

Graphs of relations and Hilbert series

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Abstract We are discussing certain combinatorial and counting problems related to quadratic algebras. First we give examples which confirm the Anick conjecture on the minimal Hilbert series for algebras given by n generators and $\frac{n(n-1)}{2}$ relations for $n \leq 7$. Then we investigate combinatorial structure of colored graph associated to relations of RIT algebra. Precise descriptions of graphs (maps) corresponding to algebras with maximal Hilbert series are given in certain cases. As a consequence it turns out, for example, that RIT algebra may have a maximal Hilbert series only if components of the graph associated to each color are pairwise 2-isomorphic.

Keywords: Quadratic algebras; Hilbert series; Gröbner basis; Colored graph

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1 Introduction

Let $\mathcal{A}(n, r)$ be the class of all graded quadratic algebras on n generators and r relations:

$$A = k\langle x_1, \dots, x_n \rangle / \text{id}\{p_i : i = 1, \dots, r\},$$

where $p_i = \sum_{k,j=1}^n \alpha_i^{k,j} x_k x_j$, $\alpha_i^{k,j} \in k$.

We deal with an arbitrary field k of char 0. Only on the way (section 2.1) we restrict ourself with \mathbb{C} for a while (to get more general statement), but it will not influence further results.

These algebras are endowed with a natural filtration $A = \bigcup_{m=0}^{\infty} U_m$, where U_m is the linear span of monomials on a_i of degree not exceeding m , a_i are the images of the variables x_i under the canonical map from $k\langle x_1, \dots, x_n \rangle$ to A and the degree of $a_{i_1} \dots a_{i_d}$ equals d . Since the generating polynomials are homogeneous, the algebra $A \in \mathcal{A}$ also possesses a canonical grading $A = \bigoplus_{i=0}^{\infty} A_i$, where A_i is the linear span of monomials of degree exactly i . This grading has a finiteness property: $\dim_k A_i < \infty$ for any i , since algebra is finitely generated. This allow to associate to the series of dimensions various generating functions. The one which reflects most straightforward properties of the algebra will be considered here.

Definition 1.1 *The Hilbert series of a graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is the generating function of the series of dimensions of graded components $d_i = \dim_k A_i$ of the following shape: $H_A(t) = \sum_{i=0}^{\infty} d_i t^i$.*

We are going to confirm Anick's conjecture [1] saying that a lower bound for the Hilbert series of an algebra with $\frac{n(n-1)}{2}$ quadratic relations given by the series $\left| \left(1 - nt + \frac{n(n-1)}{2} t^2 \right)^{-1} \right|$ is attained, for the small number of variables $n \leq 7$. Here the sign of modulus stands for the series where n th coefficient equals to the n th coefficient of initial series if this is positive together with all previous coefficients and is zero otherwise.

After notices on minimal and generic series for quadratic algebras we turn to the main subject of our investigation. We consider subclass $\mathcal{R}(m, n) \subset \mathcal{A}(g, \frac{g(g-1)}{2})$, where $g = n + m$, called RIT algebras (it was introduced and studied in papers [4, 3, 2]). Class $\mathcal{R}(m, n)$ defined as consisting of algebras with presentation of the form

$$R = k\langle x_1, \dots, x_m, y_1, \dots, y_n \rangle / F,$$

where

$$F = id \begin{cases} [x_i, x_j] = 0, \\ [y_i, y_j] = 0, \\ [x_i, y_j] = y_{f(i,j)} y_j, \end{cases} \quad (1)$$

and f is a map $f : M \times N \rightarrow N$, $M = \{1, \dots, m\}$, $N = \{1, \dots, n\}$.

To any such algebra we associate m -colored graph with n vertices in such a way that subgraph Γ_i of color i reflects the map $\sigma_i : N \rightarrow N$ defined by $\sigma_i(n) = f(i, n)$.

We formulate conditions (Theorem 3.1) on the above maps which mean that defining relations of algebra form a Gröbner basis, or equivalently that the algebra has a lexicographically maximal Hilbert series. Then we attack more subtle question on how to describe precisely the combinatorial structure of those maps. This is done explicitly in the Theorem 3.8 for a pair of maps. As a consequence interesting necessary condition is obtained for an algebra to have a maximal Hilbert series. Corollary 3.9 says that Hilbert series of algebra could be maximal only if graphs of all maps $\sigma_i, i = 1, \bar{m}$ are 2-isomorphic. We

call graphs *2-isomorphic* if they become isomorphic after gluing pairs of vertices in common cycles of length two.

Described combinatorial conditions (Theorem 3.8) also imply that algebra is Koszul and obeys a commutator generalization of Yang-Baxter equation. Another consequences for RIT algebras from being presented by a quadratic Gröbner basis are that in this case they are Auslander regular and Cohen-Macaulay. Hence we get combinatorial conditions on graphs sufficient also for obeying these properties.

In the section 4 we present the complete list of Hilbert series and corresponding non isomorphic colored graphs for RIT algebras of rank up to 4.

2 The Anick conjecture for $n \leq 7$

2.1 Series in general position

We remind here the proof of minimality of general series because it is essential for the next section. Version we present deals with the notion of general position in the Lebesgue sense, so we suppose for this section that the field is $k = \mathbb{C}$. Essentially the knowledge on this matter is due to Anick [1] and explanations in simplified form one can find in the survey of Ufnarovskii [17]. In [15] one can also find a remark on the minimality of n th component of the series in general position in the Zarisskii sense. But as it is pointed out in [17] (remark before the theorem 3 in I.4.2), since the infinite union of proper affine varieties may not be contained in a proper affine variety, one can not state that the minimal series (in case it is infinite) is in general position in the Zarisskii sense. There could be several ways to avoid this problem, for example in spirit of theorem 3, I.4.5. in [17]. But over \mathbb{C} most natural and easy way is to use the topology defined by the Lebesgue measure, so we present this version here.

