Non-trivial stably free modules over crossed products

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Abstract

We consider the class of crossed products of noetherian domains with universal enveloping algebras of Lie algebras. For algebras from this class we give a sufficient condition for the existence of projective non-free modules. This class includes Weyl algebras and universal envelopings of Lie algebras, for which this question, known as noncommutative Serre's problem, was extensively studied before. It turns out that the method of lifting of non-trivial stably free modules from simple Ore extensions can be applied to crossed products after an appropriate choice of filtration. The motivating examples of crossed products are provided by the class of RIT algebras, originating in non-equilibrium physics.

Keywords: Serre's problem, stably free modules, Ore extension, crossed product with universal enveloping of Lie algebra, RIT (relativistic internal time) algebras, faithfully flat modules, type of the element, strongly completely prime subalgebra, graded domain, ordered-like semigroup

1 Introduction

In [14] J.-P.Serre posed the question on whether any finitely generated projective module over the ring of commutative polynomials $\mathbf{k}[x_1, \ldots, x_n]$ over a field $\mathbf{k}$ is

free. It was stated there in geometrical language: whether any locally trivial vector bundle over an affine space $\mathbb{A}^n_{\mathbf{k}}$ is a trivial bundle. After almost twenty years of attempts A. Suslin [17] and D. Quillen [13] independently (and using different methods) obtained an affirmative answer to the Serre question (see also [9] for a detailed study of the techniques involved).

Later on, this question was investigated for various classes of non-commutative rings. A report on some of this work also can be found in Lam's book [9], ch.VII.8. To describe briefly what has been done let us remind some definitions.

A finitely generated left A-module M is called *stably free* if $M \oplus A^n = A^m$ for some nonnegative integers n and m, clearly, it is projective then. A module which is stably free but not free will be called *non-trivial stably free*. We will need the *rank* of the stably free module M which is defined as $\mathrm{rk}M = m - n$. This definition obviously only makes sense if A is an IBN-ring, for example, we may consider noetherian rings. We will suppose throughout the paper that all rings are IBN.

Situation in non-commutative case turned out to be more involved: there were constructed counterexamples, i.e. stably free non-free modules in several classes of non-commutative algebras. By saying that there exists a counterexample in a certain class of algebras, we mean that any algebra from this class allows a finitely generated projective but non-free module.

For example, stably free non-free ideals were constructed by Webber [18] in any Weyl algebra A_n. Another counterexample was constructed by Ojanguren and Sridharan [12] in rings of polynomials on two variables over a division ring (which is not a field). In group algebras, non-free projective modules were constructed by Dunwoody and Berridge [7] for torsion free groups and by Artamonov [6] for solvable groups. Examples of this type in enveloping algebras of non-Abelian finite dimensional Lie algebras were provided by Artamonov and by Stafford, in [15] a unified way for producing non-trivial stably free right ideals was given, which virtually covers above cases.

In this note we consider the class of crossed products of noetherian domains with a universal enveloping algebra of a Lie algebra, which subsumes most of classes mentioned above, and provide a sufficient condition for the existence of stably free non-free modules in this wider class. More precisely, we show in theorem 6.2 that stably free non-free modules can be lifted from any subalgebra of the crossed product $A \star U\mathcal{G}$ which is a simple differential Ore extension $A[g, \delta]$, $g \in \mathcal{G}$, $\delta \in DerA$ ($\delta = \delta_{\bar{g}}$ is a derivation, involved in the given crossed product, associated to the element $g \in \mathcal{G}$).

As an element of our tool we prove the following (theorem 5.7): if A is a domain, endowed with a filtration by a well-ordered semigroup, such that A_0 is a faithfully flat A-module, then any non-trivial stably free ideal in A_0 could be lifted to a nontrivial stably free ideal in A. Graded version of this fact (theorem 5.3) we

prove for a graded domain A, graded by *order-like semigroup*. This is a wider class of semigroups, which however captures most essential properties of well-ordered semigroups.

Let us emphasize that all examples of non-trivial stably free modules mentioned above, and just about all known examples in the noncommutative case, are modules of rank one. Over commutative rings, there are examples of higher rank and these are typical. An example of module of minimal rank (over commutative rings) is a module of rank two over the ring $\mathbb{R}[x,y,z]/x^2+y^2+z^2=1$. The way to ensure that the right unimodular row (x,y,z) gives rise to the non-trivial stably free module has a geometrical flavour (using the theorem of the "hedgehog brushing" on a 2-sphere) and does not explain much in the line of techniques we study here.

