Ramanujan-type formulae for $1/\pi$: A second wind?*

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Abstract

In 1914 S. Ramanujan recorded a list of 17 series for $1/\pi$. We survey the methods of proofs of Ramanujan's formulae and indicate recently discovered generalizations, some of which are not yet proven.

The twentieth century was full of mathematical discoveries. Here we expose two significant contributions from that time, in reverse chronological order. At first glance, the stories might be thought of a different nature. But we will try to convince the reader that they have much in common.

1 Ramanujan and Apéry: $1/\pi$ and $\zeta(3)$

In 1978 R. Apéry showed the irrationality of $\zeta(3)$ (see [5] and [21]). His rational approximations to the number in question (known nowadays as *Apéry's constant*) have the form $v_n/u_n \in \mathbb{Q}$ for $n = 0, 1, 2, \ldots$, where the denominators $\{u_n\} = \{u_n\}_{n=0,1,\ldots}$ and numerators $\{v_n\} = \{v_n\}_{n=0,1,\ldots}$ satisfy the same polynomial recurrence

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2 + 17n + 5)u_n + n^3 u_{n-1} = 0 \tag{1.1}$$

with the initial data

$$u_0 = 1, \quad u_1 = 5, \qquad v_0 = 0, \quad v_1 = 6.$$

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Then
$$\lim_{n\to\infty}\frac{v_n}{u_n}=\zeta(3)$$
and, surprisingly, the denominators $\{u_n\}$ are integers:
$$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\in\mathbb{Z},\qquad n=0,1,2,\dots,\tag{1.2}$$
while the numerators $\{v_n\}$ are 'close' to being integers.

In 1914 S. Ramanujan [22], [7] recorded a list of 17 series for $1/\pi$, from which we indicate the simplest one
$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3}{n!^3}(4n+1)\cdot(-1)^n=\frac{2}{\pi}\tag{1.3}$$
and also two quite impressive examples
$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}(21460n+1123)\cdot\frac{(-1)^n}{882^{2n+1}}=\frac{4}{\pi},\tag{1.4}$$
$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}(26390n+1103)\cdot\frac{1}{99^{4n+2}}=\frac{1}{2\pi\sqrt{2}}\tag{1.5}$$
which produce rapidly converging (rational) approximations to π. Here
$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}a(a+1)\cdots(a+n-1) & \text{for } n\ge 1,\\ 1 & \text{for } n=0,\end{cases}$$
denotes the Pochhammer symbol (the rising factorial). The Pochhammer products occurring in all formulae of this type may be written in terms of binomial coefficients:
$$\frac{(\frac12)_n^3}{n!^3}=2^{-6n}\binom{2n}{n}^3,\qquad \frac{(\frac13)_n(\frac12)_n(\frac23)_n}{n!^3}=2^{-2n}3^{-3n}\binom{2n}{n}\frac{(3n)!}{n!^3},$$
$$\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}=2^{-8n}\frac{(4n)!}{n!^4},\qquad \frac{(\frac16)_n(\frac12)_n(\frac56)_n}{n!^3}=12^{-3n}\frac{(6n)!}{n!^3(3n)!}.$$
Ramanujan's original list was subsequently extended to several other series which we plan to touch on later in the paper. For the moment we give two more celebrated examples:
$$\sum_{n=0}^{\infty}\frac{(\frac13)_n(\frac12)_n(\frac23)_n}{n!^3}(14151n+827)\cdot\frac{(-1)^n}{500^{2n+1}}=\frac{3\sqrt{3}}{\pi},\tag{1.6}$$
$$\sum_{n=0}^{\infty}\frac{(\frac16)_n(\frac12)_n(\frac56)_n}{n!^3}(545140134n+13591409)\cdot\frac{(-1)^n}{53360^{3n+2}}=\frac{3}{2\pi\sqrt{10005}}.\tag{1.7}$$

Formula (1.6) is proven by H. H. Chan, W.-C. Liaw and V. Tan [13] and (1.7) is the Chudnovskys' famous formula [15] which enabled them to hold the record for the calculation of π in 1989–94. On the left-hand side of each formula (1.3)–(1.7) we have linear combinations of a (generalized) hypergeometric series

$$ {}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!} \tag{1.8} $$

and its derivative at a point close to the origin. The rapid convergence of the series in (1.4)–(1.7) may be used for proving the quantitative irrationality of the numbers $\pi\sqrt{d}$ with $d \in \mathbb{N}$ (see [26] for details).

