

Three examples of non-commutative boundaries of Shimura varieties

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ABSTRACT. Our modest aims in writing this paper were twofold: we first wanted to understand the linear algebra and algebraic group theoretic background of Manin's real multiplication program proposed in [Man]. Secondly, we wanted to find nice higher dimensional analogs of the non-commutative modular curve studied by Manin and Marcolli in [MM02]. These higher dimensional objects, that we call *irrational* or *non-commutative boundaries of Shimura varieties*, are double cosets spaces of the form $\Gamma \backslash G(\mathbb{R})/P(K)$, where G is a (connected) reductive \mathbb{Q} -algebraic group, $P(K) = M(K)AN \subset G(\mathbb{R})$ is a real parabolic subgroup corresponding to a rational parabolic subgroup $P \subset G$, and $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup. Along the way, it also seemed clear that the spaces

$$\Gamma \backslash G(\mathbb{R})/M(K)A$$

are of great interest, and sometimes more convenient to study. We study in this document three examples of these general spaces. These spaces describe degenerations of complex structures on tori in (multi)foliations.

Introduction

The non-commutative modular curve is the chaotic space

$$\mathrm{GL}_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{R}).$$

Its first appearance in the non-commutative geometric world arose in the work [Con80] of Connes on differential geometry of non-commutative tori. Since the action of $\mathrm{GL}_2(\mathbb{Z})$ by homographies on $\mathbb{P}^1(\mathbb{R})$ is very chaotic, the classical quotient space, whose algebra is the one of continuous complex functions on $\mathbb{P}^1(\mathbb{R})$ that are invariant by $\mathrm{GL}_2(\mathbb{Z})$, is topologically identified with a point. The philosophy of Connes' non-commutative geometry and the related philosophy of topoi (as explained by Cartier in [Car98]) tell us that this quotient space is much more than that. Connes showed that the crossed product C^* -algebra $C^*(\mathbb{P}^1(\mathbb{R})) \rtimes \mathrm{GL}_2(\mathbb{Z})$ is a good analog of the algebra of continuous functions for such a chaotic space because it is possible to calculate from it nice cohomological invariants, as K -theory and cyclic cohomology (from its C^∞ version), that have a real geometric meaning. He showed, indeed, that cyclic cohomology is a good analog of De Rham cohomology of manifolds for such chaotic spaces. This cohomology theory is a very profound tool that permitted Connes and Moscovici,

to cite one example among many others, to prove a local index formula for foliations, which is an analog of the Riemann-Roch theorem for the spaces of leaves [CM95]. This result was not accessible without non-commutative geometric intuition. Bost-Connes and Connes have also shown [BC95], [Con99], that non-commutative geometry can be very useful for arithmetic questions. We refer the reader to Marcolli [Mar04] for a nice survey on non-commutative arithmetic geometry.

The non-commutative modular curve also appeared some years ago in Connes, Douglas and Schwarz paper [CDS97] as moduli space for physical backgrounds for compactifications of string theory. It was already shown in [SW99] by Seiberg and Witten that non-commutative spaces can be good backgrounds for open string theory. The deformation quantization story of Kontsevich and Soibelman also gives related results. The author, being more informed of arithmetic geometry, will not discuss these physical motivations that are very important for future developments of non-commutative moduli spaces.

The arithmetic viewpoint of the non-commutative modular curve appeared first in the work of Manin and Marcolli [MM02], [MM01] and in Manin's real multiplication program [Man]. These works were for us the main inspirations for writing this paper.

The basic idea in our work is the following: the classical theory of Shimura varieties was completely rewritten in terms of Hodge structures by Deligne in [Del79] and it proved to be very helpful for arithmetics, for example in the theory of absolute Hodge motives made in [DMOS82] and for the construction of canonical models for all Shimura varieties, made by Milne in [Mil83]. One of the interests of this construction is to translate fine information about moduli spaces in terms of algebraic groups morphisms. The author wanted to know if it was possible to make such a translation for non-commutative boundaries of Shimura varieties.

As higher dimensional analogs of the boundary $\mathbb{P}^1(\mathbb{R})$ of the double half plane $\mathbb{H}^\pm := \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$, we chose to use the components of Satake's compactifications of symmetric spaces. These components can be written as quotients $G(\mathbb{R})/P(K)$, where G is a (connected) reductive \mathbb{Q} -algebraic group and $P(K) = M(K)AN \subset G(\mathbb{R})$ is a real parabolic subgroup¹ corresponding to a rational parabolic subgroup $P \subset G$. The higher dimensional analogs of the non-commutative modular curve $GL_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{R})$, which we call *irrational* or *non-commutative boundaries of Shimura varieties*, are given by double coset spaces

$$\Gamma \backslash G(\mathbb{R})/P(K),$$

where $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup. Along the way, it seemed also clear that the spaces

$$\Gamma \backslash G(\mathbb{R})/M(K)A,$$

which we call *irrational* or *non-commutative shores of Shimura varieties*, are of great interest, and sometimes more convenient to study from the algebraic group theoretical viewpoint.