For the class of Weyl algebras $A_n(k)$ it was proved by Stafford [16] that all stably free modules of rank two and bigger are free. In the proof, of course, simplicity of $A_n(k)$ plays a crucial role. This result was generalized to some crossed products of simple rings with supersolvable groups by Jaikin-Zapirain in [8]. Another cases where positive result holds one can find in [5].

Examples of crossed products with the universal enveloping algebra we consider are provided by the class of RIT (relativistic internal time) algebras, which we have been studying in [1], [2]. This class originated in non-equilibrium physics [3], [4] and we consider it here in the general setting of crossed products.

2 Choice of subalgebras

We start with the definition of a crossed product with a universal enveloping algebra.

Definition. Let A be a (noncommutative) $\mathbf{k}$-algebra, and $\mathcal{G}$ a Lie algebra with a basis $\{g_i | i \in I\}$ over $\mathbf{k}$. Then a $\mathbf{k}$-algebra B containing A is called a *crossed product* provided there is an embedding $\mathcal{G} \hookrightarrow B : g \mapsto \bar{g}$ of linear spaces, which satisfies:

(1) $\bar{g}r - r\bar{g} \in A$ for any $g \in G, r \in A$
and $\bar{g}r - r\bar{g} = \delta_g(r)$ is a derivation on A.

(2) $\bar{g}\bar{h} - \bar{h}\bar{g} = \overline{[g,h]}$ mod A for all $g, h \in \mathcal{G}$

(3) B is a free (right) A-module with the commutative monomials $\bar{g}_1^{j_1} ... \bar{g}_m^{j_m}$ on $\{g_i | i \in I\}$ as a basis

We denote the crossed product algebra B by $A \star U\mathcal{G}$.

It is known from [15] (see also [10]) that non-trivial stably free ideals do exist in simple differential Ore extensions of noetherian domains, which satisfy some additional condition.

In this section we suggest how to choose appropriate subalgebras in a crossed product $A \star U\mathcal{G}$ in such a way that their non-trivial stably free modules can be lifted to non-trivial stably free modules over the whole crossed product. Namely, we take subalgebras isomorphic to a simple Ore extension of the initial algebra A, thus the idea is to use subalgebras in the "intersection" of the crossed product components.

We prove several properties of these subalgebras in order to prepare the tool which allows us to lift non-trivial stably free modules from these subalgebras to the whole crossed product.

Directly from the definitions, it can be seen that $A \star U\mathcal{G}$, where $\mathcal{G} = \{g\}$ is a one-dimensional Lie algebra, is isomorphic to the simple differential Ore extension $A[x, \delta]$, where $\delta = \delta_g$ is the derivation related to g, which was defined above as: $\delta_g(r) = g\bar{r} - \bar{r}g$, for $r \in A$.

Using the defining relations (1) and (2) in the crossed product one can easily see that $A \star U\mathcal{G}_1$, where $\mathcal{G}_1 = \{g\}$ is a Lie algebra generated by any single element $g \in \mathcal{G}$, is a subalgebra in $A \star U\mathcal{G}$. Indeed, the free basis of the A-module $A \star U\mathcal{G}_1$ according to (3) consists of elements $\bar{g}^i$, $i = 0 \leq i < \infty$. Any product of two elements of the shape $a\bar{g}^i$, $a \in A$, $g \in \mathcal{G}$ is a linear combination of elements of the same shape after applying (1).

We denote by A_1 the subalgebra $A \star U\mathcal{G}_1$ in $B = A \star U\mathcal{G}$ and will consider B as a left A_1-module writing ${}_{A_1}B$.

3 Faithful flatness of ${}_{A_1}B$

Using the above notations we will prove two crucial properties of subalgebras of our choice which allow to lift nontrivial stably free modules from $A_1 = A \star U\mathcal{G}_1 = A[g, \delta]$ to the whole crossed product $A \star U\mathcal{G}$.

Starting from here we suppose that A is Noetherian domain.

The first property we need is the faithfully flatness of B as a left A_1 module. We will prove that in our situation even a stronger condition holds, namely

Lemma 3.1. *The left A_1-module ${}_{A_1}B$ is free.*

Proof. By definition of a crossed product, $B = A \star U\mathcal{G}$ and $A_1 = A \star U\mathcal{G}_1$ are free left A-modules with bases $V = \{g_1^{i_1} \dots g_n^{i_n}\}$ and $W = \{g_1^i\}$ respectively.