In both Ramanujan's and Apéry's cases, there were just hints on how the things might be proven. Rigorous proofs appeared somewhat later. We will not discuss proofs of Apéry's theorem and its further generalizations (see [17] for a review of the subject), just concentrating on the things around the remarkable Ramanujan-type series. But we will see that both Ramanujan's and Apéry's discoveries have several common grounds.

2 Elliptic proof of Ramanujan's formulae

Although Ramanujan did not indicate how he arrived at his series, he hinted that these series belong to what is now known as 'the theories of elliptic functions to alternative bases'. The first rigorous mathematical proofs of Ramanujan's series and their generalizations were given by the Borweins [11] and Chudnovskys [15]. Let us sketch, following [15], the basic ideas of those very first proofs.

One starts with an elliptic curve $y^2 = 4x^3 - g_2x - g_3$ over $\overline{\mathbb{Q}}$ with fundamental periods ω_1, ω_2 (where $\operatorname{Im}(\omega_2/\omega_1) > 0$) and corresponding quasi-periods η_1, η_2. Besides the Legendre relation

$$ \eta_1\omega_2 - \eta_2\omega_1 = 2\pi i, $$

the following *linear* relations between $\omega_1, \omega_2, \eta_1, \eta_2$ over $\overline{\mathbb{Q}}$ are available in the complex multiplication case, i.e., when $\tau = \omega_2/\omega_1 \in \mathbb{Q}[\sqrt{-d}]$ for some $d \in \mathbb{N}$:

$$ \omega_2 - \tau\omega_1 = 0, \qquad A\tau\eta_2 - C\eta_1 + (2A\tau + B)\alpha\omega_1 = 0, \tag{2.1} $$

where the integers A, B and C come from the equation $A\tau^2 + B\tau + C = 0$ defining the quadratic number τ and $\alpha \in \mathbb{Q}(\tau, g_2, g_3) \subset \overline{\mathbb{Q}}$. Equations (2.1) allow one to express ω_2, η_2 by means of ω_1, η_1 only. Substituting these expressions into (2.1), and using the hypergeometric formulae for ω_1, η_1 and also for $\omega_1^2, \omega_1\eta_1$ (which follow from Clausen's identity) one finally arrives at a formula of Ramanujan type. An important (and complicated) problem in the proof is computing the algebraic number

$$\alpha = \frac{8\pi^2}{81\omega_1^2}\left(E_2(\tau) - \frac{3}{\pi \operatorname{Im}\tau}\right), \qquad \text{where} \quad E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} e^{2\pi i n\tau}\sum_{d|n} d.$$

Note that α viewed as a function of τ is a non-holomorphic modular form of weight 2. Although the Chudnovskys attribute the knowledge of the fact that $\alpha(\tau)$ takes values in the Hilbert class field $\mathbb{Q}(\tau, j(\tau))$ of $\mathbb{Q}(\tau)$ to Kronecker (in Weil's presentation [24]), we would refer the reader to the work [8] by B. C. Berndt and H. H. Chan.

