*, i) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^S} \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)$$

Due to the regularity, we have the residue endomorphism $\operatorname{Res}_i(\mathbb{D}^\lambda)$ on ${}_{\boldsymbol{c}}E_{|D_i}$, which preserves the filtration ${}^iF$, and hence we have the induced endomorphism $\operatorname{Gr}^F\operatorname{Res}_i(\mathbb{D}^\lambda)$ of ${}^i\operatorname{Gr}^F\big({}_{\boldsymbol{c}}E\big)$. We remark that the eigenvalues of $\operatorname{Res}_i(\mathbb{D}^\lambda)$ are constant on D_i. In particular, we obtain the generalized eigen decomposition:

$${}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E) = \bigoplus_{\alpha\in\boldsymbol{C}} {}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E).$$

We put $\mathcal{KMS}\big({}_{\boldsymbol{c}}E_*, i\big) := \big\{(a,\alpha) \in]c_i - 1, c_i] \times \boldsymbol{C} \,\big|\, {}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i}) \neq 0\big\}$. Each element of $\mathcal{KMS}\big({}_{\boldsymbol{c}}E_*, i\big)$ or $\mathcal{KMS}\big(\boldsymbol{E}_*, i\big) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^S}\mathcal{KMS}\big({}_{\boldsymbol{c}}E_*, i\big)$ is called a KMS-spectrum.

2.1.3 Prolongment of flat subbundle and Mehta-Ramanathan type theorem

To begin with, we recall a well known fact about regular singularity of a connection.

Lemma 2.3 *Let E be a holomorphic bundle on a disc Δ, and let ∇ be a logarithmic connection of E on (Δ, O), i.e., $\nabla(E) \subset E \otimes \Omega^{1,0}_\Delta(\log O)$. Let f be a flat section of $E_{|\Delta^*}$. Then, f naturally gives a meromorphic section of E.* ∎

Corollary 2.4 *We put $X = \Delta_z \times \Delta^n_w$ and $D = \{0\} \times \Delta^n_w$. Let E be a holomorphic vector bundle on X and ∇ be the logarithmic connection of E on (X, D). Let e be a flat section of $E_{|X-D}$.*

- *e gives a meromorphic section of E.*
- *Assume that e is holomorphic on E and that $e_{|Q} \neq 0$ for some $Q \in D$. Then, $e_{|Q'} \neq 0$ for any $Q' \in D$.*

Proof We may assume that we have a holomorphic frame $\boldsymbol{v}$ of E. We have the expression $e = \sum f_i(z,w)\cdot v_i$. When we fix w, then $f_i(z,w)$ are meromorphic with respect to z. Thus, we have the least integer $j(w)$ such that the orders of the poles of $f_i(z,w)$ are less than $j(w)$. We put $\mathcal{S}_j := \{w \,|\, j(w) \leq j\}$. We have $D = \bigcup_j \mathcal{S}_j$. If $\mathcal{S}_j \neq D$, the measure of $\mathcal{S}_j$ is 0. Hence, we obtain $\mathcal{S}_j = D$ for some j, which means e is meromorphic. Thus, we obtain the first claim.

Assume that e is holomorphic and that $e_{|Q} \neq 0$ for some $Q \in D$. Recall that we have the induced connection ${}^D\nabla$ of $E_{|D}$. Namely, for any holomorphic section $f \in E_{|D}$, take a holomorphic $F \in E$ such that $F_{|D} = f$, and then ${}^D\nabla(f) := \nabla(F)_{|D}$ is well defined. Since we have ${}^D\nabla(e_{|D}) = 0$, we obtain the second claim. ∎

Corollary 2.5 *We put $X = \Delta^n$, $D_i = \{z_i = 0\}$ and $D = \bigcup_{i=1}^n D_i$. Let (E, ∇) be a logarithmic connection on (X, D), and let e be a flat section on $X - D$.*

- *e gives a meromorphic section of E.*
- *Assume that e is holomorphic. We put $D_i^\circ := D_i \setminus \bigcup_{j \neq i} D_j$. If $e_{|Q} \neq 0$ for some $Q \in D_i^\circ$, we have $e_{|Q'} \neq 0$ for any $Q' \in D_i^\circ$.* ∎

Let X be a complex manifold, and let D be a normal crossing divisor of X. Let (E, ∇) be a flat bundle on $X - D$. Recall that P. Deligne gave the extension $\widetilde{E}$ of E in [4], such that (i) $\widetilde{E}_{|X-D} = E$, (ii) $\nabla(\widetilde{E}) \subset \widetilde{E} \otimes \Omega^{1,0}(\log D)$, (iii) the real parts of the eigenvalues of $\mathrm{Res}_i(\nabla)$ are contained in $\{0 \leq t < 1\}$. Such an extension is unique, or in other words, it is unique as the subsheaf of $\iota_* E$, where ι denotes the inclusion $X - D \longrightarrow X$. The prolongment can also be done for λ-flat bundle $(E, \mathbb{D}^\lambda)$ on $X - D$, or more precisely, for the associated flat bundle $(E, \mathbb{D}^{\lambda\, f})$.

Lemma 2.6 *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat bundle on (X, D), and we put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Let $(\widetilde{E}, \mathbb{D}^\lambda)$ be the Deligne extension of $(E, \mathbb{D}^\lambda)$. Then, we have $\boldsymbol{E} = \widetilde{E} \otimes \mathcal{O}_X(*D)$, where $\mathcal{O}_X(*D)$ denotes the sheaf of meromorphic functions on X whose poles are contained in D.*

Proof We have the naturally defined flat section s on $Hom({}_{\boldsymbol{c}}E, \widetilde{E})_{|X-D}$. Due to Corollary 2.5, s is a meromorphic section, and hence we obtain the flat inclusion ${}_{\boldsymbol{c}}E \longrightarrow \widetilde{E} \otimes \mathcal{O}(N \cdot D)$ for some large integer N, which induce the morphism $\boldsymbol{E} = \bigcup {}_{\boldsymbol{c}}E = {}_{\boldsymbol{c}}E \otimes \mathcal{O}(*D) \longrightarrow \widetilde{E} \otimes \mathcal{O}(*D)$. Similarly, we obtain the inclusion $\widetilde{E} \longrightarrow {}_{\boldsymbol{c}}E \otimes \mathcal{O}(N \cdot D)$, and $\widetilde{E} \otimes \mathcal{O}(*D) \longrightarrow \boldsymbol{E}$. They are clearly mutually inverse. ∎

Lemma 2.7 *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf on (X, D), and let $(\widetilde{E}, \mathbb{D}^\lambda)$ be as in the previous lemma. Then, we have $\boldsymbol{E} \simeq \widetilde{E} \otimes \mathcal{O}(*D)$ naturally.*

Proof Due to Lemma 2.2 and Lemma 2.6, there exists a subset $W \subset D$ with $\mathrm{codim}_X(W) \geq 2$ such that $\boldsymbol{E}_{|X-W} \simeq \widetilde{E} \otimes \mathcal{O}(*D)_{|X-W}$. Let us fix $\boldsymbol{c}$. There exists a large integer N such that we have ${}_{\boldsymbol{c}}E_{|X-W} \subset \widetilde{E} \otimes \mathcal{O}(N \cdot D)_{|X-W}$. Since $\widetilde{E}$ is locally free, we obtain ${}_{\boldsymbol{c}}E \subset \widetilde{E} \otimes \mathcal{O}(N \cdot D)$, and thus $\boldsymbol{E} \subset \widetilde{E} \otimes \mathcal{O}(*D)$. On the other hand, there exists a large integer N' such that $\widetilde{E}_{|X-W} \subset {}_{\boldsymbol{c}}E \otimes \mathcal{O}(N' \cdot D)_{|X-W}$. Hence, $\widetilde{E} \subset {}_{\boldsymbol{c}}E^{\vee\vee} \otimes \mathcal{O}(N' \cdot D)$, where ${}_{\boldsymbol{c}}E^{\vee\vee}$ denotes the double dual of ${}_{\boldsymbol{c}}E$. Hence, we obtain $\widetilde{E} \otimes \mathcal{O}(*D) \subset {}_{\boldsymbol{c}}E^{\vee\vee} \otimes \mathcal{O}(*D)$. It is easy to see ${}_{\boldsymbol{c}}E^{\vee\vee} \otimes \mathcal{O}(*D) \simeq {}_{\boldsymbol{c}}E \otimes \mathcal{O}(*D)$. Thus we are done. ∎

Lemma 2.8 *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf on (X, D), and we put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Let E' be a λ-flat subbundle of E. Then, we have the corresponding regular filtered λ-flat subsheaf $\boldsymbol{E}'_* \subset \boldsymbol{E}_*$ such that ${}_{\boldsymbol{c}}E'$ are saturated in ${}_{\boldsymbol{c}}E$.*

Proof Let $\widetilde{E}$ denote the Deligne extension of $(E, \mathbb{D}^\lambda)$. We have the corresponding subbundle $\widetilde{E}' \subset \widetilde{E}$. Therefore, we obtain $\widetilde{\boldsymbol{E}}' := \widetilde{E}' \otimes \mathcal{O}(*D) \subset \widetilde{E} \otimes \mathcal{O}(*D) = \boldsymbol{E}$. For each $\boldsymbol{c}$, the $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E'$ is given by the intersection of ${}_{\boldsymbol{c}}E$ and $\boldsymbol{E}'$ in $\boldsymbol{E}$. Or equivalently, ${}_{\boldsymbol{c}}E'$ can be given by the intersection of ${}_{\boldsymbol{c}}E$ and $\widetilde{E'}(N \cdot D)$ in $\widetilde{E}(N \cdot D)$ for sufficiently large N. Thus, we obtain $\boldsymbol{E}'_* \subset \boldsymbol{E}_*$. ∎

Let us show the Mehta-Ramanathan type theorem for regular filtered λ-flat sheaves. Let X be a smooth projective variety with an ample line bundle L and a simple normal crossing divisor D. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf on (X, D). Let N be a sufficiently large number. We can take a generic hyper-plane section Y of $L^{\otimes N}$ satisfying the properties: (i) $D_Y := Y \cap D$ is simply normal crossing in Y, (ii) $\pi_1(Y \setminus D) \longrightarrow \pi_1(X \setminus D)$ is surjective.

Proposition 2.9 *Assume $\dim X \geq 2$. Then, $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is μ_L-stable, if and only if $(\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|Y}$ is μ_L-stable.*

Proof Let us fix $\boldsymbol{c}$. If $W \subset {}_{\boldsymbol{c}}E$ destabilizes, the restriction $W_{|Y}$ clearly destabilizes. Hence, the μ_L-stability of $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)_{|Y}$ implies the μ_L-stability of $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)$. Assume that $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)_{|Y}$ is not μ_L-stable, and let W be a subsheaf of ${}_{\boldsymbol{c}}E_{|Y}$ satisfying $\mathbb{D}^\lambda(W) \subset W \otimes \Omega_Y^{1,0}(\log D_Y)$ and $\text{par-deg}(W_*)/\operatorname{rank}(W) \geq \text{par-deg}({}_{\boldsymbol{c}}E_*)/\operatorname{rank} E$.

Let Q be any point of $X-D$. Take a path γ connecting Q and a point P of $Y\setminus D$. By the parallel transport along the path, we obtain the vector subspace $W'_Q\subset E_{|Q}$. It is independent of choices of P and γ, and we obtain the flat subbundle $W'\subset {}_{\boldsymbol{c}}E_{|X-D}$. Due to Lemma 2.8, we obtain the saturated subsheaf $\widetilde{W}'\subset {}_{\boldsymbol{c}}E$. By a general argument, it can be shown that there exists a subset $Z\subset D$ with $\operatorname{codim}_X(Z)\geq 2$ such that $\widetilde{W}'_{*|X-Z}$ is a parabolic subbundle of ${}_{\boldsymbol{c}}E_{|X-Z}$. Then, it is easy to check that $\widetilde{W}'$ destabilizes. ∎

2.1.4 Saturated regular filtered λ-flat sheaf

Let X and D be as above. Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf ($\lambda\neq 0$).

Definition 2.10 $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ *is called saturated, if the following conditions are satisfied:*

- *There exists a subset $Z\subset D$ with $\operatorname{codim}_X(Z)\geq 2$, and each ${}_{\boldsymbol{a}}E$ are determined on ${}_{\boldsymbol{a}}E_{|X-Z}$. Namely, for any open subset $U\subset X$, we have the following:*

$$ {}_{\boldsymbol{a}}E(U)={}_{\boldsymbol{a}}E(U\setminus Z)\cap\boldsymbol{E}(U). \tag{4}$$

∎

It is easy to see that a regular filtered λ-flat *bundle* is saturated.

Lemma 2.11 *Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a saturated regular filtered λ-sheaf on (X,D). Then, each $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E$ is reflexive.*

Proof Recall we have already known that ${}_{\boldsymbol{c}}\boldsymbol{E}_*$ is a filtered bundle in codimension one (Lemma 2.2). Let ${}_{\boldsymbol{c}}E^{\vee\vee}$ denote the double dual of ${}_{\boldsymbol{c}}E$. We have the naturally defined injective map ${}_{\boldsymbol{c}}E\longrightarrow{}_{\boldsymbol{c}}E^{\vee\vee}$. Due to the saturatedness, any sections of ${}_{\boldsymbol{c}}E^{\vee\vee}$ naturally gives sections of ${}_{\boldsymbol{c}}E$, i.e., ${}_{\boldsymbol{c}}E$ is isomorphic to ${}_{\boldsymbol{c}}E^{\vee\vee}$. ∎

Lemma 2.12 *A saturated regular filtered λ-flat sheaf $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ on (X,D) is a regular filtered λ-flat bundle in codimension two.*

Proof We have only to show that there exists a subset $Z\subset D$ with $\operatorname{codim}_X(Z)\geq 3$ such that ${}_{\boldsymbol{c}}E_{*\,|\,X-Z}$ is a $\boldsymbol{c}$-parabolic bundle on $(X-Z,D-Z)$ for any $\boldsymbol{c}$. Due to ${}_{\boldsymbol{c}+\boldsymbol{b}}E={}_{\boldsymbol{c}}E\otimes\mathcal{O}(\boldsymbol{b}\cdot D)$, where $\boldsymbol{b}\cdot D=\sum_{i\in S}b_i\cdot D_i$, we have only to show such a claim for finite number of tuples $\boldsymbol{c}$. Due to Lemma 2.11, there exists a subset $Z'\subset D$ with $\operatorname{codim}_X(Z')\geq 3$ such that ${}_{\boldsymbol{c}}E_{|X-Z'}$ is locally free. Hence, we can assume that ${}_{\boldsymbol{c}}E$ is locally free from the beginning.

We have the parabolic filtration ${}^iF=\{{}^iF_a\,|\,c_i-1<a\leq c_i\}$ of ${}_{\boldsymbol{c}}E_{|D_i}$. We can take the saturation ${}^i\widetilde{F}_a$ of ${}^iF_a$. Namely, we put $G_a:={}_{\boldsymbol{c}}E_{|D_i}\big/{}^iF_a$, and let $G_{a\,tor}$ denote the torsion-part of G_a. Let $\pi_a:{}_{\boldsymbol{c}}E_{|D_i}\longrightarrow G_a$ denote the projection, and we put ${}^i\widetilde{F}_a:=\pi_a^{-1}\big(G_{a\,tor}\big)$.

Lemma 2.13 ${}^i\widetilde{F}_a={}^iF_a$.

Proof By our construction, we have ${}^iF_a\subset{}^i\widetilde{F}_a$, and we also know that there exists a subset $W\subset D_i$ with $\operatorname{codim}_{D_i}(W)\geq 1$ such that ${}^iF_{a\,|\,D_i-W}={}^i\widetilde{F}_{a\,|\,D_i-W}$.

Let P be any point of D_i. Let g be a germ of a section of ${}^i\widetilde{F}_a$ at P, and let G be a local section of ${}_{\boldsymbol{c}}E$ on an open subset $U$of P in X such that the germ of the restriction of G to D_i gives g. Then, $G_{|U\setminus W}$ gives a section of ${}_{\boldsymbol{c}'}E$ on $U\setminus W$, where $\boldsymbol{c}'=(c'_j)$ is determined by $c'_j=c_j$ $(j\neq i)$ and $c_i=a$. Due to the saturatedness, G is a section of ${}_{\boldsymbol{c}'}E$ on U. Thus, g is the germ of a section of ${}^iF_a$, and ${}^iF_a={}^i\widetilde{F}_a$. Hence, we obtain Lemma 2.13. ∎

Let us return to the proof of Lemma 2.12. Due to Lemma 2.13, the associated graded vector bundle ${}^i\operatorname{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$ is torsion free. Hence, there exists a subset $Z''_i\subset D_i$ with $\operatorname{codim}_{D_i}Z''_i\geq 2$ such that ${}^iF_{|D_i\setminus Z''_i}$ is a filtration in the category of vector bundles on $D''_i\setminus Z''_i$. Then, ${}_{\boldsymbol{c}}E_{*\,|\,X-Z''}$ is a $\boldsymbol{c}$-parabolic locally free sheaf on $(X-Z'',D-Z'')$. Thus we are done. ∎

Remark 2.14 *By the correspondence of saturated regular filtered flat bundles and filtered local systems, we can obtain more concrete picture of the saturated regular filtered flat sheaves. We will see it in Section* 6. ∎

2.1.5 Canonical decomposition

Let $\big(\mathcal{E}^{(i)}_*, \mathbb{D}^{\lambda(i)}\big)$ $(i=1,2)$ be μ_L-semistable regular $\boldsymbol{c}$-parabolic λ-flat sheaves such that $\mu_L(\mathcal{E}^{(1)}_*)=\mu_L(\mathcal{E}^{(2)}_*)$. Let $f:(\mathcal{E}^{(1)}_*, \mathbb{D}^{\lambda\,(1)}) \longrightarrow (\mathcal{E}^{(2)}_*, \mathbb{D}^{\lambda\,(2)})$ be a non-trivial morphism. Let $(\mathcal{K}_*, \mathbb{D}^{\lambda}_{\mathcal{K}})$ denote the kernel of f, which is naturally equipped with the parabolic structure and the flat λ-connection. Let $\mathcal{I}$ denote the image of f, and $\widetilde{\mathcal{I}}$ denote the saturated subsheaf of $\mathcal{E}^{(2)}$ generated by $\mathcal{I}$. The parabolic structures of $\mathcal{E}^{(1)}_*$ and $\mathcal{E}^{(2)}_*$ induce the parabolic structures of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$, respectively. We denote the induced parabolic flat sheaves by $(\mathcal{I}_*, \mathbb{D}^{\lambda}_{\mathcal{I}})$ and $(\widetilde{\mathcal{I}}_*, \mathbb{D}^{\lambda}_{\widetilde{\mathcal{I}}})$. The following lemma can be shown by the same argument as the proof of Lemma 3.9 of [31].

Lemma 2.15 *$(\mathcal{K}_*, \mathbb{D}^{\lambda}_{\mathcal{K}})$, $(\mathcal{I}_*, \mathbb{D}^{\lambda}_{\mathcal{I}})$ and $\big(\widetilde{\mathcal{I}}_*, \mathbb{D}^{\lambda}_{\widetilde{\mathcal{I}}}\big)$ are also μ_L-semistable such that $\mu_L(\mathcal{K}_*)=\mu_L(\mathcal{I}_*)=\mu_L(\widetilde{\mathcal{I}}_*)=\mu_L(\mathcal{E}^{(i)}_*)$. Moreover, $\mathcal{I}_*$ and $\widetilde{\mathcal{I}}_*$ are isomorphic in codimension one.* ∎

Lemma 2.16 *Let $\big(\mathcal{E}^{(i)}_*, \mathbb{D}^{\lambda\,(i)}\big)$ $(i=1,2)$ be μ_L-semistable reflexive saturated regular parabolic λ-flat sheaves such that $\mu_L(\mathcal{E}^{(1)}_*)=\mu_L(\mathcal{E}^{(2)}_*)$. Assume either one of the following:*

1. *One of $(\mathcal{E}^{(i)}_*, \mathbb{D}^{\lambda\,(i)})$ is μ_L-stable, and $\mathrm{rank}(\mathcal{E}^{(1)})=\mathrm{rank}(\mathcal{E}^{(2)})$ holds.*
2. *Both of $(\mathcal{E}^{(i)}_*, \mathbb{D}^{\lambda\,(i)})$ are μ_L-stable.*

If there is a non-trivial map $f:(\mathcal{E}^{(1)}_*, \mathbb{D}^{\lambda\,(1)}) \longrightarrow (\mathcal{E}^{(2)}_*, \mathbb{D}^{\lambda\,(2)})$, then f is isomorphic.

Proof If $(\mathcal{E}^{(1)}_*, \mathbb{D}^{\lambda\,(1)})$ is μ_L-stable, the kernel of f is trivial due to Lemma 2.15. If $(\mathcal{E}^{(2)}_*, \mathbb{D}^{\lambda\,(2)})$ is μ_L-stable, the image of f and $\mathcal{E}^{(2)}$ are the same at the generic point of X. Thus, we obtain that f is generically isomorphic in any case. Then, we obtain that f is isomorphic in codimension one, due to Lemma 3.7 of [31]. Since both of $\mathcal{E}^{(i)}_*$ are reflexive and saturated, we obtain that f is isomorphic. ∎

Corollary 2.17 *Let $(\mathcal{E}_*, \mathbb{D}^{\lambda})$ be a μ_L-polystable reflexive saturated regular parabolic λ-flat sheaf. Then, we have the unique decomposition:*

$$(\mathcal{E}_*, \mathbb{D}^{\lambda}) = \bigoplus_j \big(\mathcal{E}^{(j)}_*, \mathbb{D}^{\lambda\,(j)}\big) \otimes \boldsymbol{C}^{m(j)}.$$

Here, $(\mathcal{E}^{(j)}_*, \mathbb{D}^{\lambda\,(j)})$ are μ_L-stable with $\mu_L(\mathcal{E}^{(j)}_*)=\mu(\mathcal{E}_*)$, and they are mutually non-isomorphic. It is called the canonical decomposition in the rest of the paper. ∎

2.1.6 Perturbation of parabolic structure

Let X be a smooth projective *surface* with an ample line bundle L, and D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S} D_i$. Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^{\lambda})$ be a regular $\boldsymbol{c}$-parabolic λ-flat bundle over (X,D) for some $\boldsymbol{c}\in \boldsymbol{R}^S$. Assume $\lambda\neq 0$. We also assume $c_i \notin \mathcal{P}ar({}_{\boldsymbol{c}}E, \boldsymbol{F}, i)$ for each $i\in S$, for simplicity. Let $\mathcal{N}_i$ denote the nilpotent part of the induced endomorphism $\mathrm{Gr}^F \mathrm{Res}_i(\mathbb{D}^{\lambda})$ on ${}^{i}\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$. Before proceeding, we give a definition of graded semisimplicity, as in the Higgs case.

Definition 2.18 *The λ-flat $\boldsymbol{c}$-parabolic bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^{\lambda})$ is called graded semisimple, if the nilpotent parts $\mathcal{N}_i$ are 0 for any $i\in S$.* ∎

We would like to consider perturbation of parabolic structure, as in Subsection 3.4 of [31]. First, we will recall general construction. Then, we will give two kinds of perturbations.

Let η be a generic point of D_i. We have the weight filtration W_η of the nilpotent map $\mathcal{N}_{i,\eta}$ on ${}^{i}\mathrm{Gr}^F\big({}_{\boldsymbol{c}}E\big)_\eta$, which is indexed by $\mathbb{Z}$. Then, we can extend it to the filtration W of ${}^{i}\mathrm{Gr}^F\big({}_{\boldsymbol{c}}E\big)$ in the category of vector bundles on D_i due to $\dim D_i=1$. By our construction, $\mathcal{N}_i(W_k)\subset W_{k-2}$. The endomorphism $\mathrm{Res}_i(\mathbb{D}^{\lambda})$ preserves the filtration W on ${}^{i}\mathrm{Gr}^F({}_{\boldsymbol{c}}E)$, and the nilpotent part of the induced endomorphisms on $\mathrm{Gr}^W\big({}^{i}\mathrm{Gr}^F({}_{\boldsymbol{c}}E)\big)$ are trivial. Recall that the flat λ-connection $\mathbb{D}^{\lambda}$ *locally* induces the λ-connection ${}^{i}\mathbb{D}^{\lambda}$ of the vector bundle ${}_{\boldsymbol{c}}E_{|D_i}$ on D_i. Since ${}^{i}\mathrm{Gr}^F({}^{i}\mathbb{D}^{\lambda})$ commutes with $\mathrm{Res}_i\,\mathbb{D}^{\lambda}$, it preserves the filtration W.

Let us take the refinement of the filtration ${}^iF$. For any $a \in]c_i-1, c_i]$, we have the surjection $\pi_a : {}^iF_a({}_{\boldsymbol{c}}E_{|D_i}) \longrightarrow {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$. We put ${}^i\widetilde{F}_{a,k} := \pi_a^{-1}(W_k)$. We use the lexicographic order on $]c_i-1, c_i] \times \mathbb{Z}$. Thus, we obtain the increasing filtration ${}^i\widetilde{F}$ indexed by $]c_i-1, c_i] \times \mathbb{Z}$. Obviously, the set $\widetilde{S}_i := \bigl\{(a,k) \in]c_i-1, c_i] \times \mathbb{Z} \,\big|\, {}^i\mathrm{Gr}^{\widetilde{F}}_{(a,k)} \neq 0\bigr\}$ is finite.

We explain the perturbation of the weight for the parabolic structure. Let $\varphi_i : \widetilde{S}_i \longrightarrow]c_i-1, c_i]$ be the increasing map such that $|\varphi_i(a,k) - a| \leq C\cdot\epsilon$ for some $C > 0$. (Since we are interested in the family of the filtrations $\boldsymbol{F}^{(\epsilon)}$ ($\epsilon > 0$), this condition makes sense.) Then, ${}^i\widetilde{F}$ and φ_i give the $\boldsymbol{c}$-parabolic filtration $\boldsymbol{F}^{(\epsilon)} = \bigl({}^iF^{(\epsilon)} \,\big|\, i \in S\bigr)$. Thus, we obtain the regular $\boldsymbol{c}$-parabolic λ-flat bundle $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda\bigr)$, which are called the ϵ-perturbation of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^\lambda)$. By construction, we have the following convergence in $H^*(X, \boldsymbol{R})$.

$$\lim_{\epsilon\to 0} \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}), \qquad \lim_{\epsilon\to 0} \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F})$$

The following proposition is standard. (See Proposition 3.28 of [31], for example.)

Proposition 2.19 *Assume that $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^\lambda\bigr)$ is μ_L-stable. If ϵ is sufficiently small, then the ϵ-perturbation $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda\bigr)$ is also μ_L-stable.* ∎

We will use two kinds of perturbations φ_i of parabolic weights.

(I) The image of φ_i is contained in $\boldsymbol{Q}$ for each $i \in S$. This perturbation will be used to obtain the formula for the parabolic characteristic numbers.

(II) For simplicity, we assume $\epsilon = m^{-1}$ and $0 < 10\,\mathrm{rank}\,E\cdot\epsilon < \mathrm{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F})$. (See Subsection 3.1 of [31] for gap.) Let $i \in S$. For each $a \in \mathcal{P}ar({}_{\boldsymbol{c}}E, \boldsymbol{F})$, we take $a'(\epsilon, i) \in m^{-1}\cdot\mathbb{Z}$ such that $|a'(\epsilon,i) - a| < m^{-1}$. Let $L(\epsilon, i) \in \boldsymbol{R}$ be determined by the following:

$$L(\epsilon,i)\cdot\mathrm{rank}(E) := \sum (a(\epsilon,i) - a)\cdot \mathrm{rank}\,{}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$$

Then, we put $a(\epsilon,i) := a'(\epsilon,i) - L(\epsilon,i)$ and $\varphi(a,k) := a(\epsilon,i) + k\cdot\epsilon$. By construction, we have the following equality:

$$\sum_{a,k} \varphi(a,k)\cdot\mathrm{rank}\bigl({}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)\bigr) = \sum_a a\cdot\mathrm{rank}\bigl({}^i\mathrm{Gr}^{\widetilde{F}}_{a,k}({}_{\boldsymbol{c}}E)\bigr)$$

Hence, we have $\text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}) = \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)})$. For each i, we also have some $-1/m < \gamma_i \leq 0$ such that $\mathcal{P}ar({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, i)$ is contained in $\bigl\{c_i + \gamma_i + p/m \,\big|\, p \in \mathbb{Z}_{\leq 0},\ -1 < \gamma_i + p/m \leq 0\bigr\}$.

Remark 2.20 *The construction given in this subsection is valid, when the base manifold X is a curve.* ∎

2.1.7 Remarks about the terminology and the notation

We give some remarks about the terminology "parabolic structure". Let X be a complex manifold, and let D be a simple normal crossing divisor of X with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. We often discuss a regular $\boldsymbol{c}$-parabolic λ-flat bundle on (X, D) for some $\boldsymbol{c} \in \boldsymbol{R}^S$. In our most arguments, a choice of $\boldsymbol{c}$ are not relevant. In fact, $\boldsymbol{c}$ is fixed to be $(0,\ldots,0)$ in many references where the parabolic structure is discussed. But, it is sometimes convenient to avoid the case $c_i \in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)$, for example, when we consider a perturbation of the parabolic structure. That is the main reason why we consider general $\boldsymbol{c}$-parabolic structure.

In the following argument, we often assume $c_i \notin \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)$ implicitly, and we often omit to distinguish $\boldsymbol{c}$, and use the terminology "parabolic structure" instead of "$\boldsymbol{c}$-parabolic structure", when we do not have to care about a choice of $\boldsymbol{c}$. The author hopes that there will be no risk of confusion and that it will reduce unnecessary complexity of the description.

Relatedly we have the remark about the notation to denote parabolic bundles. We often use the notation $({}_{\boldsymbol{c}}E, \boldsymbol{F})$ or ${}_{\boldsymbol{c}}E_*$ to denote a $\boldsymbol{c}$-parabolic bundle, when we would like to distinguish $\boldsymbol{c}$. The notation "${}_{\boldsymbol{c}}E$" is also appropriate and useful, when we regard it as a prolongment of the locally free sheaf E on $X - D$. But, in some case, a vector bundle is given not only on $X - D$ but also on X from the beginning. And, as is said above, we will not care about a choice of $\boldsymbol{c}$. In such a case, we often prefer to using the notation $(E, \boldsymbol{F})$ or E_* for simplicity of the description.

2.2 Generality for λ-connection in the C^∞-category

We will give some generality for λ-connections. They are straightforward generalization of the argument for Higgs bundles or flat bundles given in Simpson's papers (for example [36] and [38]), and hence we will often omit to give a detailed proof. For simplicity, we will assume $\lambda \neq 0$.

2.2.1 The induced operators

Let X be a complex manifold, and $(E, \mathbb{D}^\lambda)$ be a flat λ-connection on X. We have the decomposition of $\mathbb{D}^\lambda$ into the $(0,1)$-part d''_E and the $(1,0)$-part d'_E. The holomorphic structure of E is given by d''_E. Recall that the twisted Leibniz rule $d'_E(f\cdot v) = \lambda\cdot\partial_X(f)v + f\cdot d'_E v$ holds for $f \in C^\infty(X)$ and $v \in C^\infty(X,E)$. Let h be a hermitian metric of E. From d''_E and h, we obtain the $(1,0)$-operator $\delta'_{E,h}$ determined by $\overline{\partial} h(u,v) = h(d''_E u, v) + h(u, \delta'_{E,h}v)$. From d'_E and h, we obtain the $(0,1)$-operator $\delta''_{E,h}$ determined by $\lambda\partial h(u,v) = h(d'_E u, v) + h(u, \delta''_{E,h}v)$. We remark $\delta''_{E,h}(f\cdot v) = \overline{\lambda}\cdot\overline{\partial}_X f\cdot v + f\cdot\delta''_{E,h}(v)$. We obtain the following operators:

$$\begin{aligned}
\overline{\partial}_{E,h} &:= \frac{1}{1+|\lambda|^2}(d''_E + \lambda\delta''_{E,h}), \quad \partial_{E,h} := \frac{1}{1+|\lambda|^2}(\overline{\lambda}d'_E + \delta'_{E,h}),\\
\theta^\dagger_{E,h} &:= \frac{1}{1+|\lambda|^2}(\overline{\lambda}d''_E - \delta''_{E,h}), \quad \theta_{E,h} := \frac{1}{1+|\lambda|^2}(d'_E - \lambda\delta'_{E,h}).
\end{aligned} \tag{5}$$

It is easy to see that the following Leibniz rule holds:

$$\overline{\partial}_{E,h}(fs) = \overline{\partial}_X f\cdot s + f\cdot\overline{\partial}_{E,h}s, \quad \partial_{E,h}(fs) = \partial_X f\cdot s + f\cdot\partial_{E,h}s.$$

On the other hand, θ and $\theta^\dagger$ give the sections of $\mathrm{End}(E)\otimes\Omega^{1,0}$ and $\mathrm{End}(E)\otimes\Omega^{0,1}$ respectively. We also have the formulas:

$$d''_E = \overline{\partial}_{E,h} + \lambda\theta^\dagger_{E,h}, \quad d'_E = \lambda\partial_{E,h} + \theta_{E,h}, \quad \delta'_{E,h} = \partial_{E,h} - \overline{\lambda}\theta_{E,h}, \quad \delta''_{E,h} = \overline{\lambda\partial}_{E,h} - \theta^\dagger_{E,h}.$$

Remark 2.21 *The index "E,h" is attached to emphasize the bundle E and the metric h. We will often omit them if there are no risk of confusion.* ∎

Remark 2.22 *We have the hermitian product $(\cdot,\cdot)_h : \big(E\otimes\Omega^\cdot\big)\otimes\big(E\otimes\Omega^\cdot\big) \longrightarrow \Omega^\cdot$ induced by h. For a section A of $\mathrm{End}(E)\otimes\Omega^{p,q}$, let $A^\dagger_h$ denote the section of $\mathrm{End}(E)\otimes\Omega^{q,p}$ which is the adjoint of A with respect to h in the sense $\big(A\cdot u, v\big)_h = \big(u, A^\dagger_h v\big)_h$. The above $\theta^\dagger_h$ is the adjoint of θ_h in this sense.* ∎

We put $\mathbb{D}^{\lambda\star}_h := \delta'_h - \delta''_h = \partial_h + \theta^\dagger_h - \overline{\lambda}(\overline{\partial}_h + \theta_h)$. We have the following formula:

$$\overline{\partial}_h + \theta_h = \frac{\mathbb{D}^\lambda - \lambda\mathbb{D}^{\lambda\star}_h}{1+|\lambda|^2}, \quad \partial_h + \theta^\dagger_h = \frac{\mathbb{D}^{\lambda\star}_h + \overline{\lambda}\mathbb{D}^\lambda}{1+|\lambda|^2}.$$

We recall that h is called a pluri-harmonic metric if $(\overline{\partial}_h + \theta_h)^2 = 0$ holds, i.e., $(E, \overline{\partial}_h, \theta_h)$ is a Higgs bundle. The condition is equivalent to $\big[\mathbb{D}^\lambda, \mathbb{D}^{\lambda\star}_h\big] = 0$. In the following, a λ-flat bundle with pluri-harmonic metric is called a harmonic bundle.

Let us consider the case where X is provided with a Kahler form ω. For a differential operator A of $E\otimes\Omega^\cdot$ of degree one, i.e., $A : C^\infty(X, E\otimes\Omega^i) \longrightarrow C^\infty(X, E\otimes\Omega^{i+1})$, let A^* denote a formal adjoint with respect to ω and h, i.e., $\int_X (Au,v)_{h,\omega}\,\mathrm{dvol}_\omega = \int_X (u, A^* v)_{h,\omega}\,\mathrm{dvol}_\omega$ hold for any C^∞-sections u and v with compact supports. Here, $(\cdot,\cdot)_{h,\omega}$ denotes the Hermitiann inner product of appropriate vector bundles induced by h and ω.

Lemma 2.23 $\big(\mathbb{D}^{\lambda\star}\big)^* = \sqrt{-1}\big[\Lambda_\omega, \mathbb{D}^\lambda\big]$ *and* $\big(\mathbb{D}^\lambda\big)^* = -\sqrt{-1}\big[\Lambda_\omega, \mathbb{D}^{\lambda\star}\big]$.

Proof It follows from the relations $\partial^* = \sqrt{-1}[\Lambda_\omega, \overline{\partial}_E]$, $\overline{\partial}^* = -\sqrt{-1}[\Lambda_\omega, \partial_E]$, $\theta^* = -\sqrt{-1}[\Lambda_\omega, \theta^\dagger]$ and $(\theta^\dagger)^* = \sqrt{-1}[\Lambda_\omega, \theta]$. ∎

The Laplacian $\Delta^\lambda_{h,\omega} : C^\infty(X,E) \longrightarrow C^\infty(X,E)$ is defined by $\Delta^\lambda_{h,\omega} := \sqrt{-1}\Lambda_\omega\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}$.

Remark 2.24 *For the differential operators of functions,* $\Delta^{\lambda}_{\omega} := \sqrt{-1}\Lambda(\overline{\partial}+\lambda\partial)\circ(\partial-\overline{\lambda\partial}) = (1+|\lambda|^2)\sqrt{-1}\Lambda\overline{\partial}\partial = (1+|\lambda|^2)\Delta''_{\omega}$*, where* Δ''_{ω} *denotes the usual Laplacian* $\sqrt{-1}\Lambda_{\omega}\overline{\partial}\partial$*.* ∎

Lemma 2.25 *When* $\lambda \neq 0$*, we have* $\overline{\lambda}^{-1}\partial_h^2 + \lambda^{-1}\theta_h^2 = 0$ *and* $\lambda^{-1}\overline{\partial}_h^2 + \overline{\lambda}^{-1}(\theta_h^{\dagger})^2 = 0$*.*

Proof From the flatness $(\mathbb{D}^{\lambda})^2 = 0$, we obtain the following formulas:

$$(\overline{\partial}_h + \lambda\theta_h^{\dagger})^2 = \overline{\partial}_h^2 + \lambda\overline{\partial}_h\theta_h^{\dagger} + \lambda^2(\theta_h^{\dagger})^2 = 0, \tag{6}$$

$$(\lambda\partial_h + \theta_h)^2 = \lambda^2\partial_h^2 + \lambda\partial_h\theta_h + \theta_h^2 = 0, \tag{7}$$

$$\big[\overline{\partial}_h + \lambda\theta_h^{\dagger},\, \lambda\partial_h + \theta_h\big] = \lambda\Big(\big[\overline{\partial}_h\,, \partial_h\big] + \big[\theta_h^{\dagger}, \theta_h\big]\Big) + \overline{\partial}_h\theta_h + \lambda^2\partial_h\theta_h^{\dagger} = 0. \tag{8}$$

It is easy to see $(\overline{\partial}_h^2)_h^{\dagger} = -\partial_h^2$, $(\overline{\partial}_h\theta_h^{\dagger})^{\dagger} = \partial_h\theta_h$ and $(\theta_h^{\dagger})^2 = -(\theta_h^2)^{\dagger}$. Therefore, we obtain the following equality from (6):

$$-\partial_h^2 + \overline{\lambda}\big(\partial_h\theta_h\big) - \overline{\lambda}^2\theta_h^2 = 0. \tag{9}$$

From (7) and (9), we obtain $\big(\lambda + \overline{\lambda}^{-1}\big)\partial_h^2 + \big(\lambda^{-1} + \overline{\lambda}\big)\theta_h^2 = (1+|\lambda|^2)\big(\overline{\lambda}^{-1}\partial_h^2 + \lambda^{-1}\theta_h^2\big) = 0$, which gives the first formula in the lemma. The second formula can be obtained by taking the adjoint. ∎

Lemma 2.26 *When* $\lambda \neq 0$*, we have* $\overline{\lambda}^{-1}\cdot\partial_h\theta_h^{\dagger} + \lambda^{-1}\cdot\overline{\partial}_h\theta_h = 0$ *and* $\big[\partial_h\,, \overline{\partial}_h\big] + \big[\theta_h\,, \theta_h^{\dagger}\big] = 0$*.*

Proof It is easy to check $[\partial_h, \overline{\partial}_h]_h^{\dagger} = -[\partial_h, \overline{\partial}_h]$, $[\theta_h, \theta_h^{\dagger}]_h^{\dagger} = -[\theta_h, \theta_h^{\dagger}]$ and $(\overline{\partial}_h\theta_h)_h^{\dagger} = \partial_h\theta_h^{\dagger}$. Hence, we obtain the following equality from (8):

$$-[\overline{\partial}_h, \partial_h] - [\theta_h^{\dagger}, \theta_h] + \overline{\lambda}^{-1}\cdot\partial_h\theta_h^{\dagger} + \overline{\lambda}\cdot\overline{\partial}_h\theta_h = 0. \tag{10}$$

The claim of the lemma immediately follows from (8) and (10). ∎

Corollary 2.27 *When* $\lambda \neq 0$*, the pluri-harmonicity of the metric* h *is equivalent to the vanishings* $\theta_h^2 = 0$ *and* $\overline{\partial}_h\theta_h = 0$*.* ∎

2.2.2 Local expression

Let $(E, \mathbb{D}^{\lambda})$ be a flat λ-connection, and let h be a C^{∞}-metric. Let $\boldsymbol{v} = (v_1, \ldots, v_r)$ be a holomorphic frame of E. Let $H = H(h, \boldsymbol{v})$ denote the hermitian matrix valued function of h with respect to $\boldsymbol{v}$, i.e., $H_{i,j} = h(v_i, v_j)$. Let us see the local expression of the induced operators.