We prove that ${}_{A_1}B$ is generated by the set $\Omega = g_2^{j_2} \dots g_n^{j_n}$ and this set forms a free basis of this A_1-module.

The first part of the statement, saying that Ω is generating system is obvious. To show that this is a free basis it is enough to check that if $\Sigma a_i b_i = 0$ in B, for $a_i \in A_1, b_i \in \Omega$, then all $a_i = 0$, since A_1 has no zero divisors. The equivalence follows from our condition that A is a domain. To ensure this let us first write

elements $a_i \in A_1$ from the sum $\Sigma a_i b_i$ above as follows: $a_i = a_i(g_1) = \Sigma \alpha_k^{(i)} g_1^k$, with $\alpha_k^{(i)} \in A$.

Now fulfill the multiplication in the above sum and gather terms near each $v_i \in V$. We get $\Sigma \beta_j v_j = 0$, where $\beta_j = \Sigma_{i,k:\, v_j = g_1^k v_i} \alpha_k^{(i)}$. From this it follows that $\beta_i = 0$, since V is a free generating system for ${}_A B$. Since V is a free basis for A-module B, given the fixed numbers i and j, there is only one k such that $v_j = g_1^k v_i$. Hence the set $\{\beta_j\}$ just coincides with the set $\{\alpha_k^{(i)}\}$. So, together with all $\beta_i = 0$ we have all $\alpha_k^{(i)} = 0$, and hence $a_i = 0$ for all i. □

4 Strongly completely prime subalgebras

Before we start the discussion of the second main lemma we should introduce the notion of *strongly completely prime subalgebra*, or *s.c.p.-subalgebra* for short.

Let us consider the following two properties of subalgebra.

Definition 1. We say that a *subalgebra* A_1 is *completely prime* in A if for any two non-zero elements a and b from A, $ab \in A_1$ implies $a \in A_1$ or $b \in A_1$.

Definition 2. We say that a *subalgebra* A_1 is *strongly completely prime (s.c.p.)* in A if for any two non-zero elements a and b from A, $ab \in A_1$ implies $a \in A_1$ and $b \in A_1$.

In case A_1 is an ideal in A, the first definition just coincide with the definition of a completely prime ideal, e.i. an ideal such that the quotient is a domain. (this explains the origin of our term).

The second definition degenerate in case A_1 is an ideal. Indeed, suppose $A_1 \triangleleft A$, $A_1 \neq \{0\}$ and $A \backslash A_1 \neq \emptyset$. Then take an element $b \in A \backslash A_1$, since A_1 is a (right) ideal, for arbitrary nonzero element $a \in A_1$ we have $ab \in A_1$ and the property from the definition 2 doesn't hold. If $A_1 = \emptyset$ then formally property of being *s.c.p.* always holds in a domain.

So, property of being *s.c.p.* is clearly a feature of subalgebras and should be considered only in this case (rather then for ideals).

Let us discuss now the notion of type of an element in the crossed product.

Firstly, we associate with any product (monomial) $w = a g_{i_1} \dots g_{i_n}$, $i_k \in I$, $a \in A$ in the crossed product $B = A * \mathcal{U}(\mathcal{G})$ its type on variables $g_{i_1} \dots g_{i_n}$. By definition *the type'* $t(w)$ of the element w is a tuple of nonnegative integers $(j_1, \dots, j_r)$, where j_k is the number of variables g_k in the monomial w for any $k \in I$. (In case $j_l = 0$ for all $l > r$, $j_l = 0$, we just omit zero terms in the sequence $j_1, \dots, j_r, \dots$ starting from j_{r+1} to get $t(w)$). One can also consider the type of a product on any subset of variables $\{g_{i_k},\ i_k \in I' \subset I\}$.

Let us fix an order on monomials $w = a g_{i_1} \dots g_{i_n} \in B$ using the degree lexicographical ordering on commutative words $t(w)$. Namely, we say that $w > w'$

for $w = ag_{i_1} \dots g_{i_n}$ and $w' = a'g_{i'_1} \dots g_{i'_n}$ if $t(w) \underset{dl}{>} t(w')$. The latter means that if $t(w) = (j_1, \dots, j_r)$ and $t(w') = (j'_1, \dots, j'_s)$, then either $r > s$ or $r = s$ and $j_t > j'_t$ for some t, such that $j_l = j'_l$ for all $l \leq t$.