Let we define now more precisely what is meant by an algebra in general position in the Zarisskii and in the Lebesgue sense. Algebras from $\mathcal{A}(n, r)$ are naturally labeled by the points of k^{rn^2} , corresponding to the coefficients $\alpha_i^{k,j}$ of the relations. Given a property P of quadratic algebras, we say that P is satisfied for $A \in \mathcal{A}(n, r)$ in general position in the Zarisskii sense if the set of the coefficient vectors corresponding to those $A \in \mathcal{A}(n, r)$, which obey the property P , is a non-empty Zarisskii-open subset of k^{rn^2} . We also say that P is satisfied for $A \in \mathcal{A}(n, r)$ in general position in the Lebesgue sense if the set of the coefficient vectors corresponding to those $A \in \mathcal{A}(n, r)$, which do not obey the property P has rn^2 -dimensional Lebesgue measure zero. Since the set of zeros of any non-zero polynomial has the Lebesgue measure zero, we see that as far as arbitrary property P satisfied for $A \in \mathcal{A}(n, r)$ in general position in the Zarisskii sense it is also satisfied for $A \in \mathcal{A}(n, r)$ in general position in the Lebesgue sense. Define the minimal series in the class $\mathcal{A}(n, r)$ componentwise:

$$H_{\min}^{n,r}(t) = \sum_{i=0}^{\infty} b_i t^i,$$

where $b_i = \min_{A \in \mathcal{A}(n,r)} \dim A_i$, A_i being the i th homogeneous component in the grading of A . It is not clear *a priori*, whether there exists an algebra $A \in \mathcal{A}(n, r)$ whose Hilbert series coincides with $H_{\min}^{n,r}$. This follows however from the statement below.

Proposition 2.1 *For $A \in \mathcal{A}(n, r)$ in general position in the Lebesgue sense, the equality $H_A = H_{\min}^{n,r}$ is satisfied.*

Proof. Denote the ideal generated by $\{p_i : 1 \leq i \leq r\}$ by I and its d th homogeneous component by I_d . Obviously

$$I_d = \text{span}_k \{up_iv : u, v \in \langle x_1, \dots, x_n \rangle, \deg u + \deg v = d - 2\}.$$

Here $\langle x_1, \dots, x_n \rangle$ stands for the free semigroup generated by $\{x_1, \dots, x_n\}$. Let w_1, \dots, w_m be all monomials of degree d in the free algebra $k\langle x_1, \dots, x_n \rangle$. Since it is a linear basis in the d th homogeneous component of $k\langle x_1, \dots, x_n \rangle$, we can uniquely express the above polynomials up_iv as a linear combinations of w_l :

$$up_iv = \sum_{l=1}^m \lambda_{u,v,i}^l w_l.$$

The dimension of I_d is exactly the rank of the rectangular matrix $\Lambda = \{\lambda_{u,v,i}^l\}$, whose rows of length m are labeled by the triples (u, v, i) , where $i = 1, r$ and u, v are monomials in x_1, \dots, x_n satisfying $\deg u + \deg v = d - 2$.

Obviously $\lambda_{u,v,i}^l$ are linear functions of the coefficients $\alpha_i^{k,l}$ of the polynomials p_i . Condition that the dimension of A_d is minimal is equivalent to the condition that $\dim I_d = \text{rk } \Lambda$ is maximal. Denote the maximal rank of Λ by D . Thus, the dimension of A_d is minimal if and only if there is a non-zero minor of the matrix Λ of the size D . The family of the minors of Λ of the size D is a finite family of polynomials \mathcal{P}_l on the coefficients $\alpha_i^{k,l}$ and some of these polynomials are non-zero. This means that the set of $A \in \mathcal{A}(n, r)$ with minimal $\dim A_d$ corresponds to the complement of the union of the sets of zeros of finitely many non-zero polynomials. Any such set is a non-empty Zarisskii open set and its complement has zero Lebesgue measure. The set of algebras $A \in \mathcal{A}(n, r)$ satisfying $H_A = H_{\min}^{n,r}$ is then a countable intersection of non-empty Zarisskii open sets and therefore its complement has zero Lebesgue measure as a countable union of sets with the Lebesgue measure zero. This completes the proof of the proposition. ■

Remark Let us mention that in case when the minimal series $H_{\min}^{n,r}$ is finite, the countable union from the proof of the Proposition 2.1 is in fact finite and the equality $H_A = H_{\min}^{n,r}$ is satisfied for $A \in \mathcal{A}(n, r)$ in general position in the Zarisskii sense as well and over an arbitrary field.