Let A denote the $M(r)$-valued $(1,0)$-form of $\mathbb{D}^{\lambda}$ with respect to $\boldsymbol{v}$, i.e., $\mathbb{D}^{\lambda}\boldsymbol{v} = \boldsymbol{v}\cdot A$, in other words, $\mathbb{D}^{\lambda}v_i = \sum A_{j\,i}\cdot v_j$. Let B denote the $(1,0)$-form of δ'_h with respect to $\boldsymbol{v}$, i.e., $\delta'_h\boldsymbol{v} = \boldsymbol{v}\cdot B$, and then we have $\overline{\partial}h(v_i, v_j) = h\big(v_i, \delta'_h v_j\big) = \sum h\big(v_i, B_{k,j}v_k\big)$. Hence, $\overline{\partial}H = H\cdot\overline{B}$, i.e., we obtain $B = \overline{H}^{-1}\partial\overline{H}$. Let C denote the $(0,1)$-form of δ''_h with respect to $\boldsymbol{v}$, i.e., $\delta''_h\boldsymbol{v} = \boldsymbol{v}\cdot C$, and then we have $\lambda\cdot\partial h(v_i, v_j) = h(d'v_i, v_j) + h(v_i, \delta''_h v_j) = \sum_k h(A_{k,i}v_k,\, v_j) + \sum_k h(v_i,\, C_{k,j}v_k)$. Hence, $\lambda\partial H = {}^tAH + H\overline{C}$, i.e., we obtain $C = \overline{\lambda}\cdot\overline{H}^{-1}\overline{\partial H} - \overline{H}^{-1}\,{}^t\overline{AH}$. Thus, we obtain the following:

$$\theta_h\boldsymbol{v} = \boldsymbol{v}\cdot\frac{1}{1+|\lambda|^2}(A - \lambda\overline{H}^{-1}\partial\overline{H}), \quad \overline{\partial}_h\boldsymbol{v} = \boldsymbol{v}\cdot\frac{\lambda}{1+|\lambda|^2}(\overline{\lambda}\cdot\overline{H}^{-1}\overline{\partial H} - A_h^{\dagger}).$$

Here, $A^{\dagger}$ denote the adjoint of A with respect to h, i.e., $A_h^{\dagger} = \overline{H}^{-1}\cdot{}^t\overline{A}\cdot\overline{H}$.

2.2.3 Pseudo curvature and the Hermitian-Einstein condition

Assume $\lambda \neq 0$. For a flat λ-connection $(E, \mathbb{D}^{\lambda})$ with a hermitian metric h, the pseudo curvature $G(h, \mathbb{D}^{\lambda})$ is defined as follows:

$$G(h, \mathbb{D}^{\lambda}) := \big[\mathbb{D}^{\lambda}, \mathbb{D}_h^{\lambda\star}\big] = -\frac{(1+|\lambda|^2)^2}{\lambda}(\overline{\partial}_h + \theta_h)^2.$$

Then, a hermitian metric h is a pluri-harmonic metric for $(E, \mathbb{D}^{\lambda})$, if and only if $G(h, \mathbb{D}^{\lambda}) = 0$ holds. We will often use the notation $G(h)$ or G_h instead of $G(h, \mathbb{D}^{\lambda})$ if there are no risk of confusion.

When X is provided with a Kahler form ω, a Hermitian-Einstein condition for h is $\Lambda_{\omega}G(h, \mathbb{D}^{\lambda})^{\perp} = 0$, where "$\perp$" means the trace free part.

2.2.4 Some relations between curvature and pseudo curvature

By the construction of δ'_h, the operator $d''+\delta'_h$ is a unitary connection of (E,h). The curvature of $d''+\delta'_h$ is denoted by $R(d'',h)$. We have the following expression of $R(d'',h)$ due to $[d'',d']=0$:

$$R(d'',h)=\big[d'',\delta'_h\big]=\big[d'',\lambda^{-1}d'\big]-\frac{1+|\lambda|^2}{\lambda}\big[d'',\theta_h\big]=-\frac{1+|\lambda|^2}{\lambda}\big(\overline{\partial}_h\theta_h+\lambda[\theta_h^{\dagger},\theta_h]\big). \tag{11}$$

Lemma 2.28 *The following equality holds:*

$$\operatorname{tr}R(d'',h)=\frac{1}{1+|\lambda|^2}\operatorname{tr}G(\mathbb{D}^{\lambda},h)=-\frac{1+|\lambda|^2}{\lambda}\overline{\partial}\operatorname{tr}\theta_h. \tag{12}$$

Proof From (11), we obtain $\operatorname{tr}R(d'',h)=-(1+|\lambda|^2)\lambda^{-1}\cdot\overline{\partial}\operatorname{tr}\theta_h$. On the other hand, we have the following:

$$\operatorname{tr}G(h,\mathbb{D}^{\lambda})=-\frac{\big(1+|\lambda|^2\big)^2}{\lambda}\operatorname{tr}\big(\overline{\partial}_h^2+\overline{\partial}_h\theta_h+\theta_h^2\big)=-\frac{\big(1+|\lambda|^2\big)^2}{\lambda}\overline{\partial}\operatorname{tr}\theta_h.$$

Here, we have used $\operatorname{tr}(\theta_h^2)=0$, which implies $\operatorname{tr}(\overline{\partial}_h^2)=0$ due to Lemma 2.25. Thus we are done. ∎

Lemma 2.29 *In the case* $\dim X=2$*, we have the following formula:*

$$\operatorname{tr}\big(R(d'',h)^2\big)=\frac{1}{(1+|\lambda|^2)^2}\operatorname{tr}\big(G(h,\mathbb{D}^{\lambda})^2\big)-\frac{(1+|\lambda|^2)^2}{\lambda}\overline{\partial}\operatorname{tr}(\theta_h^2\cdot\theta_h^{\dagger}).$$

Proof We have the following:

$$\operatorname{tr}\big(G(h,\mathbb{D}^{\lambda})^2\big)=\frac{(1+|\lambda|^2)^4}{\lambda^2}\Big(\operatorname{tr}\big((\overline{\partial}_h\theta_h)^2\big)+2\operatorname{tr}\big(\overline{\partial}_h^2\cdot\theta_h^2\big)\Big)$$

$$\operatorname{tr}\big(R(h,d'')^2\big)=\frac{(1+|\lambda|^2)^2}{\lambda^2}\Big(\operatorname{tr}\big((\overline{\partial}_h\theta_h)^2\big)+2\lambda\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot[\theta_h,\theta_h^{\dagger}]\big)+\lambda^2\operatorname{tr}\big([\theta_h,\theta_h^{\dagger}]^2\big)\Big).$$

Since we have $\operatorname{tr}\big([\theta_h,\theta_h^{\dagger}]^2\big)=-2\operatorname{tr}\big(\theta_h^2\theta_h^{\dagger\,2}\big)$ and $(\overline{\partial}_h+\lambda\theta_h^{\dagger})^2=\overline{\partial}_h^2+\lambda\overline{\partial}_h\theta_h^{\dagger}+\lambda^2\theta_h^{\dagger\,2}=0$, we obtain the following:

$$\lambda^2\operatorname{tr}\big([\theta_h,\theta_h^{\dagger}]^2\big)=-2\operatorname{tr}\big(\lambda^2\cdot\theta_h^2\cdot\theta_h^{\dagger\,2}\big)=2\operatorname{tr}\Big(\overline{\partial}_h^2\cdot\theta_h^2+\lambda\cdot\overline{\partial}_h\theta_h^{\dagger}\cdot\theta_h^2\Big).$$

Hence, we have the following equality:

$$\operatorname{tr}\big(R(h,d'')^2\big)=\left(\frac{1+|\lambda|^2}{\lambda}\right)^2\Big(\operatorname{tr}\big((\overline{\partial}_h\theta_h)^2\big)+2\lambda\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot[\theta_h,\,\theta_h^{\dagger}]\big)+2\operatorname{tr}\big(\overline{\partial}_h^2\cdot\theta_h^2\big)+2\lambda\operatorname{tr}\big(\overline{\partial}_h\theta_h^{\dagger}\cdot\theta_h^2\big)\Big).$$

We also remark the following:

$$\begin{aligned}\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot[\theta_h,\theta_h^{\dagger}]\big)+\operatorname{tr}(\theta_h^2\cdot\overline{\partial}_h\theta_h^{\dagger})&=\operatorname{tr}\big((\overline{\partial}_h\theta_h)\cdot\theta_h\cdot\theta_h^{\dagger}\big)+\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot\theta_h^{\dagger}\cdot\theta_h\big)-\operatorname{tr}\big(\theta_h\cdot\overline{\partial}_h\theta_h^{\dagger}\cdot\theta_h\big)\\&=\overline{\partial}\operatorname{tr}\big(\theta_h\cdot\theta_h^{\dagger}\cdot\theta_h\big)=-\overline{\partial}\operatorname{tr}(\theta_h^2\cdot\theta_h^{\dagger}).\end{aligned} \tag{13}$$

Then, the claim of the lemma immediately follows. ∎

2.2.5 Change of hermitian metrics

Let h_i $(i=1,2)$ be hermitian metrics of E. The endomorphism s is determined by $h_2=h_1\cdot s$, i.e., $h_2(u,v)=h_1\big(s\cdot u,v\big)=h_1\big(u,s\cdot v\big)$, which is self-adjoint with respect to both of h_i. Then, we have the relations $\delta'_{h_2}=\delta'_{h_1}+s^{-1}\delta'_{h_1}s$ and $\delta''_{h_2}=\delta''_{h_1}+s^{-1}\delta''_{h_1}s$. Therefore, we have the following relations from (5):

$$\overline{\partial}_{h_2}=\overline{\partial}_{h_1}+\frac{\lambda}{1+|\lambda|^2}s^{-1}\delta''_{h_1}s,\quad \partial_{h_2}=\partial_{h_1}+\frac{1}{1+|\lambda|^2}s^{-1}\delta'_{h_1}s,$$

$$\theta_{h_2}^{\dagger}=\theta_{h_1}^{\dagger}-\frac{1}{1+|\lambda|^2}s^{-1}\delta''_{h_2}s,\quad \theta_{h_2}=\theta_{h_1}-\frac{\lambda}{1+|\lambda|^2}s^{-1}\delta'_{h_1}s.$$

We also have $\mathbb{D}_{h_2}^{\lambda\star}=\mathbb{D}_{h_1}^{\lambda\star}+s^{-1}\mathbb{D}_{h_1}^{\lambda\star}s$, and thus $\left[\mathbb{D}^{\lambda},\mathbb{D}_{h_2}^{\lambda\star}\right]=\left[\mathbb{D}^{\lambda},\mathbb{D}_{h_1}^{\lambda\star}\right]+\mathbb{D}^{\lambda}(s^{-1})\cdot\mathbb{D}_{h_1}^{\lambda\star}s+s^{-1}\mathbb{D}^{\lambda}\mathbb{D}_{h_1}^{\lambda\star}s$. Then, we obtain the following formula:

$$\Delta_{h_1,\omega}^{\lambda}s=s\sqrt{-1}\big(\Lambda_\omega G(h_2)-\Lambda_\omega G(h_1)\big)+\sqrt{-1}\Lambda_\omega\mathbb{D}^{\lambda}s\cdot s^{-1}\mathbb{D}^{\lambda\star}s. \tag{14}$$

In particular, we obtain the following formula by taking the trace:

$$\Delta_\omega^{\lambda}\operatorname{tr}(s)=\operatorname{tr}\Big(s\sqrt{-1}\big(\Lambda_\omega G(h_2)-\Lambda_\omega G(h_1)\big)\Big)-\big|\mathbb{D}^{\lambda}(s)s^{-1/2}\big|_{h_1,\omega}^2. \tag{15}$$

As in Lemma 3.1 of [36], we can derive the following inequality:

$$\Delta_\omega^{\lambda}\log\operatorname{tr}(s)\leq\big|\Lambda_\omega G(h_1)\big|_{h_1}+\big|\Lambda_\omega G(h_2)\big|_{h_2} \tag{16}$$

2.3 Review of existence result of a Hermitian-Einstein metric due to Simpson

2.3.1 Analytic stability of flat λ-bundle

Let X be a complex manifold with a Kahler form ω. In this subsection, we impose the following condition as in [36].

Condition 2.30

1. *The volume of X with respect to ω is finite.*
2. *There exists a C^∞-function $\phi: X\longrightarrow \boldsymbol{R}_{\geq 0}$ with the following properties:*
   - $\{x\in X\,|\,\phi(x)\leq a\}$ *is compact for any a.*
   - $0\leq\sqrt{-1}\partial\overline{\partial}\phi\leq C\cdot\omega$, *and $\overline{\partial}\phi$ is bounded with respect to ω.*
3. *There exists a continuous increasing function $a:[0,\infty[\longrightarrow[0,\infty[$ with the following properties:*
   - $a(0)=0$ *and* $a(t)=t$ *for* $t\geq 1$.
   - *Let f be a positive bounded function on X such that $\Delta_\omega f\leq B$ for some $B\in\boldsymbol{R}$. Then, there exists a constant $C(B)$, depending only on B, such that* $\sup_X|f|\leq C(B)\cdot a\left(\int_X|f|\cdot\mathrm{dvol}_\omega\right)$. *Moreover, $\Delta_\omega(f)\leq 0$ implies $\Delta_\omega(f)=0$.* ∎

Let $(E,\mathbb{D}^{\lambda})$ be a λ-flat bundle on X. There are two conditions on the finiteness of the pseudo curvature of $(E,\mathbb{D}^{\lambda},h)$. The stronger one is the following:

$$\sup\big|G(h,\mathbb{D}^{\lambda})\big|_{h,\omega}<\infty. \tag{17}$$

The finiteness (17) implies the weaker one:

$$\sup\big|\Lambda_\omega G(h,\mathbb{D}^{\lambda})\big|_{h,\omega}<\infty. \tag{18}$$

When we are given a hermitian metric h of E satisfying the finiteness (18), the degree $\deg_\omega(E,h)$ is defined as follows:

$$\deg_\omega(E,h):=\frac{\sqrt{-1}}{2\pi}\int_X\frac{\operatorname{tr}G(h,\mathbb{D}^{\lambda})}{1+|\lambda|^2}\cdot\omega^{n-1}=\frac{\sqrt{-1}}{2\pi}\int_X\operatorname{tr}R(h,d'')\cdot\omega^{n-1}.$$

Here, we have used (12). For any λ-flat bundle $(V,\mathbb{D}_V^{\lambda})\subset(E,\mathbb{D}^{\lambda})$, the restriction $h_V:=h_{|V}$ induces $\deg_\omega(V,h_V)$. As in Lemma 3.2 of [36], we have the Chern-Weil formula. The proof is same.

Lemma 2.31 *Let π_V denote the orthogonal projection of E onto V. Then, the following equality holds:*

$$\deg_\omega(V, h_V) = \frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int_X \mathrm{tr}\big(\pi_V \circ G(h, \mathbb{D}^\lambda)\big)\cdot\omega^{n-1} - \int_X \big|\mathbb{D}^\lambda \pi_V\big|^2_{h,\omega}\right)$$

The value is finite or $-\infty$, when (18) is satisfied. ∎

Definition 2.32 *$(E, \mathbb{D}^\lambda, h)$ is defined to be analytically stable with respect to ω, if the inequality*

$$\frac{\deg_\omega(V, h_V)}{\mathrm{rank}\, V} < \frac{\deg_\omega(E, h)}{\mathrm{rank}\, E}$$

holds for any $(V, \mathbb{D}^\lambda_V) \subset (E, \mathbb{D}^\lambda)$. ∎

2.3.2 Existence theorem of Simpson and some consequence

Proposition 2.33 *Let (X, ω) be a Kahler manifold satisfying Condition 2.30, and let $(E, \mathbb{D}^\lambda, h_0)$ be a metrized flat λ-connection satisfying (17). Assume that $(E, \mathbb{D}^\lambda, h_0)$ is analytically stable with respect to ω. Then, there exists a hermitian metric $h = h_0 \cdot s$ satisfying the following conditions:*

- *h and h_0 are mutually bounded.*
- $\det(h) = \det(h_0)$.
- *$\mathbb{D}^\lambda(s)$ is L^2 with respect to h_0 and ω.*
- *It satisfies the Hermitian Einstein condition $\Lambda_\omega G(h)^\perp = 0$, where $G(h)^\perp$ denotes the trace free part of $G(h)$.*
- *The following equalities hold:*

$$\int_Y \mathrm{tr}\Big(G(h)^2\Big)\cdot\omega^{n-2} = \int_Y \mathrm{tr}\Big(G(h_0)^2\Big)\cdot\omega^{n-2}, \qquad \int_Y \mathrm{tr}\Big(G(h)^{\perp\,2}\Big)\cdot\omega^{n-2} = \int_Y \mathrm{tr}\Big(G(h_0)^{\perp\,2}\Big)\cdot\omega^{n-2}.$$

We do not give a proof of this proposition, because we need only minor modification of the proof of Theorem 1, Proposition 3.5 and Lemma 7.4 of [36]. Indeed, we have only to replace D'', D' and $F(h)$ with $\mathbb{D}^\lambda$, $\mathbb{D}^{\lambda\star}$ and $G(h)$, and to make some obvious modification of positive constant multiplications, as was mentioned by Simpson himself. (See the page 754 of [37], for example. Remark that "D^c" corresponds to our $-\mathbb{D}^{\lambda\star}$, and hence our $G(h)$ is slightly different from his.) The author recommends the reader to read a quite excellent discussion in [36]. However, we will use some results related with the Donaldson functional, which are obtained from the proof. Hence, we recall a brief outline of the proof of Proposition 2.33. We will use the notation in Subsection 2.4.

Let h_0 be a metric for $(E, \mathbb{D}^\lambda)$ satisfying the finiteness (18). Let us consider the heat equation for the self adjoint endomorphisms s_t with respect to h_0:

$$s_t^{-1}\frac{ds_t}{dt} = -\sqrt{-1}\Lambda_\omega G(h_t)^\perp. \tag{19}$$

A detailed argument to solve (19) is given in Section 6 of [36]. Moreover, $\Lambda_\omega G(h_t)$ is shown to be uniformly bounded. We do not reproduce them here.

Then, we would like to show the existence of an appropriate subsequence $t_i \to \infty$ such that $\{s_{t_i}\}$ converges to s_∞ weakly in L_2^p locally on X, and we would like to show that $h_\infty = h_0 \cdot s_\infty$ gives the desired Hermitian-Einstein metric. For that purpose, Simpson used the Donaldson functional $M\big(h_0, h_0 s_{t_i}\big)$. (We recall the definition and some fundamental property in Subsection 2.4, below.) He showed that there exist positive constants C_i $(i = 1, 2)$ such that the following holds: (Proposition 5.3 of [36]. We review it in Proposition 2.41. We will use the notation there in the following.)

$$\sup|\log s_t| \leq C_1 + C_2 \cdot M(h_0, h_0 s_t). \tag{20}$$

He also showed (Lemma 7.1 of [36]) that $M(h_0, h_0 s_t)$ is C^1 with respect to t, and that the following formula holds:

$$\frac{d}{dt} M\big(h_0, h_0 s_t\big) = -\int_X \big|\Lambda_\omega G(h_t)^{\perp}\big|^2_{h_t,\omega} \leq 0. \tag{21}$$

Since we have $M(h_0, h_0) = 0$ by definition, we obtain $M(h_0, h_0 s_t) \leq 0$ from (21). Then, we obtain the boundedness of s_t from (20). For the solution of (19), we have $\det(s_t) = 1$. Hence, we also obtain the boundedness of s_t^{-1}. We also obtain the existence of a subsequence $\{t_i'\}$ such that $\big|\Lambda_\omega G(h'_{t_i})\big|_{L^2} \longrightarrow 0$.

From the uniform boundedness of s_t and $\Lambda_\omega G(h_t)$, we obtain the lower bound of $M\big(h_0, h_0 s_t\big)$. (See Corollary 2.40 in this paper, for example.) Moreover, we obtain the uniform bound of $\int_X \big|\mathbb{D}^\lambda u_t\big|^2_{h_0}$ due to the positivity of Ψ given in (26), where $s_t = \exp(u_t)$. Due to the boundedness of s_t and s_t^{-1}, we also obtain the boundedness of $\int_X \big|\mathbb{D}^\lambda s_t\big|^2_{h_0}$. Then, we obtain the L_1^2 boundedness. Hence, we can take a subsequence $\{t_i''\}$ such that $s_{t_i''}$ converges to some s_∞ weakly in L_1^2 locally on $X - D$. Due to some more excellent additional argument given in the page 895 of [36], it can be shown that the convergence is weakly L_2^p locally on $X - D$, for any p. As a result, we obtain the Hermitian-Einstein metric.

By the above argument, we can derive the following lemma, which we would like to use in the later discussion.

Lemma 2.34 *Let h_0 be the hermitian metric satisfying* (17). *Let h_{HE} be the Hermitian-Einstein metric obtained in Proposition* 2.33. *Then, we have $M\big(h_0, h_{HE}\big) \leq 0$.*

Proof Recall that h_{HE} is obtained as the limit $h_0 \cdot s_\infty$ of some sequence $\{h_0 s_{t_i}\}$, and we have $M(h_0, h_0 \cdot s_{t_i}) \leq 0$. We use the formula (25). Let Z be any compact subset of X. The sequence $\{s_{t_i}\}$ converges to s_∞ in C^0 on Z. The sequence $\{\Lambda_\omega G(h_{t_i})\}$ converges to $\Lambda_\omega G(h_{HE})$ weakly in L^2 on Z. Therefore, we have the convergence:

$$\lim_{t_i \to \infty} \int_Z \mathrm{tr}\big(u_{t_i} \cdot \Lambda_\omega G(h_{t_i})\big)\, \mathrm{dvol}_\omega = \int_Z \mathrm{tr}\big(u_\infty \cdot \Lambda_\omega G(h_{HE})\big)\, \mathrm{dvol}_\omega\,.$$

Here, u_t are given by $\exp(u_t) = s_t$. Since $\sup_X |s_t|$ and $\sup_X |\Lambda G(h_t)|$ are bounded independently of t, we can easily obtain the convergence:

$$\lim_{t_i \to \infty} \int_X \mathrm{tr}\big(u_{t_i} \cdot \Lambda_\omega G(h_{t_i})\big)\, \mathrm{dvol}_\omega = \int_X \mathrm{tr}\big(u_\infty \cdot \Lambda_\omega G(h_{HE})\big)\, \mathrm{dvol}_\omega\,.$$

We have the C^0-convergence of the sequence $\{\mathbb{D}^\lambda u_{t_i}\}$ to $\mathbb{D}^\lambda u_\infty$. Hence, we have the following inequality due to Fatou's lemma:

$$\int_X \big(\Psi(u_\infty)\mathbb{D}^\lambda u_\infty,\ \mathbb{D}^\lambda u_\infty\big)\, \mathrm{dvol}_\omega \leq \underline{\lim} \int_X \big(\Psi(u_{t_i})\mathbb{D}^\lambda u_{t_i},\ \mathbb{D}^\lambda u_{t_i}\big)\, \mathrm{dvol}_\omega\,.$$

Then, we obtain the desired inequality. ∎

2.3.3 Uniqueness

The following proposition can be shown by an argument similar to the proof of Proposition 2.6 of [31] via the method in [36]. We state it for the reference in the later discussion.

Proposition 2.35 *Let (X, ω) be a complete Kahler manifold satisfying Condition* 2.30, *and $(E, \mathbb{D}^\lambda)$ be a λ-flat bundle on X. Let h_i $(i = 1, 2)$ be hermitian metrics of E such that $\Lambda_\omega G(h_i) = 0$. We assume that h_i $(i = 1, 2)$ are mutually bounded. Then, the following holds:*

- *We have the decomposition of λ-flat bundles $(E, \mathbb{D}^\lambda) = \bigoplus (E_a, \mathbb{D}^\lambda_a)$ which is orthogonal with respect to both of h_i $(i = 1, 2)$.*
- *The restrictions of h_i to E_a are denoted by $h_{i,a}$. Then, there exist positive numbers b_a such that $h_{1,a} = b_a \cdot h_{2,a}$.*

Proof Let s be determined by $h_2 = h_1 \cdot s$. We can show $\mathbb{D}^\lambda s = 0$ by the argument explained in the proof of Proposition 2.6 of [31]. Note we are considering the case $\lambda \neq 0$. Hence, the eigen decomposition of s is $\mathbb{D}^\lambda$-flat, which gives the desired decomposition. ∎

2.4 Review of Donaldson functional

We recall the Donaldson functional, by following Donaldson and Simpson ([5] and [36]).

2.4.1 Functions of self-adjoint endomorphisms

Let V be a vector space over $\boldsymbol{C}$ with a hermitian metric h. Let $S(V,h)$ denote the set of the endomorphisms of V which are self-adjoint with respect to h. Let $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be a continuous function. Then, $\varphi(s)$ is naturally defined for any $s \in S(V,h)$. Namely, let $v_1,\ldots,v_r$ be the orthogonal base which consists of the eigen vectors of s, and let $v_1^{\vee},\ldots,v_r^{\vee}$ be the dual base. Then, we have the description $s = \sum \kappa_i \cdot v_i^{\vee} \otimes v_i$, and we put $\varphi(s) := \sum \varphi(\kappa_i) \cdot v_i^{\vee} \otimes v_i$. Thus, we obtain the induced map $\varphi : S(V,h) \longrightarrow S(V,h)$, which is well known to be continuous. To see the continuity, for example, we can argue as follows: Let $U(h)$ denote the unitary group with respect to h. Take $\boldsymbol{e} = (e_1,\ldots,e_r)$ be an orthogonal base of V. Let T denote the set of endomorphisms of V which is diagonal with respect to the base $\boldsymbol{e}$. Then, we have the continuous surjective map $\pi : U(h) \times T \longrightarrow S(V,h)$ given by $(u,t) \longmapsto u \cdot t \cdot u^{-1}$. It is easy to check the continuity of the composite $\varphi \circ \pi$. Since the topology of $S(V,h)$ is same as the induced topology via π, we obtain the continuity. When φ is real analytic given by the convergent power series $\sum a_j \cdot t^j$, then $\varphi(s) = \sum a_j \cdot s^j$. The induced map is real analytic in this case.

Let $\Psi : \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be a continuous function. For a self-adjoint map $s \in S(V,h)$, let $v_1,\ldots,v_r$ and $v_1^{\vee},\ldots,v_r^{\vee}$ be as above. Then, we put $\Psi(s)(A) = \sum \Psi(\kappa_i,\kappa_j) \cdot A_{i,j} \cdot v_i^{\vee} \otimes v_j$ for any endomorphism $A = \sum A_{i,j} \cdot v_i^{\vee} \otimes v_j$ of V. Thus, we obtain $\Psi : S(V,h) \longrightarrow S(\mathrm{End}(V),h)$, which is also well known to be continuous. Here, $S(\mathrm{End}(V),h)$ denotes the set of the self-adjoint endomorphisms of $\mathrm{End}(V)$ with respect to the metric induced by h. To see the continuity, we can use the same argument as above. When Ψ is real analytic given by a power series, $\sum b_{m,n} t_1^m t_2^n$, then we have $\Psi(s)(A) = \sum b_{m,n} s^m \cdot A \cdot s^n$, and the induced map is real analytic.

Let $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be C^1, and let $d\varphi : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2$ denote the continuous function given by $d\varphi(t_1,t_2) = (t_1-t_2)^{-1}\bigl(\varphi(t_1)-\varphi(t_2)\bigr)$ $(t_1 \neq t_2)$ and $d\varphi(t_1,t_1) = \varphi'(t_1)$. In this case, the induced map $\varphi : S(V,h) \longrightarrow S(V,h)$ is also C^1, and the derivative at s is given by $d\varphi(s)$. To see it, we can argue as follows: When φ is real analytic, the claim can be checked by a direct calculation. In general, we can take an approximate sequence $\varphi_i \longrightarrow \varphi$ by real analytic functions on an appropriate compact neighbourhoods of the eigenvalues of $s \in S(V,h)$. The induced maps $\varphi_i : S(V,h) \longrightarrow S(V,h)$ and $d\varphi_i : S(V,h) \longrightarrow S(\mathrm{End}(V),h)$ uniformly converge to φ and $d\varphi$ on an appropriate compact neighbourhoods of s. Then, we can derive that φ is the integral of the form $d\varphi$ by a general fact.

The construction can be done on manifolds. Namely, let E be a C^{∞}-vector bundle with a hermitian metric h. Let $S_h(E)$ (or simply S_h) be the bundle of the self-adjoint endomorphisms of (E,h), and let $S_h(\mathrm{End}(E))$ be the bundle of the self-adjoint endomorphisms of $(\mathrm{End}(E),h)$. Then, a continuous function $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ induces $\varphi : S_h(E) \longrightarrow S_h(E)$, and $\Psi : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ induces $\Psi : S_h(E) \longrightarrow S_h(\mathrm{End}(E))$. We have $\mathbb{D}^{\lambda}\varphi(s) = d\varphi(s)\bigl(\mathbb{D}^{\lambda}s\bigr)$, when φ is C^1.

2.4.2 A closed one form

Let (X,ω) and $(E,\mathbb{D}^{\lambda})$ be as in Subsection 2.3.1. Following Simpson [36], we introduce the space $P(S_h)$, which consists of sections s of $S_h(E)$ satisfying the following finiteness:

$$\|s\|_{h,\omega,P} := \sup_X |s|_h + \|\mathbb{D}^{\lambda}s\|_{2,h,\omega} + \|\Delta^{\lambda}_{h,\omega}s\|_{1,h,\omega} < \infty.$$

Here, $\|\cdot\|_{p,h,\omega}$ denote the L^p-norm with respect to (h,ω). We will omit to denote ω and h, when there are no risk of confusion. The following lemma corresponds to Proposition 4.1 (d) in [36]. The proof is same.

Lemma 2.36 *Let φ and Ψ are analytic functions on $\boldsymbol{R}$ with infinite radius of convergence. Then, $\varphi : P(S_h) \longrightarrow P(S_h)$ and $\Psi : P(S_h) \longrightarrow P(S_h(\mathrm{End}(E)))$ are analytic.* ∎

Let h be a metric satisfying (18). Let $\mathcal{P}_+(S_h)$ denote the set of the self-adjoint positive definite endomorphisms s with respect to h such that $\|s\|_{h,P} < \infty$ and $\|s^{-1}\|_{h,P} < \infty$. Note $\|s\|_{h,P} < \infty$ and $\sup|s^{-1}|_h < \infty$ imply $\|s^{-1}\|_{h,P} < \infty$. We put $\mathcal{P}_h := \bigl\{h \cdot s \bigm| s \in \mathcal{P}_+(S_h)\bigr\}$. It is easy to see that any $h_1 \in \mathcal{P}_h$ also satisfies (18) due to (14). It is also easy to see $\mathcal{P}_h = \mathcal{P}_{h_1}$ for $h_1 \in \mathcal{P}_h$.

Let $\mathcal{P}(S_h)$ denote the space of the self-adjoint endomorphisms s with respect to h such that $\|s\|_{P,h} < \infty$. It is easy to see that $\mathcal{P}_+(S_h)$ is open in $\mathcal{P}(S_h)$. In particular, we obtain the Banach manifold structure of $\mathcal{P}_+(S_h)$. By the natural bijection $\mathcal{P}_h \simeq \mathcal{P}_+(S_{h_1})$ for $h_1 \in \mathcal{P}_h$, we also obtain the Banach manifold structure of $\mathcal{P}_h$, which is independent of a choice of $h_1 \in \mathcal{P}_h$. We have the map $\mathcal{P}(S_{h_1}) \longrightarrow \mathcal{P}_+(S_{h_1})$ given by $s \longmapsto e^s$ (Lemma 2.36). It gives a diffeomorphism around $0 \in \mathcal{P}(S_{h_1})$ and $1 \in \mathcal{P}_+(S_{h_1})$. Therefore, the map $\mathcal{P}(S_{h_1}) \longrightarrow \mathcal{P}_h$ by $s \longmapsto h_1 \cdot e^s$ gives a diffeomorphism around 0 and h_1. In particular, the tangent space $T_{h_1}\mathcal{P}_h$ can be naturally identified with $\mathcal{P}(S_{h_1})$ for any $h_1 \in \mathcal{P}_h$. We also have the natural isomorphism $\mathcal{P}(S_{h_1}) \simeq \mathcal{P}(S_h)$ given by $t \longmapsto u \cdot t$ for $h_1 = h \cdot u \in \mathcal{P}_h$, which gives the local trivialization of the tangent bundle.

For any $h_1 \in \mathcal{P}_h$ and $s \in T_{h_1}\mathcal{P}_h$, we put as follows:

$$\Phi_{h_1}(s) := \int_X \Phi'_{h_1}(s)\,\mathrm{dvol}_\omega \in \boldsymbol{C}, \quad \Phi'_{h_1}(s) := \sqrt{-1}\,\mathrm{tr}\big(s \cdot \Lambda_\omega G(\mathbb{D}^\lambda, h_1)\big).$$

Then, Φ' gives the $L^1(X, \Omega_X^{1,1})$-valued one form on $\mathcal{P}_h$, and Φ gives the one form of $\mathcal{P}_h$. The differentiability of Φ is easy to see.

Lemma 2.37 *Φ is a closed one form.*

Proof In the following argument, we use the notation $\mathbb{D}^{\lambda\star}$ instead of $\mathbb{D}_h^{\lambda\star}$. Let $k_1, k_2 \in \mathcal{P}_h$. They naturally give the vector field by addition. At any point $h_1 \in \mathcal{P}_h$, they give the tangent vectors $\sigma = h_1^{-1}k_1$ and $\tau = h_1^{-1}k_2$ in $T_{h_1}\mathcal{P}_h = \mathcal{P}(S_{h_1})$. Hence, we have the following at $h + \epsilon k_1$:

$$\Phi_{h+\epsilon k_1}(k_2) = \sqrt{-1}\int \mathrm{tr}\Big((h+\epsilon k_1)^{-1} \cdot k_2 \cdot G(h+\epsilon k_1)\Big) \cdot \omega^{n-1}.$$

We have $(h+\epsilon k_1)^{-1}k_2 = (1+\epsilon\sigma)^{-1}\tau = \tau - \epsilon\sigma\tau + (1+\epsilon\sigma)^{-2}\epsilon^2\sigma^2\tau$. Remark $\sigma^2\tau$ is bounded. We also have the following:

$$\begin{aligned}(1+\epsilon\sigma)\big(G(h+\epsilon k_1) - G(h)\big) &= \mathbb{D}^\lambda\mathbb{D}^{\lambda\star}(1+\epsilon\sigma) - \mathbb{D}^\lambda(1+\epsilon\sigma)\cdot(1+\epsilon\sigma)^{-1}\mathbb{D}^{\lambda\star}(1+\epsilon\sigma) \\ &= \epsilon\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma - \epsilon^2\mathbb{D}^\lambda\sigma\cdot(1+\epsilon\sigma)^{-1}\mathbb{D}^{\lambda\star}\sigma. \quad (22)\end{aligned}$$

Hence, we have $G(h+\epsilon k_1) - G(h) = \epsilon\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma + \epsilon^2 R_0(\epsilon,\sigma,\tau)$, where $R_0(\epsilon,\sigma,\tau)$ is an L^1-section of $\mathrm{End}(E)\otimes\Omega^2$, and the L^1-norm is bounded independently from ϵ. Therefore, we obtain the following:

$$\begin{aligned}\Phi_{h+\epsilon k_1}(k_2) - \Phi_h(k_2) &= \sqrt{-1}\int \mathrm{tr}\big((h+\epsilon k_1)^{-1}\cdot k_2\cdot G(h+\epsilon k_1)\big)\cdot\omega^{n-1} - \sqrt{-1}\int\mathrm{tr}\big(h^{-1}\cdot k_2\cdot G(h)\big)\cdot\omega^{n-1} \\ &= \sqrt{-1}\int\mathrm{tr}\big(\tau G(h+\epsilon k_1) - \tau G(h)\big)\cdot\omega^{n-1} - \epsilon\sqrt{-1}\int\mathrm{tr}\big(\sigma\tau G(h+\epsilon k_1)\big)\cdot\omega^{n-1} + \epsilon\cdot R_1(\epsilon,\sigma,\tau) \\ &= \epsilon\left(\sqrt{-1}\int\mathrm{tr}\big(\tau\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma\big)\cdot\omega^{n-1} - \sqrt{-1}\int\mathrm{tr}\big(\sigma\cdot\tau\cdot G(h)\big)\cdot\omega^{n-1}\right) + \epsilon R_2(\epsilon,\sigma,\tau). \quad (23)\end{aligned}$$

Here, we have $R_i(\epsilon,\sigma,\tau) \longrightarrow 0$ $(i=1,2)$ in $\epsilon \to 0$, due to $\|\sigma\|_P < \infty$ and $\|\tau\|_P < \infty$. Hence, we obtain the following equality:

$$d_h\Phi(\sigma,\tau) = \sqrt{-1}\int\Big(\mathrm{tr}\big(\tau\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma\big) - \mathrm{tr}\big(\sigma\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\tau\big)\Big)\cdot\omega^{n-1} - \sqrt{-1}\int\mathrm{tr}\big([\sigma,\tau]\cdot G(h)\big)\cdot\omega^{n-1}.$$

We have the following equality, due to $[\mathbb{D}^\lambda, \mathbb{D}^{\lambda\star}] = G(h)$:

$$\begin{aligned}(-\overline{\lambda\partial}+\partial)\,\mathrm{tr}(\tau\mathbb{D}^\lambda\sigma) + (\lambda\partial+\overline{\partial})\,\mathrm{tr}(\sigma\mathbb{D}^{\lambda\star}\tau) &= \mathrm{tr}(\mathbb{D}^{\lambda\star}\tau\mathbb{D}^\lambda\sigma) + \mathrm{tr}(\tau\mathbb{D}^{\lambda\star}\mathbb{D}^\lambda\sigma) + \mathrm{tr}(\mathbb{D}^\lambda\sigma\mathbb{D}^{\lambda\star}\tau) + \mathrm{tr}(\sigma\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\tau) \\ = -\,\mathrm{tr}(\tau\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma) + \mathrm{tr}(\tau\cdot[G(h),\sigma]) + \mathrm{tr}(\sigma\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\tau) &= -\,\mathrm{tr}\big(\tau\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma\big) + \mathrm{tr}(\sigma\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\tau) + \mathrm{tr}\big([\sigma,\tau]\cdot G(h)\big) \quad (24)\end{aligned}$$

Hence, we obtain $d_h\Phi(\sigma,\tau) = -\sqrt{-1}\int_X\Big((-\overline{\lambda\partial}+\partial)\,\mathrm{tr}(\tau\mathbb{D}^\lambda\sigma) + (\lambda\partial+\overline{\partial})\,\mathrm{tr}(\sigma\mathbb{D}^{\lambda\star}\tau)\Big)\cdot\omega^{n-1}$. By using $\|\sigma\|_P < \infty$ and $\|\tau\|_P < \infty$, we obtain the vanishing of $d_h\Phi(\sigma,\tau)$, due to Lemma 5.2 of [36]. ∎

2.4.3 Donaldson functional

For $h_1, h_2 \in \mathcal{P}_h$, take a differentiable path $\gamma : [0,1] \longrightarrow \mathcal{P}_h$ such that $\gamma(0) = h_1$ and $\gamma(1) = h_2$, and the Donaldson functional is defined to be $M(h_1, h_2) := \int_\gamma \Phi$. It is independent of a choice of a base metric ω, in the case $\dim X = 1$. We have $M(h_1, h_2) + M(h_2, h_3) = M(h_1, h_3)$ by the construction.