3 Modular proof of Ramanujan's formulae

An understanding of the complication of the above proof came in 2002 with T. Sato's discovery of the formula

$$\sum_{n=0}^{\infty} u_n \cdot (20n + 10 - 3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{20\sqrt{3} + 9\sqrt{15}}{6\pi} \tag{3.1}$$

of Ramanujan type, involving Apéry's numbers (1.2). The modular argument was essentially simplified by H. H. Chan with his collaborators and later by Y. Yang to produce a lot of new identities like (3.1) based on a not necessarily hypergeometric series $F(z) = \sum_{n=0}^{\infty} u_n z^n$. Examples are

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k} \cdot (5n+1)\frac{1}{64^n} = \frac{8}{\pi\sqrt{3}} \tag{3.2}$$

due to H. H. Chan, S. H. Chan and Z.-G. Liu [12];

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}(-1)^{n-k}3^{n-3k}\frac{(3k)!}{k!^3}\binom{n}{3k}\binom{n+k}{k} \cdot (4n+1)\frac{1}{81^n} = \frac{3\sqrt{3}}{2\pi} \tag{3.3}$$

due to H. H. Chan and H. Verrill (2005);

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^4 \cdot (4n+1)\frac{1}{36^n} = \frac{18}{\pi\sqrt{15}} \tag{3.4}$$

due to Y. Yang (2005).

It should be mentioned that Picard–Fuchs differential equations (of order 3) satisfied by the series $F(z)$ always have very nice arithmetic properties [25]. Therefore, it is not surprising that $F(z)$ admits a modular parametrization: $f(\tau) = F(z(\tau))$ is a modular form of weight 2 for a modular (uniformizing) substitution $z = z(\tau)$.

Let us follow Yang's argument to show the basic ideas of the new proof in the example of (3.1). Our choice is

$$z(\tau)=\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \qquad f(\tau)=\frac{\eta(2\tau)^7\eta(3\tau)^7}{\eta(\tau)^5\eta(6\tau)^5},$$

which are modular forms of level 6; the expressions were obtained by F. Beukers in his proof of Apéry's theorem using modular forms [9]. Here

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})$$

is the Dedekind eta-function. The function $g(\tau)=(2\pi i)^{-1}f'(\tau)/f(\tau)$ satisfies the functional equation

$$g(\gamma\tau)=\frac{c(c\tau+d)}{\pi i}+(c\tau+d)^2g(\tau) \qquad \text{for} \quad \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(6)+w_6,$$

where w_6 denotes the Atkin–Lehner involution. Taking

$$\gamma=\frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -1\\ 6 & 0\end{pmatrix}, \qquad \tau=\tau_0=\frac{i}{\sqrt{30}}$$

we obtain

$$g(\tau_0)+5g(5\tau_0)=\frac{\sqrt{30}}{\pi}. \tag{3.5}$$

On the other hand, $h(\tau)=g(\tau)-5g(5\tau)$ is a modular form of weight 2 and level 30 (on $\Gamma_0(30)+\langle w_5,w_6\rangle$). This implies that $h(\tau)/f(\tau)$ is an algebraic function of $z(\tau)$ and after explicit evaluations at $\tau=\tau_0$ we arrive at

$$g(\tau_0)-5g(5\tau_0)=h(\tau_0)=\frac{900\sqrt{2}-402\sqrt{10}}{5}f(\tau_0)=(900\sqrt{2}-402\sqrt{10})f(5\tau_0). \tag{3.6}$$

Combining (3.5) and (3.6) we deduce that

$$\frac{\sqrt{30}}{\pi}=(900\sqrt{2}-402\sqrt{10})f(5\tau_0)+10g(5\tau_0),$$

and it only remains to use the expansion

$$g(5\tau_0)=z\frac{\mathrm{d}f/\mathrm{d}z}{f}\cdot\frac{1}{2\pi i}\frac{z'(\tau)}{z(\tau)}\bigg|_{\tau=5\tau_0}=(108\sqrt{2}-48\sqrt{10})\sum_{n=0}^{\infty}nu_n\cdot z(5\tau_0)^n$$

(since $z'(\tau)/(2\pi i)$ and $f(\tau)$ are modular forms of weight 2 on $\Gamma_0(6)+w_6$, the function

$$\frac{1}{f}\cdot\frac{1}{2\pi i}\frac{z'}{z}$$

is an algebraic function of z) and the evaluation

$$z(5\tau_0) = z(\tau_0) = 161 - 72\sqrt{5} = \left(\frac{\sqrt{5}-1}{2}\right)^{12}.$$

As pointed out to us by H. H. Chan, the main difficulty one meets in the above proof is to prove the algebraicity evaluations rigorously (cf. [8]).