We can define a *normal form* (with respect to g_i, $i \in I$) of an element in $B = A * \mathcal{U}(\mathcal{G})$. We say that an element $f = \sum a_{\bar{i}} g_{i_1} \dots g_{i_n}$ is *in the normal form* if $i_1 \leq i_2 \leq \dots \leq i_n$, that is, all monomials have the form $g_1^{j_1} \dots g_r^{j_r}$ with coefficients from A. It is clear from the relations in the definition of crossed product that any element from B can be presented in a normal form, since these relations allow to commute $r \in A$ with $g \in \mathcal{G}$ and elements from $\mathcal{G}$ between each other. In both cases we might get new terms, which have a lower degree in g_i, $i \in I$. Since there is no infinite chain of words in g_i of strictly increasing degree, in certain step we will get an element equal to w in a normal form. This element we will call a *normal form of* $w \in B = A * \mathcal{U}(\mathcal{G})$ and denote it by $\mathcal{N}(w)$. Property (3) in the definition of the crossed product (PBW - property) ensures that the normal form of the element in B is unique.

This allows us to introduce the notion of the type of an element $b \in B$.

Definition. By the *type* of an arbitrary element $b \in B$, we call the type of the highest monomial in the normal form of b.

Having in hands the notion of the type of an element in $B = A \star U\mathcal{G}$ we actually have a natural filtration on B. Namely $B = \underset{\bar{i} \in \Sigma_n}{\cup} B_{\bar{i}}$, where Σ_n is a semigroup of tuples $(i_1, ..., i_n)$ with the componentwise operation and $B_{\bar{i}}$ is a linear span over $\mathbf{k}$ of elements of the type $(i_1, ..., i_n)$, in particular, $B_0 = A$.

The existence of such a filtration force us to develop a general machinery for the graded and filtered case and then apply it to the situation of crossed products, using however a filtration different from the above.

5 Semigroup graded and filtered case

For this section we break our agreement that A is a domain, in some statements here we will ask only for A being a graded domain (that is, there are no zero divisors among homogeneous elements with respect to a given grading).

The main theorem in the graded setting will have the form.

Theorem 5.1. *Let $A = \bigoplus_{j \in \mathbb{Z}_+} A_j$ be a $\mathbb{Z}_+$-graded domain, where A is a flat A_0-module. Then any stably free non-free module over A_0 can be lifted to A.*

This theorem can be further generalized in a sense that one can consider gradings more general than $\mathbb{Z}_+$-gradings. We shall prove a theorem in that bigger generality, so Theorem 5.1 will follow from Theorem 5.3.

Definition 5.2. *We call a semigroup $(G,+)$ ordered-like if it has no invertible elements except 0 and for any two finite subsets S_1, S_2 of G such that $S = S_1+S_2 \neq \{0\}$, there exists $c \in S$ with*

$$\nu(c) = |\{(a,b) : a \in S_1,\ b \in S_2,\ a+b=c\}| = 1.$$

Most common example of a semigroup with such a property is a well-ordered semigroup, where there exists a linear order compatible with an operation: $a < b \Longrightarrow a+c < b+c$. In this case as an element $c \in S$ with unique presentation as a sum of elements from S_1 and S_2 ($|\nu(c)| = 1$) will serve a sum of maximal elements of S_1 and S_2. But there are other examples where this property doesn't come from well-ordering.

Theorem 5.3. *Let $A = \bigoplus_{\sigma\in G} A_\sigma$ be a domain graded by an ordered-like semigroup G, where A is faithfully flat as a left A_0-module. Then any stably free non-free right ideal in A_0 can be lifted to A.*

The proof is based on the following lemmas.