Lemma 2.38 *When $h_2 = h_1 \cdot e^s$ for $s \in \mathcal{P}(S_{h_1})$, we have the following formula:*

$$M\bigl(h_1, h_2\bigr) = \sqrt{-1}\int_X \mathrm{tr}\bigl(s\Lambda_\omega G(h_1)\bigr)\,\mathrm{dvol}_\omega + \int_X \bigl(\Psi(s)\mathbb{D}^\lambda s, \mathbb{D}^\lambda s\bigr)_{\omega,h_1}\,\mathrm{dvol}_\omega\,. \tag{25}$$

Here, $(\cdot,\cdot)_{\omega,h_1}$ denotes the hermitian product induced by ω and h_1, and Ψ is given as follows:

$$\Psi(t_1, t_2) = \frac{e^{t_2 - t_1} - (t_2 - t_1) - 1}{(t_2 - t_1)^2}. \tag{26}$$

See Subsection 2.4.1 *for the meaning of $\Psi(s)(\mathbb{D}^\lambda s)$.*

Proof Let $M'(h_1, h_2)$ denote the right hand side of (25). The following formula immediately follows from the definition:

$$\frac{\partial}{\partial u} M'\bigl(h_1 e^{ts}, h_1 e^{(t+u)s}\bigr)_{|u=0} = \int_X \sqrt{-1}\,\mathrm{tr}\bigl(s\Lambda_\omega G(h_1 e^{ts})\bigr).$$

We also have the following equalities:

$$\frac{\partial^2}{\partial t \partial u} M'\bigl(h_1 e^{ts}, h_1 e^{(t+u)s}\bigr)_{|u=0} = \frac{\partial^2}{\partial t^2} M'\bigl(h_1, h_1 e^{ts}\bigr)_{|u=0} = \frac{\partial^2}{\partial t \partial u} M'\bigl(h_1, h_1 e^{(t+u)s}\bigr)_{|u=0}. \tag{27}$$

The second equality can be shown formally. The first equality can be shown by the argument in the page 883 of [36]. We also have the obvious equality:

$$\frac{\partial}{\partial u} M'(h_1 e^{ts}, h_1 e^{(t+u)s})_{|t=0,u=0} = \frac{\partial}{\partial u} M'(h_1, h_1 e^{(t+u)s})_{|t=0,u=0}.$$

Hence, we obtain the following:

$$\frac{\partial}{\partial t} M'(h_1, h_1 e^{ts}) = \int_X \sqrt{-1}\,\mathrm{tr}\bigl(s\Lambda_\omega G(h_1 e^{ts})\bigr).$$

Thus, $M'(h_1, h_1 e^s)$ is the integral of Φ' along the path $\gamma(t) = h_1 e^{ts}$, and hence $M'(h_1, h_2) = M(h_1, h_2)$. ∎

Remark 2.39 *In* [36]*, the formula* (25) *is adopted to be the definition of the functional. We follow the original definition of Donaldson* [5]. ∎

We obtain the following corollary due to the positivity of the function Ψ.

Corollary 2.40 *If $\sup |\Lambda_\omega G(h)|_h < B$ is satisfied, we have the following inequality:*

$$M(h, he^s) \geq \sqrt{-1}\int \mathrm{tr}\bigl(s\Lambda_\omega G(h)\bigr)\cdot \mathrm{dvol}_\omega \geq -B\int |s|_h \cdot \mathrm{dvol}_\omega\,.$$

In particular, the upper bound of s gives the lower bound of $M(h, he^s)$. ∎

2.4.4 Main estimate

The following key estimate is the counterpart of Proposition 5.3 in [36]. The proof is same.

Proposition 2.41 *Fix $B>0$. Let $(E,\mathbb{D}^{\lambda})$ be a flat λ-connection. Let h be a hermitian metric of E such that $\sup\bigl|\Lambda_{\omega}G(h,\mathbb{D}^{\lambda})\bigr|_h\leq B$. Let $(E,\mathbb{D}^{\lambda},h)$ be analytically stable with respect to ω. Then, there exist positive constants C_i $(i=1,2)$ with the following property:*

- *Let s be any self-adjoint endomorphism satisfying $\|s\|_{P,h}<\infty$, $\mathrm{tr}(s)=0$ and $\sup\bigl|\Lambda_{\omega}G(h\cdot e^{s},\mathbb{D}^{\lambda})\bigr|\leq B$. Then, the following inequality holds:*

$$\sup_X |s|_h\leq C_1+C_2\cdot M(h,he^{s})$$

(Sketch of the proof) The excellent argument given in [36] works in the case of λ-connection without any essential change. Since we would like to use some minor variants of the proposition (Subsections 2.4.5–2.4.6), we recall an outline of the proof for the convenience of the reader. To begin with, we remark that we have only to show the following inequality due to Corollary 2.40:

$$\sup_X |s|_h\leq C_1'+C_2'\cdot\max\bigl\{0,M(h,he^{s})\bigr\},$$

As is noticed in Subsection 2.2.5, the inequality $\Delta_{\omega}^{\lambda}\log\mathrm{tr}(e^{s})\leq\bigl|\Lambda G(h)\bigr|_h+\bigl|\Lambda G(he^{s})\bigr|_{he^{s}}\leq 2B$ holds. Hence, there exist some constants C_i $(i=3,4)$ such that the inequality $\log\mathrm{tr}(e^{s})\leq C_3+C_4\cdot\int\log\mathrm{tr}(e^{s})$ holds for any s as above, due to Condition 2.30. Since we have $C_5+C_6\cdot|s|_h\leq\log\mathrm{tr}\,e^{s}\leq C_7+C_8\cdot|s|_h$ for some positive constants C_i $(i=5,6,7,8)$, there exist some constants C_i $(i=9,10)$ such that the following holds for any s as above:

$$\sup|s|_h\leq C_9+C_{10}\cdot\int|s|_h. \tag{28}$$

Assume that the claim of the proposition does not hold, and we will derive a contradiction. Under the assumption, either one of the following occurs:

Case 1. There exists a sequence $\{s_i\in\mathcal{P}(S_h)\,|\,i=1,2,\cdots,\}$ such that $\sup|s_i|_h\longrightarrow\infty$ and $M(h,he^{s_i})\leq 0$.

Case 2. There exist sequences $\{s_i\in\mathcal{P}(S_h)\}$ and $\{C_{2,i}\in\boldsymbol{R}\}$ with the following properties:

$$\sup_X|s_i|\longrightarrow\infty,\quad C_{2,i}\longrightarrow\infty,\quad(i\longrightarrow\infty)$$

$$M(h,he^{s_i})>0,\quad \sup|s_i|_h\geq C_{2,i}M(h,he^{s_i})$$

In both cases, we have $\|s_i\|_{L^1}\longrightarrow\infty$ due to (28). We put $\ell_i:=\|s_i\|_{L^1}$ and $u_i:=s_i/\ell_i$. Clearly we have $\|u_i\|_{L^1}=1$, and uniform boundedness $\sup_X|u_i|<C$ due to (28). In the following, let $L^2(S_h)$ (resp. $L_1^2(S_h)$) denote the space of L^2-sections (resp. L_1^2-sections) of S_h. The following lemma is one of the keys in the proof of Proposition 2.41.

Lemma 2.42 *After going to an appropriate subsequence, $\{u_i\}$ weakly converges to some $u_{\infty}\neq 0$ in $L_1^2(S_h)$. Moreover, we have the following inequality, for any C^{∞}-function $\Phi:\boldsymbol{R}\times\boldsymbol{R}\longrightarrow\boldsymbol{R}_{\geq 0}$ such that $\Phi(y_1,y_2)\leq(y_1-y_2)^{-1}$ for $y_1>y_2$:*

$$\sqrt{-1}\int\mathrm{tr}\bigl(u_{\infty}\Lambda_{\omega}G(h)\bigr)+\int_X\bigl(\Phi(u_{\infty})\mathbb{D}^{\lambda}u_{\infty},\mathbb{D}^{\lambda}u_{\infty}\bigr)_{h,\omega}\leq 0.$$

Proof By considering $\Phi-\epsilon$ for any small positive number ϵ, we have only to consider the case $\Phi(y_1,y_2)<(y_1-y_2)^{-1}$ for $y_1>y_2$. In the both cases, we have the inequalities for some positive constant C, from the formula (25):

$$\ell_i\sqrt{-1}\int_X\mathrm{tr}\bigl(u_i\Lambda_{\omega}G(h,\mathbb{D}^{\lambda})\bigr)+\ell_i^2\int\bigl(\Psi(\ell_iu_i)\mathbb{D}^{\lambda}u_i,\mathbb{D}^{\lambda}u_i\bigr)_h\leq\ell_i\cdot\frac{C}{C_{2,i}}.$$

(In the case 1, we take any sequence $\{C_{2,i}\}$ such that $C_{2,i} \longrightarrow \infty$). Let Φ be as above. Due to the uniform boundedness of u_i, we may assume that Φ has the compact support. Then, if ℓ is sufficiently large, we have $\Phi(\lambda_1,\lambda_2) < \ell\Psi(\ell\lambda_1,\ell\lambda_2)$. Therefore, we obtain the following inequality:

$$\sqrt{-1}\int_X \mathrm{tr}\big(u_i\Lambda_\omega G(h,\mathbb{D}^\lambda)\big) + \int_X \big(\Phi(u_i)\mathbb{D}^\lambda u_i, \mathbb{D}^\lambda u_i\big)_{h,\omega} \leq \frac{C}{C_{2,i}}.$$

Since $\sup_X |u_i|$ is bounded independently of i, there exists a function Φ as above which satisfies $\Phi(u_i) = c\cdot \mathrm{id}$, moreover, for some small positive number $c>0$. Therefore, we obtain the boundedness of $\{u_i\}$ in L_1^2. By taking an appropriate subsequence, $\{u_i\}$ is weakly convergent in L_1^2. Let u_∞ denote the weak limit. Let Z be any compact subset of X. Then, $\{u_i\}$ is convergent to u_∞ on Z in L^2, and hence $\int_Z |u_i| \to \int_Z |u_\infty|$. Since $\sup|u_i|$ are uniformly bounded, we obtain $\int_Z |u_\infty| \neq 0$, if the volume of $X-Z$ is sufficiently small. Thus, $u_\infty \neq 0$. Similarly, we can show the convergence $\int \mathrm{tr}\big(u_i\Lambda G(h,\mathbb{D}^\lambda)\big) \longrightarrow \int \mathrm{tr}\big(u_\infty\Lambda G(h,\mathbb{D}^\lambda)\big)$. Since $\{u_i\}$ are weakly convergent to u_∞ in L_1^2, we have the almost everywhere convergence of $\{u_i\}$ and $\{\mathbb{D}^\lambda u_i\}$ to u_∞ and $\mathbb{D}^\lambda u_\infty$ respectively. Therefore, the sequence $\{\Phi(u_i)\mathbb{D}^\lambda u_i\}$ converges to $\Phi(u_\infty)\mathbb{D}^\lambda u_\infty$ almost everywhere. Hence, we have

$$\int \big(\Phi(u_\infty)\mathbb{D}^\lambda u_\infty, \mathbb{D}^\lambda u_\infty\big)_{h,\omega} \leq \underline{\lim} \int \big(\Phi(u_i)\mathbb{D}^\lambda u_i, \mathbb{D}^\lambda u_i\big)_{h,\omega}$$

due to Fatou's lemma. Thus, we obtain the desired inequality, and the proof of Lemma 2.42 is finished. ∎

We reproduce the rest of the excellent argument given in [36] just for the completeness. We do not use it in the later argument. The point is that we can derive a contradiction from the existence of the non-trivial section u_∞ as in Lemma 2.42.

Lemma 2.43 *The eigenvalues of u_∞ are constant, and u_∞ has at least two distinct eigenvalues.*

Proof To show the constantness of the eigenvalues, we have only to show the constantness of $\mathrm{tr}\big(\varphi(u_\infty)\big)$ for any C^∞-function $\varphi: \boldsymbol{R} \longrightarrow \boldsymbol{R}$. We have $\big(\overline{\partial}+\lambda\partial\big)\,\mathrm{tr}\,\varphi(u_\infty) = \mathrm{tr}\big(\mathbb{D}^\lambda\varphi(u_\infty)\big) = \mathrm{tr}\big(d\varphi(u_\infty)\mathbb{D}^\lambda u_\infty\big)$. Let N be any large number. We can take a C^∞-function $\Phi: \boldsymbol{R}\times\boldsymbol{R} \longrightarrow \boldsymbol{R}$ such that $\Phi(y_1,y_1) = d\varphi(y_1,y_1)$ and $N\Phi^2(y_1,y_2) < (y_1-y_2)^{-1}$ for $y_1 > y_2$. We obtain $\mathrm{tr}\big(d\varphi(u_\infty)(\mathbb{D}^\lambda u_\infty)\big) = \mathrm{tr}\big(\Phi(u_\infty)\mathbb{D}^\lambda u_\infty\big)$ due to the first condition. We obtain the following inequality from Lemma 2.42:

$$\int_X |\Phi(u_\infty)\mathbb{D}^\lambda u_\infty|^2 \leq -\frac{\sqrt{-1}}{N}\int_X \mathrm{tr}\big(u_\infty\Lambda G(h)\big).$$

Therefore, $\big|(\overline{\partial}+\lambda\partial)\,\mathrm{tr}\,\varphi(u_\infty)\big|_{L^2}^2 = 0$. Thus, the eigenvalues of u_∞ are constant. Since $\mathrm{tr}(u_\infty)=0$ and $u_\infty\neq 0$, u_∞ has at least two distinct eigenvalues. ∎

Let $\kappa_1 < \kappa_2 < \cdots < \kappa_w$ denote the constant distinct eigenvalues of u_∞. Then, $\varphi(u_\infty)$ and $\Phi(u_\infty)$ depend only on the values $\varphi(\kappa_i)$ and $\varphi(\kappa_i,\kappa_j)$ respectively.

Lemma 2.44 *Let $\Phi: \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ be a C^∞-function such that $\Phi(\kappa_i,\kappa_j) = 0$ for $\kappa_i > \kappa_j$. Then, $\Phi(u_\infty)(\mathbb{D}^\lambda u_\infty) = 0$.*

Proof We may replace Φ with Φ_1 satisfying $\Phi_1(\kappa_i,\kappa_j) = 0$ for $\kappa_i > \kappa_j$ and $N\Phi_1^2(y_1,y_2) < (y_1-y_2)^{-1}$ for $y_1 > y_2$. Then, we obtain $\big\|\Phi_1(u_\infty)\mathbb{D}^\lambda u_\infty\big\|_{L^2}^2 \leq C/N$ due to Lemma 2.42, and hence we obtain $\Phi(u_\infty)\mathbb{D}^\lambda u_\infty = \Phi_1(u_\infty)\mathbb{D}^\lambda u_\infty = 0$. ∎

Let γ_i denote the open interval $]\kappa_i,\kappa_{i+1}[$. Let $p_\gamma: \boldsymbol{R} \longrightarrow [0,1]$ be any decreasing C^∞-function such that $p_\gamma(\kappa_i) = 1$ and $p_\gamma(\kappa_{i+1}) = 0$. We put $\pi_\gamma = p_\gamma(u_\infty)$. It is easy to see that π_γ is L_1^2. Due to $p_\gamma^2 = p_\gamma$, we have $\pi_\gamma^2 = \pi_\gamma$. We have $\mathbb{D}^\lambda\pi_\gamma = dp(u_\infty)\mathbb{D}^\lambda u_\infty$. We put $\Phi_\gamma(y_1,y_2) = (1-p_\gamma)(y_2)\cdot dp_\gamma(y_1,y_2)$, and then we have $(1-\pi_\gamma)\circ\mathbb{D}^\lambda\pi_\gamma = \Phi_\gamma(u_\infty)\circ\mathbb{D}^\lambda u_\infty$. On the other hand, since we have $\Phi_\gamma(\kappa_i,\kappa_j) = 0$ ($\kappa_i > \kappa_j$), we obtain $\Phi_\gamma(u_\infty)\mathbb{D}^\lambda u_\infty = 0$ due to Lemma 2.44. Therefore, we obtain $(1-\pi_\gamma)\circ\mathbb{D}^\lambda\pi_\gamma = 0$.

From $(1-\pi_\gamma)d''\pi_\gamma = 0$, we obtain a saturated coherent subsheaf V_γ such that π_γ is the orthogonal projection on V_γ due to the result of Uhlenbeck-Yau [45]. From $(1-\pi_\gamma)d'\pi_\gamma = 0$, the bundle V_γ is $\mathbb{D}^\lambda$-invariant. Since we consider the case $\lambda \neq 0$, it is easy to see that V_γ is indeed a subbundle of E. Namely, we obtain the λ-flat subbundle $(V_\gamma, \mathbb{D}^\lambda_{V_\gamma}) \subset (E, \mathbb{D}^\lambda)$.

Let us show $\deg_\omega(V_\gamma, h_\gamma)/\operatorname{rank} V_\gamma \geq \deg_\omega(E,h)/\operatorname{rank} E$ for some γ, which contradicts the stability assumption of $(E, \mathbb{D}^\lambda, h)$, where $h_\gamma := h_{|V_\gamma}$. From Lemma 2.31, we have

$$\deg(V_\gamma) = \frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int \operatorname{tr}\bigl(\pi_\gamma G(h)\bigr) - \int |\mathbb{D}^\lambda \pi_\gamma|^2\right).$$

We have $u_\infty = \kappa_w \cdot \mathrm{id}_E - \sum |\gamma| \cdot \pi_\gamma$, where $|\gamma|$ denotes the length of γ. We put

$$W = \kappa_w \deg(E) - \sum |\gamma| \cdot \deg(V_\gamma) = \frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int \operatorname{tr}\bigl(u_\infty \Lambda G(h)\bigr) + \int \sum |\gamma| \cdot \bigl|\mathbb{D}^\lambda \pi_\gamma\bigr|^2\right).$$

Since $\mathbb{D}^\lambda \pi_\gamma = dp_\gamma(u_\infty)\mathbb{D}^\lambda u_\infty$, we have

$$W = \frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int \operatorname{tr}\bigl(u_\infty \Lambda G(h)\bigr) + \int \Bigl(\sum |\gamma| \cdot dp_\gamma(u_\infty)^2 \cdot \mathbb{D}^\lambda u_\infty,\ \mathbb{D}^\lambda u_\infty\Bigr)\right).$$

We can check $\sum |\gamma|(dp_\gamma)(\kappa_i,\kappa_j)^2 = (\kappa_i - \kappa_j)^{-1}$ for $\kappa_i > \kappa_j$ by a direct argument. Therefore, we obtain $W \leq 0$, due to Lemma 2.42. Namely we obtain $\kappa_w \cdot \deg E \leq \sum |\gamma| \cdot \deg(V_\gamma)$. On the other hand, we have $0 = \operatorname{tr}(u_\infty) = \kappa_w \cdot \operatorname{rank} E - \sum |\gamma| \cdot \operatorname{rank} V_\gamma$. Therefore, we obtain $\deg(V_\gamma)/\operatorname{rank} V_\gamma \geq \deg(E)/\operatorname{rank} E$ for at least one of γ, which contradicts with the stability of $(E, \mathbb{D}^\lambda, h)$. Thus, the proof of Proposition 2.41 is finished. ∎

2.4.5 Variant 1 of Proposition 2.41

Let C be a smooth projective curve, and D be a simple divisor. Let $(E, \mathbb{D}^\lambda, \boldsymbol{F})$ be a λ-flat bundle on (C,D). Let η be a sufficiently small positive number such that $10 \cdot \eta < \operatorname{gap}(E, \boldsymbol{F})$. Let ϵ_0 be a sufficiently smaller number than η, for example $10 \operatorname{rank}(E)\epsilon_0 < \eta$. Let ω_ϵ $(0 \leq \epsilon < \epsilon_0)$ be a Kahler metric of $C - D$ with the following conditions:

- Let $P \in D$. Let (U, z) be a holomorphic coordinate around P such that $z(P) = 0$. Then, the following holds for some positive constants C_i $(i = 1, 2)$:

$$C_1 \cdot \omega_\epsilon \leq \epsilon^2 |z|^{2\epsilon} \frac{dz \cdot d\overline{z}}{|z|^2} + \eta^2 |z|^{2\eta} \frac{dz \cdot d\overline{z}}{|z|^2} \leq C_2 \cdot \omega_\epsilon$$

- $\omega_\epsilon \longrightarrow \omega_0$ for $\epsilon \to 0$ in the C^∞-sense locally on $C - D$.

Let $\boldsymbol{F}^{(\epsilon)}$ be an ϵ-perturbation of $\boldsymbol{F}$. See Subsection 2.1.6 for the notion of ϵ-perturbation. We discuss the surface case there, but it can be applied in the curve case. Suppose that we are given hermitian metrics $h^{(\epsilon)}$ for $(E, \boldsymbol{F}^{(\epsilon)})$ with the following properties:

- $\bigl|\Lambda_{\omega_\epsilon} G(h^{(\epsilon)}, \mathbb{D}^\lambda)\bigr|_{h^{(\epsilon)}} \leq C_1$, where the constant C_1 is independent of ϵ.
- $\{h^{(\epsilon)}\}$ converges to $h^{(0)}$ for $\epsilon \to 0$ in the C^∞-sense locally on $C - D$.

Lemma 2.45 *Let $s^{(\epsilon)}$ be self-adjoint endomorphisms of $(E, h^{(\epsilon)})$ satisfying $\operatorname{tr} s^{(\epsilon)} = 0$ and the following properties:*

- $\|s^{(\epsilon)}\|_{P, h^{(\epsilon)}, \omega_\epsilon} < \infty$. *But we do not assume the uniform boundedness.*
- $\bigl|\Lambda_{\omega_\epsilon} G(h^{(\epsilon)} e^{s^{(\epsilon)}}, \mathbb{D}^\lambda)\bigr|_{h^{(\epsilon)}} \leq C_1$. *The constant C_1 is independent of ϵ.*

Then, there exist constants C_i $(i = 3, 4)$, which are independent of ϵ, with the following property:

$$\sup |s^{(\epsilon)}|_{h^{(\epsilon)}} \leq C_3 + C_4 \cdot M(h^{(\epsilon)},\ h^{(\epsilon)} e^{s^{(\epsilon)}}).$$

(Sketch of a proof) The argument is essentially same as the proof of Proposition 2.41. We assume that the claim does not hold, and we will derive a contradiction. After going to an appropriate subsequence, either one of the following holds:

Case 1. $M(h^{(\epsilon)}, h^{(\epsilon)}e^{s^{(\epsilon)}}) \leq 0$ and $\sup_{C-D}|s^{(\epsilon)}|_{h^{(\epsilon)}} \longrightarrow \infty$ for $\epsilon \to 0$.

Case 2. $M\big(h^{(\epsilon)}, h^{(\epsilon)}e^{s^{(\epsilon)}}\big) > 0$, $\sup|s^{(\epsilon)}| \geq C_2^{(\epsilon)} M\big(h^{(\epsilon)}, h^{(\epsilon)}e^{s^{(\epsilon)}}\big)$, $\sup_{C-D}\big|s^{(\epsilon)}\big|_{h^{(\epsilon)}} \longrightarrow \infty$ and $C_2^{(\epsilon)} \longrightarrow \infty$ for $\epsilon \to 0$.

By using Lemma 2.47 (given below) and the argument given in the first part of Proposition 2.41, we can show that there exist positive constants C_i $(i = 5, 6)$, which are independent of ϵ, with the following property:

$$\sup_{C-D}|s^{(\epsilon)}|_{h^{(\epsilon)}} \leq C_5 + C_6 \cdot \int |s^{(\epsilon)}|_{h^{(\epsilon)}} \,\mathrm{dvol}_{\omega_\epsilon}.$$

We put $\ell^{(\epsilon)} := \|s^{(\epsilon)}\|_{L^1}$ and $u^{(\epsilon)} := s^{(\epsilon)}/\ell^{(\epsilon)}$. The following lemma is the counterpart of Lemma 2.42.

Lemma 2.46 *We have a non-trivial L_1^2-section u_∞ of $S_{h^{(0)}}$ with the following property:*

- *The following inequality holds for any C^∞-function $\Phi : \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}_{\geq 0}$ such that $\Phi(y_1, y_2) \leq (y_1 - y_2)^{-1}$ for $y_1 > y_2$:*

$$\sqrt{-1}\int_{C-D} \mathrm{tr}\big(u_\infty \Lambda_{\omega_0} G(h^{(0)})\big)\,\mathrm{dvol}_{\omega_0} + \int_{C-D} \big(\Phi(u_\infty)\mathbb{D}^\lambda u_\infty, \mathbb{D}^\lambda u_\infty\big)_{h^{(0)},\omega_0}\,\mathrm{dvol}_{\omega_0} \leq 0.$$

Proof The argument is essentially same as the proof of Lemma 2.42. We have the following for some positive constant C_5:

$$\sqrt{-1}\int_{C-D} \mathrm{tr}\big(u^{(\epsilon)} \Lambda_{\omega_\epsilon} G(h^{(\epsilon)})\big)\,\mathrm{dvol}_{\omega_\epsilon} + \int_{C-D} \big(\Phi(u^{(\epsilon)})\mathbb{D}^\lambda u^{(\epsilon)}, \mathbb{D}^\lambda u^{(\epsilon)}\big)_{h^{(\epsilon)},\omega_\epsilon}\,\mathrm{dvol}_{\omega_\epsilon} \leq \frac{C_5}{C_2^{(\epsilon)}}.$$

(In the case 1, we take any sequence $\{C_2^{(\epsilon)}\}$ such that $C_2^{(\epsilon)} \longrightarrow \infty$.) From this, we obtain the following boundedness as in the proof of Lemma 2.42:

$$\int_{C-D} \big|\mathbb{D}^\lambda u^{(\epsilon)}\big|^2_{h^{(\epsilon)}}\,\mathrm{dvol}_{\omega_\epsilon} < C_{10}.$$

Let us take a sequence of C^∞-isometries $F_\epsilon : (E, h^{(\epsilon)}) \longrightarrow (E, h^{(0)})$ which converges to the identity of E, in the C^∞-sense locally on $C - D$. Remark that the sequence $\{F_\epsilon(\mathbb{D}^\lambda)\}$ converges to $\mathbb{D}^\lambda$ for $\epsilon \to 0$ in the C^∞-sense locally on $C - D$. The sequence $\{F_\epsilon(u^{(\epsilon)})\}$ is bounded on L_1^2 locally on $C - D$. By going to an appropriate subsequence, we may assume that the sequence $\{u^{(\epsilon)}\}$ is weakly convergent in L_1^2 locally on $C - D$, and hence it is convergent in L^2 on any compact subset $Z \subset C - D$. Let u_∞ denote the weak limit. We have $\int_Z |u^{(\epsilon)}| \longrightarrow \int_Z |u_\infty|$. Hence $\int_Z |u_\infty| \neq 0$, when the volume of $C - Z \cup D$ is sufficiently small, due to the boundedness of $\big\{\sup|u^{(\epsilon)}| \,\big|\, \epsilon > 0\big\}$. In particular, $u_\infty \neq 0$. Similarly, we obtain $\int_{C-D} \mathrm{tr}(u^{(\epsilon)} G(h^{(\epsilon)})) \longrightarrow \int_{C-D} \mathrm{tr}(u_\infty G(h^{(0)}))$. Since we can derive the almost everywhere convergence $\Phi(u^{(\epsilon)})\mathbb{D}^\lambda u^{(\epsilon)} \longrightarrow \Phi(u_\infty)\mathbb{D}^\lambda u_\infty$ and $u^{(\epsilon)} \longrightarrow u_\infty$, we obtain $\int_{C-D}\big(\Phi(u_\infty)\mathbb{D}^\lambda u_\infty, \mathbb{D}^\lambda u_\infty\big) \leq \underline{\lim} \int_{C-D}\big(\Phi(u^{(\epsilon)})\mathbb{D}^\lambda u^{(\epsilon)}, \mathbb{D}^\lambda u^{(\epsilon)}\big)$ due to Fatou's lemma. Thus, the proof of Lemma 2.46 is finished. ∎

The rest of the proof of Lemma 2.45 is completely same as the argument for Proposition 2.41. ∎

We have used the following lemma in the proof.

Lemma 2.47 *For any positive number B, there exist positive constants C_i $(i = 1, 2)$ with the following property:*

- *Let ϵ be any positive number such that $\epsilon < 1/2$. Let f be any non-negative bounded C^∞-function on $C - D$ such that $\Delta_{\omega_\epsilon} f \leq B$. Then, the inequality $\sup(f) \leq C_1 + C_2 \int f \cdot \mathrm{dvol}_{\omega_\epsilon}$ holds.*

Proof Let (U_P, z) be as above for $P \in D$, and $U_P^* := U_P - \{z = 0\}$. On U_P^*, the inequality $\Delta_{\omega_\epsilon} f \leq B$ is equivalent to the following:

$$\Delta_{g_0} f \leq B \cdot \left(\epsilon^2 \frac{|z|^{2\epsilon}}{|z|^2} + \eta^2 \frac{|z|^{2\eta}}{|z|^2} \right). \tag{29}$$

Here, $g_0 := dz \cdot d\overline{z}$. Because of the boundedness of f, (29) holds on U_P. (See the proof of Proposition 2.2 of [36].) Then, we obtain the following inequality on U_P:

$$\Delta_{g_0}\big(f - B \cdot \phi\big) \leq 0, \quad \phi = |z|^{2\epsilon} + |z|^{2\eta}.$$

For any point $Q \in \Delta(P, 1/2)$, we have the following:

$$\big(f - B \cdot \phi\big)(Q) \leq \frac{4}{\pi} \int_{\Delta(Q,1/2)} \big(f - B \cdot \phi\big) \cdot \mathrm{dvol}_{g_0}.$$

Therefore, there exist some constants C_i $(i = 3, 4)$ which are independent of ϵ, such that the following holds:

$$f(Q) \leq C_3 + C_4 \int f \cdot \mathrm{dvol}_{\omega_\epsilon}.$$

Thus, we obtain the upper bound of $f(Q)$, when Q is close to a point of D. We can obtain such an estimate when Q is far from D, similarly and more easily. ∎

2.4.6 Variant 2 of Proposition 2.41

We will use another variant. Let $\pi : \mathcal{C} \longrightarrow \Delta$ be a holomorphic family of smooth projective curves. Let $\mathcal{D} \subset \mathcal{C}$ be a relative divisor. Let $(E, \mathbb{D}^\lambda, \boldsymbol{F})$ be a logarithmic parabolic λ-flat bundle on $(\mathcal{C}, \mathcal{D})$. We denote the fiber $\pi^{-1}(t)$ by $\mathcal{C}_t$ for $t \in \Delta$. The restriction $(E, \mathbb{D}^\lambda, \boldsymbol{F})_{|\mathcal{C}_t}$ is denoted by $(E_t, \mathbb{D}_t^\lambda, \boldsymbol{F}_t)$. Let ω be a metric of the relative tangent bundle of $\mathcal{C}/\Delta$ such that $\omega \sim \eta^2 |z|^{2\eta-2} dz \cdot d\overline{z}$ around $\mathcal{D}$. Here, η denotes a small positive number such that $10\,\mathrm{rank}(E) \cdot \eta < \mathrm{gap}(E, \boldsymbol{F})$, and z is holomorphic function such that $z^{-1}(0) = \mathcal{D}$ and $dz \neq 0$. The restriction $\omega_{|\mathcal{C}_t}$ is denoted by ω_t for $t \in \Delta$. Let h be a C^∞-hermitian metric of E adapted to $\boldsymbol{F}$ such that $\big|\Lambda_{\omega_t} G(\mathbb{D}_t^\lambda, h_t)\big|_{h_t} \leq C_1$ for any $t \in \Delta$, where a constant C_1 is independent of t, and h_t denotes the restriction $h_{|\mathcal{C}_t}$. The following lemma can be shown by an argument similar to the proof of Lemma 2.47.

Lemma 2.48 *There exist positive constants C_i $(i = 3, 4)$, which are independent of t, with the following property.*

- *Let $s^{(t)}$ be an element of $\mathcal{P}_{h_t}(E_t)$ satisfying $\mathrm{tr}\, s^{(t)} = 0$, $\|s^{(t)}\|_{h_t,P} < \infty$ and $\big|\Lambda_{\omega_t} G(\mathbb{D}_t^\lambda, h_t e^{s^{(t)}})\big| \leq C_1$. Then, the inequality $\sup |s^{(t)}| \leq C_3 + C_4 \cdot M(h_t, h_t e^{s^{(t)}})$ holds.* ∎

2.5 Regular filtered λ-flat bundles associated to tame harmonic bundles

2.5.1 Tame pluri-harmonic metric

Let X be a complex manifold with a simple normal crossing divisor D. Let $(E, \mathbb{D}^\lambda)$ be a λ-flat bundle on $X - D$. Let h be a pluri-harmonic metric of $(E, \mathbb{D}^\lambda)$. Then, we have the induced Higgs bundle $(E, \overline{\partial}_h, \theta_h)$. Recall the tameness of pluri-harmonic metric. Let P be any point of X, and let $(U_P, z_1, \ldots, z_n)$ be a holomorphic coordinate around P such that $D \cap U_P = \bigcup_{i=1}^{l} \{z_i = 0\}$. Then, we have the expression:

$$\theta = \sum_{i=1}^{l} f_i \cdot \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} g_j \cdot dz_j.$$

The pluri-harmonic metric h is called tame, if the coefficients of the characteristic polynomials $\det(t - f_i)$ and $\det(t - g_j)$ are holomorphic on U_P for any P. A λ-flat bundle with tame pluri-harmonic metric is called a tame harmonic bundle. Recall that the curve test for tameness is valid.

Proposition 2.49 (Corollary 8.7 of [30]) *A pluri-harmonic metric h for $(E,\mathbb{D}^{\lambda})$ is tame, if and only if $h_{|C}$ is tame for any closed curve $C \subset X$ transversal with D.* ▮

From a holomorphic vector bundle E with a hermitian metric h, we obtain the filtered sheaf $\boldsymbol{E}_*(h) := \bigl({}_{\boldsymbol{c}}E \,\big|\, \boldsymbol{c} \in \boldsymbol{R}^S\bigr)$ as explained in Subsection 3.5 of [31]. We recall the following proposition.

Proposition 2.50 (Theorem 8.58, Theorem 8.59 and Corollary 8.89 of [30]) *Let $(E,\mathbb{D}^{\lambda},h)$ be a tame harmonic bundle on $X-D$. Then, $\bigl(\boldsymbol{E}_*(h),\mathbb{D}^{\lambda}\bigr)$ is a regular filtered λ-flat bundle.* ▮

2.5.2 One dimensional case

In the one dimensional case, Simpson established the Kobayashi-Hitchin correspondence for parabolic flat bundles and the parabolic Higgs bundles, i.e., λ-flat bundles in the case $\lambda = 0,1$. His result can be generalized for any λ.

Proposition 2.51 (Simpson, [37]) *Let X be a smooth projective curve, and D be a simple divisor of X. Let $\bigl(\boldsymbol{E}_*,\mathbb{D}^{\lambda}\bigr)$ be a regular filtered λ-flat bundle on (X,D). We put $E = {}_{\boldsymbol{c}}E_{|X-D}$. The following conditions are equivalent:*

- *$(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ is poly-stable with* $\text{par-deg}(\boldsymbol{E}_*) = 0$.
- *There exists a harmonic metric h of $(E,\mathbb{D}^{\lambda})$, which is adapted to the parabolic structure of $\boldsymbol{E}_*$, i.e., $\boldsymbol{E}_* \simeq \boldsymbol{E}_*(h)$.*

Moreover, such a metric is unique up to obvious ambiguity. Namely, let h_i $(i=1,2)$ be two harmonic metrics as above. Then, we have the decomposition of Higgs bundles $(E,\mathbb{D}^{\lambda}) = \bigoplus(E_a,\mathbb{D}^{\lambda}_a)$ satisfying the following:

- *The decomposition is orthogonal with respect to both of h_i.*
- *The restrictions of h_i to E_a are denoted by $h_{i,a}$. Then, there exist positive numbers b_a such that $h_{1,a} = b_a \cdot h_{2,a}$.* ▮

2.5.3 The projective case

Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor of X with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. Let $(E,\mathbb{D}^{\lambda},h)$ be a tame harmonic bundle on $X-D$.

Proposition 2.52 *Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be as above.*

- *$(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ is μ_L-polystable with* $\text{par-deg}_L(\boldsymbol{E}_*) = 0$.
- *Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda}) = \bigoplus_j (\boldsymbol{E}_{j\,*},\mathbb{D}^{\lambda}_j)\otimes \boldsymbol{C}^{p(j)}$ be the canonical decomposition of μ_L-polystable regular filtered λ-flat bundle. Then, we have the corresponding decomposition of the metric $h = \bigoplus h_i \otimes g_i$, where h_i denote pluri-harmonic metrics of $(E_i,\mathbb{D}^{\lambda}_i)$ adapted to the parabolic structure, and g_i denote metrics of $\boldsymbol{C}^{p(i)}$.*
- *We have the vanishings of characteristic numbers:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = \int_X \text{par-c}^2_{1,L}(\boldsymbol{E}_*) = 0.$$

Proof The first two claims can be shown by the same argument as the proof of Proposition 5.1 of [31]. The third claim can be shown by an argument similar to the proof of Proposition 5.3 of [31], which we explain briefly. We have only to consider the case $\dim X = 2$. Since h is pluri-harmonic, we have the equalities $\operatorname{tr} R(d'',h) = (1+|\lambda|^2)^{-1}\operatorname{tr} G(h,\mathbb{D}^{\lambda}) = 0$ and $\operatorname{tr}\bigl(R(d'',h)^2\bigr) = (1+|\lambda|^2)^{-2}\cdot\operatorname{tr}\bigl(G(h,\mathbb{D}^{\lambda})^2\bigr) = 0$, due to Lemma 2.28 and Lemma 2.29 on $X-D$. We also have the norm estimate for the holomorphic sections of ${}_{\boldsymbol{c}}E$. (It is explained in Subsection 2.5 of [31] for $\lambda = 0$. Similar claims hold for any λ, as shown in Subsection 13.3 of [30].) Then, the same argument as the proof of Proposition 5.3 works. ▮

Proposition 2.53 *Let* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *be a regular filtered* λ*-flat bundle. We put* $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. *Let* h_a $(a = 1, 2)$ *be pluri-harmonic metrics of* $(E, \mathbb{D}^\lambda)$ *on* $X - D$ *which is adapted to the parabolic structure. Then, we have the decomposition* $(E, \mathbb{D}^\lambda) = \bigoplus (E_i, \mathbb{D}^\lambda)$ *with the following properties:*

- *The decomposition is orthogonal with respect to both of* h_a $(a = 1, 2)$. *Hence, we have the decomposition* $h_a = \bigoplus_i h_{a,i}$.
- *There exist positive numbers* b_i *such that* $h_{1,i} = b_i \cdot h_{2,i}$.

The decomposition on $X - D$ *is prolonged to the decomposition* $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = \bigoplus (\boldsymbol{E}_{i\,*}, \mathbb{D}^\lambda)$ *on* X.

Proof Similar to Proposition 5.2 of [31]. ∎

2.6 Some integral for non-flat λ-connection on a curve

Let Y be a smooth projective curve, and let D be a divisor. Let $(E, \boldsymbol{F})$ be a parabolic bundle on (Y, D). Let $\mathbb{D}^\lambda$ be a C^∞ λ-connection on $E_{|Y-D}$. In this subsection, we do not assume $\mathbb{D}^\lambda$ is flat, i.e., $(\mathbb{D}^\lambda)^2$ may not be 0. But, it is assumed to be flat around an appropriate neighbourhood U_P of each $P \in D$, and $(E, \boldsymbol{F}, \mathbb{D}^\lambda)_{|U_P}$ is a parabolic λ-flat bundle. In particular, we have $\mathrm{Res}_P(\mathbb{D}^\lambda) \in \mathrm{End}(E_{|P})$. We assume that it is graded semisimple, for simplicity, i.e., the induced endomorphism on $\mathrm{Gr}^F(E_{|P})$ is semisimple for each $P \in D$. (By using ϵ-perturbation in Subsection 2.1.6, we can drop the condition.)