¹See the book [BL01] for the definition and decomposition of these real parabolic subgroups.

The plan of this document is the following. In the first part, we study special geodesics on the modular curve that are good analogs of elliptic curves with complex multiplication. We explain how to relate the counting of these geodesics to number theoretical considerations. In the second part, we construct the moduli space and universal family of non-commutative tori in a way analogous to the construction of the moduli space of elliptic curves and its universal family. In the third part, we give two higher dimensional examples of non-commutative shores of Shimura varieties that parametrize some degenerations of complex structures on tori in multi-foliations.

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Notations

DEFINITION 0.1. A pair (P, X) consisting of a \mathbb{Q} -algebraic group P and a left $P(\mathbb{R})$ -space X is called a *pre-Shimura datum*². A *morphism of pre-Shimura data* $(P_1, X_1) \rightarrow (P_2, X_2)$ is a pair (ϕ, ψ) consisting of a morphism $\phi : P_1 \rightarrow P_2$ of groups and a $P_1(\mathbb{R})$ -equivariant map $\psi : X_1 \rightarrow X_2$. If (P, X) is a pre-Shimura datum and $K \subset P(\mathbb{A}_f)$ is a compact open subgroup, the set

$$\mathrm{Sh}_K(P, X) := P(\mathbb{Q}) \backslash (X \times P(\mathbb{A}_f) / K)$$

is called the *pre-Shimura set of level K* for (P, X) , and the set

$$\mathrm{Sh}(P, X) := \varprojlim_K \mathrm{Sh}_K(P, X),$$

where K runs over all compact open subgroups in $P(\mathbb{A}_f)$, is called the *pre-Shimura tower* associated to (P, X) .

²Most of the pre-Shimura data that will appear in this document will be constructed using conjugacy classes of morphisms. However, the target group will not always be P and some useful morphisms between them will not be morphisms of conjugacy classes but just equivariant morphisms. This is why we use such a weak definition.

We warn the reader that even if X has a nontrivial C^∞ -structure, the corresponding Shimura space, viewed as a quotient topological space, can be very degenerate (even trivial). In such cases, it may be more interesting to study the crossed product algebra

$$C^\infty(X \times P(\mathbb{A}_f)/K) \rtimes P(\mathbb{Q}).$$

These are essentially the kind of non-commutative spaces we will consider in this paper.

1. Special geodesics on the modular curve

The modular curve is one of the basic examples of a Shimura variety. Its interpretation as a moduli space of Hodge structures (or simpler of complex structures) allows one to nicely define *special points* of this curve as those whose Mumford-Tate group is a torus.

We propose an analogous construction for the space $\mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2(\mathbb{R}) / D(\mathbb{R})$ of leaves of the geodesic foliation (where $D(\mathbb{R})$ is the subgroup of diagonal matrices in $\mathrm{GL}_2(\mathbb{R})$). This naive translation gives another point of view of the strong analogy between the modular curve and the space of leaves of the geodesic flow, which was already known to Gauss. Being closer to the modern (i.e., adelic) point of view of Shimura varieties increases its chances to be generalized to boundaries of higher dimensional Shimura varieties.

1.1. Mumford-Tate groups of elliptic curves. Let E be a complex elliptic curve. Let $M = H^1(E, \mathbb{Q})$ be its first singular homology group. We have the Hodge decomposition

$$M_{\mathbb{C}} = H^1(E, \mathbb{C}) \cong H^0(E, \Omega_E^1) \oplus H^1(E, \mathcal{O}_E).$$

Let $\mathbb{G}_{m, \mathbb{C}}$ be the multiplicative group of \mathbb{C} as an algebraic group. We let $(x, y) \in (\mathbb{C}^*)^2$ act on $M_{\mathbb{C}}$ by multiplication by x on $H^0(E, \Omega_E^1)$ and y on $H^1(E, \mathcal{O}_E)$. This defines a natural morphism $h : \mathbb{G}_{m, \mathbb{C}}^2 \rightarrow \mathrm{GL}(M_{\mathbb{C}})$. The *Mumford-Tate group* of E is the smallest \mathbb{Q} -algebraic subgroup of $\mathrm{GL}(M)$ that contains the image of h over \mathbb{C} . The following proposition serves us as a guide to define Mumford-Tate groups of geodesics.