4 Creative telescoping

There is yet another method of proof, but applicable only to a small number of Ramanujan-type series. It is based on the algorithm of creative telescoping, due to Gosper–Zeilberger. Note that an essential part of the first proof of Apéry's theorem [21], namely, the proof of the recurrence (1.1), was given by D. Zagier also using a telescoping argument. D. Zeilberger (and his automatic collaborator S. B. Ekhad) could prove the simplest Ramanujan's identity (1.3) in the following way [16]. One verifies the (terminating) identity

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^2(-k)_n}{n!^2(3/2+k)_n}(4n+1)(-1)^n = \frac{\Gamma(3/2+k)}{\Gamma(3/2)\Gamma(1+k)} \tag{4.1}$$

for all *non-negative* integers k. To do this, divide both sides of (4.1) by the right-hand side and denote the summand on the left by $F(n,k)$:

$$F(n,k) = (4n+1)(-1)^n \frac{(1/2)_n^2(-k)_n}{n!^2(3/2+k)_n} \frac{\Gamma(3/2)\Gamma(1+k)}{\Gamma(3/2+k)};$$

then take

$$G(n,k) = \frac{(2n+1)^2}{(2n+2k+3)(4n+1)} F(n,k)$$

with the motive that $F(n,k+1) - F(n,k) = G(n,k) - G(n-1,k)$, hence $\sum_n F(n,k)$ is a constant, which is seen to be 1 by plugging in $k=0$. Finally, to deduce (1.3) one takes $k = -1/2$, which is legitimate in view of Carlson's theorem [6, Section 5.3].

5 Guillera's series for $1/\pi^2$

If one wishes to use the latter method of proof for other Ramanujan-type formulae, ingenuity is required in order to put the new parameter k in the right place. This was done only recently by J. Guillera [18, 20], who used the method to prove some other identities of Ramanujan (in those cases when z

has only 2 and 3 in its prime decomposition). If the reader doubts the applicability of the method, then take into account that the purely hypergeometric origin of the method and its independence from the elliptic and modular stuff allowed Guillera [18, 19, 20] to prove new generalizations of Ramanujan-type series, namely,

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}}=\frac{8}{\pi^2}, \tag{5.1}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^5}{n!^5}(820n^2+180n+13)\frac{(-1)^n}{2^{10n}}=\frac{128}{\pi^2}, \tag{5.2}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3(\frac14)_n(\frac34)_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}}=\frac{32}{\pi^2}, \tag{5.3}$$

and also to find experimentally [19] four additional formulae

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n(\frac16)_n(\frac56)_n}{n!^5}(1640n^2+278n+15)\frac{(-1)^n}{2^{10n}}=\frac{256\sqrt3}{3\pi^2}, \tag{5.4}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n(\frac13)_n(\frac23)_n}{n!^5}(252n^2+63n+5)\frac{(-1)^n}{48^n}=\frac{48}{\pi^2}, \tag{5.5}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac13)_n(\frac23)_n(\frac16)_n(\frac56)_n}{n!^5}(5418n^2+693n+29)\frac{(-1)^n}{80^{3n}}=\frac{128\sqrt5}{\pi^2}, \tag{5.6}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5}(1920n^2+304n+15)\frac{1}{7^{4n}}=\frac{56\sqrt7}{\pi^2}. \tag{5.7}$$

As Guillera notices, the series in (5.5)–(5.7) are closely related to the series

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(28n+3)\frac{(-1)^n}{48^n}=\frac{16}{\pi\sqrt3},$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac16)_n(\frac56)_n}{n!^3}(5418n+263)\frac{(-1)^n}{80^{3n}}=\frac{640\sqrt{15}}{3\pi},$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(40n+3)\frac{1}{7^{4n}}=\frac{49}{3\pi\sqrt3},$$

respectively, proven by the methods in Sections 2 and 3. However, there is no obvious way to deduce any of formulae (5.1)–(5.7) by modular means; the problem lies in the fact that the (Zariski closure of the) projective monodromy group for the corresponding series $F(z)=\sum_{n=0}^{\infty}u_nz^n$ is always $O_5(\mathbb{R})$ (this is an immediate consequence of a general result of F. Beukers and G. Heckman [10]), which is essentially 'richer' than $O_3(\mathbb{R})$ for classical Ramanujan's series.