2.2 The Anick conjecture holds for $n \leq 7$

Now we are back to arbitrary basic field k of char 0. We consider the question whether the minimal series is finite for the case $r = n(n-1)/2$. It was raised in the paper of D. Anick [1], where a lower bound for the Hilbert series for algebras from $\mathcal{A}(n, \frac{n(n-1)}{2})$ was discovered. It was established that

$$H_{\min}^{n, n(n-1)/2} \geq \left| \left(1 - nt + \frac{n(n-1)}{2} t^2 \right)^{-1} \right|,$$

where \geq is a componentwise inequality, i.e. it holds if each coefficient of the first series is greater or equal then the corresponding coefficient of the second one and $|f(t)|$ stands for the positive part of the series $f \in k[[t]]$. More precisely, if $f(t) = a_0 + a_1 t + a_2 t^2 \dots$, then $|f(t)| = b_0 + b_1 t + b_2 t^2 \dots$, where $b_m = a_m$ for $m \in \{i \mid a_j \geq 0 \forall j \leq i\}$ and $b_m = 0$ otherwise. There was a question raised whether this lower bound is attained.

Since we know from the theorem 2.1 that the algebras of $\mathcal{A}(n, r)$ in general position have minimal Hilbert series, to prove that this estimate attained it

would be enough to be able to write down generic coefficients of the relations and calculate the Hilbert series.

Example 1. The algebra A over the field $k = \mathbb{Z}_{17}$ given by the relations

$$A = k\langle a, b, c \rangle / \left\{ \begin{array}{l} ac + 2ba + 9b^2 + 3ca + 9cb + 8c^2, \\ 3ab + 5ac + 7ba + b^2 + 8bc + 4ca + cb + 2c^2, \\ 10a^2 + 2ab + 11ac + 2ba + 8b^2 + 4bc + 9ca + 7cb + 5c^2 \end{array} \right.$$

has the Hilbert series $H_A = 1 + 3t + 6t^2 + 9t^3 + 9t^4 = |(1 - 3t + 3t^2)^{-1}|$.¹

By this method we are able to confirm Anick's conjecture for small number of generators.

Proposition 2.2 *The lower bound for the Hilbert series of an algebra $A \in \mathcal{A}(n, \frac{n(n-1)}{2})$ over a field k of char 0 given by $\left| \left(1 - nt + \frac{n(n-1)}{2}t^2 \right)^{-1} \right|$ is achieved for $n \leq 7$.*

Proof. In Example 1 we have been calculating over the field k of characteristic $p = 17$. Since the series $|(1 - 3t + 3t^2)^{-1}|$ which is known to be the lower bound coincide with the result of our calculations, we actually have shown that for any term of the series, rank of matrix $\Lambda = \{\lambda_{u,v,i}^l\}$ formed as above in the proof of the proposition 2.1, with $\lambda_{u,v,i}^l \in \mathbb{Z}_p$ is maximal. We now can see that rank of the same matrix considered over k is also maximal. Indeed, the reduction from \mathbb{Z}_p to \mathbb{Z} we have because $\text{rk}M(\mathbb{Z}_p) \leq \text{rk}M(\mathbb{Z})$. Then since $\text{char}k = 0$, we have \mathbb{Z} embedded into k and the same matrix has maximal rank over k . So, if over the field \mathbb{Z}_p for some p , the rank is maximal, then it is maximal over k .

Similarly, we have got an examples of algebras with the Hilbert series $1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3 + n^2(n - \frac{(n^2-1)}{2})t^4$ for $n = 4$ and $1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3$ for $n=5,6,7$, which are coincide with the series $|(1 + nt + \frac{n(n-1)}{2}t^2)^{-1}|$ for these values of n . ■

3 RIT algebras and maps of the finite set

3.1 The class of RIT algebras

Here we consider a subclass \mathcal{R} of the above class of quadratic algebras $\mathcal{A}(n, \frac{n(n-1)}{2})$. The class \mathcal{R} of RIT (relativistic internal time) algebras consists of homogeneous finitely generated quadratic algebras given by relations of type (1).

It turned out that if relations form a Gröbner basis, the algebras from \mathcal{R} are so called "geometric rings", more precisely they are Auslander regular, Cohen-Macaulay, $\text{gldim} R = \text{GKdim} R = n$ (number of generators) for them. We have proved this in [9] using combinatorial techniques related to the notion of an I -type algebra introduced by J.Tate and M.Van den Bergh in [16]. These arguments appeared due to inspiring question of M.Van den Bergh on whether RIT algebras obey these properties. It becomes clear later on that Auslander regularity *et.al.* could also be proved without employing the I -type property, in more general context, using arguments involving associated graded structures with respect to appropriate grading (see [13], [12]).

The origin of the class of RIT algebras could be described in such a way. The Lie algebra RIT was introduced in [4] as a modification of the Poincare algebra $\mathcal{P}_4 = \mathcal{L}_4 + \mathcal{U}$. Here $\mathcal{L}_4 = O(3, 1)$ is a Lorenz algebra but the space

¹The computations were done using bunch of programs GRAAL (Graded Algebras) written in Uljanovsk by A.Kondratyev under the guide of A.Verevkin.

\mathcal{U} changed by addition of a new variable T (related to the relativistic internal time) to the set of initial variables. The corresponding commutation relations containing T were derived. Taking an enveloping algebra of \mathcal{P}_4 and considering the associated graded algebra we obtain the associative RIT algebra which gives rise to the class under consideration.

Let us mention that the simplest algebra from this class $R = R_{1,1} = k\langle x, y \rangle / (xy - yx - y^2)$ is one of the two Auslander regular algebras of global dimension 2, the second one is the usual quantum plane $k\langle x, y \rangle / (xy - qyx)$ (this follows from the Artin, Shelter classification [5]). We have been studying finite dimensional representations of it in [10].