For each $P \in D$, we have the generalized eigen decomposition $E_{|P} := \bigoplus {}^P\mathbb{E}_\alpha$ of $\mathrm{Res}_P(\mathbb{D}^\lambda)$. We also have the filtration ${}^P F$ of $E_{|P}$. Let us take a holomorphic frame $\boldsymbol{v}$ of $E_{|U_P}$, which is compatible with $({}^P\mathbb{E}, {}^P F)$. We put $\alpha(v_i) := \deg^{\mathbb{E}}(v_i)$ and $a(v_i) := \deg^F(v_i)$. Let h be a C^∞-metric of $E_{|Y-D}$ such that $h(v_i, v_j) = |z|^{-2a(v_i)}$ $(i = j)$ and 0 $(i \neq j)$. Let us decompose $\mathbb{D}^\lambda = d'' + d'$. Let us take a $(1,0)$-operator d'_0 such that $d'' + d'_0$ is C^∞ λ-connection of E on Y, not only on $Y - D$. We also assume $d'_0\boldsymbol{v} = 0$. We put $A := d' - d'_0$, which is a C^∞-section of $\mathrm{End}(E) \otimes \Omega^{1,0}(\log D)$ on Y, and holomorphic around D. We have $\mathrm{tr}\,\mathrm{Res}_P(A) = \mathrm{tr}\,\mathrm{Res}_P(\mathbb{D}^\lambda)$.

Let h_0 be a C^∞-metric of E on Y such that $h_0(v_i, v_j)$ is 1 $(i = j)$ or 0 $(i \neq j)$ on U_P $(P \in D)$. Let s be the endomorphism determined by $h = h_0 \cdot s$. Then, s is described by the diagonal matrix $\mathrm{diag}\big(|z|^{-2a(v_1)}, \ldots, |z|^{-2a(v_r)}\big)$ with respect to the frame $\boldsymbol{v}$ on U_P.

Although $\mathbb{D}^\lambda$ is not necessarily flat, we obtain the operators δ'_h, δ''_h, $\overline{\partial}_h$, ∂_h, θ_h and $\theta_h^\dagger$ as in Subsection 2.2.1. We put $\mathrm{wt}(E, \boldsymbol{F}, P) := \sum_{a \in \mathcal{P}ar(E,P)} a \cdot \mathrm{rank}\big({}^P\mathrm{Gr}^F_a(E)\big)$.

Lemma 2.54 *We have the following formula:*

$$\frac{\sqrt{-1}}{2\pi} \int_Y \overline{\partial}\,\mathrm{tr}\,\theta = \frac{\lambda}{1 + |\lambda|^2} \sum_P \Big(\lambda^{-1} \cdot \mathrm{tr}\,\mathrm{Res}_P\,\mathbb{D}^\lambda + \mathrm{wt}(E, \boldsymbol{F}, P)\Big). \tag{30}$$

Proof Let δ'_{h_0} denote the $(1,0)$-operator obtained from d'' and h_0 as in Subsection 2.2.1. Then, we have

$$\theta_h = \frac{1}{1 + |\lambda|^2}(d' - \lambda \cdot \delta'_h) = \frac{1}{1 + |\lambda|^2}(d'_0 - \lambda \cdot \delta'_{h_0}) + \frac{1}{1 + |\lambda|^2}(A - \lambda \cdot s^{-1}\delta'_{h_0}s).$$

We would like to apply the Stokes formula to the integral of $\overline{\partial}\,\mathrm{tr}\,\theta_h$. If we do so, $d'_0 - \lambda\delta'_{h_0}$ does not contribute, because it is the C^∞-section of $\mathrm{End}(E) \otimes \Omega^{1,0}$. We have

$$\frac{\sqrt{-1}}{2\pi} \int_Y \overline{\partial}\,\mathrm{tr}(A) = \sum_P \mathrm{tr}\,\mathrm{Res}_P\,\mathbb{D}^\lambda.$$

Since $s^{-1}\delta'_{h_0}s$ is described by $\mathrm{diag}\big(-a(v_1), \ldots, -a(v_r)\big) \cdot dz/z$ with respect to $\boldsymbol{v}$ on U_P $(P \in D)$, we have

$$\frac{\sqrt{-1}}{2\pi} \int_Y \overline{\partial}\,\mathrm{tr}(s^{-1}\delta'_{h_0}s) = \sum_P \sum_{i=1}^{\mathrm{rank}\,E} -a(v_i) = -\sum_P \mathrm{wt}(E, \boldsymbol{F}, P).$$

Therefore, we obtain the following formula:

$$\frac{\sqrt{-1}}{2\pi}\frac{1+|\lambda|^2}{\lambda}\int\overline{\partial}\operatorname{tr}\theta_h=\sum_P\Bigl(\lambda^{-1}\operatorname{tr}\operatorname{Res}_P\mathbb{D}^\lambda+\operatorname{wt}(E,\boldsymbol{F},P)\Bigr). \tag{31}$$

Thus, we obtain (30). ∎

3 Ordinary metric and some consequences

We give a construction of an ordinary metric for a graded semisimple parabolic λ-flat bundle on a surface satisfying SPW-condition, and we give the estimate for the pseudo curvature. Then, we obtain the existence result of Hermitian-Einstein metric if such a parabolic λ-flat bundle is stable.

3.1 Around the intersection of the divisor

3.1.1 Some estimates

We put $X:=\Delta_z^2$, $D_i:=\{z_i=0\}$, and $D:=D_1\cup D_2$. Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a *graded semisimple* regular filtered λ-flat bundle on (X,D). Let $\boldsymbol{c}=(c_1,c_2)\in\boldsymbol{R}^2$ such that $c_i\notin\mathcal{P}ar(\boldsymbol{E}_*,i)$. We assume the following:

(SPW) We have a positive integer m and $\gamma_i\in\boldsymbol{R}$ with $-1/m<\gamma_i\leq 0$, such that $\mathcal{P}ar\bigl({}_{\boldsymbol{c}}E_*,i\bigr)$ is contained in $\bigl\{c_i+\gamma_i+p/m\,\big|\,p\in\mathbb{Z},\ -1<\gamma_i+p/m<0\bigr\}$. (The condition $-1/m<\gamma_i\leq 0$ is not essential.)

We put $\widetilde{X}:=\Delta_\zeta^2$, $\widetilde{D}_i:=\{\zeta_i=0\}$ and $\widetilde{D}=\widetilde{D}_1\cup\widetilde{D}_2$. Let $\varphi:\widetilde{X}\longrightarrow X$ be the ramified covering given by $\varphi(\zeta_1,\zeta_2)=(\zeta_1^m,\zeta_2^m)$. Let $\operatorname{Gal}(\widetilde{X}/X)$ denote the Galois group of $\widetilde{X}/X$. Recall the construction in [15]. For any $\boldsymbol{a}\in\boldsymbol{R}^2$, let ${}_{\boldsymbol{a}}\widetilde{E}$ denote the subsheaf of $\widetilde{\boldsymbol{E}}:=\varphi^*(\boldsymbol{E})$ given as follows:

$${}_{\boldsymbol{a}}\widetilde{E}:=\bigcup_{\boldsymbol{n}+m\boldsymbol{d}\leq\boldsymbol{a}}\varphi^*\bigl({}_{\boldsymbol{d}}E\bigr)\cdot\prod_{i=1,2}\zeta_i^{-n_i}$$

Then, it is easy to see that $\widetilde{\boldsymbol{E}}_*=\bigl({}_{\boldsymbol{a}}\widetilde{E}\,\big|\,\boldsymbol{a}\in\boldsymbol{R}^2\bigr)$ is a filtered bundle, and the induced flat λ-connection $\widetilde{\mathbb{D}}^\lambda$ is regular. We put $\widetilde{c}_i:=m\cdot(\gamma_i+c_i)$. By the assumption, we have $\mathcal{P}ar\bigl(\widetilde{\boldsymbol{E}}_*,i\bigr)=\bigl\{p+\widetilde{c}_i\,\big|\,p\in\mathbb{Z}\bigr\}$.

We have the generalized eigen decompositions ${}_{\boldsymbol{c}}E_{|D_i}=\bigoplus{}^i\mathbb{E}_\alpha$ with respect to $\operatorname{Res}_i(\mathbb{D}^\lambda)$. We have the parabolic filtration ${}^iF$ of ${}_{\boldsymbol{c}}E$. Let $\boldsymbol{v}$ be a frame of ${}_{\boldsymbol{c}}E$ compatible with ${}^iF$ and ${}^i\mathbb{E}$ $(i=1,2)$. We put as follows:

$$a_i(v_j):={}^i\deg^F(v_j)-(c_i+\gamma_i)\in\frac{1}{m}\cdot\mathbb{Z}_{\leq 0}$$

Let $\alpha_i(v_j)\in\boldsymbol{C}$ denote the complex number determined by $v_{j|D_i}\in{}^i\mathbb{E}_{\alpha_i(v_j)}$. We put $\widetilde{v}_j:=\varphi^*(v_j)\cdot\prod_{i=1,2}\zeta_i^{ma_i(v_j)}$. Then, $\widetilde{\boldsymbol{v}}=(\widetilde{v}_j)$ gives the frame of ${}_{\widetilde{\boldsymbol{c}}}\widetilde{E}$. We put $\beta_i(v_j):=m\bigl(\lambda\cdot a_i(v_j)+\alpha_i(v_j)\bigr)$. Let Γ be the diagonal matrix whose (j,j)-entry is $\sum_{i=1,2}\beta_i(v_j)\cdot d\zeta_i/\zeta_i$. Let A be determined by $\widetilde{\mathbb{D}}^\lambda\widetilde{\boldsymbol{v}}=\widetilde{\boldsymbol{v}}\cdot A$, and let $A_0:=A-\Gamma$. In the following, let $F_\Gamma\in\operatorname{End}\bigl({}_{\widetilde{\boldsymbol{c}}}\widetilde{E}\bigr)\otimes\Omega^1(\log\widetilde{D})$ be determined by $F_\Gamma(\widetilde{\boldsymbol{v}})=\widetilde{\boldsymbol{v}}\cdot\Gamma$. We put $\widetilde{\mathbb{D}}_0^\lambda:=\widetilde{\mathbb{D}}^\lambda-F_\Gamma$.

Let $A_0=\sum_{i=1,2}A_0^i\cdot d\zeta_i$. If m is sufficiently large, we may assume the following:

(A): $A_0^i=O(\zeta_i^2)$. Moreover, $(A_0^1)_{j,k}=O\bigl(\zeta_1^2\cdot\zeta_2^2\bigr)$ in the case $\beta_2(v_j)\neq\beta_2(v_k)$, and $(A_0^2)_{j,k}=O\bigl(\zeta_1^2\cdot\zeta_2^2\bigr)$ in the case $\beta_1(v_j)\neq\beta_1(v_k)$.

Let $\widetilde{h}$ be the hermitian metric of ${}_{\widetilde{\boldsymbol{c}}}\widetilde{E}$ determined by $\widetilde{h}(\widetilde{v}_i,\widetilde{v}_j)=\delta_{i,j}\cdot|\zeta_1|^{-2\widetilde{c}_1}\cdot|\zeta_2|^{-2\widetilde{c}_2}$.

Let $\widetilde{\theta}$ (resp. $\widetilde{\theta}_0$) be the section of $\operatorname{End}(\widetilde{E})\otimes\Omega^1$ on $\widetilde{X}-\widetilde{D}$ induced by $\widetilde{h}$ and $\widetilde{\mathbb{D}}^\lambda$ (resp. $\widetilde{\mathbb{D}}_0^\lambda$) as in Subsection 2.2.1. Let $\widetilde{\theta}^\dagger$ and $\widetilde{\theta}_0^\dagger$ denote the adjoint of $\widetilde{\theta}$ and $\widetilde{\theta}_0$, respectively. Let $\widetilde{g}$ denote the Euclidean metric of $\widetilde{X}$.

Lemma 3.1

- $\bigl[\widetilde{\theta},\widetilde{\theta}^\dagger\bigr]$ *is bounded with respect to* $\widetilde{h}$ *and* $\widetilde{g}$.

- $\widetilde{\theta}^2 = O(z_1 \cdot z_2) \cdot dz_1 \cdot dz_2$.

Proof We have the relations $\widetilde{\theta} = \widetilde{\theta}_0 + (1+|\lambda|^2)^{-1} F_\Gamma$ and $\widetilde{\theta}^\dagger = \widetilde{\theta}_0^\dagger + (1+|\lambda|^2)^{-1} F_\Gamma^\dagger$. Hence, we have the following:

$$\big[\widetilde{\theta}, \widetilde{\theta}^\dagger\big] = \big[\widetilde{\theta}_0, \widetilde{\theta}_0^\dagger\big] + (1+|\lambda|^2)^{-1}\big[\widetilde{\theta}_0, F_\Gamma^\dagger\big] + (1+|\lambda|^2)^{-1}\big[\widetilde{\theta}_0^\dagger, F_\Gamma\big] \tag{32}$$

We have the decomposition of $\widetilde{\theta}_0$ into the sum $\lambda(1+|\lambda|^2)^{-1}\sum \widetilde{c}_i \cdot d\zeta_i/\zeta_i + \widetilde{\theta}_0'$, where $\widetilde{\theta}_0'$ is the C^∞-section of $\mathrm{End}({}_{\widetilde{\boldsymbol{c}}}\widetilde{E}) \otimes \Omega^1_{\widetilde{X}}$ on $\widetilde{X}$. Hence, $\big[\widetilde{\theta}_0, \widetilde{\theta}_0^\dagger\big]$ is the C^∞-section of $\mathrm{End}({}_{\widetilde{\boldsymbol{c}}}\widetilde{E}) \otimes \Omega^2$ on $\widetilde{X}$. By Condition (A), $\big[\widetilde{\theta}_0, F_\Gamma^\dagger\big]$ and $\big[F_\Gamma, \widetilde{\theta}_0^\dagger\big]$ is also bounded. We have $\widetilde{\theta}^2 = \widetilde{\theta}_0^2 + 2\big[\widetilde{\theta}_0, F_\Gamma\big]$. Then, we obtain the desired estimate for $\widetilde{\theta}^2$ by Condition (A). ∎

Lemma 3.2 *We have the boundedness of $G(\widetilde{\mathbb{D}}^\lambda, \widetilde{h})$ and $\widetilde{\theta}^2 \cdot \widetilde{\theta}^\dagger$ with respect to $\widetilde{h}$ and $\widetilde{g}$.*

Proof The boundedness of $\widetilde{\theta}^2 \cdot \widetilde{\theta}^\dagger$ follows from the estimate for $\widetilde{\theta}^2$. We have the following equality (See Subsection 2.2.4):

$$G(\widetilde{\mathbb{D}}^\lambda, \widetilde{h}) = (1+|\lambda|^2) \cdot R(\widetilde{h}, d'') - \frac{(1+|\lambda|^2)^2}{\lambda}\big(\overline{\partial}_{\widetilde{h}}^2 + \widetilde{\theta}^2 - \lambda\big[\widetilde{\theta}, \widetilde{\theta}^\dagger\big]\big)$$

We have the vanishing of the curvature $R(\widetilde{h}, d'') = 0$, and the relation $\lambda^{-1}\overline{\partial}_{\widetilde{h}}^2 = \overline{\lambda}^{-1}(\widetilde{\theta}^\dagger)^2$. Hence, we obtain the boundedness of $G(\widetilde{\mathbb{D}}^\lambda, \widetilde{h})$ from Lemma 3.1. ∎

Since $\widetilde{h}$ is $\mathrm{Gal}(\widetilde{X}/X)$-equivariant, we obtain the induced metric h of E on $X - D$. Clearly, h is given by $h(v_i, v_j) = \delta_{i,j} \cdot |z_1|^{-2a(v_i)} \cdot |z_2|^{-2a(v_j)}$. Let θ be the section of $\mathrm{End}(E) \otimes \Omega^{1,0}$ on $X - D$ induced by $\mathbb{D}^\lambda$ and h, and let $\theta^\dagger$ denote the adjoint of θ. Let g_m denote the metric of $X - D$ given by $g_m = \sum |z_i|^{2(-1+1/m)} \cdot dz_i \cdot d\overline{z}_i$.

Corollary 3.3 *We have the boundedness of $G(\mathbb{D}^\lambda, h)$ and $\theta^2 \cdot \theta^\dagger$ with respect to g_m and h.* ∎

3.1.2 The induced metric and the λ-connection on the divisors

For simplicity, we assume $c_i = \gamma_i = 0$ $(i = 1, 2)$ in this subsection. Let $(a, \alpha) \in \mathcal{KMS}({}^\diamond E, i)$. Let ρ be a C^∞-function on X such that $\rho > 0$. We put $\chi := \rho \cdot |z_1|^2$. Let $D_i^\circ := D_i - (D_1 \cap D_2)$. We discuss the induced hermitian metric and the induced λ-connection of ${}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_{|D_i^\circ}$ $(i = 1, 2)$, depending on the choice of ρ. Let us consider the case $i = 1$. Let u_j $(j = 1, 2)$ be sections of ${}^1\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)$. We take sections u_j' $(j = 1, 2)$ of ${}^\diamond E$ which induce u_j. Then, it can be shown that $\big(\chi^a \cdot h_0(u_1', u_2')\big)_{|D_1^\circ}$ is independent of the choice of u_j', which is denoted by $h_{a,\alpha}(u_1, u_2)$.

We have the frame $\boldsymbol{v}_{(a,\alpha)}$ induced by $\boldsymbol{v}$ above. By construction, $h_{a,\alpha}(v_i, v_j) = \rho^a \cdot \delta_{i,j} \cdot |z_2|^{-2a_2(v_i)}$. Hence, the following equality can be checked by a direct calculation:

$$\mathrm{tr}\, R(h_{a,\alpha}) - a \cdot \mathrm{rank}\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E) \cdot \overline{\partial}\partial \log \rho = 0 \tag{33}$$

Let F_0 denote the C^∞-section of $\mathrm{End}({}^\diamond E) \otimes \Omega^{1,0}_X(\log D)$ determined by $F_0(v_j) = v_j \cdot \alpha_1(v_j) \cdot \partial \log \chi$. Then, $\mathbb{D}^\lambda - F_0$ is C^∞ around D_1°, whose restriction preserves the filtration ${}^1F$ and the decomposition ${}^1\mathbb{E}$. Hence, we obtain the induced λ-connection $\mathbb{D}^\lambda_{a,\alpha}$ of ${}^1\mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}({}^\diamond E)$. We have $\theta_{a,\alpha}$ induced by $\mathbb{D}^\lambda_{a,\alpha}$ and $h_{a,\alpha}$.

Lemma 3.4 *The following holds:*

$$\overline{\partial}\,\mathrm{tr}\,\theta_{a,\alpha} + \frac{\lambda a + \alpha}{1+|\lambda|^2}\,\mathrm{rank}\Big({}^1\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)\Big) \cdot \overline{\partial}\partial \log \rho = 0 \tag{34}$$

Proof Let $\mathbb{D}^\lambda_{a,\alpha,1}$ and $\theta_{a,\alpha,1}$ denote the operator, and let $h_{a,\alpha,1}$ denote the metric in the case where ρ is constantly 1. Since $\theta_{a,\alpha,1}$ is holomorphic, we have $\overline{\partial}\,\mathrm{tr}\,\theta_{a,\alpha,1} = 0$. Note that we have $\mathbb{D}^\lambda_{a,\alpha} = \mathbb{D}^\lambda_{a,\alpha,1} - \alpha \cdot \partial \log \rho$ and $h_{a,\alpha} = h_{a,\alpha,1} \cdot \rho^a$. Then, we obtain $\theta_{a,\alpha,1} = \theta_{a,\alpha} + (1+|\lambda|^2)^{-1}\big(\lambda \cdot a + \alpha\big) \cdot \partial \log \rho$. Thus, we obtain (34). ∎

3.2 Around the smooth part of the divisor

3.2.1 Construction of the metric and some estimates

Let $X=\Delta^2$ and $D=\{z_1=0\}$. Let ρ be a positive C^{∞}-function on X, and we put $\chi:=\rho\cdot|z_1|^2$. Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be a graded semisimple regular filtered λ-flat bundle on (X,D) with rational weights. We take $c\in\boldsymbol{R}$ such that $c\notin\mathcal{P}ar(\boldsymbol{E}_*)$. We assume the following:

(SPW) We have a positive integer m and $\gamma\in\boldsymbol{R}$ with $-1/m<\gamma\leq 0$ such that $\mathcal{P}ar\bigl({}_cE_*\bigr)$ is contained in $\bigl\{c+\gamma+p/m\,\big|\,p\in\mathbb{Z},\ -1<\gamma+p/m<0\bigr\}$.

Let $\widetilde{X}:=\Delta^2_{\zeta}$ and $D=\{\zeta_1=0\}$. Let $\varphi:\widetilde{X}\longrightarrow X$ be given by $\varphi(\zeta_1,\zeta_2)=(\zeta_1^m,\zeta_2)$. We have the induced filtered λ-flat bundle $(\widetilde{\boldsymbol{E}}_*,\widetilde{\mathbb{D}}^{\lambda})$ on $(\widetilde{X},\widetilde{D})$ as in Subsection 3.1.1. We put $\widetilde{c}:=m\cdot(c+\gamma)$. Then, $\mathcal{P}ar\bigl({}_{\widetilde{c}}\widetilde{E}_*\bigr)$ is contained in $\bigl\{\widetilde{c}+p\,\big|\,p\in\mathbb{Z}\bigr\}$.

We have the generalized eigen decomposition ${}_cE_{|D}=\bigoplus\mathbb{E}_{\alpha}$. We have the filtration F of ${}_cE_{|D}$. Let $\boldsymbol{v}$ be a frame of ${}_cE$ compatible with F and $\mathbb{E}$. We put $a(v_j):=\deg^F(v_j)-(c+\gamma)$. Let $\alpha(v_j)\in\boldsymbol{C}$ be determined by $v_{j|D}\in\mathbb{E}_{\alpha(v_j)}$. We put $\widetilde{v}_j:=\varphi^*(v_j)\cdot\zeta_1^{m\cdot a(v_j)}$. Then, $\widetilde{\boldsymbol{v}}=(\widetilde{v}_j)$ gives the frame of ${}_{\widetilde{c}}\widetilde{E}$. Let Γ be the diagonal matrix whose (j,j)-entry is given by the following:

$$\alpha(v_j)\cdot\partial\log(\varphi^*\chi)+\lambda\cdot m\cdot a(v_j)\cdot\frac{d\zeta_1}{\zeta_1}$$

Let A be determined by $\widetilde{\mathbb{D}}^{\lambda}\widetilde{\boldsymbol{v}}=\widetilde{\boldsymbol{v}}\cdot A$, and let $A_0:=A-\Gamma$. Let F_{Γ} be the C^{∞}-section of $\mathrm{End}(\widetilde{E})\otimes\Omega^1$ on $\widetilde{X}-\widetilde{D}$, determined by $F_{\Gamma}\widetilde{\boldsymbol{v}}=\widetilde{\boldsymbol{v}}\Gamma$. We put $\widetilde{\mathbb{D}}_0:=\widetilde{\mathbb{D}}-F_{\Gamma}$.

Let $A_0=A_0^1\cdot d\zeta_1+A_0^2\cdot d\zeta_2$. If m is sufficiently large, the following holds:

(A) $A_0^1=O\bigl(|\zeta_1|^2\bigr)$. Moreover, $(A_0^2)_{k,l}=O\bigl(|\zeta_1|^2\bigr)$ in the case $\bigl(a(v_k),\alpha(v_k)\bigr)\neq\bigl(a(v_l),\alpha(v_l)\bigr)$.

Let $\widetilde{h}_1$ be the $\mathrm{Gal}(\widetilde{X}/X)$-equivariant hermitian metric of ${}^{\diamond}\widetilde{E}$ such that $\widetilde{h}_1(v_i,v_j)=O\bigl(|\zeta_1|^2\bigr)$ in the case $\bigl(a(v_i),\alpha(v_i)\bigr)\neq\bigl(a(v_j),\alpha(v_j)\bigr)$. Then, let $\widetilde{h}:=\varphi^*(\chi^{-c-\gamma})\cdot\widetilde{h}_1$.

Let $\widetilde{\theta}$ (resp. $\widetilde{\theta}_0$) be the section of $\mathrm{End}(\widetilde{E})\otimes\Omega^1$ on $\widetilde{X}-\widetilde{D}$ induced by $\widetilde{h}$ and $\widetilde{\mathbb{D}}^{\lambda}$ (resp. $\widetilde{\mathbb{D}}^{\lambda}_0$) as in Subsection 2.2.1. Let $\widetilde{\theta}^{\dagger}$ and $\widetilde{\theta}^{\dagger}_0$ denote the adjoint of $\widetilde{\theta}$ and $\widetilde{\theta}_0$, respectively. Let $\widetilde{g}$ denote the Euclidean metric of $\widetilde{X}$.

Lemma 3.5

- $\bigl[\widetilde{\theta},\widetilde{\theta}^{\dagger}\bigr]$ *is bounded with respect to* $\widetilde{h}$ *and* $\widetilde{g}$.
- $\widetilde{\theta}^2=O\bigl(|z_1|\bigr)\cdot dz_1\cdot dz_2$.

Proof Similar to Lemma 3.1. ∎

Lemma 3.6 *We have the boundedness of* $G\bigl(\widetilde{\mathbb{D}}^{\lambda},\widetilde{h}\bigr)$ *and* $\widetilde{\theta}^2\cdot\widetilde{\theta}^{\dagger}$ *with respect to* $\widetilde{h}$ *and* $\widetilde{g}$.

Proof It follows from Lemma 3.5. See the proof of Lemma 3.2. ∎

We have the induced hermitian metric h of E on $X-D$. It is adapted to the parabolic structure of E. Let θ denote the section of $\mathrm{End}(E)\otimes\Omega^1_{X-D}$ induced by h and $\mathbb{D}^{\lambda}$, and let $\theta^{\dagger}$ denote the adjoint. Let g_m denote the metric of $X-D$ given by $g_m=|z_1|^{-2+2/m}\cdot dz_1\cdot d\overline{z}_1+dz_2\cdot d\overline{z}_2$.

Corollary 3.7 *We have the boundedness of* $G(\mathbb{D}^{\lambda},h)$ *and* $\theta^2\cdot\theta^{\dagger}$ *with respect to* h *and* g_m. ∎

3.2.2 The induced metric and the λ-connections

For simplicity, we assume $c = \gamma = 0$ in this subsection. Let $(a, \alpha) \in \mathcal{KMS}({}^\diamond E_*)$. We discuss the induced hermitian metric and the induced λ-connection of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)$. Let u_j $(j = 1, 2)$ be sections of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)$. We take sections u'_j $(j = 1, 2)$ of ${}^\diamond E$ which induce u_j. Then, it can be shown that $\bigl(\chi^a \cdot h_0(u'_1, u'_2)\bigr)_{|D}$ is independent of the choice of u'_j, which is denoted by $h_{a,\alpha}(u_1, u_2)$.

On the other hand, let $U_{a,\alpha}$ be the subbundle of ${}^\diamond\widetilde{E}$ generated by $\widetilde{v}_j$ such that $\bigl(a(v_j), \alpha(v_j)\bigr) = (a, \alpha)$. It is easy to see that the restriction $U_{a,\alpha|\widetilde{D}}$ are independent of the choice of the frame $\boldsymbol{v}$, and $U_{a,\alpha|\widetilde{D}}$ are orthogonal with respect to $\widetilde{h}_{|\widetilde{D}}$. The induced metric of $U_{a,\alpha|\widetilde{D}}$ is denoted by $h'_{a,\alpha}$.

Lemma 3.8 *Let $R(h_{a,\alpha})$ and $R(h'_{a,\alpha})$ denote the curvatures of $\bigl(\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E), h_{a,\alpha}\bigr)$ and $\bigl(U_{a,\alpha|\widetilde{D}}, h'_{a,\alpha}\bigr)$. Then, we have the following relation:*

$$\mathrm{tr}\bigl(R(h'_{a,\alpha})\bigr) = \mathrm{tr}\bigl(R(h_{a,\alpha})\bigr) - a \cdot \mathrm{rank}\, \mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E) \cdot \overline{\partial}\partial \log \rho \tag{35}$$

Proof We take the isomorphism $\Phi : \mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E) \simeq U_{a,\alpha|\widetilde{D}}$ given as follows. Let v be a section of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)$. Let v' be a section of ${}^\diamond E$ which induces v. Then, $\Phi(v) := \bigl(\varphi^*(v') \cdot z_1^{m\cdot a}\bigr)_{|\widetilde{D}}$ is contained in $U_{a,\alpha|\widetilde{D}}$, and independent of the choice of v'. Under the isomorphism, we have $h'_{a,\alpha} = h_{a,\alpha} \cdot \rho^{-a}$. Then, (35) follows. ∎

We have the induced λ-connection, once we fix χ. (See [2].) Let f be any section of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)$. Let $\widetilde{f}$ be a lift of f to ${}^\diamond E$. We put $\mathbb{D}^\lambda f - \alpha \cdot \log \chi \cdot f =: G_1 \cdot (dz_1/z_1) + G_2 \cdot dz_2$. Then, $G_{1|D}$ is contained in $F_{<a}(E)$. Hence, $G_2 \cdot dz_2$ induces the well defined section of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E) \otimes \Omega^1_D$, which is $\mathbb{D}_{a,\alpha}(f)$. We have the induced section $\theta_{a,\alpha}$ of $\mathrm{End}\bigl(\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)\bigr) \otimes \Omega^1_D$.

Lemma 3.9 *We have the following relation:*

$$\mathrm{tr}\bigl(R(h'_{a,\alpha})\bigr) = -\frac{1+|\lambda|^2}{\lambda}\left(\overline{\partial}\,\mathrm{tr}\,\theta_{a,\alpha} + \frac{(a\lambda+\alpha)\cdot\overline{\partial}\partial\log\rho}{1+|\lambda|^2}\,\mathrm{rank}\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)\right) \tag{36}$$

Proof We have the relation:

$$R(\widetilde{h}) = -\frac{1+|\lambda|^2}{\lambda} d''\widetilde{\theta} = -\frac{1+|\lambda|^2}{\lambda}\left(d''\widetilde{\theta}_0 + \frac{1}{1+|\lambda|^2} d'' F_\Gamma\right) \tag{37}$$

Let $\mathbb{D}^{\lambda\prime}_{a,\alpha}$ be the induced λ-connection of $U_{a,\alpha|\widetilde{D}}$. Let $\theta'_{a,\alpha}$ be the section of $\mathrm{End}(U_{a,\alpha|\widetilde{D}}) \otimes \Omega^1_{\widetilde{D}}$ induced by $\mathbb{D}^{\lambda\prime}_{a,\alpha}$ and $h'_{a,\alpha}$. Then, we obtain the following equality from (37):

$$\mathrm{tr}\bigl(R(h'_{a,\alpha})\bigr) = -\frac{1+|\lambda|^2}{\lambda}\left(\overline{\partial}\,\mathrm{tr}\,\theta'_{a,\alpha} + \frac{1}{1+|\lambda|^2}\cdot\alpha\cdot\overline{\partial}\partial\log\rho\cdot\mathrm{rank}\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)\right) \tag{38}$$

Under the isomorphism Φ in the proof of Lemma 3.8, we have the $\mathbb{D}^{\lambda\prime}_{a,\alpha} = \mathbb{D}^\lambda_{a,\alpha}$. Because of $h'_{a,\alpha} = h_{a,\alpha} \cdot \rho^{-a}$, we have $\theta'_{a,\alpha} = \theta_{a,\alpha} + a\lambda(1+|\lambda|^2)^{-1}\partial\log\rho$. Therefore, the right hand side of (38) is the same as (36). ∎

3.3 An ordinary metric

3.3.1 Setting

Let X be a smooth projective surface, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. Let L be an ample line bundle on X, and ω be a Kahler form which represents $c_1(L)$. We take a hermitian metric g_i of $\mathcal{O}(D_i)$. The canonical section $\mathcal{O} \longrightarrow \mathcal{O}(D_i)$ is denoted by σ_i.

Let ϵ be any number such that $0 < \epsilon < 1/2$. Let us fix a sufficiently large number N, for example $N > 10$. We put as follows, for some positive number $C > 0$:

$$\omega_\epsilon := \omega + \sum_i C \cdot \epsilon^N \cdot \sqrt{-1}\partial\overline{\partial}|\sigma_i|^{2\epsilon}_{g_i}. \tag{39}$$

It can be shown that ω_ϵ are Kahler metrics of $X - D$ for any $0 < \epsilon < 1/2$, if C is sufficiently small.

Remark 3.10 *The factor ϵ^N is added for our later discussion (Subsection 5.1).* ∎

Remark 3.11 *Let τ be a closed 2-form on $X-D$ which is bounded with respect to ω_ϵ. Then, the following formula holds:*

$$\int_{X-D}\omega\cdot\tau=\int_{X-D}\omega_\epsilon\cdot\tau.$$

In particular, we also have $\int_{X-D}\omega^2=\int_{X-D}\omega_\epsilon^2$. ∎

In the case $\epsilon=1/m$ for some positive integer m, it can be shown that the metric ω_ϵ satisfies Condition 2.30. The Kahler forms ω_ϵ behave as follows around any point of D, which is clear from the construction:

- Let P be any point of $D_i\cap D_j$, and (U_P,z_i,z_j) be a holomorphic coordinate around P such that $D_i\cap U_P=\{z_i=0\}$ and $D_j\cap U_P=\{z_j=0\}$. Then, there exist positive constants C_i $(i=1,2)$ such that the following holds on U_P, for any $0<\epsilon<1/2$:

$$C_1\cdot\omega_\epsilon\leq\sqrt{-1}\cdot\epsilon^{N+2}\cdot\left(\frac{dz_i\cdot d\bar{z}_i}{|z_i|^{2-2\epsilon}}+\frac{dz_j\cdot d\bar{z}_j}{|z_j|^{2-2\epsilon}}\right)+\sqrt{-1}\big(dz_i\cdot d\bar{z}_i+dz_j\cdot d\bar{z}_j\big)\leq C_2\cdot\omega_\epsilon.$$

- Let Q be any point of $D_i\setminus\bigcup_{j\neq i}D_j$, and (U,w_1,w_2) be a holomorphic coordinate around Q such that $U\cap D_i=\{w_1=0\}$. Then, there exist positive constants C_i $(i=1,2)$ such that the following holds for any $0<\epsilon<1/2$ on U:

$$C_1\cdot\omega_\epsilon\leq\sqrt{-1}\cdot\epsilon^{N+2}\cdot\left(\frac{dw_1\cdot d\bar{w}_1}{|w_1|^{2-2\epsilon}}\right)+\sqrt{-1}\big(dw_1\cdot d\bar{w}_1+dw_2\cdot d\bar{w}_2\big)\leq C_2\cdot\omega_\epsilon.$$

3.3.2 Construction and some property

Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a graded semisimple parabolic λ-flat bundle. For simplicity, we consider only the case $\lambda\neq 0$. We take $\boldsymbol{c}\in\boldsymbol{R}^S$ such that $c_i\notin\mathcal{P}ar(\boldsymbol{E}_*,i)$ for each $i\in S$. We assume the following:

(SPW) We have a positive integer m and $\gamma_i\in\boldsymbol{R}$ $(i\in S)$ with $-1/m<\gamma_i\leq 0$, such that $\mathcal{P}ar\big({}_{\boldsymbol{c}}E_*,i\big)$ is contained in $\big\{c_i+\gamma_i+p/m\,\big|\,p\in\mathbb{Z},\ -1<\gamma_i+p/m<0\big\}$.

Let $\epsilon=m^{-1}$. Let h_0 be a C^∞-hermitian metric of E on $X-D$ as in Subsection 3.1 around the intersection points of D, and as in Subsection 3.2 around the smooth points of D. Let θ_0 denote the section of $\mathrm{End}(E)\otimes\Omega^{1,0}$ on $X-D$ induced by $\mathbb{D}^\lambda$ and h_0, and let $\theta_0^\dagger$ denote the adjoint.

Lemma 3.12 *We have the boundedness of $G(\mathbb{D}^\lambda,h_0)$ and $\theta_0^2\cdot\theta_0^\dagger$ with respect to h_0 and ω_ϵ.*

Proof It follows from Corollary 3.3 and Corollary 3.7. ∎

Corollary 3.13 *The following equality holds:*

$$\int_{X-D}\mathrm{tr}\big(R(h_0)^2\big)=\frac{1}{(1+|\lambda|^2)^2}\int_{X-D}\mathrm{tr}\big(G(h_0)^2\big).$$

As a result, we have the following formula:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\frac{1}{(1+|\lambda|^2)^2}\int_{X-D}\mathrm{tr}\big(G(h_0)^2\big)=2\int_X\text{par-ch}_2(\boldsymbol{E}_*).\qquad(40)$$

Proof The second equality follows from the first equality and the equality (36) in the proof of Proposition 4.18 of [31]. Due to Lemma 2.29, we have only to show the vanishing $\int\overline{\partial}\,\mathrm{tr}\big(\theta_0^2\cdot\theta_0^\dagger\big)=0$, which follows from the estimate of $\theta_0^2\cdot\theta_0^\dagger$ in Lemma 3.12. ∎

We can also show the following equality by using Lemma 4.16 of [31] and the equality $\operatorname{tr} G(h_0) = (1+|\lambda|^2)\cdot \operatorname{tr} R(h_0)$:

$$\Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^2 \int_{X-D} \Bigl(\frac{\operatorname{tr} G(h_0)}{1+|\lambda|^2}\Bigr)^2 = \Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^2 \int_{X-D} \bigl(\operatorname{tr} R(h_0)\bigr)^2 = \int_X \text{par-c}_1(\boldsymbol{E}_*)^2.$$

Let $V \subset E$ be a λ-flat subbundle. Because of $\lambda \neq 0$ and the regularity, we have the saturated filtered λ-flat subsheaf $\boldsymbol{V}_* \subset \boldsymbol{E}_*$. Let h_V be the metric of V induced by h_0.

Lemma 3.14 *We have* $\deg_{\omega_\epsilon}(V, h_V) = \text{par-deg}_\omega(\boldsymbol{V}_*)$. *In particular,* $\deg_{\omega_\epsilon}(E, h_0) = \text{par-deg}_\omega(\boldsymbol{E}_*)$.

Proof It can be shown by the same argument as the proof of Lemma 4.20 of [31]. ∎

3.3.3 The induced metric and the λ-connection on D_i°

For simplicity, we assume $c_i = \gamma_i = 0$ $(i \in S)$ in this subsection. We put $\mathcal{S}(D_i) := D_i \cap \bigcup_{j\neq i} D_j$ and $D_i^\circ := D_i \setminus \mathcal{S}(D_i)$. Let $(a,\alpha) \in \mathcal{KMS}\bigl({}^\diamond E, \boldsymbol{F}, i\bigr)$. We have the naturally induced parabolic flat bundle ${}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_*$ on $\bigl(D_i, \mathcal{S}(D_i)\bigr)$. By using the functions $|\sigma_i|^2_{g_i}$, we obtain the induced hermitian metric ${}^i h_{a,\alpha}$ and the λ-connection ${}^i\mathbb{D}^\lambda_{a,\alpha}$ of ${}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_{|D_i^\circ}$, as explained in Subsection 3.2.2. (See also Subsection 3.1.2.) Let $\tau_i := \overline{\partial}\partial \log|\sigma_i|^2_{g_i}$.

Lemma 3.15 *We have the following equality:*

$$\operatorname{tr}\bigl(R(h_{a,\alpha})\bigr) = -\frac{1+|\lambda|^2}{\lambda}\overline{\partial}\bigl({}^i\theta_{a,\alpha}\bigr) - \lambda^{-1}\alpha\cdot\tau_i\cdot\operatorname{rank}{}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)$$

Proof It follows from (33), (34), (35) and (36). ∎

Corollary 3.16 *We have the following equalities:*

$$\text{par-deg}_{D_i}\Bigl({}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_*\Bigr) = -\sum_{P\in\mathcal{S}(D_i)}\Bigl(\operatorname{Re}\bigl(\lambda^{-1}\operatorname{tr}\bigl(\operatorname{Res}_P({}^i\mathbb{D}^\lambda_{a,\alpha})\bigr)\bigr) + \operatorname{wt}\bigl({}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_*, P\bigr)\Bigr) - \operatorname{Re}(\lambda^{-1}\alpha)\cdot\operatorname{rank}{}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)\cdot[D_i]^2 \quad (41)$$

$$0 = \sum_{P\in\mathcal{S}(D_i)}\operatorname{Im}\Bigl(\lambda^{-1}\operatorname{tr}\bigl(\operatorname{Res}_P({}^i\mathbb{D}^\lambda_{a,\alpha})\bigr)\Bigr) + \operatorname{Im}(\lambda^{-1}\alpha)\cdot\operatorname{rank}{}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)\cdot[D_i]^2 \quad (42)$$

Proof It follows from Lemma 3.15 and (30). ∎

Remark 3.17 *Although we have assumed that graded semisimplicity and (SPW)-condition for* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$, *the formulas* (41) *and* (42) *without the assumption, because the general case can be reduced to the above special case by perturbation explained in Subsection* 2.1.6. ∎

3.4 Preliminary existence result of a Hermitian-Einstein metric

3.4.1 Hermitian-Einstein metric for graded semisimple λ-flat parabolic bundle on surface

We use the setting in Subsection 3.3. Let X be a smooth projective surface with an ample line bundle L and a simple normal crossing divisor D. Let ω be a Kahler form representing $c_1(L)$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a graded semisimple regular filtered λ-flat bundle on (X, D). We assume the (SPW)-condition in Subsection 3.3.2. Let $\epsilon = m^{-1}$, and let ω_ϵ be the Hermitian metric given in (39). We have an ordinary metric h_0 constructed in Subsection 3.3.2.