There exists also the higher-dimensional identity

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^7}{n!^7}(168n^3+76n^2+14n+1)\frac{1}{2^{6n}}=\frac{32}{\pi^3},$$

discovered by B. Gourevich in 2002 (using an integer relations algorithm). Guillera also found experimentally an analogue of Sato's series:

$$\begin{gathered}\sum_{n=0}^{\infty}v_n\cdot(36n^2+12n+1)\frac{1}{2^{10n}}=\frac{32}{\pi^2},\\ \text{where}\quad v_n=\binom{2n}{n}^2\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2.\end{gathered}\tag{5.8}$$

6 Transformations of hypergeometric series

As we have seen, Ramanujan's original formulae as well as Guillera's formulae (5.1)–(5.7) involve classical hypergeometric series (1.8), while series like (3.1)–(3.4) and (5.8) are based on double hypergeometric series. A natural way to pass from one formula to another is by algebraic transformations of the hypergeometric series involved. For instance, formula (3.1) may be deduced from the transformation

$$\sum_{n=0}^{\infty}u_nz^n=\frac{1}{2+2z-\sqrt{1-34z+z^2}}\cdot{}_3F_2\left(\left.\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix}\,\right|\,256t(z)\right)$$

where

$$\begin{aligned}t(z)=\frac{z}{2(1+14z+z^2)^4}\big(&1-36z+199z^2+184z^3+199z^4-36z^5+z^6\\ &+(1+z)(1-z)^2(1-18z+z^2)\sqrt{1-34z+z^2}\,\big),\end{aligned}$$

given by Y. Yang (2005), together with the Ramanujan-type formula for the ${}_3F_2$-series on the right-hand side specialized at the point

$$\left(\frac{5+4\sqrt2}{7\sqrt3}\right)^4=256t\bigg(\Big(\frac{\sqrt5-1}{2}\Big)^{12}\bigg).$$

A similar argument is used by M. D. Rogers in [23] to deduce some further identities for $1/\pi$ of Ramanujan–Sato type.

Using the quadratic transformation $z \mapsto -4z/(1-z)^2$ of the hypergeometric series, we were able to produce from (5.1), (5.2) two more series of the latter type [27]:

$$\sum_{n=0}^{\infty} w_n \frac{(4n)!}{n!^2(2n)!}(18n^2-10n-3)\frac{1}{(2^8 5^2)^n} = \frac{10\sqrt{5}}{\pi^2},$$

$$\sum_{n=0}^{\infty} w_n \frac{(4n)!}{n!^2(2n)!}(1046529n^2+227104n+16032)\frac{1}{(5^4 41^2)^n} = \frac{5^4 41\sqrt{41}}{\pi^2},$$

where the sequence of integers

$$w_n = \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k} 2^{4(n-k)}, \qquad n = 0, 1, 2, \dots,$$

satisfies the recurrence relation

$$(n+1)^3 w_{n+1} - 8(2n+1)(8n^2+8n+5)w_n + 4096n^3 w_{n-1} = 0, \qquad n = 1, 2, \dots.$$

In [28] we show that a huge family of formulae for $1/\pi^2$ (as well as for $1/\pi^3$, $1/\pi^4$, etc) can be derived by taking powers of Ramanujan-type formulae for $1/\pi$. For instance, the square of the Chudnovskys' formula (1.7) takes the monstrous form

$$\sum_{n=0}^{\infty} w_n \frac{(3n)!}{n!^3}(222883324273153467n^2 + 16670750677895547n + 415634396862086)\frac{(-1)^n}{640320^{3n+3}} = \frac{1}{64\pi^2}.$$