3.2 Condition on maps and Gröbner basis

The presentation (1) of the algebra gives us a set of maps $\sigma_i : N \rightarrow N$ defined as

$$\sigma_i(j) = f(i, j) \quad \forall j \in N = \{1, \dots, n\}, \quad i \in M = \{1, \dots, m\}.$$

We are interested to relate the properties of algebras to the properties of these maps.

In particular, we will clarify combinatorial conditions on the associated colored graph (=set of maps) which mean that relations form a Gröbner basis. This gives at the same time a condition equivalent to the maximality of the Hilbert series. Here we mean a lexicographical order on the series.

To speak about the Gröbner basis we have to fix an ordering on the set of variables, let $x_i > y_j$ for any i, j and $x_i > x_j$ for $i > j$, $y_i > y_j$ for $i > j$. On the monomials of variables x_i, y_j the order is supposed to be degree-lexicographical.

Then we rewrite relations (1) in the form

$$F = \begin{cases} [x_i, x_j] = 0, \forall i > j \\ [y_i, y_j] = 0, \forall i > j \\ [x_i, y_j] = |y_{f(i,j)}y_j|, \forall i, j \end{cases} \quad (2)$$

Here $|y_{f(i,j)}y_j|$ stands for the normal form of this monomial, i.e.

$$|y_{f(i,j)}y_j| = \begin{cases} y_{f(i,j)}y_j & \text{if } f(i, j) \leq j \\ y_j y_{f(i,j)} & \text{if } f(i, j) > j \end{cases} \quad (3)$$

Theorem 3.1 *The relations of the type (2) form a reduced Gröbner basis if and only if the function $f(i, j)$ defines the set of actions σ_i with the following property. For any pair of maps σ_i, σ_k , $k > i$ one of the two conditions is satisfied in each point $j \in N$:*

- either (1). $\sigma_k(j) = \sigma_i(j)$ and $\sigma_i\sigma_k(j) = \sigma_k\sigma_i(j)$
or (2). $\sigma_k(j) = \sigma_k\sigma_i(j)$ and $\sigma_i(j) = \sigma_i\sigma_k(j)$.*

Let us mention that the second condition implies kind of strong version of braid type relations on σ_i : $\sigma_k = \sigma_k\sigma_i\sigma_k$ and $\sigma_i = \sigma_i\sigma_k\sigma_i$ for $k > i$.

Proof. The proof is a direct application of Gröbner bases technique due to Buchberger [7] and Bergman [6].

To find out that relations form a Gröbner basis we have to check that all ambiguities are solvable. Possible ambiguities in our case are of four types:

1. $x_i x_j x_k$, $i > j > k$,
2. $y_i y_j y_k$, $i > j > k$,
3. $x_i y_j y_l$, $\forall i, j > l$,
4. $x_l x_i y_j$, $\forall j, l > i$.

The first two types are trivially solvable. Ambiguities of the type three are also solvable:

$$\begin{array}{ccc}
y_j \underline{x_i y_l} + y_{(i,j)} y_j y_l & \longleftarrow \underline{x_i y_j y_l} & \longrightarrow \underline{x_i y_l y_j} \\
\downarrow & & \downarrow \\
y_j y_l x_i + y_j y_{(i,l)} y_l & & y_l \underline{x_i y_j} + y_{(i,l)} y_l y_j \\
& & \downarrow \\
& & y_l y_j x_i + y_l y_{(i,j)} y_j
\end{array}$$

Ambiguities of the type four are solvable if and only if the following two-element (non-ordered) sets are coincide:

$$\{f(i, j); f(l, f(i, j))\} = \{f(i, f(l, j)); f(l, j)\},$$

for any j and $l > i$.

Indeed:

$$\begin{array}{ccc}
& x_i \underline{x_l y_j} & \longleftarrow \underline{x_l x_i y_j} & \longrightarrow & \underline{x_l y_j x_i} + x_l |y_{(i,j)} y_j| \\
& \downarrow & & & \downarrow \\
& x_i y_j x_l + x_i |y_{(l,j)} y_j| & & & y_j x_l x_i + y_{(l,j)} y_j x_i + y_{(i,j)} \underline{x_l y_j} + \\
& \downarrow & & & + y_{(l,(i,j))} y_{(i,j)} y_j \\
y_j x_i x_l + y_{(i,j)} y_j x_l + y_{(l,j)} \underline{x_i y_j} + & & & & \downarrow \\
+ y_{(i,(l,j))} y_{(l,j)} y_j & & & & y_{(i,j)} y_j x_l + y_{(i,j)} y_{(l,j)} y_j \\
& \downarrow & & & \\
& y_{(l,j)} y_j x_i + y_{(l,j)} y_{(i,j)} y_j & & &
\end{array}$$

In some places above we write for example $|y_{(i,j)} y_j|$ in stead of $y_{(i,j)} y_j$. These are those places where order on y_i essential for the future reductions (namely we have some x_j before y_i). We actually had to check all possibilities for the pairs $|y_{(l,j)} y_j|$, $|y_{(i,j)} y_j|$ appearing at the above sequences of reductions and all of them via different cancelations gave the same result.

The coincidence of above mentioned sets means in the language of maps σ_i that

$$\{\sigma_i(j); \sigma_l \sigma_i(j)\} = \{\sigma_i \sigma_l(j); \sigma_l(j)\},$$

for any j and $l > i$.