Lemma 3.18 *We can construct a hermitian metric* h_{in} *for* $E_{|X-D}$ *which satisfies the following conditions:*

- h_{in} *is adapted to* $\boldsymbol{E}_*$. *More strongly,* $h_{in} = h_0 \cdot e^\chi$ *for some function* χ *such that* χ, $\partial\chi$ *and* $\overline{\partial}\partial\chi$ *are bounded with respect to* ω_ϵ.

- $G(h_{in}, \mathbb{D}^\lambda)$ *is bounded with respect to* h_{in} *and* ω_ϵ.
- *Let* $\boldsymbol{V}_*$ *be a* λ*-flat filtered subsheaf of* $\boldsymbol{E}_*$. *Let* $V := \boldsymbol{V}_{|X-D}$ *and let* $h_{in,V}$ *denote the induced metric of* V. *Then, we have* $\text{par-deg}_\omega(\boldsymbol{V}_*) = \deg_{\omega_\epsilon}(V, h_{in,V})$.
- $\operatorname{tr} G(h_{in}, \mathbb{D}^\lambda) \cdot \omega_\epsilon = (1+|\lambda|^2) \cdot a \cdot \omega_\epsilon^2$ *for some constant* a. *The constant* a *is determined by the following condition:*

$$a \cdot \frac{\sqrt{-1}}{2\pi} \int_{X-D} \omega_\epsilon^2 = a \cdot \frac{\sqrt{-1}}{2\pi} \int_X \omega^2 = \text{par-deg}_\omega(\boldsymbol{E}_*). \tag{43}$$

- *The following equalities hold:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \frac{\operatorname{tr}\bigl(G(h_{in})^2\bigr)}{(1+|\lambda|^2)^2} = \int_X 2\,\text{par-ch}_2(\boldsymbol{E}_*), \quad \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \frac{\operatorname{tr}\bigl(G(h_{in})\bigr)^2}{(1+|\lambda|^2)^2} = \int_X \text{par-c}_1^2(\boldsymbol{E}_*).$$

- *Let* s *be determined by* $h_{in} = h_0 \cdot s$. *Then,* s *and* s^{-1} *are bounded, and* $\mathbb{D}^\lambda s$ *is* L^2 *with respect to* h_0 *and* ω_ϵ.

Due to the third condition, (E, h_{in}, θ) *is analytic stable with respect to* ω_ϵ, *if and only if* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *is* μ_L*-stable. The metric* h_{in} *is called an initial metric.*

Proof Let χ be a positive-valued function χ such that $\operatorname{tr} G(h_0) \cdot \omega_\epsilon = a \cdot \omega_\epsilon^2$ holds. We put $h_{in} := h_0 \cdot e^\chi$. By construction, the fourth condition is satisfied. The other property can be reduced to the property for h_0, as in Lemma 6.3 of [31]. ∎

Proposition 3.19 *There exists a hermitian metric* h_{HE} *of* $(E, \mathbb{D}^\lambda)$ *with respect to* ω_ϵ *satisfying the following properties:*

- *Hermitian-Einstein condition* $\Lambda_{\omega_\epsilon} G(h_{HE}) = a$ *holds for the constant* a *determined by* (43).
- $\text{par-deg}_L(\boldsymbol{E}_*) = \deg_\omega(E, h_{HE})$.
- *We have the following formulas:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \frac{\operatorname{tr}\bigl(G(h_{HE})^{\perp\,2}\bigr)}{(1+|\lambda|^2)^2} = \int_X \left(2\,\text{par-ch}_2(\boldsymbol{E}_*) - \frac{\text{par-c}_1^2(\boldsymbol{E}_*)}{\operatorname{rank} E}\right) \tag{44}$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \frac{\operatorname{tr}\bigl(G(h_{HE})^2\bigr)}{(1+|\lambda|^2)^2} = \int_X 2\,\text{par-ch}_2(\boldsymbol{E}_*). \tag{45}$$

- h_{HE} *is adapted to* $\boldsymbol{E}_*$, *i.e.,* $\boldsymbol{E}_*(h_{HE}) \simeq \boldsymbol{E}_*$. *More strongly, let* s *be determined by* $h_{HE} = h_{in} \cdot s$. *Then,* s *and* s^{-1} *are bounded with respect to* h_{in}, *and* $\mathbb{D}^\lambda s$ *is* L^2 *with respect to* h_{in} *and* ω_ϵ.

Proof It follows from Lemma 3.18 and Proposition 2.33. ∎

3.4.2 Bogomolov-Gieseker inequality

Let Y be a smooth projective variety of *any* dimension. Let L be an ample line bundle on Y, and let D be a simple normal crossing divisor.

Corollary 3.20 *Let* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *be a* μ_L*-stable regular filtered* λ*-flat bundle on* (Y, D) *in codimension two. Then, Bogomolov-Gieseker inequality holds for* $\boldsymbol{E}_*$. *Namely, we have the following inequality:*

$$\int_Y \text{par-ch}_{2,L}(\boldsymbol{E}_*) \le \frac{\int_Y \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2 \operatorname{rank} E}.$$

Proof Similar to Theorem 6.1 of [31]. Namely, since we have the Mehta-Ramanathan type theorem (Proposition 2.9), we have only to prove the claim in the case $\dim Y = 2$. Due to the method of perturbation of parabolic structure, we have only to prove the inequality in the case $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is a graded semisimple μ_L-stable regular filtered λ-flat bundle on (Y, D), satisfying (SPW)-condition. Then we can take a Hermitian-Einstein metric h_{HE} as in Proposition 3.19, for which we have the standard inequality (See Proposition 3.4 of [36]):

$$\int_{Y-D} \operatorname{tr}\bigl(G(h_{HE}, \mathbb{D}^\lambda)^{\perp\, 2}\bigr) \geq 0. \tag{46}$$

Here $G(h_{HE}, \mathbb{D}^\lambda)^\perp$ denotes the trace free part of $G(h_{HE}, \mathbb{D}^\lambda)$. Hence we obtain the desired inequality from (46). ∎

3.5 Some formula and vanishing of characteristic numbers

3.5.1 Formula for $\int_X \text{par-ch}_2(\boldsymbol{E}_*)$

Let X be a smooth projective surface, and let D be a simple normal crossing divisor of X. We will derive some formulas and vanishings for the characteristic numbers of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$.

Remark 3.21 *To begin with, we remark that we have only to show such formulas for regular filtered λ-flat bundles satisfying the following conditions due to the method of perturbation of the parabolic structure (Subsection* 2.1.6*).*

- graded semisimple.
- $\mathcal{P}ar\bigl(\boldsymbol{E}_*, i\bigr) \subset \boldsymbol{Q}$.
- $0 \notin \mathcal{P}ar\bigl(\boldsymbol{E}_*, i\bigr)$ *for any* $i \in S$.

We will use it without mention in the following argument.

We restrict ourselves to the case $\dim X = 2$ *just for simplicity. The formula can be obviously generalized for* $\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*)$ *of regular λ-flat parabolic bundles $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on (X, D) in codimension two for* $\dim X > 2$*, where L denotes a line bundle on X.* ∎

For simplicity of the description, we put as follows, for $u =\in \mathcal{KMS}(i) := \mathcal{KMS}\bigl({}^\diamond E, i\bigr)$:

$$r(i,u) := \operatorname{rank}_{D_i}\bigl({}^i \operatorname{Gr}_u^{F,\mathbb{E}}({}^\diamond E)\bigr)$$

For any point $P \in D_i \cap D_j$ and $(u_i, u_j) \in \mathcal{KMS}(P) := \mathcal{KMS}\bigl({}^\diamond E, P\bigr)$, we put as follows:

$$r(P, u_i, u_j) := \operatorname{rank}\bigl({}^P \operatorname{Gr}_{u_i,u_j}^{F,\mathbb{E}}(E)\bigr)$$

Proposition 3.22 *We have the following equality:*

$$\begin{aligned}\int_X 2\,\text{par-ch}_2(\boldsymbol{E}_*) = &\sum_{i\in S} \sum_{u \in \mathcal{KMS}(i)} \bigl(\operatorname{Re}(\lambda^{-1}\alpha) + a\bigr)^2 \cdot r(i,u) \cdot [D_i]^2 \\ &+ \sum_{i \in S} \sum_{\substack{j \neq i \\ P \in D_i \cap D_j}} \sum_{(u_i,u_j) \in \mathcal{KMS}(P)} \bigl(\operatorname{Re}\lambda^{-1}\alpha_i + a_i\bigr)\bigl(\operatorname{Re}\lambda^{-1}\alpha_j + a_j\bigr) \cdot r(P, u_i, u_j).\end{aligned} \tag{47}$$

We also have the following equalities:

$$2\,\text{par-ch}_2(\boldsymbol{E}_*) = \sum_{i \in S} \sum_{u \in \mathcal{KMS}(i)} \operatorname{Re}\bigl(\lambda^{-1}\alpha + a\bigr) \cdot \Bigl(-\,\text{par-deg}\bigl({}^i \operatorname{Gr}_{a,\alpha}^{F,\mathbb{E}}({}^\diamond E)_*\bigr) + a \cdot r(i,u) \cdot [D_i]^2\Bigr). \tag{48}$$

$$0 = \sum_{i \in S} \sum_{u \in \mathcal{KMS}(i)} \operatorname{Im}(\lambda^{-1}\alpha) \cdot \Bigl(-\,\text{par-deg}_{D_i}\bigl({}^i \operatorname{Gr}_u^{F,\mathbb{E}}({}^\diamond E)_*\bigr) + a \cdot r(i,u) \cdot [D_i]^2\Bigr) \tag{49}$$

Proof Let $X_\delta := \bigcap\bigl\{|\sigma_i| \geq \delta\bigr\}$ and $Y_{\delta,i} := X_\delta \cap \{|\sigma_i| = \delta\}$. We have $R(h_0) = -\lambda^{-1}(1+|\lambda|^2)\cdot d''\theta$. Hence, we have the following equality:

$$\lim_{\delta\to 0}\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X_\delta}\mathrm{tr}\bigl(R(h_0)^2\bigr) = -\frac{1+|\lambda|^2}{\lambda}\lim_{\delta\to 0}\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{\partial X_\delta} d\,\mathrm{tr}\bigl(\theta\cdot R(h_0)\bigr) \tag{50}$$

By using the estimates in Subsections 3.1–3.2, the contribution of $Y_{\delta,i}$ to (50) is the following:

$$-\frac{1}{m}\sum_{(a,\alpha)\in\mathcal{KMS}(i)} m\bigl(a+\lambda^{-1}\alpha\bigr)\frac{\sqrt{-1}}{2\pi}\int_{D_i}\Bigl(\mathrm{tr}\bigl(R({}^ih_{a,\alpha})\bigr) - a\cdot r\bigl(i,(a,\alpha)\bigr)\cdot\tau_i\Bigr)$$
$$= -\sum_{(a,\alpha)\in\mathcal{KMS}(i)}(a+\lambda^{-1}\alpha)\cdot\Bigl(\deg_{D_i}\bigl({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_*\bigr) - a\cdot r\bigl(i,(a,\alpha)\bigr)\cdot[D_i]^2\Bigr) \tag{51}$$

By taking the real part, we obtain (48). By taking the imaginary part, we obtain (49). The equality (47) follows from (48) and Lemma 3.15 by a formal calculation. ▮

Lemma 3.23 *For any C^∞ 2-form τ, we have the following:*

$$\int_X \text{par-c}_1(\boldsymbol{E}_*)\cdot\tau = \frac{\sqrt{-1}}{2\pi}\int_X \mathrm{tr}\,R(h_0)\cdot\tau = -\sum_{i\in S}\sum_{(a,\alpha)\in\mathcal{KMS}(i)}\mathrm{Re}(\lambda^{-1}\alpha+a)\cdot r\bigl(i,(a,\alpha)\bigr)\cdot(D_i,\tau) \tag{52}$$

Namely, the cohomology class of $\mathrm{tr}\,R(h_0)$ *is* $\text{par-c}_1(\boldsymbol{E}_*)$. *In particular, we also have the following equality:*

$$\text{par-deg}_\omega(\boldsymbol{E}_*) = -\sum_{i\in S}\sum_{(a,\alpha)\in\mathcal{KMS}(i)}\mathrm{Re}(\lambda^{-1}\alpha+a)\cdot r\bigl(i,(a,\alpha)\bigr)\cdot(D_i,\omega) \tag{53}$$

Proof Recall we have $R(h_0) = \lambda^{-1}(1+|\lambda|^2)\cdot d''\theta_0$. Then, we obtain (52) by using the estimates in Subsections 3.1–3.2. ▮

Remark 3.24 *We considered the KMS-spectra of ${}^\diamond E$. But, we have the following equality for any $\boldsymbol{c}\in\boldsymbol{R}^S$ and $i\in S$:*

$$\bigl\{\mathrm{Re}(\lambda^{-1}\alpha)+a \,\big|\, (a,\alpha)\in\mathcal{KMS}({}^\diamond E,i)\bigr\} = \bigl\{\mathrm{Re}(\lambda^{-1}\alpha)+a \,\big|\, (a,\alpha)\in\mathcal{KMS}({}_{\boldsymbol{c}}E,i)\bigr\}$$

We also have such comparison for $\mathcal{KMS}\bigl({}^\diamond E,P\bigr)$ and $\mathcal{KMS}\bigl({}_{\boldsymbol{c}}E\bigr)$ for $\boldsymbol{c}\in\boldsymbol{R}^S$ and $P\in D_i\cap D_j$. Namely, the choice ${}^\diamond E$ is not essential. (See also Section 6.) ▮

3.5.2 Remark on the vanishing of the parabolic Chern character numbers

Recall the formulas for $\int_X \text{par-ch}_2(\boldsymbol{E}_*)$ in Proposition 3.22 and the formula for $\text{par-deg}_\omega(\boldsymbol{E}_*)$ in Lemma 3.23. Then, we immediately obtain the following corollary.

Corollary 3.25 *If $a+\mathrm{Re}\,\lambda^{-1}\alpha = 0$ holds for any KMS-spectrum (a,α) of $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$, the characteristic numbers $\text{par-deg}_\omega(\boldsymbol{E}_*)$ and $\int_X \text{par-ch}_2(\boldsymbol{E}_*)$ automatically vanish.* ▮

Remark 3.26 *Let E be a vector bundle on $X-D$ with a flat connection ∇. We have the Deligne extension $(\widetilde{E},\nabla)$. (See Subsection 2.1.3, for example.) We have the canonically defined parabolic structure $\boldsymbol{F}$ such that $\mathrm{Re}\,\alpha+a=0$ for any KMS-spectrum. In that case, the stability of $(\widetilde{E},\boldsymbol{F},\nabla)$ and the semisimplicity of (E,∇) is equivalent. The corollary means $\int_X \text{par-c}_2(\widetilde{E},\boldsymbol{F}) = \text{par-deg}_\omega(\widetilde{E},\boldsymbol{F}) = 0$.*

When (E,∇) is semisimple, we know that there exists the Corlette-Jost-Zuo metric of (E,∇) which is a pure imaginary tame pluri-harmonic metric adapted to the parabolic structure $\boldsymbol{F}$ (See [3] for the case $D=\emptyset$ and [16] for the general case. See also [30].) To show such an existence theorem from the Kobayashi-Hitchin correspondence, we have to show the vanishing of the characteristic numbers which is "the obstruction on the way from harmonicity to pluri-harmonicity". Corollary 3.25 clarifies the point. ▮

4 Continuity of some families of harmonic metrics

4.1 Statements

In this section, we will show continuity of two kinds of families of harmonic metrics on curves, i.e., Proposition 4.1 and Proposition 4.2. We will give a detailed proof of the first one. Because the second one can be proved similarly and more easily, we just give some remarks in the end of this section.

4.1.1 Continuity for ϵ-perturbation

Let C be a smooth projective curve with a simple divisor D. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a parabolic flat λ-connection over (C, D), which is stable and par-deg$(E, \boldsymbol{F}) = 0$. Let $\boldsymbol{F}^{(\epsilon)}$ be the ϵ-perturbation of the parabolic structures, explained in (II) of Subsection 2.1.6. We remark $\det(E, \boldsymbol{F}) = \det(E, \boldsymbol{F}^{(\epsilon)})$. Let $h^{(\epsilon)}$ be the harmonic metric for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$. Let $\theta^{(\epsilon)}$ denote the Higgs fields for the harmonic bundles $(E, \mathbb{D}^\lambda, h^{(\epsilon)})$.

Proposition 4.1 *The sequences $\{h^{(\epsilon)} \,|\, \epsilon > 0\}$ and $\{\theta^{(\epsilon)}\}$ converge to $h^{(0)}$ and $\theta^{(0)}$ respectively, in the C^∞-sense locally on $C - D$.*

The proof is given in Subsection 4.5 after the preparation given in Subsections 4.2–4.4.

4.1.2 Continuity for a holomorphic family

Before going into the proof of Proposition 4.1, we give a similar statement for another family. Let $\mathcal{C} \longrightarrow \Delta$ be a holomorphic family of smooth projective curve, and $\mathcal{D} \longrightarrow \Delta$ be a relative divisor. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a parabolic flat bundle on $(\mathcal{C}, \mathcal{D})$. Let t be any point of Δ. We denote the fibers by $\mathcal{C}_t$ and $\mathcal{D}_t$, and the restriction of $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ to $(\mathcal{C}_t, \mathcal{D}_t)$ is denoted by $(E_t, \boldsymbol{F}_t, \mathbb{D}^\lambda_t)$. We assume par-deg$(E_t, \boldsymbol{F}_t) = 0$ and that $(E_t, \boldsymbol{F}_t, \mathbb{D}^\lambda_t)$ is stable for each t. For simplicity, we also assume that we are given a pluri harmonic metric $h_{\det(E)}$ of $\det(E, \mathbb{D}^\lambda)_{|\mathcal{C}-\mathcal{D}}$ which is adapted to the induced parabolic structure.

Let $h_{H,t}$ be a harmonic metric of $(E_t, \boldsymbol{F}_t, \mathbb{D}^\lambda_t)$ such that $\det(h_{H,t}) = h_{\det(E)\,|\,\mathcal{C}_t}$. They give the metric h_H of E. Let $\theta_{H,t}$ be the Higgs field obtained from $(E_t, \mathbb{D}^\lambda, h_{H,t})$, which is a section of $\mathrm{End}(E_t) \otimes \Omega^{1,0}_{\mathcal{C}_t}(\log \mathcal{D}_t)$. They give the section θ_H of $\mathrm{End}(E) \otimes \Omega^{1,0}_{\mathcal{C}/\Delta}(\log \mathcal{D})$, where $\Omega^{1,0}_{\mathcal{C}/\Delta}(\log \mathcal{D})$ denotes the sheaf of the logarithmic relative $(1,0)$-forms.

Proposition 4.2 *h_H and θ_H are continuous. Their derivatives of any degree along the fiber directions are continuous.*

Since Proposition 4.2 can be proved similarly and more easily, we will not give a detailed proof. See Remark 4.15.

4.2 Preliminary from elementary calculus

For any $z \in \Delta^* = \{z \in \boldsymbol{C} \,\big|\, |z| < 1\}$ and $\epsilon > 0$, we put as follows:

$$L_\epsilon(z) := \frac{|z|^{-\epsilon} - |z|^{\epsilon}}{\epsilon}, \quad K_\epsilon(z) := \frac{|z|^{-\epsilon} + |z|^{\epsilon}}{2}, \quad M_\epsilon(z) := |z|^{4\epsilon}(1 - \log |z|^{4\epsilon}).$$

We also put $L_0(z) := -\log |z|^2$, $K_0(z) = 1$ and $M_0(z) = 1$. Then, they are continuous with respect to $(z, \epsilon) \in \Delta^* \times \boldsymbol{R}_{\geq 0}$.

Lemma 4.3 *For any $(z, \epsilon) \in \Delta^* \times \boldsymbol{R}_{\geq 0}$, we have $L_0(z) \leq L_\epsilon(z)$.*

Proof We put $g(\epsilon) := a^{-\epsilon} - a^{\epsilon} + \epsilon \cdot \log a^2$ for $0 < a < 1$ and $0 \leq \epsilon$. Taking the derivative with respect to ϵ, we obtain the following:

$$g'(\epsilon) = -\big(a^{-\epsilon} + a^{\epsilon}\big) \log a + \log a^2, \quad g''(\epsilon) = (a^{-\epsilon} - a^{\epsilon})(\log a)^2 \geq 0.$$

Since we have $g(0) = g'(0) = 0$, the claim of the lemme follows. ∎

Lemma 4.4 *$(K_\epsilon(z)-1)\cdot\bigl(L_\epsilon(z)^2\cdot\epsilon^2\cdot|z|^\epsilon\bigr)^{-1}$ are bounded on Δ^*, independently of ϵ. We also have $K_\epsilon(z)-1\geq 0$.*

Proof The second claim is clear. Let us check the first claim. We put as follows, for $0<a<1$ and $0\leq\epsilon\leq 1$:

$$g_1(\epsilon):=a^{-\epsilon}-2+a^{\epsilon},\quad g_2(\epsilon):=(a^{-\epsilon}-a^{\epsilon})^2a^{\epsilon}=a^{-\epsilon}-2a^{\epsilon}+a^{3\epsilon}.$$

We have only to show that $g_2(\epsilon)\geq g_1(\epsilon)$. We put $g(\epsilon):=g_2(\epsilon)-g_1(\epsilon)=2+a^{3\epsilon}-3a^{\epsilon}$. By taking the derivative with respect to ϵ, we obtain the following:

$$g'(\epsilon)=3a^{3\epsilon}\cdot\log a-3a^{\epsilon}\cdot\log a=3(-a^{3\epsilon}+a^{\epsilon})(-\log a)\geq 0.$$

Since we have $g(0)=0$, we obtain $g(\epsilon)\geq 0$. Thus we are done. ∎

Lemma 4.5 *$\bigl(1-M_\epsilon(z)\bigr)\cdot\bigl(L_\epsilon(z)^2\cdot\epsilon^2\cdot|z|^\epsilon\bigr)^{-1}$ are bounded on Δ^*, independently of ϵ. We also have $1-M_\epsilon(z)\geq 0$.*

Proof We have only to show the following inequalities for $0<a<1$ and $0\leq\epsilon<1$:

$$0\leq 1-a^{4\epsilon}(1-\log a^{4\epsilon})\leq 3(a^{-\epsilon}-a^{\epsilon})^2a^{\epsilon}.$$

To show the left inequality, we put $h(\epsilon):=1-a^{4\epsilon}(1-\log a^{4\epsilon})$. By taking the derivative with respect to ϵ, we have $h'(\epsilon)=-a^{4\epsilon}\log a^4(1-\log a^{4\epsilon})+a^{4\epsilon}\log a^4=\epsilon a^{4\epsilon}(\log a^4)^2\geq 0$. We also have $h(0)=0$. Hence, we obtain $h(\epsilon)\geq 0$. To show the right inequality, we put as follows:

$$g(\epsilon):=a^{-4\epsilon}\Bigl(3(a^{-\epsilon}-a^{\epsilon})^2a^{\epsilon}-\bigl(1-a^{4\epsilon}(1-\log a^{4\epsilon})\bigr)\Bigr)=3(a^{-5\epsilon}-2a^{-3\epsilon}+a^{-\epsilon})+(1-\log a^{4\epsilon})-a^{-4\epsilon}.$$

By taking the derivative with respect to ϵ, we obtain the following:

$$g'(\epsilon)=3\bigl(a^{-5\epsilon}(-5\log a)-2a^{-3\epsilon}(-3\log a)+a^{-\epsilon}(-\log a)\bigr)-4\log a-a^{-4\epsilon}(-4\log a)$$

$$g''(\epsilon)=\bigl(75a^{-5\epsilon}-16a^{-4\epsilon}-54a^{-3\epsilon}+3a^{-\epsilon}\bigr)\cdot(\log a)^2.$$

It is easy to check $g''(\epsilon)\geq 0$ by using $a^{-5\epsilon}\geq a^{-k\epsilon}$ $(k=3,4)$. Since we have $g'(0)=g(0)=0$, we obtain $g(\epsilon)\geq 0$. Thus we are done. ∎

Lemma 4.6 *Let $P(t)$ be a polynomial with variable t, and let b be any fixed positive number. Then, we have the boundedness of $|z|^{b\epsilon}\cdot P\bigl(\epsilon L_0(z)\bigr)$ on Δ^*, independently of $0\leq\epsilon\leq 1/2$.*

Proof We put $u:=|z|^\epsilon$, and then $|z|^{b\epsilon}P(\epsilon L_0(z))=u^b\cdot P\bigl(L_0(u)\bigr)$. Hence, we have only to show the boundedness of $u^b\cdot P\bigl(L_0(u)\bigr)$ when $0<u<1$, but it is easy. ∎

4.3 A family of the metrics for a logarithmic λ-flat bundle of rank two on a disc

4.3.1 Construction of a family of metrics

We put $X=\Delta=\{z\,\big|\,|z|<1\}$. Let O denote the origin, and we put $X^*:=X-\{O\}$. We use the Kahler form $\omega_\epsilon:=(\epsilon^2|z|^{\epsilon-2}+1)\cdot dz\cdot d\bar{z}$ in this subsection. We will use the notation in Subsection 4.2.

To begin with, we recall an example of a harmonic bundle on a punctured disc. Let $E=\mathcal{O}_X\cdot v_1\oplus\mathcal{O}_X\cdot v_2$ be a holomorphic vector bundle on a disc. Let θ be a Higgs bundle such that $\theta\cdot v_1=v_2\cdot dz/z$ and $\theta\cdot v_2=0$. Let h be the metric of $E_{|X^*}$ such that $h(v_1,v_1)=L_0$, $h(v_2,v_2)=L_0^{-1}$ and $h(v_i,v_j)=0$ $(i\neq j)$. Recall that the tuple $(E,\overline{\partial}_E,\theta,h)$ is a harmonic bundle. Let us see the associated λ-connection. We put $u_1:=v_1$ and $u_2:=v_2-\lambda\cdot L_0^{-1}\cdot v_1$. Then, we can show $(\overline{\partial}_E+\lambda\theta^\dagger)u_i=0$ $(i=1,2)$, $\mathbb{D}^\lambda u_1=u_2\cdot dz/z$ and $\mathbb{D}^\lambda u_2=0$ by a direct calculation. We also have the following:

$$h(u_1,u_1)=L_0,\quad h(u_2,u_2)=(1+|\lambda|^2)\cdot L_0^{-1},\quad h(u_1,u_2)=-\overline{\lambda},\quad h(u_2,u_1)=-\lambda.$$

Motivated by this example, we consider the following family of the metrics h_ϵ on the λ-connection $(E,\mathbb{D}^\lambda)$ given as follows:

$$h_\epsilon(u_1,u_1)=L_\epsilon,\quad h_\epsilon(u_2,u_2)=\bigl(1+|\lambda|^2\bigr)\cdot L_\epsilon^{-1},\quad h_\epsilon(u_1,u_2)=-\overline{\lambda}\cdot M_\epsilon,\quad h_\epsilon(u_2,u_1)=-\lambda\cdot M_\epsilon.$$

The λ-connection $\mathbb{D}^\lambda$ and the metric h_ϵ induce the operators $\overline{\partial}_\epsilon$ and θ_ϵ (Subsection 2.2.1). The main purpose of this subsection is to show the following proposition.

Proposition 4.7 *There exists a some positive constant C such that $\left|\overline{\partial}_{\epsilon}\theta_{\epsilon}\right|_{h_{\epsilon},\omega_{\epsilon}} \leq C$ for any $0 \leq \epsilon < 1/2$.*

Although the proof of the proposition is just a calculation, we will give the detail in the rest of this subsection.

Remark 4.8 *Let h'_{ϵ} be the metric determined by $h'_{\epsilon}(u_1,u_1)=L_{\epsilon}$, $h'_{\epsilon}(u_2,u_2)=L_{\epsilon}^{-1}$ and $h'_{\epsilon}(u_i,u_j)=0$ $(i\neq j)$. Then, there exist positive constants C_i $(i=1,2)$ such that $C_1\cdot h'_{\epsilon}\leq h_{\epsilon}\leq C_2\cdot h'_{\epsilon}$ for any $0\leq\epsilon\leq 1/2$. Hence, we have only to consider the norms for h'_{ϵ} instead of those for h_{ϵ}.* ∎

4.3.2 Preliminary

Let H_{ϵ} be the hermitian matrix valued function given by $H_{\epsilon}:=H(h_{\epsilon},\boldsymbol{u})$, i.e.,

$$H_{\epsilon}:=\begin{pmatrix} L_{\epsilon} & -\overline{\lambda}\cdot M_{\epsilon} \\ -\lambda\cdot M_{\epsilon} & (1+|\lambda|^2)L_{\epsilon}^{-1}\end{pmatrix}.$$

Let N be determined by $\mathbb{D}^{\lambda}\boldsymbol{u}=\boldsymbol{u}\cdot N\cdot dz/z$, and let $N_{\epsilon}^{\dagger}$ denote the adjoint of N with respect to the metric H_{ϵ}, i.e.,

$$N=\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix},\quad N_{\epsilon}^{\dagger}=\overline{H}_{\epsilon}^{-1}\cdot {}^{t}\overline{N}\cdot\overline{H}_{\epsilon}=\frac{1}{1+|\lambda|^2(1-M_{\epsilon}^2)}\begin{pmatrix} -\overline{\lambda}(1+|\lambda|^2)L_{\epsilon}^{-1}M_{\epsilon} & (1+|\lambda|^2)^2L_{\epsilon}^{-2} \\ -\overline{\lambda}^2M_{\epsilon}^2 & \overline{\lambda}(1+|\lambda|^2)M_{\epsilon}L_{\epsilon}^{-1}\end{pmatrix}.$$

Recall the calculation given in Subsection 2.2.2. Then, $\overline{\partial}_{\epsilon}$ and θ_{ϵ} can be described with respect to $\boldsymbol{u}$ as follows:

$$\overline{\partial}_{\epsilon}\boldsymbol{u}=\boldsymbol{u}\cdot\frac{\lambda}{1+|\lambda|^2}\Big(\overline{\lambda}\cdot\overline{H}_{\epsilon}^{-1}\overline{\partial}\overline{H}_{\epsilon}-N_{\epsilon}^{\dagger}\frac{d\overline{z}}{\overline{z}}\Big),\quad \theta_{\epsilon}\boldsymbol{u}=\boldsymbol{u}\frac{1}{1+|\lambda|^2}\Big(N\frac{dz}{z}-\lambda\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}\Big).$$

Therefore, $\overline{\partial}_{\epsilon}(\theta_{\epsilon})$ is described by the following 2×2-matrix valued 2-form with respect to $\boldsymbol{u}$:

$$\frac{1}{1+|\lambda|^2}\overline{\partial}\Big(-\lambda\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}\Big)+\frac{\lambda}{(1+|\lambda|^2)^2}\left(\Big[\overline{\lambda}\cdot\overline{H}_{\epsilon}^{-1}\overline{\partial}\overline{H}_{\epsilon},N\frac{dz}{z}\Big]-\Big[N_{\epsilon}^{\dagger}\frac{d\overline{z}}{\overline{z}},\,N\frac{dz}{z}\Big]+\Big[N_{\epsilon}^{\dagger}\frac{d\overline{z}}{\overline{z}},\,\lambda\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}\Big]\right). \quad (54)$$

Here we have used $\big[\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon},\,\overline{H}_{\epsilon}^{-1}\overline{\partial}\overline{H}_{\epsilon}\big]=0$, which can be checked easily.

Lemma 4.9 *To show Proposition 4.7, we have only to show the uniform boundedness of $(1,1)$-entry, $(2,2)$-entry, $L_{\epsilon}\times$ $(1,2)$-entry and $L_{\epsilon}^{-1}\times$ $(2,1)$-entry, in the matrix valued function (54).*

Proof It follows from Remark 4.8. ∎

In the following calculation, we often use the notation L and M instead of L_{ϵ} and M_{ϵ}, if there are no risk of confusion. Let us see $\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}$. We have the following equality:

$$\overline{H}_{\epsilon}^{-1}=\frac{1}{1+|\lambda|^2(1-M_{\epsilon}^2)}\begin{pmatrix}(1+|\lambda|^2)\cdot L_{\epsilon}^{-1} & \lambda\cdot M_{\epsilon} \\ \overline{\lambda}\cdot M_{\epsilon} & L_{\epsilon}\end{pmatrix},\quad \partial\overline{H}_{\epsilon}=\begin{pmatrix}\partial L_{\epsilon} & -\lambda\cdot\partial M_{\epsilon} \\ -\overline{\lambda}\cdot\partial M_{\epsilon} & (1+|\lambda|^2)\cdot\partial L_{\epsilon}^{-1}\end{pmatrix}.$$

Then, we obtain the following formula for $\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}$:

$$\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}=\frac{1}{1+|\lambda|^2(1-M_{\epsilon}^2)}\begin{pmatrix}(1+|\lambda|^2)L^{-1}\partial L-|\lambda|^2M\partial M & \lambda(1+|\lambda|^2)\big(-L^{-1}\partial M+M\partial L^{-1}\big) \\ \overline{\lambda}(M\partial L-L\partial M) & (1+|\lambda|^2)L\partial L^{-1}-|\lambda|^2M\cdot\partial M\end{pmatrix}. \quad (55)$$

We also have a similar formula for $\overline{H}_{\epsilon}^{-1}\overline{\partial}\overline{H}_{\epsilon}$. We obtain the following formula for $\overline{\partial}\big(\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}\big)$:

$$\overline{\partial}\big(\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}\big)=\frac{2|\lambda|^2M\overline{\partial}M}{\big(1+|\lambda|^2(1-M^2)\big)^2}\overline{H}_{\epsilon}^{-1}\partial\overline{H}_{\epsilon}$$
$$+\frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix}(1+|\lambda|^2)\overline{\partial}\partial\log L-2^{-1}|\lambda|^2\overline{\partial}\partial M^2 & \lambda(1+|\lambda|^2)(M\overline{\partial}\partial L^{-1}-L^{-1}\overline{\partial}\partial M) \\ \overline{\lambda}(M\overline{\partial}\partial L-L\overline{\partial}\partial M) & (1+|\lambda|^2)\overline{\partial}\partial\log L^{-1}-2^{-1}|\lambda|^2\overline{\partial}\partial M^2\end{pmatrix}. \quad (56)$$

The commutator of $\overline{H}_\epsilon^{-1}\overline{\partial H}_\epsilon$ and $N\cdot dz/z$ is as follows:

$$\Big[\overline{H}_\epsilon^{-1}\overline{\partial H}_\epsilon, N\cdot\frac{dz}{z}\Big]=\frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M^2)}\begin{pmatrix}\lambda(-L^{-1}\overline{\partial}M+M\overline{\partial}L^{-1}) & 0\\ 2L\overline{\partial}L^{-1} & -\lambda(-L^{-1}\overline{\partial}M+M\overline{\partial}L^{-1})\end{pmatrix}\frac{dz}{z}. \tag{57}$$

Let us see the commutator of $\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon$ and $N_\epsilon^\dagger$. By direct calculations, we have the following equality:

$$\begin{aligned}\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\cdot N_\epsilon^\dagger&=\frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix}-\overline{\lambda}(1+|\lambda|^2)L^{-2}M\partial L & (1+|\lambda|^2)^2L^{-3}\partial L\\ \overline{\lambda}^2\cdot M\partial M & -\overline{\lambda}(1+|\lambda|^2)L^{-1}\partial M\end{pmatrix}\\ &+\frac{1}{\big(1+|\lambda|^2(1-M^2)\big)^2}\begin{pmatrix}2|\lambda|^2\overline{\lambda}(1+|\lambda|^2)M^2L^{-1}\partial M & -2|\lambda|^2(1+|\lambda|^2)^2ML^{-2}\partial M\\ 2M^3\partial M\overline{\lambda}^2|\lambda|^2 & -2\overline{\lambda}|\lambda|^2(1+|\lambda|^2)M^2L^{-1}\partial M\end{pmatrix}.\end{aligned} \tag{58}$$

We also have the following:

$$N_\epsilon^\dagger\cdot\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon=\frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix}-\overline{\lambda}(1+|\lambda|^2)L^{-1}\partial M & (1+|\lambda|^2)^2L^{-1}\partial L^{-1}\\ -\overline{\lambda}^2M\partial M & \overline{\lambda}(1+|\lambda|^2)M\partial L^{-1}\end{pmatrix}. \tag{59}$$

Therefore, we obtain the following formula:

$$\begin{aligned}&\Big[N_\epsilon^\dagger\frac{d\overline{z}}{\overline{z}},\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\Big]\\ &=\frac{1}{1+|\lambda|^2(1-M^2)}\frac{d\overline{z}}{\overline{z}}\begin{pmatrix}-\overline{\lambda}(1+|\lambda|^2)(L^{-1}\partial M-L^{-2}M\partial L) & -2(1+|\lambda|^2)^2L^{-3}\partial L\\ -2\overline{\lambda}^2M\partial M & \overline{\lambda}(1+|\lambda|^2)(M\partial L^{-1}+L^{-1}\partial M)\end{pmatrix}\\ &\qquad-\frac{2|\lambda|^2}{\big(1+|\lambda|^2(1-M^2)\big)^2}\frac{d\overline{z}}{\overline{z}}\begin{pmatrix}\overline{\lambda}(1+|\lambda|^2)M^2L^{-1}\partial M & -(1+|\lambda|^2)^2ML^{-2}\partial M\\ \overline{\lambda}^2M^3\partial M & -\overline{\lambda}(1+|\lambda|^2)M^2L^{-1}\partial M\end{pmatrix}\end{aligned} \tag{60}$$

The commutator of N and $N_\epsilon^\dagger$ is as follows:

$$\big[N_\epsilon^\dagger,N\big]=\frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix}(1+|\lambda|^2)^2L^{-2} & 0\\ 2\overline{\lambda}(1+|\lambda|^2)ML^{-1} & -(1+|\lambda|^2)^2L^{-2}\end{pmatrix}. \tag{61}$$

4.3.3 Estimate

We have the following:

$$\partial L_\epsilon=-K_\epsilon\frac{dz}{z},\quad \partial K_\epsilon=-\frac{\epsilon^2}{4}L_\epsilon\frac{dz}{z},\quad \partial M_\epsilon=4\epsilon^2\cdot|z|^{4\epsilon}\cdot L_0\cdot\frac{dz}{z}. \tag{62}$$

In particular, we have the following estimate:

$$M_\epsilon\partial M_\epsilon=O\Big(\epsilon^2\cdot|z|^{8\epsilon}\cdot L_0\cdot\big(1+\epsilon L_0\big)\frac{dz}{z}\Big).$$

Let us see the first term in the right hand side of (56):

$$\frac{2|\lambda|^2M_\epsilon\overline{\partial}M_\epsilon}{\big(1+|\lambda|^2(1-M_\epsilon^2)\big)^2}\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon \tag{63}$$

For the $(1,1)$-entry and $(2,2)$-entry, we have the following estimates:

$$M_\epsilon\overline{\partial}M_\epsilon\cdot L_\epsilon^{-1}\partial L_\epsilon=O\left(\epsilon^2\cdot L_0\cdot|z|^{8\epsilon}(1+\epsilon L_0)\frac{K_\epsilon}{L_\epsilon}\right)\frac{d\overline{z}\cdot dz}{|z|^2}=O\left(|z|^{5\epsilon}(1+\epsilon L_0)\frac{L_0}{L_\epsilon}\right)\cdot\omega_\epsilon$$

$$M_\epsilon \overline{\partial} M_\epsilon \cdot M_\epsilon \partial M_\epsilon = O\Big(\epsilon^4 \cdot |z|^{16\epsilon} \cdot (1+\epsilon L_0)^2 L_0^2\Big) \frac{dz \cdot d\overline{z}}{|z|^2} = O\Big(|z|^{15\epsilon} \cdot (1+\epsilon L_0)^2 (\epsilon L_0)^2\Big) \cdot \omega_\epsilon.$$