Lemma 5.4. *Let $A = \bigoplus_{\sigma\in G} A_\sigma$ be a graded domain, with G being ordered-like semigroup. Then A_0 is a completely prime subalgebra of A.*

Proof. To prove that A_0 is completely prime it is enough to ensure that for any two elements $a, b \in A$, from $a, b \notin A_0$ it follows that $ab \notin A_0$. For any element a of A let us denote by $S(a)$ the subset $S(a) = \{\sigma \in G : a_\sigma \neq 0\}$ of the semigroup G, where $a = \sum\limits_{\sigma\in G} a_\sigma$, $a_\sigma \in A_\sigma$ is the graded decomposition of a. Clearly $a = \sum\limits_{\sigma\in S(a)} a_\sigma$. Since $a, b \notin A_0$, the sets $S(a)$ and $S(b)$ contain non-zero elements. Since G has no non-zero invertible elements, we have $S = S(a) + S(b) \neq \{0\}$. Taking into account that G is ordered-like, we can find $\gamma \in S \setminus \{0\}$ such that $\nu(\gamma) = |\{(\sigma,\tau) : \sigma \in S(a),\ \tau \in S(b),\ \sigma+\tau = \gamma\}| = 1$. For the γ-th graded component of ab we will have $(ab)_\gamma = a_\sigma b_\tau$. Since $a_\sigma \neq 0$, $b_\tau \neq 0$ and A is a graded domain, we get $(ab)_\gamma \neq 0$ for $\gamma \neq 0$. So, $ab \notin A_0$. □

The following fact is true for grading by any semigroup, not necessarily with the ordered-like property.

Lemma 5.5. *Let $A = \bigoplus_{\sigma\in G} A_\sigma$ be a domain graded by an arbitrary abelian semigroup G. Then the property of A_0 to be completely prime implies the property of A_0 to be strongly completely prime.*

Proof. To ensure this we should show that if $a, b \in A \setminus \{0\}$ and $ab \in A_0$ implies $a \in A_0$, then we also have $b \in A_0$.

Indeed, let $b = \sum\limits_{g\in S(b)} b_g$ be the graded decomposition of b. Then $ab = \sum\limits_{g\in S(b)} ab_g$. Here $(ab)_g = ab_g \in A_0A_g \subseteq A_g$. On the other hand, $ab \in A_0$ and therefore

$(ab)_g = ab_g = 0$ for any $g \neq 0$. Since $a \neq 0$ and A is a graded domain, this implies that $b_g = 0$ for any $g \neq 0$. That is, $b \in A_0$. □

As a corollary of Lemmas 5.4 and 5.5 we have that the subalgebra A_0 of A is strongly completely prime. Using this we can proceed with the proof of the theorem5.3 by analogy with [15].

Proof. (of the Theorem 5.3)

Let K be a nontrivial stably free right ideal in A_0. We will show that the induced ideal $KA = K \otimes_{A_0} A$ is also stably free but not free.

Since A is flat as A_0-module, $KA = K \otimes_{A_0} A$ and is also projective as A-module, hence stably free. The essential part is to prove that it is not free. We have $KA \oplus A = A \oplus A$, thus we have to show that KA is not cyclic.

Suppose this is not the case, i.e. $KA = yA$ for some $0 \neq y \in A$. (In case $y = 0$ we will have a contradiction immediately: $A = yA \oplus A = A \oplus A$ and this contradicts with the condition we suppose to holds throughout the paper that all rings have the IBN property).

Since K sitting inside KA (A has a unit and all modules are unital) $K \subset KA = yA$, we can take a nonzero element $p \in K$, which is $p = yb$ for some nonzero $b \in A$. But K is an ideal in A_0 and we can use primeness of A_0: if the element $p \in A_0$ and $p = yb$ for nonzero $y, b \in A$ then it should imply $y \in A_0$.

Now, for any right ideals $I \subsetneqq J \lhd_r A_0$, due to faithfully flatness of A we have $IK \subsetneqq JK \lhd_r A$.

Suppose that the following inclusion of right ideals in A_0 holds: $K \subsetneqq KA \cap A_0$. Then applying the above observation we get $KA \subsetneqq (KA \cap A_0)A$. But in fact $(KA \cap A_0)A = KA$: $K \subset KA \cap A_0$ implies $KA \subset (KA \cap A_0)A$, while $KAA \cup A_0A \subset KA \cap A = KA$, hence $K = KA \cap A_0$. This contradiction shows that $K = KA \cap A_0$.