These sets are coincide if and only if for any fixed $l > i$ in each point j we have either $\sigma_i(j) = \sigma_l(j)$ and $\sigma_l \sigma_i(j) = \sigma_i \sigma_l(j)$ or $\sigma_i(j) = \sigma_i \sigma_l(j)$ and $\sigma_l \sigma_i(j) = \sigma_l(j)$. By this we are done. ■

It is of course very natural and important question, when given presentation of an algebra form a Gröbner basis. In RIT case these conditions take a specific shape of description of defining maps σ_i , obtained above. Conditions for that were formulated also for example for the class of G-algebras in [11] under the name of non-degeneracy condition. Now we turn to more difficult matter of clarifying a precise combinatorial structure of maps obeying conditions of the theorem 3.1. As a first step we consider few particular cases, which we will use later on to prove the general fact.

3.3 Representations of the semigroup $\langle x_i | x_i = x_i x_j \rangle$

Here we consider the case when all elements of N obey conditions (2) from the theorem 3.1. That is we have $\sigma_k(j) = \sigma_k \sigma_i(j)$ and $\sigma_i(j) = \sigma_i \sigma_k(j)$ for any $k > i, j \in N$. This means that σ_i s form a representation by actions on the finite set of the semigroup $\Omega = \langle x_i | x_i = x_i x_j, 1 \leq i \neq j \leq m \rangle$. From these relations it

follows that all x_i are idempotents. Reduction of the first subword $x_i x_j$ in $x_i x_j x_i$ gives $x_i x_j x_i = x_i x_i$, of the second one: $x_i x_j x_i = x_i x_j$, but then $x_i x_j = x_i$. Hence we could also write relations just like $\Omega = \langle x_i | x_i = x_i x_j, i, j = 1, \bar{m} \rangle$, without the condition $i \neq j$. Note, that this semigroup consists in fact of $m + 1$ elements: any word in this semigroup is equal to its first letter.

What is the structure of maps which form representations then?

Theorem 3.2 *Any representation of the semigroup $\Omega = \langle x_i | x_i x_j = x_i \rangle$ has the following structure. The set of representation N is decomposed into a disjoint union of subsets. In each of them there are m fixed points (not necessarily different), such that the maps $\sigma_k, k = 1, \bar{m}$ send the entire subset to the k th of these points.*

Proof. Let $\{\sigma_k\}_{k=1}^m$ be a representation of the semigroup Ω on the set N . That is, $\sigma_j \sigma_k = \sigma_j$ for any $1 \leq j, k \leq m$. First note that since σ_k are idempotents, they are identity on their images: $\sigma_k(m) = m$ for each $m \in R_k = \text{Im } \sigma_k$. Define the equivalence relation on N corresponding to σ_1 : $m_1 \sim_{\sigma_1} m_2$ if $\sigma_1(m_1) = \sigma_1(m_2)$. Then N splits into the union of equivalence classes $\mathcal{O}_{s_1}, \dots, \mathcal{O}_{s_k}$, where each class contains a unique element s_j from $R_j = \text{Im } \sigma_j$, so we can enumerate these classes by these elements. Consider the restriction of the maps σ_k to an arbitrary class \mathcal{O}_{s_j} . Since $\sigma_j \sigma_k = \sigma_j$, each σ_k leaves the set \mathcal{O}_{s_j} invariant. Indeed, $\sigma_j(\sigma_k(r)) = \sigma_j(r) = s_j$ for each $r \in \mathcal{O}_{s_j}$ and therefore $\sigma_k(r) \in \mathcal{O}_{s_j}$. Moreover, since $\sigma_k \sigma_j = \sigma_k$, the set $\sigma_k(\mathcal{O}_{s_j})$ consists of one element $\sigma_k(s_j)$. Indeed, $\sigma_k(r) = \sigma_k(\sigma_j(r)) = \sigma_k(s_j)$ for each $r \in \mathcal{O}_{s_j}$. Hence the structure of these maps is the following: the set N is decomposed into a disjoint union of subsets, in each of which m points are chosen and the maps σ_k map the entire subset to the k th of these points. Some of these points could coincide. ■

Obviously, the other way around, if one take any set of maps $\{\sigma_i\}_{i=1, \bar{m}}$ with the described structure, then they form a representation of the semigroup Ω , that is satisfy the relations $\sigma_k \sigma_i = \sigma_k, \forall k, i = 1, \bar{m}$.

Thus there exists 1-1 correspondence between representations of Ω on a finite set and maps described in the theorem 3.2.

Let we mention that the same is true for representations on an infinite set, our arguments work there without any change.

It is natural to ask when there exists a faithful representation.

Corollary 3.3 *For any $n \geq m$ there exists a faithful representation of Ω on the set of size n .*

Proof. The image of the semigroup Ω in the set of maps consists just of the maps $\sigma_1, \dots, \sigma_m$, which are images of the generators x_1, \dots, x_m of the semigroup. This follows from the relations. Hence if we can just take m different maps of the required nature, then they form a faithful representation. It is certainly possible if $n \geq m$: take $\sigma_1(j) = r_1, \dots, \sigma_m(j) = r_m, r_i \in N$. For different r_i we get different maps. ■

It is possible to find faithful representations of smaller dimension. For example take a subsets from theorem 3.2 of size 3. Namely, let $m = 3^d$ and our representation set consists of the pairs $N = \{(k, \varepsilon) | k = 0, \dots, d-1, \varepsilon = 0, 1, 2\}$. Maps define as follows: $\sigma_i(k, \varepsilon) = (k, \varepsilon_k(i))$, where $\varepsilon_k(i)$ is an i th coefficient in presentation of i in base 3: $i = \varepsilon_0(i) + 3\varepsilon_1(i) + \dots + 3^{d-1}\varepsilon_{d-1}(i)$. We have then a faithful representation on the set N of size $3 \lceil \log_3 m \rceil$.