7 Further observations and open problems

It is worth mentioning that identities like (4.1) are valid for all non-negative *real* values of k. This fact has several other curious implications; for instance, the series

$$G(k) = \sum_{n=0}^{\infty} \frac{(1/2+k)_n^5}{(1+k)_n^5}\big(820(n+k)^2+180(n+k)+13\big)\frac{(-1)^n}{2^{10n}} \tag{7.1}$$

has a closed-form evaluation at $k=0$ and $k=1/2$:

$$G(0) = \frac{128}{\pi^2} \qquad \text{and} \qquad G(\tfrac{1}{2}) = 256\zeta(3),$$

where the first formula follows from (5.2) while the second one was given by T. Amdeberhan and D. Zeilberger [4]. Guillera has conjectured (and proven) evaluations for series like (7.1) viewed as functions of the continuous (complex or real) parameter k.

It seems to be a challenge to develop a modular-like theory for proving Guillera's identities and finding a (more or less) general pattern of them. For the moment, we have only speculations in this respect on a relationship to mirror symmetry, namely, to the linear differential equations for the periods of certain Calabi–Yau threefolds. A standard example here is the hypergeometric series (cf. (5.1) and (5.2))

$$F(z) = {}_5F_4\left(\left.\begin{matrix}\frac12, \frac12, \frac12, \frac12, \frac12\\ 1, 1, 1, 1\end{matrix}\,\right|\, 2^{10}z\right) = \sum_{n=0}^{\infty} \binom{2n}{n}^5 z^n,$$

which satisfies the 5th-order linear differential equation

$$\bigl(\theta^5 - 32z(2\theta+1)^5\bigr)Y = 0, \qquad \text{where} \quad \theta = z\frac{\mathrm{d}}{\mathrm{d}z}.$$

If $G(z)$ is another solution of the latter equation of the form $F(z)\log z + F_1(z)$ with $F_1(z) \in z\mathbb{Q}[[z]]$, then

$$\widetilde{F}(z) = (1-2^{10}z)^{-1/2} \det\begin{pmatrix} F & G \\ \theta F & \theta G\end{pmatrix}^{1/2}$$

(the sharp normalization factor $(1-2^{10}z)^{-1/2}$ is due to Y. Yang) satisfies the 4th-order equation

$$\bigl(\theta^4 - 16z(128\theta^4 + 256\theta^3 + 304\theta^2 + 176\theta + 39) + 2^{20}z^2(\theta+1)^4\bigr)Y = 0 \tag{7.2}$$

(entry #204 in [3, Table A]). For a quadratic transformation of the new function $\widetilde{F}(z)$ we have the following explicit formula [2]:

$$\frac{1+z}{(1-z)^2}\widetilde{F}\left(\frac{-z}{(1-z)^2}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 4^{n-k}\binom{2k}{k}^2\binom{2n-2k}{n-k}\right)^2 z^n, \tag{7.3}$$

where the right-hand side is the Hadamard square of the series

$$\begin{aligned}\frac{1}{1-16z}\,{}_2F_1\left(\left.\begin{matrix}\frac12, \frac12\\ 1\end{matrix}\,\right|\, \frac{-16z}{1-16z}\right) &= \sum_{n=0}^{\infty}\binom{2n}{n}^2 \frac{(-1)^n z^n}{(1-16z)^{n+1}} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 4^{n-k}\binom{2k}{k}^2\binom{2n-2k}{n-k}\right) z^n\end{aligned}$$

which admits a modular uniformization (cf. Section 3). It is worth mentioning that (7.2) and the differential equation of order 4 for the right-hand side of (7.3) are of Calabi–Yau type [1], [14], i.e., they imitate all properties of a differential equation for the periods of a Calabi–Yau threefold. Are there analogues of Hilbert class fields for this and similar situations? Can formulae (5.1)–(5.8) be deduced from formulae for $1/\pi$ by means of algebraic transformations of hypergeometric series? There is still some work to do in the subject originated by Ramanujan's note [22] almost 100 years ago.

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