Now we have $K = KA \cap A_0 = yA \cap A_0 \subset yA_0$, due to $yb \in yA$ belongs to A_0 we again use the fact that A_0 is a prime subalgebra, this implies $b \in A_0$, in case $b \neq 0$ (obviously in case $b = 0$ we also have $b \in A_0$). Hence $K = yA \cap A_0 = yA_0$, this contradiction with cyclicity completes the proof. □

Remark. Let us mention here that for the question on existence of non-trivial stably free modules it is enough to look only at ideals. In other words the existence of non-trivial stably free (right) module is equivalent to the existence of non-trivial stably free (right) ideal. This follows from the simple observation that if we have a f.g. projective module, which is non-free, then we also have a projective non-free ideal. Indeed, let P be a f.g. projective R-module and $R^n = P \oplus Q$. For any $x \in R^n$ we have a unique decomposition $x = x_P + x_Q$. Consider a submodule of R^n of the form $(0) \times ... \times (0) \times R \times (0) \times ... \times (0) = R_j \subset R^n$, and define with respect to the above decomposition submodules $P_j = \{x_P \,|\, x \in R_j\} \subset P$ and $Q_j = \{x_Q \,|\, x \in Q_j\} \subset Q$. Clearly, we have an isomorphism $R_j = P_j \oplus Q_j$. On the other hand from the definition of R_j it is clear that $R_j = R$. Moreover we have $P = P_1 \oplus ... \oplus P_n$ and if P is not free, then one of P_j is not free. But P_j is a

submodule of R, i.e. an ideal in R. So we get a projective ideal which is non-free. This remark shows why it is enough to consider in the theorem only behavior of ideals under the extension of the base ring.

Now we will formulate filtered versions which will be used for the results about crossed products. Here we restrict ourselves by an arbitrary well-ordered semigroup.

Lemma 5.6. *Let $A = \cup_{\sigma\in\Sigma} U_\sigma$ be a domain, endowed with a filtration by a well-ordered semigroup Σ. Then U_0 is a s.c.p. – subalgebra of A.*

Theorem 5.7. *Let $A = \cup_{\sigma\in\Sigma} U_\sigma$ be a domain, endowed with a filtration by a well-ordered semigroup Σ, and A is faithfully flat as a left U_0-module. Then any stably free non-free right ideal in U_0 can be lifted to A.*

Proofs are analogous to those of lemmas 5.4, 5.5 and Theorem 5.3.

6 Back to crossed products

Now we can prove the second lemma we need in the cross product case.

Lemma 6.1. *A subalgebra $A_1 = A \star U\mathcal{G}_1$ in $B = A \star U\mathcal{G}$, where $\mathcal{G}_1 \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$ generated by one (nonzero) element, is a s.c.p. - subalgebra.*

Proof. Most essential point in this proof is an appropriate choice of filtration on B. After that we apply lemma 5.6. Instead of using a natural filtration on B mentioned at the end of section 4, we suggest the following one. Let

$$B = \bigcup_{\bar{i}\in\Sigma_{n-1}} B_{\bar{i}},$$

where

$$B_{\bar{i}} = A[g_1]U_{\bar{i}},$$

for

$$U_{\bar{i}} = \mathrm{Sp}\langle g_2^{i_2}...g_n^{i_n} \mid \bar{i} = (i_2, ..., i_n) \in \Sigma_{n-1}\rangle_{\mathbf{k}},$$

in particular, $B_0 = A[g_1]$ is a polynomial algebra over A on one variable g_1, where $\mathcal{G}_1$ generated by g_1 (as we set in section 2).

Note that it is a filtration by well-ordered semigroup Σ_{n-1} and an order on it is degree-lexicographical (the same as we used for ordering of types in section 4, but this time with respect to $n-1$ variables $g_2, ..., g_n$). It is an easy exercise then to check that this is indeed a filtration.

□

Using tools provided by the lemmas 3.1 and 6.1 we can lift nontrivial stably free modules from subalgebras of the type $A_1 = A \star U\mathcal{G}_1 = A[g, \delta]$ in a crossed product.

Theorem 6.2. *Let $B = A \star U\mathcal{G}$. Let K be a nontrivial stably free right ideal in $A_1 = A \star U\mathcal{G}_1 = A[g, \delta]$, for some $g \in \mathcal{G}$. Then the induced right ideal $K \otimes_{A_1} B$ in B is stably free, but not free.*

Proof. We use the same filtration as in the previous lemma and apply theorem 5.7 together with lemmas 3.1 and 6.1 for it. □

The lifting technique could be applied whenever we have a *s.c.p.*–subalgebra D in B, such that ${}_DB$ is a faithfully flat module. Lemmas 3.1 and 6.1 ensure that it is always the case for the crossed product algebra $B = A \star U\mathcal{G}$, if we choose as a subalgebra D a simple Ore extension $A[g, \delta]$ of A.