It is not difficult to show that asymptotically this strategy gives the best possible result, so asymptotically minimal size of faithful representation is $3 \log_3 m$.

3.4 Combinatorial description of maps corresponding to maximal Hilbert series

Here we give a combinatorial description of the maps σ_i , $i = 1, 2$ satisfying condition (1) or (2) as they appear in the theorem 3.1 above, that is those maps which define an algebra with maximal Hilbert series.

We also prove as a consequence that if Hilbert series is maximal then all maps σ_i , $i = 1, \dots, m$ coming from defining relations of arbitrary RIT algebra have pairwise 2-isomorphic graphs.

Consider maps σ_i and σ_k . Suppose they obey conditions described in theorem 3.1. Let us define the set Y_0 as a set of elements $j \in N$ where σ_i and σ_k coincide:

$$Y_0 = \{j \in N \mid \sigma_i(j) = \sigma_k(j)\}.$$

Lemma 3.4 Y_0 is invariant under the action of both σ_i and σ_k

Proof. Let j be from Y_0 . If in point j σ_i and σ_k obey condition (1) from theorem 3.1, then $\sigma_i(j) \in Y_0$ and $\sigma_k(j) \in Y_0$ due to the second part of (1). Indeed, values of maps σ_i and σ_k on their images should coincide, hence these images are again in Y_0 . Suppose in point j condition (2) is fulfilled. Since $j \in Y_0$ we have $\sigma_i(j) = \sigma_k(j) = r$, but due to (2): $r = \sigma_k(j) = \sigma_k\sigma_i(j) = \sigma_k(r)$ and $r = \sigma_i(j) = \sigma_i\sigma_k(j) = \sigma_i(r)$. Which means that actually (1) holds also for this point j . So for each point $j \in Y_0$ condition (1) is satisfied. From this it easy follows that $\sigma_i(Y_0) \subset Y_0$ and $\sigma_k(Y_0) \subset Y_0$. ■

Let now consider an element $j \notin Y_0$ and its images $j_1 = \sigma_i(j)$ and $j_2 = \sigma_k(j)$. Since images of σ_i and σ_k are different in j , condition (1) could not holds in this point, thus we have condition (2) there. This gives us: $j_2 = \sigma_k(j) = \sigma_k\sigma_i(j) = \sigma_k(j_1)$ and $j_1 = \sigma_i(j) = \sigma_i\sigma_k(j) = \sigma_i(j_2)$. So we have that σ_k maps j_1 to j_2 and σ_i maps j_2 to j_1 .

Using this information let us clarify how the element from outside Y_0 could get to Y_0 . Suppose $j \notin Y_0$ but $j_1 = \sigma_i(j) \in Y_0$. Then $\sigma_k(j_1) = \sigma_i(j_1) = j_2$. This means that not only j_2 goes to j_1 under σ_i , but also the other way around, σ_i maps j_1 to j_2 . On Y_0 condition (1) from the theorem 3.1 always holds and $j_1 \in Y_0$ therefore for j_2 which is image of j_1 under σ_i we have $\sigma_i(j_2) = \sigma_k(j_2)$ (due to the second part of condition (1) in point j_1). Since $j_1 = \sigma_i(j_2) \in Y_0$, j_2 is also in Y_0 .

We have proved

Lemma 3.5 Let $j \notin Y_0$ but $\sigma_i(j) \in Y_0$. Then images of j under σ_i and σ_k both are in Y_0 and σ_i as well as σ_k maps them to each other.

Lemma 3.6 If $j \notin Y_0$ but $j_1 = \sigma_i(j) \in Y_0$, then there is no such element from N , which has an image j under σ_i or σ_k .

Proof. Suppose there exists $m \in N$, such that say $\sigma_i(m) = j$. Obviously $m \notin Y_0$, since Y_0 is invariant and then j should be in Y_0 , but it is not. For points which are not in Y_0 condition (1) could not holds, thus we have condition (2) in m . This leads to the following contradiction: $\sigma_k(m) = \sigma_k\sigma_i(m) = \sigma_k(j) = j_2$, $\sigma_i(m) = \sigma_i\sigma_k(m) = \sigma_i(j_2) = j_1$ hence $\sigma_i(m) = j$ and $\sigma_i(j) = j_1$, but j_1 can not be equal to j just because one is from Y_0 and another is not. ■

We are now in a position to define a bigger set \tilde{Y}_0 :

$$\tilde{Y}_0 = Y_0 \cup \{j \in N \mid \sigma_i(j) \in Y_0\},$$

which satisfy the following nice property.

Lemma 3.7 *The set N splits on two disjoint subsets which are invariant under σ_k and σ_i :*

$$N = \tilde{Y}_0 \oplus P$$

where $P = N \setminus \tilde{Y}_0$. Moreover the structure of maps on P is precisely as it was described in the theorem 3.2: P is a disjoint union of subsets on which two points are picked such that σ_i maps entire subset to one of them and σ_k to another.

Proof. The fact that σ_i and σ_k preserve Y_0 was proved above, invariance of \tilde{Y}_0 then follows from its definition. Invariance of complement P of \tilde{Y}_0 comes from the statement of lemma 3.6. ■

Let us define one more subset: $Z = Y_0 \setminus \bigcup_{j=i,k} \sigma_j(\tilde{Y}_0 \setminus Y_0)$.