They are bounded with respect to ω_ϵ due to Lemma 4.3 and Lemma 4.6. Hence, the $(1,1)$-entry and the $(2,2)$-entry of (63) are bounded independently of ϵ. Let us see the $(1,2)$-entry. Recall Lemma 4.9. Hence, we have only to see the following:

$$L_\epsilon \times (M_\epsilon \overline{\partial} M_\epsilon) \cdot \big(L_\epsilon^{-1} \partial M_\epsilon - M_\epsilon \partial L_\epsilon^{-1}\big) = M_\epsilon \overline{\partial} M_\epsilon \partial M_\epsilon + M_\epsilon^2 \overline{\partial} M_\epsilon L_\epsilon^{-1} \partial L_\epsilon.$$

Both terms in the right hand side can be estimated as in the previous case, by using Lemma 4.3 and Lemma 4.6:

$$M_\epsilon \overline{\partial} M_\epsilon \partial M_\epsilon = O\Big(|z|^{10\epsilon}(1+\epsilon L_0)(\epsilon L_0)^2\Big) \cdot \omega_\epsilon = O(1) \cdot \omega_\epsilon$$

$$M_\epsilon^2 \overline{\partial} M_\epsilon L_\epsilon^{-1} \partial L_\epsilon = O\Big(|z|^{11\epsilon}(1+\epsilon L_0)^2 \frac{L_0}{L_\epsilon}\Big) \cdot \omega_\epsilon = O(1) \cdot \omega_\epsilon$$

The $(2,1)$-entry can be estimated similarly:

$$L_\epsilon^{-1} \times (M_\epsilon \overline{\partial} M_\epsilon)\big(M_\epsilon \partial L_\epsilon - L_\epsilon \partial M_\epsilon\big) = M_\epsilon^2 L_\epsilon^{-1} \overline{\partial} M_\epsilon \partial L_\epsilon - M_\epsilon \cdot \overline{\partial} M_\epsilon \partial M_\epsilon = O(1) \cdot \omega_\epsilon.$$

Let us see the second term in the right hand side of (56):

$$\frac{1}{1+|\lambda|^2(1-M^2)} \begin{pmatrix} (1+|\lambda|^2)\overline{\partial}\partial \log L - 2^{-1}|\lambda|^2 \overline{\partial}\partial M^2 & \lambda(1+|\lambda|^2)(M\overline{\partial}\partial L^{-1} - L^{-1}\overline{\partial}\partial M) \\ \overline{\lambda}(1+|\lambda|^2)(M\overline{\partial}\partial L - L\overline{\partial}\partial M) & (1+|\lambda|^2)\overline{\partial}\partial \log L^{-1} - 2^{-1}|\lambda|^2 \overline{\partial}\partial M^2 \end{pmatrix}. \tag{64}$$

It is easy to see the following estimate:

$$\overline{\partial}\partial M_\epsilon^2 = O\Big(\epsilon^2 \cdot |z|^{6\epsilon}(1+\epsilon L_0)^2\Big) \cdot \omega_\epsilon = O(\epsilon^2) \cdot \omega_\epsilon. \tag{65}$$

Hence, it is bounded with respect to ω_ϵ independently of ϵ. We remark that $L_\epsilon^{-1} M_\epsilon \overline{\partial}\partial L_\epsilon$ is also bounded independently of ϵ:

$$L_\epsilon^{-1} M_\epsilon \cdot \overline{\partial}\partial L_\epsilon = \frac{\epsilon^2}{4} M_\epsilon \cdot \frac{d\overline{z} \cdot dz}{|z|^2} = O(1) \cdot \omega_\epsilon.$$

Hence, we have the following, modulo the uniformly bounded term with respect to $(h_\epsilon, \omega_\epsilon)$:

$$\overline{\partial}\big(\overline{H}_\epsilon^{-1} \partial \overline{H}_\epsilon\big) \equiv \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M_\epsilon^2)} \begin{pmatrix} \overline{\partial}\partial \log L_\epsilon & \lambda M_\epsilon \overline{\partial}\partial L_\epsilon^{-1} \\ 0 & -\overline{\partial}\partial \log L_\epsilon \end{pmatrix}. \tag{66}$$

Let us see (57). By the same argument, we have the following uniform boundedness:

$$L_\epsilon^{-1} \overline{\partial} M_\epsilon \cdot \frac{dz}{z} = O\left(\epsilon^2 |z|^{4\epsilon} \frac{L_0}{L_\epsilon}\right) \cdot \frac{dz \cdot d\overline{z}}{|z|^2} = O(1) \cdot \omega_\epsilon.$$

Hence, we have the following, modulo the uniformly bounded terms with respect to $(h_\epsilon, \omega_\epsilon)$:

$$\left[\overline{H}_\epsilon^{-1} \overline{\partial} \overline{H}_\epsilon,\ N \cdot \frac{dz}{z}\right] \equiv \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M_\epsilon^2)} \begin{pmatrix} \lambda M_\epsilon \overline{\partial} L_\epsilon^{-1} & 0 \\ 2L_\epsilon \overline{\partial} L_\epsilon^{-1} & -\lambda M_\epsilon \overline{\partial} L_\epsilon^{-1} \end{pmatrix} \cdot \frac{dz}{z}. \tag{67}$$

Let us see (60). We remark the following, for any $k \geq 1$:

$$\frac{d\overline{z}}{\overline{z}} \frac{M_\epsilon^k \partial M_\epsilon}{L_\epsilon} = O\left(\epsilon^2 |z|^{4(k+1)\epsilon}(1+\epsilon L_0)^k \frac{L_0}{L_\epsilon}\right) \cdot \frac{d\overline{z} \cdot dz}{|z|^2} = O(1) \cdot \omega_\epsilon.$$

Hence, the terms containing ∂M in the right hand side of (60) can be ignored. Hence, we obtain the following, modulo the uniformly bounded terms with respect to $(h_\epsilon, \omega_\epsilon)$:

$$\left[N_\epsilon^\dagger \frac{d\overline{z}}{\overline{z}},\ \overline{H}_\epsilon^{-1} \partial \overline{H}_\epsilon\right] \equiv \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M_\epsilon^2)} \frac{d\overline{z}}{\overline{z}} \begin{pmatrix} \overline{\lambda} L_\epsilon^{-2} M_\epsilon \partial L_\epsilon & -2(1+|\lambda|^2) L_\epsilon^{-3} \partial L_\epsilon \\ 0 & \overline{\lambda} M_\epsilon \partial L_\epsilon^{-1} \end{pmatrix}. \tag{68}$$

In all, (54) is same as the following, modulo uniformly bounded terms due to (61), (66), (67) and (68):

$$\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} -\lambda\overline{\partial}\partial\log L_\epsilon & -\lambda^2 M_\epsilon\cdot\overline{\partial}\partial L_\epsilon^{-1} \\ 0 & \lambda\overline{\partial}\partial\log L_\epsilon \end{pmatrix}$$
$$+\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\frac{|\lambda|^2}{1+|\lambda|^2}\frac{d\overline{z}\cdot dz}{|z|^2}\begin{pmatrix} \lambda\cdot M_\epsilon\cdot K_\epsilon\cdot L_\epsilon^{-2} & 0 \\ 2K_\epsilon\cdot L_\epsilon^{-1} & -\lambda\cdot M_\epsilon\cdot K_\epsilon\cdot L_\epsilon^{-2} \end{pmatrix}$$
$$+\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\frac{\lambda^2}{1+|\lambda|^2}\frac{d\overline{z}\cdot dz}{|z|^2}\begin{pmatrix} -\overline{\lambda}\cdot M_\epsilon\cdot K_\epsilon\cdot L_\epsilon^{-2} & 2(1+|\lambda|^2)L_\epsilon^{-3}\cdot K_\epsilon \\ 0 & \overline{\lambda}\cdot M_\epsilon\cdot K_\epsilon\cdot L_\epsilon^{-2} \end{pmatrix}$$
$$-\frac{\lambda}{1+|\lambda|^2(1-M_\epsilon^2)}\frac{d\overline{z}\cdot dz}{|z|^2}\begin{pmatrix} L_\epsilon^{-2} & 0 \\ 2\overline{\lambda}(1+|\lambda|^2)^{-1}M_\epsilon\cdot L_\epsilon^{-1} & -L_\epsilon^{-2} \end{pmatrix}. \quad (69)$$

The summation of the last three term in (69) is as follows:

$$\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\frac{d\overline{z}\cdot dz}{|z|^2}\begin{pmatrix} -\lambda L_\epsilon^{-2} & 2\lambda^2 L_\epsilon^{-3}K_\epsilon \\ 2|\lambda|^2(1+|\lambda|^2)^{-1}(K_\epsilon-M_\epsilon)L_\epsilon^{-1} & \lambda L_\epsilon^{-2} \end{pmatrix}. \quad (70)$$

By a direct calculation, we can show the following equalities:

$$\overline{\partial}\partial\log L_\epsilon = -\frac{1}{L_\epsilon^2}\frac{d\overline{z}\cdot dz}{|z|^2}, \qquad \overline{\partial}\partial L_\epsilon^{-1} = \frac{2}{L_\epsilon^3}\frac{d\overline{z}\cdot dz}{|z|^2} - \frac{\epsilon^2}{2}\frac{1}{L_\epsilon}\frac{d\overline{z}\cdot dz}{|z|^2}.$$

Therefore, (69) can be rewritten as follows:

$$\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} 0 & 2\lambda^2 L_\epsilon^{-3}(K_\epsilon-M_\epsilon) \\ 2|\lambda|^2(1+|\lambda|^2)^{-1}L_\epsilon^{-1}(K_\epsilon-M_\epsilon) & 0 \end{pmatrix}\cdot\frac{d\overline{z}\cdot dz}{|z|^2}$$
$$+\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} 0 & \lambda^2\epsilon^2 M_\epsilon(2L_\epsilon)^{-1} \\ 0 & 0 \end{pmatrix}\cdot\frac{d\overline{z}\cdot dz}{|z|^2}. \quad (71)$$

Due to $M_\epsilon = O\big(|z|^{4\epsilon}(1+\epsilon L_0)\big)$, the second term in (71) can be ignored. Due to Lemma 4.5 and Lemma 4.4, we have the uniform boundedness of $(M_\epsilon-1)\cdot L_\epsilon^{-2}\cdot dz\cdot d\overline{z}/|z|^2$ and $(K_\epsilon-1)\cdot L_\epsilon^{-2}\cdot dz\cdot d\overline{z}/|z|^2$. Thus, the proof of Proposition 4.7 is finished. ∎

4.4 A family of metrics of a parabolic flat bundle on a disc

4.4.1 Simple case

We put $X := \Delta = \{z \in \boldsymbol{C} \,|\, |z| < 1\}$ and $X^* := \Delta - \{O\}$. Let V_l be a vector space over $\boldsymbol{C}$ with a base $\boldsymbol{e} = (e_1,\ldots,e_l)$, and let N_l be the nilpotent endomorphism of V_l given by $N_l\cdot e_{i+1} = e_i$ for $i = 1,\ldots,l-1$ and $N_l\cdot e_l = 0$. We put $E_l := \mathcal{O}_X\otimes V_l$. Then, e_i naturally induce the frame of E_l, which we denote by $\boldsymbol{v} = (v_1,\ldots,v_l)$. The fiber $E_{|O}$ is naturally identified with V, and we have $\boldsymbol{v}_{|O} = \boldsymbol{e}$. We have the logarithmic λ-connection $\mathbb{D}_l^\lambda$ of E_l given by $\mathbb{D}_l^\lambda v_i = v_{i+1}\cdot dz/z$ for $i = 1,\ldots,l-1$ and $\mathbb{D}_l^\lambda v_l = 0$. The residue $\mathrm{Res}(\mathbb{D}^\lambda)$ is given by N_l. We have the weight filtration W of $E_{|O}$ with respect to N_l.

We have the trivial parabolic structure F of E_l. Take a sufficiently small positive number ϵ. We consider the ϵ-perturbation $F^{(\epsilon)}$ given by $F^{(\epsilon)}_{k\epsilon} = W_k$ for $k = -l+1, -l+3\ldots, l-1$ in this case.

Let us fix a sufficiently small positive number ϵ_0 such that $\mathrm{rank}\,E\cdot\epsilon_0 < \eta/10$. In the previous subsection, we have constructed a family of metrics $h_2^{(\epsilon)}$ $(0\le\epsilon\le\epsilon_0)$. It induces the metric of $\mathrm{Sym}^{l-1}(E_2,\mathbb{D}_2^\lambda)\simeq(E_l,\mathbb{D}_l)$, which we denote by $h_l^{(\epsilon)}$. The following property can be shown by reducing to the case $l=2$.

- $h_l^{(0)}$ is the harmonic metric.
- $h_l^{(\epsilon)} \longrightarrow h_l^{(0)}$ for $\epsilon\to 0$, in the C^∞-sense locally on X^*.
- $\big|\Lambda_{\omega_\epsilon}G(h_l^{(\epsilon)})\big|_{h_l^{(\epsilon)}} < C$.

- $h_l^{(\epsilon)}$ is adapted to the parabolic structure $F_l^{(\epsilon)}$.
- Let $t_\epsilon := \det(h_l^{(\epsilon)})\big/\det(h_l^{(0)})$. Then, t_ϵ and t_ϵ^{-1} are bounded, independently of ϵ.

Lemma 4.10 *Let $H_\epsilon = \big(h^{(\epsilon)}(v_i, v_j)\big)$. Then, we have the following estimate on $\{0 < |z| < 1/2\}$ with respect to $h_l^{(\epsilon)}$:*

$$\overline{H}_\epsilon^{-1}\cdot\big(\overline{\partial}+\lambda\partial\big)\overline{H}_\epsilon = O(1)\cdot\frac{dz}{z}+O(1)\cdot\frac{d\overline{z}}{\overline{z}}$$

Proof We see only $\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon$. The term $\overline{H}_\epsilon^{-1}\overline{\partial H}_\epsilon$ can be discussed in the same way. We have only to check the case $l = 2$. As in Lemma 4.9, we have only to see the $(1,1)$-entry, $(2,2)$-entry, $L_\epsilon\times$ $(1,2)$-entry and $L_\epsilon^{-1}\times$ $(2,1)$-entry in the matrix valued function (55). As is seen in Subsection 4.3.3, the term containing ∂M_ϵ is bounded with respect to ω_ϵ, and the estimate is uniform for ϵ. Hence, we can ignore them. Therefore, we have only to show that $L_\epsilon^{-1}\partial L_\epsilon = -L_\epsilon\partial L_\epsilon^{-1}$ is $O(1)\cdot dz/z$, but it can be checked by a direct calculation. ∎

4.4.2 General case

Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a parabolic flat λ-connection on (X, O). Take a positive number η such that $10\cdot\eta < \mathrm{gap}(E, \boldsymbol{F})$. We will use the metrics:

$$\omega_\epsilon = \epsilon^2|z|^\epsilon\frac{dz\cdot d\overline{z}}{|z|^2}+|z|^{2\eta}\frac{dz\cdot d\overline{z}}{|z|^2}. \tag{72}$$

Here, ϵ will be m^{-1} for some $m \in \mathbb{Z}_{>0}$ such that $10\,\mathrm{rank}(E)\cdot\epsilon < \eta$. We take the ϵ-perturbation $\boldsymbol{F}^{(\epsilon)}$ as in (II) of Subsection 2.1.6. Let $a(\epsilon)$ be the numbers which is denoted by $a(\epsilon, i)$ in the explanation there.

We have the endomorphism $\mathrm{Res}(\mathbb{D}^\lambda)$ of $\mathrm{Gr}^F_a(E)$. It induces the generalized eigen decomposition $\mathrm{Gr}^F_a(E) = \bigoplus_{\alpha\in\boldsymbol{C}}\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)$. On $\mathrm{Gr}^{F,\mathbb{E}}_u(E)$, the endomorphism $\mathrm{Res}(\mathbb{D}^\lambda)$ is decomposed as $\alpha\cdot\mathrm{id}+N_u$, where $u = (a,\alpha) \in \boldsymbol{R}\times\boldsymbol{C}$. Let W be the weight filtration of N_u on $\mathrm{Gr}^{F,\mathbb{E}}_u(E)$. They induce the filtration W of $\mathrm{Gr}^F_a(E)$.

For $u \in \boldsymbol{R}\times\boldsymbol{C}$, we put $V_u := \mathrm{Gr}^{F,\mathbb{E}}_u(E)$ with the induced nilpotent map N_u. Then, we can take an isomorphism:

$$(V_u, N_u) \simeq \bigoplus_{i=1}^{m(u)}\big(V_{l(u,i)}, N_{l(u,i)}\big).$$

We put $(E_u, \mathbb{D}^\lambda_u) := \bigoplus\big(E_{l(u,i)}, \mathbb{D}^\lambda_{l(u,i)}\big)$. Let $h_u'^{(\epsilon)}$ denote the metric of E_u induced by $h^{(\epsilon)}_{l(u,i)}$ $(i = 1,\ldots,m(u))$. (See Subsection 4.4.1).

Let $Q(u)$ denote the logarithmic λ-flat bundle of rank one for $u = (a,\alpha)$, which is given by $\mathcal{O}_X\cdot e$ with the λ-connection $\mathbb{D}^\lambda e = e\cdot\alpha\cdot dz/z$. It is equipped with the family of the harmonic metrics $h''^{(\epsilon)}_{u,\epsilon}(e,e) = |z|^{-2a(\epsilon)}$. Then, we obtain the vector bundle E_0 with the λ-connection $\mathbb{D}^\lambda_0$ and the parabolic structure F, as follows:

$$(E_0, \mathbb{D}^\lambda_0) = \bigoplus_u\big(E_u, \mathbb{D}^\lambda_u\big)\otimes Q(u), \qquad F_b(E_{0\,|\,O}) = \bigoplus_{a\le b}E_{(a,\alpha)|O}\otimes Q(a,\alpha)_{|O}.$$

The metrics $h_u'^{(\epsilon)}$ and $h_u''^{(\epsilon)}$ induce the metric $h_u^{(\epsilon)}$ of $E_u\otimes Q(u)$. Let $h_0^{(\epsilon)}$ denote the direct sum of them. We can take a holomorphic isomorphism $\Psi : E_0 \longrightarrow E$ satisfying the following conditions:

- It preserves the filtration F.
- $\mathrm{Gr}^F(\Psi)\circ\mathrm{Gr}^F\mathrm{Res}\,\mathbb{D}^\lambda = \mathrm{Gr}^F\mathrm{Res}\,\mathbb{D}^\lambda_0\,\mathrm{Gr}^F(\Psi)$.

We identify E_0 and E via Ψ. The naturally induced metric of E is denoted by the same notation $h_0^{(\epsilon)}$.

Lemma 4.11 *The family $\big\{h_0^{(\epsilon)}\,\big|\,0 \le \epsilon \le \epsilon_0\big\}$ of the hermitian metrics has the following properties:*

- *$G(\mathbb{D}^\lambda, h_0^{(\epsilon)})$ is uniformly bounded with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$.*
- *$\{h_0^{(\epsilon)}\,|\,\epsilon > 0\}$ converges to $h_0^{(0)}$ in the C^∞-sense locally on X^*.*

- $h_0^{(\epsilon)}$ *is adapted to the perturbed parabolic structure* $F^{(\epsilon)}$.
- *Let* t_ϵ *be determined by* $\det(h_0^{(\epsilon)})\big/\det(h_0^{(0)})$. *Then,* t_ϵ *and* t_ϵ^{-1} *are bounded, independently from* ϵ.

Proof We check only the first claim. The other claims are easy to see. Let f be determined by $f\cdot dz/z = \mathbb{D}^\lambda - \mathbb{D}_0^\lambda$, and we put $f_\epsilon^\dagger := f_{h^{(\epsilon)}}^\dagger$. We put $\mathbb{D}_\epsilon^{\lambda\star} := \mathbb{D}_{h^{(\epsilon)}}^{\lambda\star}$ and $\mathbb{D}_{0,\epsilon}^{\lambda\star} := \mathbb{D}_{0,h^{(\epsilon)}}^{\lambda\star}$. Then, we have the following:

$$G(\mathbb{D}^\lambda, h_0^{(\epsilon)}) = \big[\mathbb{D}^\lambda, \mathbb{D}_\epsilon^{\lambda\star}\big] = \Big[\mathbb{D}_0^\lambda + f\frac{dz}{z},\ \mathbb{D}_{0,\epsilon}^{\lambda\star} + f_\epsilon^\dagger\frac{d\overline{z}}{\overline{z}}\Big] \\ = G(\mathbb{D}_0^\lambda, h_0^{(\epsilon)}) + \mathbb{D}_{0,\epsilon}^{\lambda\star}(f)\frac{dz}{z} + \mathbb{D}_0^\lambda(f_\epsilon^\dagger)\frac{d\overline{z}}{\overline{z}} + [f, f_\epsilon^\dagger]\frac{dz\cdot d\overline{z}}{|z|^2}. \quad (73)$$

We have the decomposition $f = \sum f_{u,u'}$, where $f_{u,u'} \in Hom\big(E_u\otimes Q(u),\ E_{u'}\otimes Q(u')\big)$. We have $f_{u,u'\,|\,O} = 0$ unless $\alpha = \alpha'$ and $a > a'$. Hence, there exist positive constants C and N such that the following holds for $0 < \epsilon < \epsilon_0$:

$$|f|_{h_0^{(\epsilon)}} \leq C\cdot|z|^{10\eta}L_\epsilon^N,$$

Here $N\cdot\epsilon$ is sufficiently smaller than η. Hence, we have the following:

$$|f|_{h_0^{(\epsilon)}} \leq C\cdot|z|^{9\eta}, \qquad [f, f_\epsilon^\dagger] = O\big(|z|^{18\eta}\big).$$

We have the induced frames $\boldsymbol{v}_u$ of $E_u\otimes Q(u)$. They induce the frame $\boldsymbol{v}$ of E_0. Let B and A_0 be determined by $f\boldsymbol{v} = \boldsymbol{v}\cdot B\cdot dz/z$ and $\mathbb{D}_0^\lambda\boldsymbol{v} = \boldsymbol{v}A_0\cdot dz/z$. Then, we have the following:

$$\big[\mathbb{D}_0^\lambda, f^\dagger\big]\boldsymbol{v} = \boldsymbol{v}\left(\mathcal{D}B_\epsilon^\dagger\frac{d\overline{z}}{\overline{z}} + [A_0, B_\epsilon^\dagger]\frac{dz\cdot d\overline{z}}{|z|^2}\right).$$

Here we put $\mathcal{D} = \overline{\partial} + \lambda\partial$ and $B_\epsilon^\dagger = \overline{H}_\epsilon^{-1}\cdot{}^t\overline{B}\cdot\overline{H}_\epsilon$, where $H_\epsilon = H(h_0^{(\epsilon)}, \boldsymbol{u})$. Since $B_\epsilon^\dagger$ is sufficiently small with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$, $[A_0, B_\epsilon^\dagger]$ is also sufficiently small. Corresponding to the decomposition $f = \sum f_{u,u'}$, we have $B = \sum B_{u,u'}$. Then, the following holds:

$$\big(B_\epsilon^\dagger\big)_{u,u'} = \overline{H}_{u',\epsilon}^{-1}\,{}^t\overline{B}_{u',u}\overline{H}_{u,\epsilon}.$$

Here $H_{u,\epsilon} := H(h_u^{(\epsilon)}, \boldsymbol{v}_u)$. Hence, we obtain the following:

$$\big(\mathcal{D}B_\epsilon^\dagger\big)_{u,u'}\frac{d\overline{z}}{\overline{z}} = \overline{H}_{u',\epsilon}^{-1}\cdot(\mathcal{D}^t\overline{B}_{u',u})\cdot\overline{H}_{u,\epsilon} - \overline{H}_{u',\epsilon}^{-1}\mathcal{D}\overline{H}_{u',\epsilon}\cdot(B_\epsilon^\dagger)_{u,u'} + (B_\epsilon^\dagger)_{u,u'}\cdot\overline{H}_{u,\epsilon}^{-1}\mathcal{D}\overline{H}_{u,\epsilon}.$$

Since B is holomorphic, we have $\overline{H}_{u',\epsilon}^{-1}\cdot\big(\mathcal{D}^t\overline{B}_{u',u}\big)\cdot\overline{H}_{u,\epsilon}\cdot d\overline{z}/\overline{z} = 0$. We put $H'_{u\,\epsilon} := H(h_u'^{(\epsilon)}, \boldsymbol{v}_u)$. Then, we have $H_{u,\epsilon} = |z|^{-2a}H'_{u,\epsilon}$, and the following holds with respect to $h_0^{(\epsilon)}$ due to Lemma 4.10:

$$\overline{H}_{u,\epsilon}^{-1}\mathcal{D}\overline{H}_{u,\epsilon} = -a\left(\lambda\frac{dz}{z} + \frac{d\overline{z}}{\overline{z}}\right) + \overline{H}_{u,\epsilon}'^{-1}\mathcal{D}\overline{H}'_{u,\epsilon} = O(1)\frac{dz}{z} + O(1)\frac{d\overline{z}}{\overline{z}}.$$

Since $(B_\epsilon^\dagger)_{u,u'}$ is small with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$, $(B_\epsilon^\dagger)_{u,u'}\cdot\overline{H}_{u,\epsilon}^{-1}\partial\overline{H}_{u,\epsilon}$ is also small. Therefore, $\mathbb{D}_0^\lambda f^\dagger\cdot d\overline{z}/\overline{z}$ is small with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$. It also follows that $\mathbb{D}_{0,\epsilon}^{\lambda\star}f\cdot dz/z$ is small. Thus we are done. ∎

4.5 Proof of Proposition 4.1

4.5.1 Construction of a family of initial metrics

Let η be a small positive number such that $\eta < \mathrm{gap}(E, \boldsymbol{F})/10$. Let ϵ_0 be a small positive number such that $10\,\mathrm{rank}\,E\cdot\epsilon_0 < \eta$. For any $0 \leq \epsilon < \epsilon_0$, let us take ω_ϵ be the Kahler forms of $C - D$ with the following properties:

- Let (U_P, z) be a holomorphic coordinate around $P \in D$ such that $z(P) = 0$, and then ω_ϵ is given by (72).

- $\omega_\epsilon \longrightarrow \omega_0$ for $\epsilon \longrightarrow 0$ in the C^∞-sense locally on $X - D$.

Lemma 4.12 *We can construct a family of metrics $h_0^{(\epsilon)}$ of $E_{|C-D}$ with the following properties:*

- *$h_0^{(\epsilon)}$ is adapted to the perturbed parabolic structure $\boldsymbol{F}^{(\epsilon)}$.*
- *$h_0^{(\epsilon)} \longrightarrow h_0^{(0)}$ in the C^∞-sense locally on $C - D$.*
- *$G(h_0^{(\epsilon)})$ is uniformly bounded with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$.*
- *We put $t_\epsilon := \det(h_0^{(\epsilon)}) \big/ \det(h_0^{(0)})$. Then, t_ϵ and t_ϵ^{-1} are bounded independently from ϵ.*

Proof We construct a C^∞-metric of E on $\bigcup_{P\in D}(U_P - \{P\})$, by applying the construction given in Subsection 4.4.2 to $(E, \boldsymbol{F}, \mathbb{D}^\lambda)_{|U_P}$ for each $P \in D$, and then we prolong it to a C^∞-metric of E on $C - D$. ∎

Let $R(\det h_0^{(0)})$ denote the curvature of the metrized holomorphic bundle $\det(E, d'', h_0^{(0)})$, where d'' denote the $(0,1)$-part of $\mathbb{D}^\lambda$. Since $\det h_0^{(0)}$ gives the harmonic metric around D due to our construction, $R(\det h_0^{(0)})$ vanishes around D. We also have $\int R(\det h_0^{(0)}) = -2\pi\sqrt{-1} \cdot \text{par-deg}(E, \boldsymbol{F}) = 0$. Let us take the C^∞-function χ_0 on C and satisfies the equality $\text{rank}(E) \cdot \overline{\partial}\partial\chi_0 + R\big(\det(h_0^{(0)})\big) = 0$. We put $h_{in}^{(0)} := h_0^{(0)} \cdot \exp\big(\chi_0\big)$. Then, $R\big(\det h_{in}^{(0)}\big) = 0$, i.e., $\det h_{in}^{(0)}$ is a harmonic metric of $\det(E, \mathbb{D}^\lambda)$. Let χ_ϵ be the functions determined by $\det(h_{in}^{(0)}) = \det(h_0^{(\epsilon)}) \cdot \exp\Big(\text{rank}(E) \cdot \chi_\epsilon\Big)$. The following claims immediately follows from Lemma 4.12.

- χ_ϵ and $-\chi_\epsilon$ are bounded on C, independently from ϵ.
- $\chi_\epsilon \longrightarrow 0$ in the C^∞-sense locally on $C - D$.

We put $h_{in}^{(\epsilon)} := h_0^{(\epsilon)} \cdot \exp\big(\chi_\epsilon\big)$, which is the metric of $E_{|C-D}$.

Lemma 4.13 *The following claims are easy to check.*

- *$h_{in}^{(\epsilon)}$ is adapted to the parabolic structure $\boldsymbol{F}^{(\epsilon)}$.*
- *$h_{in}^{(\epsilon)} \longrightarrow h_{in}^{(0)}$ in the C^∞-sense locally on $C - D$.*
- *$G(h_{in}^{(\epsilon)})$ is uniformly bounded with respect to $(\omega_\epsilon, h_{in}^{(\epsilon)})$.*
- *$\det h_{in}^{(\epsilon)}$ is harmonic, and we have $\det h_{in}^{(\epsilon)} = \det h_{in}^{(0)}$.*

In other words, they give initial metrics for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ in the sense of Lemma 3.18, and their pseudo curvature satisfy some uniform finiteness. ∎

4.5.2 L_1^2-finiteness of the sequence

Due to Proposition 2.33, we obtain the harmonic metrics $h^{(\epsilon)}$ for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ such that $\det h^{(\epsilon)} = \det h_{in}^{(0)}$. Due to Lemma 2.34, we have the following inequalities for any ϵ:

$$M_{\omega_\epsilon}(h_{in}^{(\epsilon)}, h^{(\epsilon)}) \leq 0. \tag{74}$$

Let $s^{(\epsilon)}$ be determined by $h^{(\epsilon)} = h_{in}^{(\epsilon)} s^{(\epsilon)}$. Due to Lemma 2.45, (74) and $\det s^{(\epsilon)} = 1$, there exists a positive constant A which is independent on ϵ, with the following property:

$$\big|s^{(\epsilon)}\big|_{h_{in}^{(\epsilon)}} \leq A, \quad \big|s^{(\epsilon)\,-1}\big|_{h_{in}^{(\epsilon)}} \leq A. \tag{75}$$

Let $\mathbb{D}_{in}^{\lambda\star}$ be the operator obtained from $\mathbb{D}^\lambda$, ω_ϵ and $h_{in}^{(\epsilon)}$ as in Subsection 2.2.1. We have the following equalities:

$$\Delta_{\omega_\epsilon}^\lambda \operatorname{tr} s^{(\epsilon)} = -\sqrt{-1}\operatorname{tr}\big(s^{(\epsilon)}\Lambda_{\omega_\epsilon} G(h_{in}^{(\epsilon)})\big) + \sqrt{-1}\operatorname{tr}\Big(\Lambda_{\omega_\epsilon}\mathbb{D}^\lambda s^{(\epsilon)} \cdot (s^{(\epsilon)})^{-1} \cdot \mathbb{D}_{in}^{\lambda\star} s^{(\epsilon)}\Big).$$

See Remark 2.24 for $\Delta_{\omega_\epsilon}^\lambda$.

Lemma 4.14 *We have* $\int \Delta^{\lambda}_{\omega_\epsilon} \operatorname{tr} s^{(\epsilon)} \operatorname{dvol}_{\omega_\epsilon} = 0$.

Proof Let g be a C^∞-Kahler metric of C. We have only to show $\int \Delta^{\lambda}_{g} \operatorname{tr} s^{(\epsilon)} \operatorname{dvol}_g = 0$. We have the following:

$$\Delta^{\lambda}_{g} \operatorname{tr} s^{(\epsilon)} = -\sqrt{-1}\operatorname{tr}\bigl(s^{(\epsilon)}\Lambda_g G(h^{(\epsilon)}_{in})\bigr) - \bigl|\mathbb{D}^{\lambda} s^{(\epsilon)} \cdot (s^{(\epsilon)})^{-1/2}\bigr|^2_{h^{(\epsilon)},g}$$

Since $\bigl|\operatorname{tr}\bigl(s^{(\epsilon)}\Lambda_g G(h^{(\epsilon)}_{in})\bigr)\bigr|$ is $O(|z|^{\epsilon-2})$, we can take a bounded function a_ϵ such that $\Delta_g a_\epsilon = \bigl|\operatorname{tr}\bigl(s^{(\epsilon)}\Lambda_g G(h^{(\epsilon)}_{in})\bigr)\bigr|$. Hence, we obtain $\int_{X-D}\bigl|\mathbb{D}^{\lambda}s^{(\epsilon)}\cdot(s^{(\epsilon)})^{-1/2}\bigr|^2_{h^{(\epsilon)},g} < \infty$, due to Lemma 2.2 of [37]. Since $s^{(\epsilon)}$ is bounded with respect to $h^{(\epsilon)}$, we obtain $\int_{X-D}\bigl|\mathbb{D}^{\lambda}s^{(\epsilon)}\bigr|^2_{h^{(\epsilon)},g} < \infty$. Then, it is easy to obtain the vanishing $\int \Delta^{\lambda}_{\omega_\epsilon} \operatorname{tr} s^{(\epsilon)} \operatorname{dvol}_{\omega_\epsilon} = 0$ by Stokes formula and Lemma 5.2 of [36]. ∎

Then, there exists a positive constant A' such that the following holds:

$$\int |\mathbb{D}^{\lambda} s^{(\epsilon)} \cdot s^{(\epsilon)\,-1/2}|^2_{h^{(\epsilon)}_{in},\omega_\epsilon} \operatorname{dvol}_{\omega_\epsilon} \leq A'. \tag{76}$$

In particular, we obtain $\bigl\|\mathbb{D}^{\lambda}s^{(\epsilon)}\bigr\|_{L^2,\omega_\epsilon,h^{(\epsilon)}_{in}}$ is bounded for $0 < \epsilon < \epsilon_0$.

4.5.3 The end of the proof of Proposition 4.1

Let Q be a point of $C - D$. Let (U, z) be a holomorphic coordinate around Q such that $z(Q) = 0$ and $U \simeq \Delta = \{z \,|\, |z| < 1\}$. We use the standard metric $g = dz \cdot d\bar{z}$ of U. The harmonic bundle $(E, \mathbb{D}^{\lambda}, h^{(\epsilon)})$ induces the Higgs bundle $(E, \overline{\partial}_\epsilon, \theta_\epsilon)$. We have $\theta_\epsilon = f_\epsilon \cdot dz$ on U. On the other hand, we also obtain $\overline{\partial}_{in,\epsilon}$ and $\theta_{in,\epsilon}$ from $(E, \mathbb{D}^{\lambda}, h^{(\epsilon)}_{in})$, although $\overline{\partial}_{in,\epsilon}(\theta_{in,\epsilon}) = 0$ is not satisfied, in general. Let $\delta'_{in,\epsilon}$ be the $(1,0)$-operator obtained from $h^{(\epsilon)}_{in}$ and d'', as in Subsection 2.2.1. Then, we have the relation:

$$\theta_\epsilon = \theta_{in,\epsilon} - \frac{\lambda}{1+|\lambda|^2}\bigl(s^{(\epsilon)\,-1}\cdot \delta'_{in,\epsilon}s^{(\epsilon)}\bigr). \tag{77}$$

Due to (75), (76) and (77), there exists a positive constant C_0 such that $\int_U |f_\epsilon|^2_{h^{(\epsilon)}} \operatorname{dvol}_g < C_0$ holds for any $0 < \epsilon < \epsilon_0$. Hence, the following inequality holds for some positive constants C_i $(i = 1, 2, 3)$ and for any $0 < \epsilon < \epsilon_0$:

$$\int_U \log |f_\epsilon|^2_{h^{(\epsilon)}} \operatorname{dvol}_g \leq C_1 + \int_U C_2 \cdot |f_\epsilon|^2_{h^{(\epsilon)}} \operatorname{dvol}_g \leq C_3. \tag{78}$$

Recall the fundamental inequality for the Higgs field of a harmonic bundle [37]:

$$\Delta_g \log|f_\epsilon|^2_{h^{(\epsilon)}} \leq -\frac{\bigl|[f_\epsilon, f_\epsilon^\dagger]\bigr|^2_{h^{(\epsilon)}}}{|f_\epsilon|^2_{h^{(\epsilon)}}} \leq 0. \tag{79}$$

Due to (78) and (79), there exists a positive constant C_4 such that the following holds for any $Q' \in U(1/2) := \{|z| < 1/2\}$:

$$\bigl|f_\epsilon(Q')\bigr|^2_{h^{(\epsilon)}_{in}} \leq C_4. \tag{80}$$

By using (77), we obtain that $\delta'_{in,\epsilon}s^{(\epsilon)}$ is uniformly bounded with respect to $(\omega_\epsilon, h^{(\epsilon)}_{in})$ on $U(1/2)$.

Since $\theta^\dagger_\epsilon$ is the adjoint of θ_ϵ, we obtain the uniform boundedness of $\theta^\dagger_\epsilon$ on $U(1/2)$. Let $\delta''_{in,\epsilon}$ be the operator obtained from $h^{(\epsilon)}_{in}$ and d' as in Subsection 2.2.1, where d' denotes the $(1,0)$-part of $\mathbb{D}^{\lambda}$. Then, we also obtain the uniform boundedness of $\delta''_{in,\epsilon}s^{(\epsilon)}$ on $U(1/2)$. Hence, $\mathbb{D}^{\lambda\star}_{in,\epsilon}s^{(\epsilon)}$ is uniformly bounded on $U(1/2)$, where $\mathbb{D}^{\lambda\star}_{in,\epsilon} = \delta'_{in,\epsilon} - \delta''_{in,\epsilon}$. Since we have $d'' = \overline{\lambda}^{-1}\bigl(\delta''_{in,\epsilon} + (1+|\lambda|^2)\theta^\dagger_{in,\epsilon}\bigr)$ and $d' = \lambda\delta'_{in,\epsilon} + (1+|\lambda|^2)\theta_{in,\epsilon}$, we also obtain $\mathbb{D}^{\lambda}s^{(\epsilon)}$ is uniformly bounded on $U(1/2)$. Recall the formula $\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}_{in}s^{(\epsilon)} = s^{(\epsilon)}\cdot G(h^{(\epsilon)}_{in}) + \mathbb{D}^{\lambda}s^{(\epsilon)}\cdot s^{(\epsilon)\,-1}\cdot \mathbb{D}^{\lambda\star}_{in}s^{(\epsilon)}$. Thus $\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}_{in}s^{(\epsilon)}$ is also uniformly bounded on $U(1/2)$. Therefore, $\{s^{(\epsilon)}\}$ is L^p_2-bounded for any $p > 1$ and $U(1/2)$. By taking an appropriate subsequence (ϵ_i), $s^{(\epsilon_i)}$ weakly converges to some $\widetilde{s}$ in L^p_2 locally on $C - D$.

It is easy to see that $h_{in}^{(0)} \cdot \widetilde{s}$ is a harmonic metric. We have $\det \widetilde{s} = 1$. We also have the boundedness of $\widetilde{s}$ and $\widetilde{s}^{-1}$ with respect to $h_{in}^{(0)}$. Thus, we have $h_{in}^{(0)} \cdot \widetilde{s} = h^{(0)}$, i.e., the sequence $\{h^{(\epsilon_i)}\}$ converges to $h^{(0)}$ weakly in L_2^p locally on $C - D$.

Although we take a subsequence in the above argument, we can conclude that $h^{(\epsilon)}$ converges to $h^{(0)}$ weakly in L_2^p locally on $C - D$, due to a general argument. We can also obtain the C^∞-convergence by a standard bootstrapping argument. In the above argument, the convergence of $\{\theta^{(\epsilon)}\}$ is also proved. ∎

Remark 4.15 *As for the proof of Proposition 4.2, we take a C^∞-metric h_{in} of $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ such that each restriction $h_{in\,|\,C_t}$ is an initial metric. Let s be determined by $h_H = h_{in} \cdot s$. By applying the same argument, we obtain the continuity of s. Similarly for θ_H.* ∎

5 The existence of a pluri-harmonic metric

We will prove our main existence theorem of pluri-harmonic metric for parabolic λ-flat bundle, which is adapted to the parabolic structure. (See Subsection 3.3 of [31] for the adaptedness.)