Now we are in a position to state the result which gives a sufficient condition of existence of nontrivial stably free modules over crossed products.

Theorem 6.3. *Let A be a noetherian domain, $U\mathcal{G}$ - the universal enveloping of Lie algebra $\mathcal{G}$, and $B = A \star U\mathcal{G}$ a crossed product. If there exists an element $g \in \mathcal{G}$ such that $(r, g + q)$ is a unimodular row in a subalgebra $A[g, \delta]$ of B, for some $r, s \in A$, r a non-unit, then the ideal $rB \cap (g + q)B$ is a non-trivial stably free B-module.*

This result shows that nontrivial stably free modules can be lifted from the Ore extensions of the basic ring A, appeared inside the construction of the crossed product with the universal enveloping. Obviously these modules not always exist over $A \star U\mathcal{G}$.

This we can see already from the example of a simple Ore extension $A[g, \delta]$, which is also a simplest case of a crossed product. Let take A to be a commutative local ring with the maximal ideal μ, it is known [15] that a nontrivial Ore extension of A allow stably free non-free ideals if and only if at least one of the following conditions fails: 1). $\mathrm{Kdim}A = 1$ or 2). $\delta(\mu) \subseteq \mu$. Thus situation in the wider class of crossed products is not so definitive as in group algebras of solvable groups or in Weyl algebras where nontrivial stably free non-free modules always exist, so we only can give conditions when they do.

Let us mention also the following immediate corollary of the mentioned above fact and theorem 6.2.

Corollary 6.4. *A crossed product of a local commutative ring A of* $\mathrm{Kdim}A > 1$ *with $U\mathcal{G}$ for an arbitrary Lie algebra G always allow stably free non-free module. If* $\mathrm{Kdim}A \leq 1$ *then nontrivial stably free module does exist if $\mathcal{G}$ acts in such a way that for some $g \in \mathcal{G}$, $g(\mathcal{M}) \not\subset \mathcal{M}$, where $\mathcal{M}$ is a maximal ideal in A.*

7 Remark on modules of higher ranks

Here we remind some known results, just to emphasize that in the class of crossed products there are obvious examples of non-trivial stably free modules of higher ranks. They can be obtained by a slight modification of arguments for the case of 2-sphere (see [10], 11.2.3).

Namely, let us take a (commutative) ring $A = \mathbb{R}[x_1, ..., x_n]/\sum_{i=1}^{n} x_i^2 - 1$, for $n \geq 3$. Due to the nature of these relations the column $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, with entries a_i — images of variables x_i under the natural morphism $\varphi : \mathbb{R}[x_1, ..., x_n] \to A$, is unimodular, that is $Aa_1 + ... + Aa_n = A$.

Hence it defines a split monomorphism

$$\alpha : A \hookrightarrow A^n : a \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot a$$

with cokernel P, so $P \oplus A = A^n$. Suppose that P is a free A-module. This is equivalent to the fact that the column $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is extendable to an invertible matrix. That is, there exists $M \in Gl_n(\mathbb{R}), M = (\bar{r}, \bar{c}_1, ..., \bar{c}_{n-1})$, where $\bar{r}, \bar{c}_1, ..., \bar{c}_{n-1}$ denote columns of the matrix and $\bar{r} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. We can construct a continuous tangent vector field on a sphere $\mathbb{S}^{n-1}$, it is provided by the minors $v_r = M_{i2}(r)$ of matrix M, corresponding to the second column. Indeed, the scalar product $(r, v_r) = \det(\bar{r}, \bar{r}, \bar{c}_2, ..., \bar{c}_{n-1}) = 0$. On the other hand this vector field can't vanish, since there exist a vector $\bar{c}_1$, such that $(c_1, v_r) = \det(r_1, c_1, c_2, ..., c_{n-1}) = \mathrm{detM} \neq 0$.

The existence of a continuous tangent vector field on a real $n-1$ sphere which vanishes nowhere does contradict, for even n, with the well known theorem on the "brushing of a hedgehog" (or "hairy ball theorem", see for example [11]).

8 Remark on non-gradable modules

Let us mention that for the class of RIT algebras, which form a special case of crossed products, we can state that nontrivial stably free modules, we construct here, are also examples of non-gradable modules.

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