Above lemmata allow us to give the following precise description of maps σ_i , σ_k corresponding to the maximal Hilbert series.

Theorem 3.8 *Algebra $R \in \mathcal{R}(2, n)$ has a maximal Hilbert series if and only if maps σ_i , σ_k coming from defining relations has the following structure.*

The set N is a disjoint union of invariant under both maps subsets P and \tilde{Y}_0 : $N = P \oplus \tilde{Y}_0$.

The action on P is the following: P is a disjoint union of P_i , in each of them two points (not necessary different) are fixed, such that σ_i maps entire P_i to one of them and σ_j to another.

The map on the other disjoint component \tilde{Y}_0 is the following: there are three subsets $Z \subseteq Y_0 \subseteq \tilde{Y}_0$. The set $Y_0 \setminus Z$ is a disjoint union of pairs $\{j_1^{(l)}, j_2^{(l)}\}_{l \in \Sigma}$, (Σ is a finite set of indexes), such that $\forall j \in \tilde{Y}_0 \setminus Y_0 \exists l \in \Sigma : \sigma_i(j) = j_1^{(l)}, \sigma_k(j) = j_2^{(l)}$ and $\sigma_{i,k}(j_1^{(l)}) = j_2^{(l)}, \sigma_{i,k}(j_2^{(l)}) = j_1^{(l)}$. Values of σ_i and σ_k on Z are in Y_0 and coincide.

Let us say that two graphs are *2-isomorphic* if their images under gluing pairs of vertices of common cycles of length two are isomorphic.

As a consequence of the above theorem we get the following

Corollary 3.9 *If an algebra $R \in \mathcal{R}(m, n)$ has a maximal Hilbert series, then all graphs of maps σ_i coming from the defining relations of R are pairwise 2-isomorphic.*

Proof. Note that the feature of condition on σ_i formulated in theorem 3.1 is that it is satisfied for an arbitrary set $\{\sigma_i\}_{i=1, \bar{m}}$ if and only if it is satisfied for any pair σ_i, σ_k , $i < k$. So we can apply theorem 3.8 for any fixed pair σ_i, σ_k . It is clear from the above description that graphs of maps σ_i and σ_k are the same or isomorphic with the isomorphism defined by permuting of some pairs j_1, j_2 . Except one possible situation when for some pair $j_1, j_2 \in Y_0 \setminus Z$ which consists of images of $j \in \tilde{Y}_0 \setminus Y_0$ we have a point $l \in Z$ such that $\sigma_i(l) = \sigma_k(l) = j_1$. This gives a possibility for graphs of σ_i and σ_k to be nonisomorphic. We exclude this possibility by gluing vertices of common cycles of length two (j_1 and j_2) in this two graphs, so after such an operation graphs become isomorphic. ■

Corollary 3.10 *Combinatorial conditions from the theorem 3.8 on σ_i, σ_k are equivalent to the following properties of algebra $R \in \mathcal{R}(m, n)$:*

- (i) R has a quadratic Gröbner basis;
- (ii) R has a lexicographically maximal in $\mathcal{R}(m, n)$ Hilbert series;
- (iii) R is a PBW algebra (that is, have a series $H_R = \frac{1}{(1-t)^{(m+n)}}$), and imply the following properties of algebra:
- (iv) R is Koszul;

- (v) R is Auslander regular;
 (vi) R is Cohen-Macaulay.

Proof. Equivalence to the condition (i) was proved in theorems 3.1 and 3.8. Equivalence of (i),(ii) and (iii) is a direct consequence from the standard procedure of Hilbert series computation for algebras presented by Gröbner basis. Implication (i) \Rightarrow (iv) is known and could be found for example in [14] or [8]. Implications (i) \Rightarrow (v) and (i) \Rightarrow (vi) could be found in [9], [12], [13]. ■

4 Toward classification of Hilbert series

Here we give a list of all RIT algebras of rank up to 4 with non isomorphic colored graphs and the precise values of their Hilbert series.

Denotation for a single graph (map) is the following: we write (i_1, i_2, \dots, i_k) for the map $\sigma : N \longrightarrow N : j \mapsto i_j$.

We also denote by $P_d(t)$ the series $\frac{1}{(1-t)^d}$, which is a series of algebra $k[x_1, \dots, x_d]$ of commutative polynomials on d variables.

rk1 Commutative polynomials $k[x]$, with $H_R(t) = P_1(t) = \frac{1}{1-t}$.

rk2

There are three possibilities: $R \in \mathcal{R}(2, 0)$, $R \in \mathcal{R}(1, 1)$, $R \in \mathcal{R}(0, 2)$.

(2,0)

$$H_R(t) = P_2(t) = \frac{1}{(1-t)^2}$$

(1,1)

Graph: (1)

$$H_R(t) = P_2(t) = \frac{1}{(1-t)^2}$$

(0,2)

$$H_R(t) = P_2(t) = \frac{1}{(1-t)^2}$$

rk3

There are four possibilities: $R \in \mathcal{R}(3, 0)$, $R \in \mathcal{R}(2, 1)$, $R \in \mathcal{R}(1, 2)$, $R \in \mathcal{R}(0, 3)$.