5.1 Preliminary

Let C be a smooth projective curve with a simple effective divisor D. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a stable parabolic λ-flat bundle on (C, D) with par-deg$(E, \boldsymbol{F}) = 0$. For each $P \in D$, let (U_P, z) be a holomorphic coordinate around P such that $z(P) = 0$. Let $\boldsymbol{F}^{(\epsilon)}$ be an ϵ-perturbation as in (II) of Subsection 2.1.6 for $\epsilon = m^{-1}$. We have $h_0^{(\epsilon)}$ be harmonic metrics for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$. We assume $\det h_0^{(\epsilon)} = \det h_0^{(0)}$. As shown in Proposition 4.1, $h_0^{(\epsilon)}$ converges to $h_0^{(0)}$ in the C^∞-sense locally on $C - D$. Let N be a large positive number, for example $N > 10$. In this subsection, we use Kahler metrics g_ϵ $(\epsilon \geq 0)$ of $C - D$ which are as follows on U_P for each $P \in D$:

$$\big(\epsilon^{N+2}|z|^{2\epsilon} + |z|^2\big)\frac{dz \cdot d\overline{z}}{|z|^2}.$$

We assume that $\{g_\epsilon\}$ converges to g_0 for $\epsilon \longrightarrow 0$ in the C^∞-sense locally on $C - D$.

Proposition 5.1 *Let $h^{(\epsilon)}$ $(\epsilon > 0)$ be hermitian metrics of $E_{|C-D}$ with the following properties:*

1. *Let $s^{(\epsilon)}$ be determined by $h^{(\epsilon)} = h_0^{(\epsilon)} \cdot s^{(\epsilon)}$. Then, $s^{(\epsilon)}$ is bounded with respect to $h_0^{(\epsilon)}$, and we have $\det s^{(\epsilon)} = 1$. We also have the finiteness $\big\|\mathbb{D}^\lambda s^{(\epsilon)}\big\|_{2,h_0^{(\epsilon)},g_\epsilon} < \infty$. (The estimates may depend on ϵ.)*
2. *We have $\|G(h^{(\epsilon)})\|_{2,h^{(\epsilon)},g_\epsilon} < \infty$ and $\lim_{\epsilon\to 0} \|G(h^{(\epsilon)})\|_{2,h^{(\epsilon)},g_\epsilon} = 0$.*

Then, the following claims hold.

- *The sequence $\{s^{(\epsilon)}\}$ is weakly convergent to the identity in L_1^2 locally on $C - D$.*
- *$\big\{\sup_{P\in C-D} |s^{(\epsilon)}_{|P}|_{h_0^{(\epsilon)}} \,\big|\, \epsilon > 0\big\}$ and $\big\{\sup_{P\in C-D} |(s^{(\epsilon)})^{-1}_{|P}|_{h_0^{(\epsilon)}} \,\big|\, \epsilon > 0\big\}$ are bounded.*

Proof To begin with, we remark that we have only to show the existence of a subsequence $\{s^{(\epsilon_i)}\}$ with the desired properties as above. We put $\|s^{(\epsilon)}\|_{\infty,h_0^{(\epsilon)}} := \sup_{P\in C-D}\big|s^{(\epsilon)}_{|P}\big|_{h_0^{(\epsilon)}}$. For any point $P \in C - D$, let $SE(s^{(\epsilon)})(P)$ denote the maximal eigenvalue of $s^{(\epsilon)}_{|P}$. There exists a constant $0 < C_1 < 1$ such that $C_1 \cdot |s^{(\epsilon)}_{|P}|_{h_0^{(\epsilon)}} \leq SE(s^{(\epsilon)})(P) \leq |s^{(\epsilon)}_{|P}|_{h_0^{(\epsilon)}}$. We have $\det s^{(\epsilon)}_{|P} = 1$. Hence, it is easy to see $\log \operatorname{tr} s^{(\epsilon)}_{|P} \geq \log \operatorname{rank}(E) \geq 0$. We also have $SE(s^{(\epsilon)})(P) \geq 1$ for any P.

Let us take $b_\epsilon > 0$ satisfying $2 \leq b_\epsilon \cdot \sup SE(s^{(\epsilon)})(P) \leq 2 + \epsilon$. We put $\widetilde{s}^{(\epsilon)} = b_\epsilon s^{(\epsilon)}$ and $\widetilde{h}^{(\epsilon)} := h_0^{(\epsilon)} \cdot \widetilde{s}^{(\epsilon)}$. Then, $\widetilde{s}^{(\epsilon)}$ are uniformly bounded with respect to $h_0^{(\epsilon)}$. We remark $G(\widetilde{h}^{(\epsilon)}) = G(h^{(\epsilon)})$. We also remark that $h^{(\epsilon)}$ and $\widetilde{h}^{(\epsilon)}$ induce the same metric of $\operatorname{End}(E)$.

Lemma 5.2 *After going to an appropriate subsequence,* $\{\widetilde{s}^{(\epsilon_i)}\}$ *converges to a positive constant multiplication, weakly in* L_1^2 *locally on* $C-D$.

Proof We have the following (Subsection 2.2.5):

$$\Delta^{\lambda}_{g_0,h_0^{(\epsilon)}}\widetilde{s}^{(\epsilon)}=\widetilde{s}^{(\epsilon)}\sqrt{-1}\Lambda_{g_0}G(\widetilde{h}^{(\epsilon)})+\sqrt{-1}\Lambda_{g_0}\mathbb{D}^{\lambda}\widetilde{s}^{(\epsilon)}\big(\widetilde{s}^{(\epsilon)\,-1}\big)\mathbb{D}^{\lambda\star}_{h_0^{(\epsilon)}}\widetilde{s}^{(\epsilon)}. \tag{81}$$

We can show $\int\Delta^{\lambda}_{g_0}\operatorname{tr}\widetilde{s}^{(\epsilon)}\cdot\mathrm{dvol}_{g_0}=0$, by the same argument as the proof of Lemma 4.14, we obtain the following inequality from (81) and the uniform boundedness of $\widetilde{s}^{(\epsilon)}$:

$$\int\big|\mathbb{D}^{\lambda}\widetilde{s}^{(\epsilon)}\cdot\widetilde{s}^{(\epsilon)\,-1/2}\big|^2_{g_0,h_0^{(\epsilon)}}\mathrm{dvol}_{g_0}\leq A\cdot\int\big|\operatorname{tr}\Lambda_{g_0}G(\widetilde{h}^{(\epsilon)})\big|\cdot\mathrm{dvol}_{g_0}$$
$$=A\cdot\int\big|\operatorname{tr}\Lambda_{g_\epsilon}G(\widetilde{h}^{(\epsilon)})\big|\cdot\mathrm{dvol}_{g_\epsilon}\leq A'\cdot\big\|G(\widetilde{h}^{(\epsilon)})\big\|_{2,\widetilde{h}^{(\epsilon)},g_\epsilon}. \tag{82}$$

In particular, we obtain the uniform estimate $\big\|\mathbb{D}^{\lambda}\widetilde{s}^{(\epsilon)}\big\|^2_{2,g_0,h_0^{(\epsilon)}}\leq A''\cdot\big\|G(\widetilde{h}^{(\epsilon)})\big\|_{2,\widetilde{h}^{(\epsilon)},g_\epsilon}$. Therefore, the sequence $\{\widetilde{s}^{(\epsilon)}\}$ is L_1^2-bounded on any compact subset of $C-D$. By taking an appropriate subsequence, it is weakly L_1^2-convergent locally on $C-D$. Let $\widetilde{s}^{(\infty)}$ denote the weak limit. We obtain $\mathbb{D}^{\lambda}\widetilde{s}^{(\infty)}=0$. We also know that $\widetilde{s}^{(\infty)}$ is bounded with respect to $h_0^{(0)}$. Therefore, $\widetilde{s}^{(\infty)}$ gives an automorphism of $(E,\boldsymbol{F},\mathbb{D}^{\lambda})$. Due to the stability of $(E,\boldsymbol{F},\mathbb{D}^{\lambda})$, $\widetilde{s}^{(\infty)}$ is a constant multiplication.

We would like to show $\widetilde{s}^{(\infty)}\neq 0$. Let us take any point $Q_\epsilon\in C-D$ satisfying the following:

$$SE(s^{(\epsilon)})(Q_\epsilon)\geq\frac{9}{10}\cdot\sup_{P\in C-D}SE(s^{(\epsilon)})(P).$$

Then, we have $\log\operatorname{tr}\widetilde{s}^{(\epsilon)}(Q_\epsilon)\geq\log(9/5)$. By taking an appropriate subsequence, we may assume the sequence $\{Q_\epsilon\}$ converges to a point Q_∞. We have two cases (i) $Q_\infty\in D$ (ii) $Q_\infty\notin D$. We discuss only the case (i). The other case is similar and easier.

We use the coordinate neighbourhood (U,z) such that $z(Q_\infty)=0$. For any point $P\in U$, we put $\Delta(P,r):=\big\{Q\in U\,\big|\,|z(P)-z(Q)|<r\big\}$. When ϵ is sufficiently small, Q_ϵ is contained in $\Delta(Q_\infty,1/2)=\{|z|<1/2\}$. Let $g=dz\cdot d\overline{z}$ denote the standard metric of U. We have the following inequality on $U-\{Q_\infty\}$ (see Subsection 2.2.5):

$$\Delta^{\lambda}_{g}\log\operatorname{tr}\widetilde{s}^{(\epsilon)}\leq\big|\Lambda_g G(\widetilde{h}^{(\epsilon)})\big|_{\widetilde{h}^{(\epsilon)}}. \tag{83}$$

Let $B^{(\epsilon)}$ be the endomorphism of E determined as follows:

$$G(\widetilde{h}^{(\epsilon)})=G(h^{(\epsilon)})=B^{(\epsilon)}\cdot\frac{dz\cdot d\overline{z}}{|z|^2}$$

Then, we have the following estimate for some constant $A>0$ which is independent of ϵ:

$$\int\big|B^{(\epsilon)}\big|^2_{\widetilde{h}_0^{(\epsilon)}}\big(\epsilon^{N+1}|z|^{2\epsilon}+|z|^2\big)^{-1}\frac{\mathrm{dvol}_g}{|z|^2}\leq A\int\big|G(\widetilde{h}^{(\epsilon)})\big|^2_{\widetilde{h}^{(\epsilon)},g_\epsilon}\mathrm{dvol}_{g_\epsilon}.$$

Here A denotes a constant independent of ϵ. Due to Proposition 2.16 in [31], there exist $v^{(\epsilon)}$ such that the following inequalities hold for some positive constant A' which is independent of ϵ:

$$\overline{\partial}\partial v^{(\epsilon)}=\big|B^{(\epsilon)}\big|_{\widetilde{h}^{(\epsilon)}}\frac{dz\cdot d\overline{z}}{|z|^2},\qquad\big|v^{(\epsilon)}(z)\big|\leq A'\cdot\big(\epsilon^{(N-1)/2}|z|^{\epsilon}+|z|^{1/2}\big)\cdot\big\|G(\widetilde{h}^{(\epsilon)})\big\|_{2,\widetilde{h}^{(\epsilon)},g_\epsilon}$$

Then, we have $\Delta^{\lambda}_{g}\big(\log\operatorname{tr}\widetilde{s}^{(\epsilon)}-v^{(\epsilon)}\big)\leq 0$ on $U-\{Q_\infty\}$. Since $\log\operatorname{tr}\widetilde{s}^{(\epsilon)}-v^{(\epsilon)}$ is bounded from above, the inequality holds on U. Therefore, we obtain the following:

$$\log\operatorname{tr}\widetilde{s}^{(\epsilon)}(Q_\epsilon)-v^{(\epsilon)}(Q_\epsilon)\leq A''\cdot\int_{\Delta(Q_\epsilon,1/2)}\Big(\log\operatorname{tr}\widetilde{s}^{(\epsilon)}-v^{(\epsilon)}\Big)\cdot\mathrm{dvol}_g.$$

Here A'' denotes a positive constant which is independent of ϵ. Then, we obtain the following inequalities, for some positive constants C_i $(i = 1, 2)$ which are independent of ϵ:

$$\log(9/5) \leq \log \operatorname{tr} \widetilde{s}^{(\epsilon)}(Q_\epsilon) \leq C_1 \cdot \int_{\Delta(Q_\epsilon, 1/2)} \log \operatorname{tr} \widetilde{s}^{(\epsilon)} \cdot \mathrm{dvol}_g + C_2.$$

Recall that $\log \operatorname{tr} \widetilde{s}^{(\epsilon)}$ are uniformly bounded from above. Therefore, there exists a positive constant C_3 such that the following holds for any sufficiently small $\epsilon > 0$:

$$\int_{\Delta(Q_\epsilon, 1/2)} -\min(0, \log \operatorname{tr} \widetilde{s}^{(\epsilon)}) \cdot \mathrm{dvol}_g \leq C_3.$$

Due to Fatou's lemma, we obtain the following:

$$\int_{\Delta(Q_\infty, 1/2)} -\min\bigl(0, \log \operatorname{tr} \widetilde{s}^{(\infty)}\bigr) \cdot \mathrm{dvol}_g \leq C_3.$$

It means $\widetilde{s}^{(\infty)}$ is not constantly 0 on $\Delta(Q_\infty, 1/2)$. In all, we can conclude that $\widetilde{s}^{(\infty)}$ is a positive constant multiplication. Thus, the proof of Lemma 5.2 is accomplished. ∎

Let $\bigl\{\widetilde{s}^{(\epsilon_i)}\bigr\}$ be a subsequence as in Lemma 5.2. It is almost everywhere convergent to some constant multiplication. Then, we obtain that the sequence $\bigl\{\det \widetilde{s}^{(\epsilon_i)} = b_{\epsilon_i}^{\operatorname{rank} E} \cdot \mathrm{id}_{\det(E)}\bigr\}$ converges to the positive constant. In particular, $\{b_{\epsilon_i}\}$ is convergent. Therefore, the sequence $\bigl\{s^{(\epsilon_i)}\bigr\}$ is convergent to the identity. Thus we are done. ∎

Corollary 5.3

- *The sequence* $\bigl\{h^{(\epsilon)}\bigr\}$ *is convergent to* $h_0^{(0)}$ *weakly in* L_1^2 *locally on* $C - D$.
- *The sequence* $\bigl\{\mathbb{D}^\lambda s^{(\epsilon)}\bigr\}$ *is weakly convergent to* 0 *in* L^2 *locally on* $C - D$.
- *The sequence* $\{\theta^{(\epsilon)}\}$ *converges to* $\theta^{(0)}$ *is weakly convergent to* 0 *in* L^2 *locally on* $C - D$.
- *In particular, the sequences are convergent almost everywhere.* ∎

5.2 The surface case

5.2.1 Statement

Let X be a smooth projective surface with an ample line bundle L, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. We put $X^* := X - D$. Let $\boldsymbol{c}$ be any element of $\boldsymbol{R}^S$. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a μ_L-stable $\boldsymbol{c}$-parabolic λ-flat bundle on (X, D) with trivial characteristic numbers $\text{par-deg}_L(E, \boldsymbol{F}) = \int_X \text{par-ch}_2(E, \boldsymbol{F}) = 0$. Recall that we have already known $\text{par-c}_1(E, \boldsymbol{F}) = 0$ due to Bogomolov-Gieseker inequality and Hodge index theorem (See Corollary 6.2 of [31].) Hence, we can take the pluri-harmonic metric $h_{\det(E)}$ of the determinant bundle $\det(E, \boldsymbol{F}, \mathbb{D}^\lambda)$. The purpose of this subsection is to show the following existence theorem.

Theorem 5.4 *There exists a tame pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)_{|X^*}$ with $\det(h) = h_{\det E}$ which is adapted to the parabolic structure.*

The proof will be given in the rest of this subsection.

5.2.2 The sequence of Hermitian-Einstein metrics for the ϵ-perturbations

Let $\boldsymbol{F}^{(\epsilon)}$ be an ϵ-perturbation as in (II) of Subsection 2.1.6. If ϵ is sufficiently small, $(E,\boldsymbol{F}^{(\epsilon)},\mathbb{D}^{\lambda})$ is also μ_L-stable. We also have $\text{par-c}_1(E,\boldsymbol{F}^{(\epsilon)})=\text{par-c}_1(E,\boldsymbol{F})=0$. Since $(E,\boldsymbol{F}^{(\epsilon)},\mathbb{D}^{\lambda})$ is graded semisimple and satisfies (SPW)-condition, we can apply Proposition 3.19. Let $h^{(\epsilon)}$ be the Hermitian-Einstein metric for $(E,\boldsymbol{F}^{(\epsilon)},\mathbb{D}^{\lambda})$ with respect to ω_ϵ, such that $\det h^{(\epsilon)}=h_{\det(E)}$ and $\Lambda_{\omega_\epsilon}G(h^{(\epsilon)})=0$ (Proposition 3.19).

Since $h_{\det(E)}$ is pluri-harmonic, we also have $\operatorname{tr}G(h^{(\epsilon)})=0$. Therefore, we have the following convergence:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int\big|G(h^{(\epsilon)})\big|^2_{h^{(\epsilon)},\omega_\epsilon}\operatorname{dvol}_{\omega_\epsilon}=\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int\operatorname{tr}\Big(G(h^{(\epsilon)})^2\Big)=2\big(1+|\lambda|^2\big)^2\cdot\text{par-ch}_2(E,\boldsymbol{F}^{(\epsilon)})\longrightarrow 0. \quad (84)$$

We would like to discuss the limit of $h^{(\epsilon)}$ for $\epsilon\to 0$.

5.2.3 Convergence on almost every curve

Let L^m be sufficiently ample. We put $\mathbb{P}_m:=\mathbb{P}\big(H^0(X,L^m)^\vee\big)$. For any $s\in\mathbb{P}_m$, we put $X_s:=s^{-1}(0)$. Recall Proposition 2.9, and let $\mathcal{U}$ denote the Zariski open subset of $\mathbb{P}_m$ which consists of the points s with the following properties:

- X_s is smooth, and $X_s\cap D$ is a simple normal crossing divisor.
- $(E,\boldsymbol{F},\mathbb{D}^\lambda)_{|X_s}$ is μ_L-stable.

If ϵ is sufficiently small, we have $\mathcal{U}\neq\emptyset$.

We will use the notation $X_s^*:=X_s\setminus D$ and $D_s:=X_s\cap D$. We have the metric $\omega_{\epsilon,s}$ of X_s^*, induced by ω_ϵ. The induced volume form is denoted by dvol_s. We put $(E_s,\boldsymbol{F}_s,\mathbb{D}_s^\lambda):=(E,\boldsymbol{F},\mathbb{D}^\lambda)_{|X_s}$. We have the metric $h^{(\epsilon)}_{|X_s^*}$ of $E_{s\,|\,X_s^*}$. Since $(E_s,\boldsymbol{F}_s^{(\epsilon)},\mathbb{D}_s^\lambda)$ are also stable for any point $s\in\mathcal{U}$, we have the harmonic metric $h_s^{(\epsilon)}$ of $(E_s,\boldsymbol{F}_s^{(\epsilon)},\mathbb{D}_s^\lambda)$ with $\det h_s^{(\epsilon)}=h_{\det E\,|\,X_s^*}$. Let $u_s^{(\epsilon)}$ be the endomorphism of $E_{|X_s^*}$ determined by $h^{(\epsilon)}_{|X_s^*}=h_s^{(\epsilon)}\cdot u_s^{(\epsilon)}$. For a point $x\in X^*$, we put $\mathcal{U}_x:=\{s\in\mathcal{U}\,|\,x\in X_s\}$. We put $Z:=\{x\in X^*\,\big|\,\mathcal{U}_x=\emptyset\}$. We remark that Z is a finite set. Let us fix a sequence $\epsilon_i\longrightarrow 0$. We often use the notation "ϵ" instead of "ϵ_i", for simplicity of the description. Let $\mathbb{D}_s^\lambda:=\mathbb{D}^\lambda_{|X_s^*}$.

Lemma 5.5 *For almost every $s\in\mathcal{U}$, the following holds:*

- *We have the following convergence when $\epsilon\longrightarrow 0$:*

$$\int_{X_s}\big|G(h^{(\epsilon)}_{|X_s})\big|^2_{h_s^{(\epsilon)},\omega_\epsilon}\operatorname{dvol}_s\longrightarrow 0. \quad (85)$$

- *For each ϵ, we have the finiteness:*

$$\big\|\mathbb{D}_s^\lambda u_s^{(\epsilon)}\big\|_{L^2,h_s^{(\epsilon)},\omega_\epsilon}<\infty. \quad (86)$$

Let $\widetilde{\mathcal{U}}$ denote the set of s for which both of (85) *and* (86) *hold.*

Proof It can be shown by the same argument as the proof of Lemma 9.3 of [31]. ($\mathcal{Z}_2$ should be corrected to $\big\{(x,s,t)\in X\times U_1\times\mathcal{B}\,\big|\,(ts_2+(1-t)s)(x)=0\big\}$.) ∎

We obtain the following claims from Proposition 5.1 and Corollary 5.3.

Corollary 5.6 *For any $s\in\widetilde{\mathcal{U}}$, the sequence $\{h^{(\epsilon)}_{|X_s^*}\}$ converges to $h_s^{(0)}$ weakly in L_1^2 locally on X_s^*, and $\{\theta^{(\epsilon)}_{|X_s^*}\}$ converges to $\theta_s^{(0)}$ weakly in L^2 locally on X_s^*. In particular, they are almost everywhere convergent.*

Proof It follows from Lemma 5.5 and Proposition 5.1 ∎

5.2.4 The construction of a metric defined almost everywhere

Let us take any Kahler form $\omega_{\mathbb{P}_m}$ of $\mathbb{P}_m$. We put $\mathcal{Z} := \{(s,x) \in \mathcal{U} \times X^* \,|\, x \in X_s\}$. Then, we have the induced metric of $\mathcal{Z}$. The induced volume form is denoted by $\mathrm{dvol}_{\mathcal{Z}}$. Let $\mathcal{T}$ denote the set of $(s,x) \in \widetilde{\mathcal{U}} \times X$ such that $(s,x) \in \mathcal{Z}$ and $\lim_{\epsilon\to 0} h^{(\epsilon)}_{|x} = h^{(0)}_{s|x}$.

Lemma 5.7 *The measure of* $\mathcal{T}^c := \mathcal{Z} - \mathcal{T}$ *is* 0 *with respect to* $\mathrm{dvol}_{\mathcal{Z}}$.

Proof Let us consider the naturally defined fibration $\mathcal{Z} \longrightarrow \mathcal{U}$. Then, the claim follows from Corollary 5.6 and Fubini's theorem. ∎

Lemma 5.8 *For almost every* $x \in X^*$ *and almost every* $s \in \mathcal{U}_x$*, the sequence* $\{h^{(\epsilon)}_{|x}\}$ *converges to* $h^{(0)}_{s\,|\,x}$.

Proof Let us consider the naturally defined fibration $\mathcal{T} \longrightarrow X^*$. Then, the claim follows from Lemma 5.7 and Fubini's theorem. ∎

Let $\mathcal{V}$ denote the set of $x \in X^*$ such that the sequence $\{h^{(\epsilon)}_{|x}\}$ converges to $h^{(0)}_{s\,|\,x}$ for almost $s \in \mathcal{U}_x$. For any $x \in \mathcal{V}$, let $\widetilde{\mathcal{U}}_x$ denote the set of s such that $\{h^{(\epsilon)}_{|x}\}$ converges to $h^{(0)}_{s\,|\,x}$.

Lemma 5.9 *For any* $x \in \mathcal{V}$ *and for any* $s_i \in \widetilde{\mathcal{U}}_x$ $(i = 1,2)$*, we have* $h^{(0)}_{s_1\,|\,x} = h^{(0)}_{s_2\,|\,x}$.

Proof Both of them are same as the limit $\lim_{\epsilon\to 0} h^{(\epsilon)}_x$. ∎

Let us take any $x \in \mathcal{V}$ and any $s \in \widetilde{\mathcal{U}}_x$. Then, the metric h_x of $E_{|x}$ is given by $h_x := h^{(0)}_{s\,|\,x}$. Due to Lemma 5.9, it is well defined. Thus, we obtain the metric $h_{\mathcal{V}} := (h_x \,|\, x \in \mathcal{V})$ of $E_{|\mathcal{V}}$.

5.2.5 The C^1-property

We would like to show that $h_{\mathcal{V}}$ is C^1 on $X^* - Z$, in other words, we would like to show the existence of a C^1-metric h of $E_{|X^*-Z}$ such that $h = h_{\mathcal{V}}$ on $\mathcal{V}$. Let us begin with a preparation.

Lemma 5.10 *Let* $x \in X^* - Z$*. Let us take any* $s \in \mathcal{U}_x$*. Then, there exists a Lefschetz fibration* $\varphi : \widetilde{X} \longrightarrow \mathbb{P}^1$ *with the following properties:*

- x *is not a singular point of* φ.
- $\varphi^{-1}(0) = X_s$.
- *Almost every* $t \in \mathbb{P}^1$ *belongs to* $\widetilde{\mathcal{U}}$.

Proof Let $\mathcal{M}$ denote the set of the lines ℓ of $\mathbb{P}_m$ which contain s. We put as follows:

$$\widehat{\mathbb{P}}_m = \big\{(\ell, s') \in \mathcal{M} \times \mathbb{P}_m \,\big|\, s' \in \ell\big\} \subset \mathcal{M} \times \mathbb{P}_m.$$

It is the blow up of $\mathbb{P}_m$ at s. We have the projection $\pi_2 : \widehat{\mathbb{P}}_m \longrightarrow \mathbb{P}_m$. We put $\widehat{\mathcal{U}} := \pi_2^{-1}(\mathcal{U})$ and $\widehat{\widetilde{\mathcal{U}}} := \pi_2^{-1}(\widetilde{\mathcal{U}})$. Since $\mathcal{U} - \widetilde{\mathcal{U}}$ has measure 0, the measure of $\widehat{\mathbb{P}}_m - \widehat{\widetilde{\mathcal{U}}}$ is also 0. Let us consider the projection $\pi_1 : \widehat{\mathbb{P}}_m \longrightarrow \mathcal{M}$, and apply Fubini's theorem. Then, for almost every $\ell \in \mathcal{M}$ and for almost every $s_1 \in \ell$, we have $s_1 \in \widehat{\widetilde{\mathcal{U}}}$. Thus we are done. ∎

Let x be any point of $X^* - Z$. Let us take a Lefschetz fibration $\pi_i : \widetilde{X}_i \longrightarrow \mathbb{P}^1$ $(i = 1,2)$ with the following properties:

- Both of them satisfy the properties in Lemma 5.10.
- Around x, the fibers of π_1 and π_2 are transversal. Then, two fibrations give the holomorphic coordinate (z_1, z_2) of an appropriate neighbourhood U_x of x, such that $\{z_i = a\} = \pi_i^{-1}(a) \cap U_x$.

For any $t_i \in \mathbb{P}^1$, let $X_{t_i} := \pi_i^{-1}(t_i)$. If t_i are close to 0, $(E, \boldsymbol{F}, \mathbb{D}^\lambda)_{|X_{t_i}}$ are stable, and hence there exist tame harmonic bundles h_{t_i} for $(E, \boldsymbol{F}, \mathbb{D}^\lambda)_{|X_{t_i}}$ such that $\det(h_{t_i}) = h_{\det(E)|X_{t_i}}$. Let θ_{t_i} denote the operator obtained from $\mathbb{D}^\lambda_{|X_{t_i}}$ and h_{t_i} as in Subsection 2.2.1.

Let us take an appropriate neighbourhoods $B_i \subset \mathbb{P}^1$ of 0. Recall Proposition 4.2. Then, $\{h_{t_1} \,|\, t_1 \in B_1\}$ are C^∞-along z_2, and it is continuous with respect to (z_1, z_2). The family $\{\theta_{t_1} \,|\, t_1 \in B_1\}$ has a similar property. Thus, we obtain a continuous metric $h^{(1)}$ and the continuous section $\theta^{(1)}$ of $\mathrm{End}(E) \otimes \Omega^{1,0}$ around x. Similarly $\{h_{t_2} \,|\, t_2 \in B_2\}$ is C^∞ along z_1 and it is continuous with respect to (z_1, z_2). The family $\{\theta_{t_2} \,|\, t_2 \in B_2\}$ has a similar property. Thus, we obtain a continuous metric $h^{(2)}$ and the continuous section $\theta^{(2)}$ of $\mathrm{End}(E) \otimes \Omega^{1,0}$ around x.

We remark that $h^{(1)} = h_{\mathcal{V}} = h^{(2)}$ on $U_x \cap \mathcal{V}$ due to our construction of $h_{\mathcal{V}}$. Since $h^{(i)}$ are continuous, we obtain $h^{(1)} = h^{(2)}$ on U_x. Then, we obtain that $h^{(i)}$ are C^1 on U_x, due to the continuity of $\theta^{(i)}$.

Therefore, we obtain the C^1-metric h of E on $X^* - Z$ with the following properties:

- $h_{|\mathcal{V}} = h_{\mathcal{V}}$
- For any $s \in \mathcal{U}$, we have $h_{|X_s^*} = h_s$ and $\theta_{h\,|\,X_s^*} = \theta_{h_s}$.

5.2.6 Pluri-harmonicity

We would like to show that h is pluri-harmonic. By the formalism explained in Subsection 2.2.1, the operators $\overline{\partial}_h$ and θ_h are given on $X - (D \cup Z)$ from h and $\mathbb{D}^\lambda$. Let us take any C^∞ metric h' of E on $X - D$, and let s' be the endomorphism determined by $h = h' \cdot s'$. Then, s' is C^1, and we have the following relation:

$$\overline{\partial}_h = \overline{\partial}_{h'} + \frac{\lambda}{1+|\lambda|^2} s'^{-1} \delta''_{h'} s', \quad \theta_h = \theta_{h'} - \frac{\lambda}{1+|\lambda|^2} s'^{-1} \delta'_{h'} s'.$$

Then, we obtain $\overline{\partial}_h \theta_h$ as a distribution:

$$\overline{\partial}_h \theta_h = \overline{\partial}_{h'} \theta_{h'} - \frac{\lambda}{1+|\lambda|^2} \overline{\partial}_{h'} \big(s'^{-1} \delta'_{h'} s'\big) + \frac{\lambda}{1+|\lambda|^2} \big[s'^{-1} \delta''_{h'} s',\, \theta_{h'}\big] - \left(\frac{\lambda^2}{1+|\lambda|^2}\right)^2 \big[s'^{-1} \delta''_{h'} s',\ s'^{-1} \delta'_{h} s'\big].$$

Similarly, we obtain $G(h)$ as a distribution.

Lemma 5.11 *$\overline{\partial}_h \theta_h = 0$.*

Proof For any point $x \in X^* - D$, let us take the holomorphic coordinate (z_1, z_2) as before. We remark that the curves $\{z_i = a\}$ $(i = 1, 2)$, $\{z_1 + z_2 = b\}$, $\{z_1 + \sqrt{-1} z_2 = c\}$ can be regarded as parts of $X_{s'}$ for some $s' \in \mathcal{U}$. We have the expression $\theta = f_1 \cdot dz_1 + f_2 \cdot dz_2$, where f_i are continuous sections of $\mathrm{End}(E)$. We have already known $\partial f_1 / \partial \overline{z}_1 = \partial f_2 / \partial \overline{z}_2 = 0$. Thus, we have only to show $\partial f_i / \partial \overline{z}_j = 0$ for $i \neq j$. Let us consider the change of the coordinate given by $w_1 = z_1 + z_2$ and $w_2 = z_1 - z_2$. Then, we have the following:

$$f_1 \cdot dz_1 + f_2 \cdot dz_2 = \frac{1}{2}(f_1 + f_2) \cdot dw_1 + \frac{1}{2}(f_1 - f_2) \cdot dw_2.$$

Thus, we obtain the following:

$$0 = \frac{\partial}{\partial \overline{w}_1}(f_1 + f_2) = \frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1} + \frac{\partial}{\partial \overline{z}_2}\right)(f_1 + f_2) = \frac{1}{2}\left(\frac{\partial f_2}{\partial \overline{z}_1} + \frac{\partial f_1}{\partial \overline{z}_2}\right). \tag{87}$$

Let us consider the change of the coordinate given by $u_1 = z_1 + \sqrt{-1} z_2$ and $u_2 = z_1 - \sqrt{-1} z_2$. Then, we have the following:

$$f_1 \cdot dz_1 + f_2 \cdot dz_2 = \frac{1}{2}\left(f_1 + \frac{1}{\sqrt{-1}} f_2\right) du_1 + \frac{1}{2}\left(f_1 - \frac{1}{\sqrt{-1}} f_2\right) du_2.$$

Thus, we obtain the following:

$$0 = \frac{\partial}{\partial \overline{u}_1}\left(f_1 + \frac{1}{\sqrt{-1}} f_2\right) = \frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1} - \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \overline{z}_2}\right)\left(f_1 + \frac{1}{\sqrt{-1}} f_2\right) = \frac{1}{2}\left(\frac{1}{\sqrt{-1}} \frac{\partial f_2}{\partial \overline{z}_1} - \frac{1}{\sqrt{-1}} \frac{\partial f_1}{\partial \overline{z}_2}\right). \tag{88}$$

From (87) and (88), we obtain $\partial f_i/\partial \overline{z}_j = 0$ for $i \neq j$. Thus, we obtain $\overline{\partial}_h \theta_h = 0$, and the proof of Lemma 5.11 is accomplished. ∎

Lemma 5.12 *h is a harmonic metric for $(E, \mathbb{D}^{\lambda})$ with respect to ω_0 on $X^* - Z$. (Recall $Z = \{x \in X^* \,|\, \mathcal{U}_x = \emptyset\}$.)*

Proof Due to Lemma 5.11, we have $\Lambda_{\omega} G(h) = \Lambda_{\omega}(\overline{\partial}_h \theta_h) = 0$. Hence, we have only to show that h is C^{∞}. We obtain the following formula in the level of distribution, by the formalism explained in Subsection 2.2.5:

$$\Delta^{\lambda}_{h',\omega}(s') = s'\bigl(-\Lambda_{\omega} G(h')\bigr) + \sqrt{-1}\Lambda_{\omega} \mathbb{D}^{\lambda} s' \cdot s'^{-1} \cdot \mathbb{D}^{\lambda\star}_{h'} s'.$$

The right hand side is C^0. Hence, by using the elliptic regularity and the standard boot strapping argument, we obtain that s' is C^{∞}. Thus, we obtain Lemma 5.12. ∎

Lemma 5.13 *h is pluri-harmonic metric of $E_{|X^*-Z}$.*

Proof We have already shown $\overline{\partial}_h \theta_h = 0$ in Lemma 5.11. Because of Corollary 2.27, we have only to show $\theta_h^2 = 0$. Due to Corollary 5.6 and $\theta_{h\,|\,X_s} = \theta_s$, we know that the sequence $\{\theta^{(\epsilon)}\}$ converges to θ_h almost everywhere. In particular, we obtain the almost everywhere convergence of $\{\theta^{(\epsilon)\,2}\}$ to θ_h^2. On the other hand, we know the almost everywhere convergence $G(h^{(\epsilon)}) \longrightarrow 0$, due to (84). We have $G(h^{(\epsilon)}) = \overline{\partial}^{(\epsilon)\,2} + \overline{\partial}^{(\epsilon)} \theta^{(\epsilon)} + \theta^{(\epsilon)\,2}$, which is the decomposition into $(2,0)$, $(1,1)$ and $(0,2)$-forms. Therefore, we obtain $\theta_h^2 = 0$, almost everywhere. Thus, we obtain Lemma 5.13. ∎

Lemma 5.14 *h gives a pluri-harmonic metric of $E_{|X^*}$.*

Proof We have only to check that h gives a C^{∞}-metric of $E_{|X^*}$. Let Q be a point of Z. Let (U, z_1, z_2) be a holomorphic coordinate around Q such that $z_1(Q) = z_2(Q) = 0$. The pluri-harmonic metric h of $(E, \mathbb{D}^{\lambda})_{|U-\{Q\}}$ is given. We would like to show that h is naturally extended to the pluri-harmonic metric of $(E, \mathbb{D}^{\lambda})_{|U}$.

We have $\theta = f_1 \cdot dz_1 + f_2 \cdot dz_2$ defined on $U - \{Q\}$. Let us consider the characteristic polynomials $\det(t - f_i)$ for $i = 1, 2$. The coefficients are holomorphic on $U - \{Q\}$, and thus on U due to the theorem of Hartogs. Hence, the eigenvalues of f_i are bounded on U. Let us consider the restriction of $(E, \mathbb{D}^{\lambda}, h)$ to the discs $C(a_j) := \{z_j = a_j\}$ $(a_j \neq 0)$ for $j = 1, 2$. Then, it can be shown that the norms $\bigl|f_{i\,|\,C(a_j)}\bigr|_h \leq C$ $(i \neq j)$ can be dominated independently from a_j. (See Lemma 2.7 in [38], for example.) Thus, f_i are bounded with respect to h on $U - \{Q\}$. In other words, θ is bounded on $U - \{Q\}$.

Let $E' := E_{|U-\{z_1 \cdot z_2 = 0\}}$. Let us consider the sheaf ${}^{\diamond}E'$ on U of the sections satisfying the growth condition $|g|_h = O\bigl(\prod |z_i|^{-\epsilon}\bigr)$ for any $\epsilon > 0$ (Subsection 2.5.3). By using the result of the asymptotic behaviour of tame harmonic bundle at λ ([30]), ${}^{\diamond}E'$ is locally free on U. Since ${}^{\diamond}E'$ and $E_{|U-\{Q\}}$ are naturally isomorphic on $U - \{Q\}$, they are isomorphic on U. Let h' be any C^{∞}-metric of $E_{|U}$, and let s' be the endomorphism determined by $h = h' \cdot s'$. Due to the norm estimate given in [30], the metrics h and h' are mutually bounded. Hence, s' and $(s')^{-1}$ are bounded on U. Let $\delta'_{h'}$ and $\delta''_{h'}$ be obtained from $\mathbb{D}^{\lambda}$ and h' as in Subsection 2.2.1. Due to the boundedness of θ, we have the boundedness of $(s')^{-1}\delta'_{h'} s'$ on $U - \{Q\}$. Due to the boundedness of $\theta^{\dagger}$, we have the boundedness of $(s')^{-1}\delta''_{h'} s'$ on $U - \{Q\}$. Then, we can deduce that $s'^{-1}\mathbb{D}^{\lambda} s'$ is also bounded on $U - \{Q\}$. (See Subsection 2.2.5. for example.) Since we have the formula $\Delta^{\lambda}_{h',\omega_0} s' = s'(-\Lambda_{\omega_0} G(h')) + \Lambda_{\omega_0} \mathbb{D}^{\lambda}_{h'} s' \cdot s'^{-1} \cdot \mathbb{D}^{\lambda\star}_{h'} s'$, we can conclude that s' is C^{∞} due to the standard bootstrapping argument. Namely, h is extended to the C^{∞}-metric of $E_{|U}$. ∎

5.2.7 The end of the proof of Theorem 5.4

Now, we have only to show that h is tame and adapted to the parabolic structure. Since $h_{|X_s} = h_s$ for any $s \in \mathcal{U}$, the tameness immediately follows from the curve test. (See Proposition 2.49.) Then, we obtain the prolongment $\widetilde{E} := {}_{\boldsymbol{c}}E$ with the induced parabolic structure $\boldsymbol{F}$ (Subsection 2.5.3). We would like to show that $(E, \boldsymbol{F}, \mathbb{D}^{\lambda})$ and $(\widetilde{E}, \boldsymbol{F}, \mathbb{D}^{\lambda})$ are isomorphic. For that purpose, we see that the identity $E_{|X^*} \longrightarrow E_{|X^*}$ can be prolonged to the homomorphism $\Psi : E \longrightarrow \widetilde{E}$. Let Q be any smooth point of $D_i \subset D$. We take a holomorphic coordinate (U_Q, z_1, z_2) with the following property:

- The curve $z_1^{-1}(0)$ is same as $U_Q \cap D$.
- The curves $C(b) := z_2^{-1}(b)$ are parts of $X_{s(b)}$ for $s(b) \in \mathcal{U}$.