(3,0) and **(0,3)**

$$H_R(t) = P_3(t) = \frac{1}{(1-t)^3}$$

(1,2)

List of non isomorphic graphs: (1,2),(1,1),(2,1),(2,2)

All these four algebras have the series $H_R(t) = P_3(t) = \frac{1}{(1-t)^3}$ due to the theorem 3.2.

(2,1)

2-colored graph: $\sigma_1 = (1), \sigma_2 = (1)$; $H_R(t) = P_3(t) = \frac{1}{(1-t)^3}$ due to the theorem 3.2.

rk4 There are five possibilities: $R \in \mathcal{R}(4, 0)$, $R \in \mathcal{R}(3, 1)$, $R \in \mathcal{R}(2, 2)$, $R \in \mathcal{R}(1, 3)$, $R \in \mathcal{R}(0, 4)$.

(4,0) and **(0,4)**

$$H_R(t) = P_4(t) = \frac{1}{(1-t)^4}$$

(1,3)

List of non isomorphic graphs: (1,1,1),(1,1,2),(1,1,3),(1,2,3),(1,3,2),(2,1,1),(2,3,1)

All these algebras have the series $H_R(t) = P_4(t) = \frac{1}{(1-t)^4}$ due to the theorem 3.2.

(2,2)

List of non isomorphic 2-colored graphs:

$\sigma_1 = (1, 2), \sigma_2 = (1, 2)$; $\sigma_1 = (1, 1), \sigma_2 = (1, 1)$; $\sigma_1 = (1, 1), \sigma_2 = (2, 2)$;
 $\sigma_1 = (2, 1), \sigma_2 = (2, 1)$

All algebras in this part of the list has maximal series:

$H_R(t) = P_4(t) = \frac{1}{(1-t)^4}$ due to the theorem 3.2.

$\sigma_1 = (1, 2), \sigma_2 = (1, 1); \sigma_1 = (1, 2), \sigma_2 = (2, 1); \sigma_1 = (1, 1), \sigma_2 = (2, 1)$

All algebras in this part of the list has series

$H_R(t) = \frac{1+t+t^2}{1-3t+3t^2-t^3}$, which is not maximal since they does not obey conditions of the Corollary 3.9.

(3,1)

3-colored graph: $\sigma_1 = (1), \sigma_2 = (1), \sigma_3 = (1); H_R(t) = P_4(t) = \frac{1}{(1-t)^4}$ due to the theorem 3.2.

Corollary 4.1 *All RIT algebras of rank ≤ 4 are PBW (with the series $P_d(t)$, where d is a rank) except of three cases, given by maps:*

$R_1 = \{\sigma_1 = (1, 2), \sigma_2 = (1, 1)\};$

$R_2 = \{\sigma_1 = (1, 2), \sigma_2 = (2, 1)\};$

$R_3 = \{\sigma_1 = (1, 1), \sigma_2 = (2, 1)\}.$

These algebras have the series $H_{R_i}(t) = \frac{1+t+t^2}{1-3t+3t^2-t^3}$, $i = 1, 3$.

5 Remark on the Generalized Yang–Baxter Equation for RIT algebras

Once we have quadratic relations (1) of RIT algebra we could define a linear map $r : V \otimes V \rightarrow V \otimes V$, where $V = \langle x_1, \dots, x_m, y_1, \dots, y_n \rangle$ as follows:

$$\begin{aligned} r(x_i \otimes y_j) &= y_j \otimes x_i + \overline{y_{f(i,j)}} \times y_j, \\ r(x_i \otimes x_j) &= \overline{x_i} \times x_j, \\ r(y_i \otimes y_j) &= \overline{y_i} \times y_j, \end{aligned}$$

where by $\overline{y_i \times y_j}$ we denote the product $y_i \otimes y_j$ if $i < j$ and $y_j \otimes y_i$ if $i > j$. Analogously $\overline{x_i \times x_j}$ stands for $x_i \otimes x_j$ if $i < j$ and for $x_j \otimes x_i$ if $i > j$.

The map of this kind satisfies the Yang–Baxter equation if for the action on $V \otimes V \otimes V$ induced by r the following is true

$$(r_{1,2} \otimes 1)(1 \otimes r_{2,3})(r_{1,2} \otimes 1) = (1 \otimes r_{2,3})(r_{1,2} \otimes 1)(1 \otimes r_{2,3}).$$

Denote the operators on the left-hand and right-hand sides by R_{12} and R_{23} respectively. Arguments of the same kind like in the theorem 3.1 and 3.8 give us the following.

Theorem 5.1 *If the relations (2) of a RIT algebra R form a Gröbner basis, or equivalently if σ_i satisfy the conditions of Theorem 3.1, then the corresponding operator $r : V \otimes V \rightarrow V \otimes V$ defined as above from the relations, satisfies the generalized Yang–Baxter equation $[R_{12}, R_{23}] = 0$.*

Here in stead of the usual Yang–Baxter equation $R_{12} = R_{23}$ we have got a generalized commutator version: $[R_{12}, R_{23}] = 0$.

In Theorem 3.1 we formulate combinatorial conditions on the relations which mean that they form a quadratic Gröbner basis. In Theorem 5.1 we state what kind of Yang–Baxter equation is satisfied in this case. The latter has to do with the length of chain of reductions necessary to ensure that defining relations form a Gröbner basis.

In more details connection to Yang–Baxter equations will be discussed elsewhere, but we emphasize here that combinatorial description of maps in theorems 3.1 and 3.8 is also a description of the case when defining relations may be associated to a solutions of a kind of YBE.

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