Let f be a holomorphic section of $E_{|U}$. Since the restriction $h_{|X_{s(b)}}$ is the same as $h_{s(b)}$, we have $|f_{|C(b)}|_h = O(|z_1|^{-c_i-\epsilon})$ for any $\epsilon > 0$. Then, we obtain $|f|_h = O(|z_1|^{-c_i-\epsilon})$ for any $\epsilon > 0$, due to the result given in [30]. Thus, f naturally gives the section of $\widetilde{E}$ on U. Therefore, we obtain the morphism $E \longrightarrow \widetilde{E}$ on $X - \bigl(\cup_{i\neq j} D_i \cap D_j\bigr)$. It is naturally extended to the morphism $E \longrightarrow \widetilde{E}$.

Recall that the restriction of $\widetilde{E} = {}_{\boldsymbol{c}}E(h)$ to X_s is same as ${}_{\boldsymbol{c}}(E_{|X_s})(h_s)$. (See [30].) Therefore, the restrictions of Ψ to X_s are isomorphic, by construction. Hence, Ψ is isomorphic on $X - \bigl(\bigcup_{i\neq j} D_i \cap D_j\bigr)$, and thus on X. By a similar argument, we can show that the parabolic structures are also same. Thus, the proof of Theorem 5.4 is finished. ∎

5.3 Correspondences

5.3.1 Kobayashi-Hitchin correspondence in the higher dimensional case

Let X be a smooth projective variety of dimension n ($n \geq 3$) with an ample line bundle L, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a μ_L-stable regular filtered λ-flat bundle on (X, D) in codimension two with trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$, and we put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Recall $\text{par-c}_1(\boldsymbol{E}_*) = 0$ due to the Bogomolov-Gieseker inequality and the Hodge index theorem. For each $\boldsymbol{c} \in \boldsymbol{R}^S$, we have the determinant line bundle $\det({}_{\boldsymbol{c}}E)$ of torsion-free sheaf ${}_{\boldsymbol{c}}E$, on which we have the induced parabolic structure and the induced flat λ-connection. Thus, we obtain the canonically determined regular filtered λ-flat bundle $\bigl(\det \boldsymbol{E}_*, \mathbb{D}^\lambda\bigr)$ on (X, D) of rank one. We also have $\text{par-c}_1\bigl(\det \boldsymbol{E}_*\bigr) = \text{par-c}_1\bigl(\boldsymbol{E}_*\bigr) = 0$. Therefore, we can take a pluri-harmonic metric $h_{\det E}$ of $(\det(E), \mathbb{D}^\lambda)$ which is adapted to the parabolic structure of $\det \boldsymbol{E}_*$. By the assumption, we have a subset $Z \subset D$ with $\text{codim}_X(Z) \geq 3$ such that $(\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-Z}$ is a regular filtered λ-flat *bundle*.

Theorem 5.15 *There exists the unique tame pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)$ with the following properties:*

- $\det(h) = h_{\det E}$.
- *It is adapted to the parabolic structure of $\boldsymbol{E}_*$ on $X - Z$. Namely, $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)_{|X-Z} \simeq (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-Z}$, where $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)$ denotes the regular filtered λ-flat bundle on (X, D) obtained from $(E, \mathbb{D}^\lambda, h)$. (See Subsection 2.5.)*

Proof Due to Mehta-Ramanathan type theorem (Proposition 2.9), the uniqueness can be easily reduced to the $\dim X = 1$ case, by considering the restriction to the generic curves $C \subset X$. We have already known it (Proposition 2.53).

We will use the induction on the dimension n to show the existence. The case $n = 2$ has already been shown (Theorem 5.4). Assume that L^m is sufficiently ample. We put $\mathbb{P}_m := \mathbb{P}(H^0(X, L^m)^\vee)$. For any $s \in \mathbb{P}_m$, we put $X_s := s^{-1}(0)$. Recall Proposition 2.9. Let $\mathcal{U}$ be the Zariski open subset of $\mathbb{P}_m$ which consists of $s \in \mathbb{P}_m$ with the following properties:

- X_s is smooth, and $D_s := X_s \cap D$ is a normal crossing divisor.
- The codimension of $Z \cap X_s$ in X_s is larger than 3.
- $(\boldsymbol{E}, \mathbb{D}^\lambda)_{|X_s}$ is μ_L-stable.

We use the existence hypothesis in the $(n-1)$-dimensional case of the induction. Then, we may have the tame pluri-harmonic metric h_s of $(E, \mathbb{D}^\lambda)_{|X_s \setminus D}$ with $\det(h_s) = h_{\det E\,|\,X_s\setminus D}$ which is adapted to the parabolic structure on $X_s \setminus W$. We also use the uniqueness result in the $(n-2)$-dimensional case. Then, we can show the existence of a finite subset $Z' \subset X - D$ and a metric h of $E_{|X-D}$ such that $h_{s\,|\,P} = h_{|P}$. By the arguments given in Subsections 5.2.5–5.2.7, we can show that h is the desired metric. The only different point is the argument to show the vanishing of $G(h) = 0$. Due to $\dim X_s \geq 2$, it can be shown easier. ∎

Theorem 5.16 *Let X, D and L be as above. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a saturated μ_L-stable regular filtered λ-flat sheaf on (X, D) with the trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. We put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Then, there exists a pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)$ such that the induced regular filtered λ-flat bundle $\big(\boldsymbol{E}_*(h), \mathbb{D}^\lambda\big)$ is isomorphic to $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$. Such a metric is unique up to positive constant multiplication. In particular, $\boldsymbol{E}_*$ is a filtered bundle.*

Proof Since a saturated regular filtered λ-flat sheaf is a regular filtered λ-flat bundle in codimension two (Lemma 2.12), we may apply Theorem 5.15. Then, there exists a pluri-harmonic metric h and a subset $W \subset D$ with $\text{codim}_X(W) \geq 3$ such that the induced regular filtered λ-flat bundle $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)$ is isomorphic to $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on $X - W$. Since both of $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)$ and $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ are saturated, they are isomorphic on X. ∎

5.3.2 The equivalence of the categories

Let $\mathcal{C}_\lambda^{poly}$ denote the category of μ_L-stable regular filtered λ-flat bundles $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on (X, D) with the trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. Morphisms $f : (\boldsymbol{E}_{1\,*}, \mathbb{D}_1^\lambda) \longrightarrow (\boldsymbol{E}_{2\,*}, \mathbb{D}_2^\lambda)$ are defined to be $\mathcal{O}_X$-homomorphism $f : \boldsymbol{E}_1 \longrightarrow \boldsymbol{E}_2$ satisfying $\mathbb{D}_2^\lambda \circ f = f \circ \mathbb{D}_1^\lambda$ and $f\big({}_{\boldsymbol{c}}E_1\big) \subset {}_{\boldsymbol{c}}E_2$ for any $\boldsymbol{c}$.

Corollary 5.17 *Let λ_i $(i = 1, 2)$ be two complex numbers. We have the natural functor $\Xi_{\lambda_1,\lambda_2} : \mathcal{C}_{\lambda_1}^{poly} \longrightarrow \mathcal{C}_{\lambda_2}^{poly}$, which is equivalent. It preserves direct sums, tensor products and duals.*

Proof Let $(\boldsymbol{E}_*^{\lambda_1}, \mathbb{D}^{\lambda_1})$ be an object of $\mathcal{C}_{\lambda_1}^{poly}$. We put $E^{\lambda_1} := \boldsymbol{E}^{\lambda_1}_{|D}$. We have a pluri-harmonic metric h of $(E^{\lambda_1}, \mathbb{D}^{\lambda_1})$, which is adapted to the parabolic structure. Then, we obtain the operators $\overline{\partial}_h, \partial_h, \theta_h, \theta_h^\dagger$, as in Subsection 2.2.1. Note that the holomorphic structure of E^{λ_1} is given by $\overline{\partial}_h + \lambda_1 \theta_h^\dagger$. The $(0,1)$-operator $\overline{\partial}_h + \lambda_2 \theta_h^\dagger$ also gives a holomorphic structure of C^∞-bundle E^{λ_1}. To distinguish them, we use the notation E^{λ_2}, when we consider the holomorphic structure $\overline{\partial}_h + \lambda_2 \theta_h^\dagger$. We put $\mathbb{D}^{\lambda_2} := \overline{\partial}_h + \theta_h + \lambda_2(\partial_h + \theta_h^\dagger)$, which gives a flat λ_2-connection of E^{λ_2}. The metric h is pluri-harmonic for $(E^{\lambda_2}, \mathbb{D}^{\lambda_2})$. Since the corresponding Higgs bundle for $(E^{\lambda_1}, \mathbb{D}^{\lambda_1}, h)$ and $(E^{\lambda_2}, \mathbb{D}^{\lambda_2}, h)$ are same, we obtain the tameness of $(E^{\lambda_2}, \mathbb{D}^{\lambda_2}, h)$. Therefore, we obtain the prolongment $(\boldsymbol{E}^{\lambda_2}, \mathbb{D}^\lambda)$, which are μ_L-polystable regular filtered λ_2-flat bundle on (X, D) with trivial characteristic numbers (Proposition 2.52).

We remark that $(\boldsymbol{E}^{\lambda_2}, \mathbb{D}^{\lambda_2})$ is independent of a choice of h, due to the uniqueness in Proposition 2.53. Therefore, we put $\Xi_{\lambda_1,\lambda_2}(\boldsymbol{E}^{\lambda_1}, \mathbb{D}^{\lambda_1}) := (\boldsymbol{E}^{\lambda_2}, \mathbb{D}^{\lambda_2})$. It is easy to see that $\Xi_{\lambda_1,\lambda_2}$ gives a functor. It is also easy to see that $\Xi_{\lambda_2,\lambda_1} \circ \Xi_{\lambda_1,\lambda_2}(\boldsymbol{E}^{\lambda_1}, \mathbb{D}^{\lambda_1})$ is naturally isomorphic to $(\boldsymbol{E}^{\lambda_1}, \mathbb{D}^{\lambda_1})$. The compatibility with the direct sums, duals and tensor products are obtained from the corresponding compatibility statements of the prolongments for tame harmonic bundles ([30]). We also remark that the categories are semisimple. Thus, we have only to compare the objects. ∎

Remark 5.18 *From a λ_1-connection $\mathbb{D}^{\lambda_1} = d'' + d'$, a λ_2-connection is given $d'' + (\lambda_2/\lambda_1) \cdot d'$. Hence, we have the obvious functor $\text{Obv} : \mathcal{C}_{\lambda_1}^{poly} \longrightarrow \mathcal{C}_{\lambda_2}^{poly}$. This is not same as the above functor $\Xi_{\lambda_1,\lambda_2}$.* ∎

6 Filtered local system

6.1 Definition

6.1.1 Filtered structure

Let X be a complex manifold, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. We will use the notation $D^{[2]} := \bigcup_{i \neq j} D_i \cap D_j$ and $D_i^\circ := D_i \setminus \bigcup_{j \neq i} D_j$. Let $\mathcal{L}$ be a local system on $X - D$. A filtered structure of $\mathcal{L}$ at D is a tuple of increasing filtrations ${}^i\mathcal{F}$ $(i \in S)$ of $\mathcal{L}_{|U_i \setminus D}$ indexed by $\boldsymbol{R}$, where U_i denotes an appropriate open neighbourhood of D_i. Let U_i' be an open neighbourhood of D_i such that $U_i' \subset U_i$, then we have the induced filtration ${}^i\mathcal{F}_{|U_i'}$, and the filtration ${}^i\mathcal{F}$ can be reconstructed from ${}^i\mathcal{F}_{|U_i'}$. Hence, we define two filtered structures $({}^i\mathcal{F}, U_i \,|\, i \in S)$ and $({}^i\mathcal{F}', U_i' \,|\, i \in S)$ are equivalent, if there exists an open neighbourhood U_i'' of D_i such that $U_i'' \subset U_i \cap U_i'$ and ${}^i\mathcal{F}_{|U_i''} = {}^i\mathcal{F}'_{|U_i''}$. A local system $\mathcal{L}$ equipped with an equivalence class of filtered structures $({}^{i}\mathcal{F}, U_i)$ is called a filtered local system, and it is denoted by $\mathcal{L}_*$. We do not have to care about a choice of open neighbourhoods U_i.

Morphisms of filtered local systems $f : \mathcal{L}_{1*} \longrightarrow \mathcal{L}_{2*}$ are defined to be a morphism $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ of local systems preserving the filtered structures in an obvious sense. We denote by $\widetilde{\mathcal{C}}(X, D)$ the category of filtered local systems on (X, D).

6.1.2 Characteristic numbers

We put $U_i^* := U_i \setminus D$ and ${}^{i}\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}) := {}^{i}\mathcal{F}_a(\mathcal{L}_{|U_i^*}) \big/ {}^{i}\mathcal{F}_{<a}(\mathcal{L}_{|U_i^*})$. Since the local monodromy around D_i preserves the filtration ${}^{i}\mathcal{F}$, we obtain the induced endomorphism of ${}^{i}\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*})$, and thus the generalized eigen decomposition:

$$ {}^{i}\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}) = \bigoplus_{\omega} {}^{i}\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(a,\omega)}(\mathcal{L}_{|U_i^*}). $$

We put as follows:

$$ \mathcal{P}ar(\mathcal{L}_*, i) := \bigl\{ a \in \boldsymbol{R} \,\big|\, {}^{i}\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}) \neq 0 \bigr\}, \quad \mathcal{KMS}(\mathcal{L}_*, i) := \bigl\{ (a,\omega) \in \boldsymbol{R} \times \boldsymbol{C}^* \,\big|\, {}^{i}\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(a,\omega)}(\mathcal{L}_{|U_i^*}) \neq 0 \bigr\}. $$

The parabolic first Chern class is defined as follows:

$$ \text{par-c}_1(\mathcal{L}_*) := -\sum_{i \in S} \operatorname{wt}(\mathcal{L}_*, i) \cdot [D_i] \in H^2(X, \boldsymbol{R}), \qquad \operatorname{wt}(\mathcal{L}_*, i) := \sum_{a \in \mathcal{P}ar(\mathcal{L}_*, i)} a \cdot \operatorname{rank} {}^{i}\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}). \tag{89} $$

Here $[D_i]$ denotes the cohomology class representing D_i.

Let $\operatorname{Irr}(D_i \cap D_j)$ denote the set of the irreducible components of $D_i \cap D_j$. For each $P \in \operatorname{Irr}(D_i \cap D_j)$, let U_P be an appropriate open neighbourhood of P in X such that $U_P \subset U_i \cap U_j$. We put $U_P^* := U_P \setminus D$. We have the two filtrations ${}^{i}\mathcal{F}$ and ${}^{j}\mathcal{F}$ of $\mathcal{L}_{|U_P^*}$. The naturally induced graded local system is denoted as follows:

$$ {}^{P}\operatorname{Gr}^{\mathcal{F}}(\mathcal{L}_{|U_P^*}) = \bigoplus_{(a_i,a_j) \in \boldsymbol{R}^2} {}^{P}\operatorname{Gr}^{\mathcal{F}}_{(a_i,a_j)}(\mathcal{L}_{|U_P^*}), \quad {}^{P}\operatorname{Gr}^{\mathcal{F}}_{(a_i,a_j)}(\mathcal{L}_{|U_P^*}) := \frac{{}^{i}\mathcal{F}_{a_i} \cap {}^{j}\mathcal{F}_{a_j}}{\sum_{(b_i,b_j) \lneq (a_i,a_j)} {}^{i}\mathcal{F}_{b_i} \cap {}^{j}\mathcal{F}_{b_j}}. $$

Here $(b_i, b_j) \lneq (a_i, a_j)$ means "$b_i \leq a_i$, $b_j \leq a_j$ and $(b_i, b_j) \neq (a_i, a_j)$". We have the two endomorphisms induced by the local monodromies around $U_P \cap D_i$ and $U_P \cap D_j$, which are commutative. Hence, we obtain the generalized eigen decomposition:

$$ {}^{P}\operatorname{Gr}^{\mathcal{F}}_{\boldsymbol{a}}(\mathcal{L}_{|U_P^*}) = \bigoplus_{\boldsymbol{\omega} \in \boldsymbol{C}^{*\,2}} {}^{P}\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{\boldsymbol{a},\boldsymbol{\omega}}(\mathcal{L}_{|U_P^*}). $$

We put as follows:

$$ \mathcal{P}ar(\mathcal{L}_*, P) := \bigl\{ (a_i, a_j) \in \boldsymbol{R}^2 \,\big|\, {}^{P}\operatorname{Gr}^{\mathcal{F}}_{(a_i,a_j)}(\mathcal{L}_{|U_P^*}) \neq 0 \bigr\}, $$
$$ \mathcal{KMS}(\mathcal{L}_*, P) := \bigl\{ (\boldsymbol{a}, \boldsymbol{\omega}) \in \boldsymbol{R}^2 \times \boldsymbol{C}^{*\,2} \,\big|\, {}^{P}\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\omega})}(\mathcal{L}_{|U_P^*}) \neq 0 \bigr\}. $$

The parabolic second Chern character is defined as follows:

$$ \begin{aligned} \text{par-ch}_2(\mathcal{L}_*) := \frac{1}{2} \sum_{i \in S} \sum_{a \in \mathcal{P}ar(\mathcal{L}_*, i)} a^2 \cdot \operatorname{rank} {}^{i}\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}) \cdot [D_i]^2 \\ + \frac{1}{2} \sum_{i \in S} \sum_{j \neq i} \sum_{P \in \operatorname{Irr}(D_i \cap D_j)} \sum_{(a_i,a_j) \in \mathcal{P}ar(\mathcal{L}_*, P)} a_i \cdot a_j \cdot \operatorname{rank} {}^{P}\operatorname{Gr}^{\mathcal{F}}_{(a_i,a_j)}\bigl(\mathcal{L}_{|U_P^*}\bigr) \cdot [P]. \end{aligned} \tag{90} $$

When X is a smooth projective variety with an ample line bundle L, we put as follows:

$$ \text{par-deg}_L(\mathcal{L}_*) := \int_X \text{par-c}_1(\mathcal{L}_*) \cdot c_1(L)^{\dim X - 1}, \quad \mu_L(\mathcal{L}_*) := \frac{\text{par-deg}_L(\mathcal{L}_*)}{\operatorname{rank} \mathcal{L}}. $$

Then, the notion of μ_L-stability, μ_L-semistability, and μ_L-polystability for filtered local systems on (X, D) are defined in the standard manner. We also put as follows:

$$ \int_X \text{par-c}^2_{1,L}(\mathcal{L}_*) := \int_X \text{par-c}_1(\mathcal{L}_*)^2 \cdot c_1(L)^{\dim X - 2}, \quad \int_X \text{par-ch}_{2,L}(\mathcal{L}_*) := \int_X \text{par-ch}_{2,L}(\mathcal{L}_*) \cdot c_1(L)^{\dim X - 2}. $$

6.2 Correspondence

In this subsection, we give the correspondence of filtered local systems on (X,D) and saturated regular filtered λ-flat sheaves ($\lambda \neq 0$). See Subsection 2.1.4 for saturated regular filtered λ-flat sheaves. Since we have the obvious equivalence between flat λ-connection and flat 1-connection, we only discuss the case $\lambda = 1$, i.e. ordinary flat connections.

Let $\mathcal{C}_1^{sat}(X,D)$ denote the category of saturated regular filtered flat sheaves on (X,D). Let us see briefly that we have the functor $\Phi : \widetilde{\mathcal{C}}(X,D) \longrightarrow \mathcal{C}_1^{sat}(X,D)$ which gives the equivalence. Since it is given by Simpson in [37] essentially in the curve case, we give only an outline.

6.2.1 Construction of Φ

First, we give a construction of Φ. Let $\mathcal{L}_*$ be a filtered local system on (X,D). Let (E,∇) be the corresponding flat bundle on $X-D$. We have the Deligne extension $(\widetilde{E},\nabla)$ on (X,D). We put $\boldsymbol{E} := \widetilde{E}\otimes\mathcal{O}(*D)$. Thus, we have only to give the way of the construction of the $\mathcal{O}_X$-coherent submodules ${}_{\boldsymbol{a}}E \subset \boldsymbol{E}$ such that $\nabla {}_{\boldsymbol{a}}E \subset {}_{\boldsymbol{a}}E \otimes \Omega^{1,0}(\log D)$ and $\bigcup_{\boldsymbol{a}\in\boldsymbol{R}^S} {}_{\boldsymbol{a}}E = \boldsymbol{E}$. Let us consider the case $X = \Delta^n = \{(z_1,\ldots,z_n)\,|\,|z_i| < 1\}$ and $D = \{z_1 = 0\}$. Then, the construction is essentially same as that for the case $\dim X = 1$ given by Simpson [37]. We briefly recall it. Let $H(\mathcal{L})$ denote the space of the multi-valued flat sections of $\mathcal{L}$. We have the induced filtration $\mathcal{F}H(\mathcal{L})$ and the generalized eigen decomposition $H(\mathcal{L}) = \bigoplus_\omega \mathbb{E}_\omega(H(\mathcal{L}))$, which are compatible in the sense $\mathcal{F}_a = \bigoplus_\omega \mathcal{F}_a \cap \mathbb{E}_\omega$. Let $\boldsymbol{u} = (u_1,\ldots,u_r)$ be a frame compatible of $H(\mathcal{L})$, compatible with $(\mathcal{F},\mathbb{E})$. Then, for each u_i, the numbers $\omega(u_i) \in \boldsymbol{C}^*$ and $a(u_i) \in \boldsymbol{R}$ are determined by $u_i \in \mathbb{E}_{\omega(u_i)}$ and $u_i \in \mathcal{F}_{a(u_i)} - \mathcal{F}_{<a(u_i)}$. The complex number $\alpha(u_i)$ is determined by the conditions $\exp(-2\pi\alpha(u_i)) = \omega(u_i)$ and $0 \le \operatorname{Re}\alpha(u_i) < 1$. Let M^u denote the endomorphism of $H(\mathcal{L})$ or $\mathcal{L}$, which is the unipotent part of the monodromy around D, and we put $N := -(2\pi\sqrt{-1})^{-1}\log M^u$. We regard u_i as a multi-valued C^∞-section of E. Then, it is standard that $v_i := \exp\bigl(\log z_1(\alpha(u_i)+N)\bigr)\cdot u_i$ gives a holomorphic section of E. Moreover, $\boldsymbol{v} = (v_1,\ldots,v_r)$ gives a frame of the Deligne extension $\widetilde{E}$. Let b be any real number. Then, we put $n(b,u_i) := \max\bigl\{n\in\mathbb{Z}\,\big|\, a(u_i) - \operatorname{Re}\alpha(u_i) + n \le b\bigr\}$, and we put $v_i(b) := z_1^{-n(b,u_i)}\cdot v_i$. Let ${}_bE$ denote the $\mathcal{O}_X$-submodule of $\boldsymbol{E}$ generated by $v_1(b),\ldots,v_r(b)$. It is easy to check that ${}_bE$ is locally free and independent of a choice of $\boldsymbol{u}$. It is also easy to see $\boldsymbol{E} = \bigcup_{b\in\boldsymbol{R}} {}_bE$. Thus, we obtain the filtration in the case $X = \Delta^n$ and $D = \{z_1 = 0\}$. It can be checked that the filtration is independent of a choice of the coordinate $(z_1,z_2,\ldots,z_n)$ satisfying $D = \{z_1 = 0\}$.

For any $\boldsymbol{b}\in\boldsymbol{R}^S$, we obtain ${}_{\boldsymbol{b}}E$ on $X - D^{[2]}$ by gluing them. The subsheaves ${}_{\boldsymbol{b}}E$ are determined by the condition (4).

Lemma 6.1 *${}_{\boldsymbol{b}}E$ is a coherent $\mathcal{O}_X$-module. Hence, we obtain the saturated regular filtered flat sheaf $(\boldsymbol{E}_*,\nabla)$ on (X,D).*

Proof We may assume that $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $H(\mathcal{L})$ denote the space of the multi-valued flat sections of $\mathcal{L}$. We have the monodromy endomorphisms M_i $(i = 1,\ldots,\ell)$ along the loop around D_i with counter clockwise direction. They induce the decomposition

$$H(\mathcal{L}) = \bigoplus_{\boldsymbol{\omega}\in\boldsymbol{C}^\ell} \mathbb{E}_{\boldsymbol{\omega}} H(\mathcal{L}), \tag{91}$$

where each $\mathbb{E}_{\boldsymbol{\omega}}H(\mathcal{L})$ is preserved by M_i $(i=1,\ldots,\ell)$, and the eigenvalues of M_i on $\mathbb{E}_{\boldsymbol{\omega}}H(\mathcal{L})$ are ω_i. We also have the filtrations ${}^i\mathcal{F}$ $(i = 1,\ldots,\ell)$ of $H(\mathcal{L})$, corresponding to the divisor D_i. Each ${}^i\mathcal{F}$ is compatible with the decomposition (91).

Fix j such that $1 \le j \le \ell$. We take a frame $\boldsymbol{u} = (u_1,\ldots,u_r)$ of $H(\mathcal{L})$ compatible with the filtration ${}^j\mathcal{F}$ and the decomposition (91). For each u_p, the tuple $\boldsymbol{\omega}(u_p)\in\boldsymbol{C}^\ell$ is determined by $u_p\in\mathbb{E}_\omega$. Let $\alpha_i(u_p)\in\boldsymbol{C}$ $(i=1,\ldots,\ell)$ be determined by $\exp\bigl(-2\pi\alpha_i(u_p)\bigr) = \omega_i(u_p)$ and $0\le\operatorname{Re}\alpha_i(u_p) < 1$. We also have the numbers $a(u_p)\in\boldsymbol{R}$ such that $u_p\in{}^j\mathcal{F}_{a(u_p)} - {}^j\mathcal{F}_{<a(u_p)}$. We put $n(b_j,u_p) := \max\bigl\{n\in\mathbb{Z}\,\big|\,a_j(u_p) - \operatorname{Re}\alpha_j(u_p) + n \le b_j\bigr\}$. Let $N_i := -(2\pi\sqrt{-1})^{-1}\log M^u$ $(i=1,\ldots,\ell)$, where N_i denotes the logarithm of the unipotent part of M_i. We take a sufficiently large integer I. Then, we put as follows:

$$v_p := z_j^{n(b_j,u_p)}\cdot\prod_{i\neq j} z_i^I \prod_{i=1}^{\ell}\exp\bigl(\log z_i\cdot(\alpha_i(u_p)+N_i)\bigr)\cdot u_p$$

If I is sufficiently large, v_p gives the section of ${}_{\boldsymbol{b}}E$ on X. By the correspondence, we obtain the following morphism, for $j = 1, \ldots, \ell$:

$$\Phi_j : \bigoplus_{p=1}^{r} \mathcal{O}_X \cdot v_p \longrightarrow {}_{\boldsymbol{b}}E$$

The morphisms Φ_j $(j = 1, \ldots, \ell)$ induce the morphism $\Phi : \mathcal{O}^{\oplus \ell \cdot r} \longrightarrow {}_{\boldsymbol{b}}E$. The image of Φ is $\mathcal{O}_X$-coherent, and it is the same as ${}_{\boldsymbol{b}}E$ on $X - D^{[2]}$. Then, it is easy to show that ${}_{\boldsymbol{b}}E$ is the same as the double dual of the image of Φ which is $\mathcal{O}_X$-coherent. ∎

Let $f : \mathcal{L}_{1\,*} \longrightarrow \mathcal{L}_{2\,*}$ be a morphism. Let $(\boldsymbol{E}_{i\,*}, \nabla_i) := \Phi(\mathcal{L}_i)$. We have the induced map $\widetilde{f} : \boldsymbol{E}_1 \longrightarrow \boldsymbol{E}_2$. It is easy to see that ${}_{\boldsymbol{c}}E_{1\,|\,X-D^{[2]}} \longrightarrow {}_{\boldsymbol{c}}E_{2\,|\,X-D^{[2]}}$ is induced. Due to saturatedness of $(\boldsymbol{E}_{2\,*}, \nabla)$, we obtain maps ${}_{\boldsymbol{c}}E_1 \longrightarrow {}_{\boldsymbol{c}}E_2$, and thus $\Phi(f) : (\boldsymbol{E}_{1\,*}, \nabla_1) \longrightarrow (\boldsymbol{E}_{2\,*}, \nabla_2)$.

6.2.2 Equivalence

Let us show that Φ is equivalent. To begin with, we consider the case $X = \Delta^n$ and $D = \{z_1 = 0\}$. Let $\mathcal{C}_1^{vb}(X, D)$ denote the category of regular filtered flat bundles on (X, D), which is the subcategory of $\mathcal{C}_1^{sat}(X, D)$. By the construction, the image of Φ is contained in $\mathcal{C}_1^{vb}(X, D)$. The following lemma can be shown as in [37].

Lemma 6.2 *The functor Φ gives the equivalence of $\widetilde{\mathcal{C}}_1(X, D)$ and $\mathcal{C}_1^{vb}(X, D)$. It is also compatible with direct sums, duals, and tensor products.* ∎

Lemma 6.3 *In the case $X = \Delta^n$ and $D = \{z_1 = 0\}$, we have $\mathcal{C}_1^{vb}(X, D) \simeq \mathcal{C}_1^{sat}(X, D)$ naturally. In particular, Φ gives the equivalence $\widetilde{\mathcal{C}}_1(X, D) \simeq \mathcal{C}_1^{sat}(X, D)$.*

Proof Let $(\boldsymbol{E}_*, \nabla)$ be a saturated regular filtered flat sheaf on (X, D). We put $(E, \nabla) := (\boldsymbol{E}_*, \nabla)_{|X-D}$, and let $\mathcal{L}$ denote the corresponding local system on $X - D$. Let $H(\mathcal{L})$ denote the space of the multi-valued flat sections of $\mathcal{L}$.

Recall that there exists a subset $W \subset D$ with $\operatorname{codim}_X(W) \geq 3$ such that $(\boldsymbol{E}_*, \nabla)_{|X-W}$ is regular filtered flat bundle on $(X - W, D - W)$ (Lemma 2.12). Let P be any point of $D - W$, and let $(U_P, z_1, \ldots, z_n)$ be a holomorphic coordinate neighbourhood such that $z_1^{-1}(0) = U_P \cap D$ and $U_P \cap W = \emptyset$. Due to Lemma 6.2, we have the unique filtration $\mathcal{F}$ of $H(\mathcal{L}_{|U_P \setminus D}) \simeq H(\mathcal{L})$ corresponding to $(\boldsymbol{E}_*, \nabla)_{|U_P}$. Due to the uniqueness, it is independent of a choice of P and U_P.

Let $\boldsymbol{u} = (u_1, \ldots, u_r)$ be a frame of $H(\mathcal{L})$ compatible with the filtration $\mathcal{F}$ and the generalized eigen decomposition with respect to the monodromy around D. For any real number $b \in \boldsymbol{R}$, we construct $\boldsymbol{v}(b) = \big(v_1(b), \ldots, v_r(b)\big)$ as above. Then, for any $P \in D - W$, $\boldsymbol{v}(b)$ gives a holomorphic frame of ${}_bE_{|U_P}$ compatible with the filtration due to Lemma 6.3. Hence, each $v_i(b)$ gives a section of ${}_bE_{|X-W}$. Due to the saturatedness of $(\boldsymbol{E}_*, \nabla)$, $v_i(b)$ gives a section of ${}_bE$ on X. Now it is easy to see that $\boldsymbol{v}(b)$ gives a frame of ${}_bE$, and in particular, ${}_bE$ is locally free. Hence, $(\boldsymbol{E}_*, \nabla)$ is a regular filtered flat bundle on (X, D). ∎

Now, it is easy to see that Φ is equivalent for general (X, D). Let us see the fully faithfulness of Φ. The faithfulness is obvious. Let $f : \Phi(\mathcal{L}_{1\,*}) \longrightarrow \Phi(\mathcal{L}_{2\,*})$ be a morphism in $\mathcal{C}_1^{sat}(X, D)$. We have the map $g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ corresponding to f. We would like to check that g preserves the filtrations ${}^i\mathcal{F}$. Let P be any point of D_i°, and $(U, z_1, \ldots, z_n)$ be any coordinate neighbourhood such that $U \cap D = z_1^{-1}(0)$. Applying Lemma 6.3, we obtain that g preserves the filtration ${}^i\mathcal{F}$ on $U \setminus D_i$. Thus, we obtain the fully faithfulness.

Let us show the essential surjectivity. Let $(\boldsymbol{E}_*, \nabla)$ be a saturated filtered flat sheaf on (X, D). Let $\mathcal{L}$ denote the local system corresponding to $(\boldsymbol{E}_*, \nabla)_{|X-D}$. We have only to construct the appropriate filtrations ${}^i\mathcal{F}$ of $\mathcal{L}_{|U_i \setminus D}$ on appropriate neighbourhoods of D_i. Let P be any point of D_i°, and $(U_P, z_1, \ldots, z_n)$ denote any coordinate neighbourhood around P such that $z_1^{-1}(0) = U_P \cap D$. Due to Lemma 6.2, we obtain the unique filtration ${}^i\mathcal{F}$ of $\mathcal{L}_{|U_P \setminus D}$. We obtain the filtration ${}^i\mathcal{F}$ on $\bigcup_{P \in D_i^\circ} U_P$ by gluing them, due to the uniqueness. Thus, we obtain that Φ is essentially surjective, and hence equivalent.

6.2.3 The parabolic first Chern class

We have the $\mathbb{Z}$-action on $\boldsymbol{R} \times \boldsymbol{C}$ given by $n \cdot (a, \alpha) = (a + n, \alpha - n)$. It induces the action of $\mathbb{Z}$ on $\mathcal{KMS}(\boldsymbol{E}_*, i)$. The following lemma is clear from the construction of Φ.

Lemma 6.4 *We have the bijective correspondence of the sets $\mathcal{KMS}(\Phi(\mathcal{L}_*),i)/\mathbb{Z}$ and $\mathcal{KMS}(\mathcal{L}_*,i)$, which is given by $(a,\alpha)\longmapsto(b,\omega)=\Big(a+\operatorname{Re}\alpha,\exp\big(-2\pi\sqrt{-1}\alpha\big)\Big)$ for $(a,\alpha)\in\mathcal{KMS}(\Phi(\mathcal{L}_*),i)$. Moreover,* $\operatorname{rank}^i\operatorname{Gr}^{F,\mathbb{E}}_{(a,\alpha)}=\operatorname{rank}^i\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(b,\omega)}$. ∎

Corollary 6.5 *We have the equality of the parabolic first Chern class* $\operatorname{par-c}_1(\mathcal{L}_*)=\operatorname{par-c}_1(\Phi(\mathcal{L}_*))$*. In particular, when X is a smooth projective variety with an ample line bundle L, the μ_L-stability of $\mathcal{L}_*$ and μ_L-stability of $\Phi(\mathcal{L}_*)$ are equivalent.*

Proof Recall Lemma 3.23. It is shown for the case where $(\boldsymbol{E}_*,\nabla)$ is graded semisimple and $\dim X$ is two dimensional. However, the graded semisimplicity condition is not necessary as is explained in Remark 3.21. The assumption $\dim X=2$ is also not necessary, due to the Lefschetz theorem. Then, the claim of the corollary follows from Lemma 3.21 and the correspondence of the KMS-spectrums given in Lemma 6.4. ∎

6.2.4 The second parabolic Chern character

Lemma 6.6 *Let $X=\Delta^n=\{(z_1,\ldots,z_n)\,|\,|z_i|<1\}$, and $D=D_1\cup D_2$, where $D_i=\{z_i=0\}$. Let $(\boldsymbol{E}_*,\nabla)$ be a saturated regular filtered flat sheaf on (X,D).*

- *$(\boldsymbol{E}_*,\nabla)$ is a regular filtered flat bundle on (X,D).*
- *Let $\boldsymbol{c}$ be any element of $\boldsymbol{R}^2$, and let ${}_{\boldsymbol{c}}E$ denote the $\boldsymbol{c}$-truncation. Let $\mathcal{L}_*$ be the corresponding filtered local system on (X,D). Then, we have the equality:*

$$\operatorname{rank}^{\underline{2}}\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(\boldsymbol{b},\boldsymbol{\omega})}(\mathcal{L})=\operatorname{rank}^{\underline{2}}\operatorname{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}({}_{\boldsymbol{c}}E).$$

  Here the meaning of the notation is as follows:

  - *$\boldsymbol{b}=(b_1,b_2)$ and $\boldsymbol{\omega}=(\omega_1,\omega_2)$ denote elements of $\boldsymbol{R}^2$ and ${\boldsymbol{C}^*}^2$ respectively.*
  - *$\boldsymbol{a}=(a_1,a_2)$ and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ denote elements of $\boldsymbol{R}^2$ and $\boldsymbol{C}^2$ respectively, determined by the conditions $c_i-1<a_i\leq c_i$, $\exp(-2\pi\sqrt{-1}\alpha_i)=\omega_i$ and $a_i+\operatorname{Re}\alpha_i=b_i$.*

Proof Let $\mathcal{L}_*=(\mathcal{L},{}^1\mathcal{F},{}^2\mathcal{F})$ be as above. Let $\boldsymbol{u}$ be a frame of $H(\mathcal{L})$ compatible with the filtrations ${}^k\mathcal{F}$ $(k=1,2)$ and the generalized eigen decompositions of $H(\mathcal{L})$. For each u_j and the divisor D_k, the complex number $\alpha_k(u_j)$ and $a_k(u_j)$ are determined as before. For the monodromies around D_k, we obtain the nilpotent endomorphism N_k as before. The holomorphic section v_j is given by $v_j:=\exp\Big(\sum\log z_k\big(\alpha_k(u_j)+N_k\big)\Big)$. Let $n_k(u_j)$ be the numbers determined by the condition $c_k-1<n_k(u_j)+a_k(u_j)-\operatorname{Re}\alpha_k(u_j)\leq c_k$. We put $\widetilde{v}_j:=\prod z_k^{-n_k(u_j)}\cdot v_j$. Then, $\widetilde{\boldsymbol{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_r)$ gives the frame of ${}_{\boldsymbol{c}}E_{|X-(D_1\cap D_2)}$. Due to the saturatedness, $\widetilde{\boldsymbol{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_r)$ gives the frame of ${}_{\boldsymbol{c}}E$, and hence ${}_{\boldsymbol{c}}E$ are locally free. Thus, the first claim is proved. The frame $\widetilde{\boldsymbol{v}}$ is compatible with ${}^i\mathbb{E}$ and ${}^iF$, and we have ${}^k\deg^F(\widetilde{v}_j)=a_k(u_j)-\operatorname{Re}\alpha_k(u_j)+n_k(u_j)$ and $\widetilde{v}_{j\,|\,D_k}\in{}^k\mathbb{E}(\alpha_k(u_j)-n_k(u_j))$. Thus, the second claim follows. ∎

Corollary 6.7 *Let X be a projective manifold with an ample line bundle L, and let D be a simple normal crossing divisor. Let $(\boldsymbol{E}_*,\nabla)$ be a saturated regular filtered flat sheaf on (X,D), and let $\mathcal{L}_*$ denotes the corresponding filtered local system. Then, we have the equality of the parabolic second Chern character numbers* $\int_X\operatorname{par-ch}_{2,L}(\mathcal{L}_*)=\int_X\operatorname{par-ch}_{2,L}(\boldsymbol{E}_*)$. ∎

Corollary 6.8 *Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor. Let $\mathcal{L}_*$ be a μ_L-stable filtered local system on (X,D). Then, the Bogomolov-Gieseker inequality for $\mathcal{L}_*$ holds:*

$$\int_X\operatorname{par-ch}_{2,L}(\mathcal{L}_*)\leq\frac{\int_X\operatorname{par-c}^2_{1,L}(\mathcal{L}_*)}{2\operatorname{rank}\mathcal{L}}.$$

Proof Recall that saturated regular filtered flat shaves are regular filtered flat bundles in codimension two (Lemma 2.12). Hence, the claim follows from Corollary 6.5, Corollary 6.7 and Corollary 3.20. ∎

Corollary 6.9 *Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor. Let $\mathcal{C}_1^{poly}$ be the category of μ_L-polystable regular filtered flat bundle on (X,D) with trivial characteristic numbers, and let $\widetilde{\mathcal{C}}_1^{poly}$ be the category of μ_L-polystable filtered local system on (X,D) with trivial characteristic numbers. Then, the functor Φ naturally gives the equivalence of them.*

Proof We have only to remark that saturated μ_L-stable regular filtered flat sheaves with trivial characteristic numbers are regular filtered bundles (Theorem 5.16). ∎

Remark 6.10 *Due to the result in* [30] *and the existence of a pluri-harmonic metric for $\Phi(\mathcal{L}_*)$, the filtrations ${}^i\mathcal{F}$ for μ_L-stable filtered local systems $\mathcal{L}_*$ satisfy some compatibility around the intersection points of D.* ∎

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