PROPOSITION 1.1.1. *Let E be an elliptic curve over \mathbb{C} . The Mumford-Tate group of E is either GL_2 , or $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$ (i.e., the group K^\times as a \mathbb{Q} -algebraic group) with K/\mathbb{Q} an imaginary quadratic field. In this case, we say that the curve is special or with complex multiplication.*

1.2. Geodesics and analogs of Hodge structures. We recall that the space of geodesics on the modular curve can be written as a double coset space $Y := \mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2(\mathbb{R}) / D(\mathbb{R})$ with $D(\mathbb{R})$ the subgroup of diagonal matrices in $\mathrm{GL}_2(\mathbb{R})$. We denote by $\mathbb{G}_{m, \mathbb{R}}$ the multiplicative group \mathbb{R}^* viewed as an algebraic group. Let $h_0 : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow \mathrm{GL}_{2, \mathbb{R}}$ be the morphism of algebraic groups that sends a pair $(x, y) \in (\mathbb{R}^*)^2$ to the diagonal matrix $\mathrm{diag}(x, y) := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. We remark that $\mathrm{GL}_2(\mathbb{R})$ acts on the space $\mathrm{Hom}(\mathbb{G}_{m, \mathbb{R}}^2, \mathrm{GL}_{2, \mathbb{R}})$ by conjugation. Let

$$X := \mathrm{GL}_2(\mathbb{R}) \cdot h_0 := \{gh_0g^{-1}, g \in \mathrm{GL}_2(\mathbb{R})\}$$

be the conjugacy class of the morphism h_0 .

An easy computation shows that the centralizer of h_0 in $\mathrm{GL}_2(\mathbb{R})$ is the subgroup $D(\mathbb{R})$. We thus obtain an identification $X \cong \mathrm{GL}_2(\mathbb{R})/D(\mathbb{R})$. There is also a left $\mathrm{GL}_2(\mathbb{Z})$ -action on X , and this gives us an interpretation of the space of geodesics as the quotient $Y \cong \mathrm{GL}_2(\mathbb{Z}) \backslash X$.

The reader will probably ask now: what did we win in this translation? The author's answer is: a strong analogy with the modular curve.

This allows us to view Y as the moduli space of triples (M, F, \tilde{F}) , where M is a free \mathbb{Z} -module of rank 2, and $F, \tilde{F} \subset M_{\mathbb{R}}$ are two lines in the corresponding real vector space $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. To each $h \in X$, we associate a triple (\mathbb{Z}^2, F_x, F_y) with F_x the line of weight x for h (i.e., $h(x, y) \cdot v = x \cdot v$ for all $v \in F_x$), and F_y the line of weight y . It is helpful to think of the direct sum decomposition

$$M_{\mathbb{R}} = F \oplus \tilde{F}$$

as an analog of the Hodge decomposition of the first complex singular cohomology group of an elliptic curve E/\mathbb{C} :

$$H^1(E, \mathbb{C}) = H^1(E, \mathcal{O}_E) \oplus H^0(E, \Omega_E^1).$$

1.3. Examples of bad Mumford-Tate groups. In view of the analogy in the previous paragraph, we can ask: for geodesics, what is the analog of the Mumford-Tate group³ of elliptic curves? Recall that an element $h \in X$ is a morphism of algebraic groups $h : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow \mathrm{GL}_{2, \mathbb{R}}$.

We suggest two possible analogs. The first is obtained by copying the usual definition.

DEFINITION 1.3.1. Let $h \in X$. We define the *bad Mumford-Tate group* of h to be the smallest \mathbb{Q} -algebraic subgroup $\mathrm{BMT}(h) \subset \mathrm{GL}_{2, \mathbb{Q}}$ such that $h((\mathbb{R}^*)^2) \subset \mathrm{BMT}(h)(\mathbb{R})$.

Let us test this definition on some examples. It is clear that $\mathrm{BMT}(h_0) = D \cong \mathbb{G}_{m, \mathbb{Q}}^2$ is the group of rational diagonal matrices, the maximal torus of $\mathrm{GL}_{2, \mathbb{Q}}$.

Let $u \in \mathbb{R}$ and $g_u := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ be the corresponding unipotent matrix. Denote $h_u := g_u h_0 g_u^{-1} \in X$. The morphism h_u is given by the matrix $\begin{pmatrix} x & u(y-x) \\ 0 & y \end{pmatrix}$. If we suppose that u is not rational, then $\mathrm{BMT}(h_u)$ is the subgroup $B \subset \mathrm{GL}_{2, \mathbb{Q}}$ of upper triangular matrices. Otherwise, $\mathrm{BMT}(h_u) = g_u D g_u^{-1} \cong \mathbb{G}_{m, \mathbb{Q}}^2$ is a maximal torus.

Let h' be the conjugate of h_0 by the matrix $g' := \begin{pmatrix} 1 & 1 \\ u & -u \end{pmatrix}$, with $u = e \in \mathbb{R}$. Then we get $h' \in X$ such that $\mathrm{BMT}(h') = \mathrm{GL}_{2, \mathbb{Q}}$.

Now let us consider a square free positive integer $d > 1$. Let $g := \begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \end{pmatrix}$. If we conjugate h_0 by g , we obtain the matrix $h' = \begin{pmatrix} a & b \\ db & a \end{pmatrix}$ with $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2\sqrt{d}}$. The group of matrices of the form $\begin{pmatrix} a & b \\ db & a \end{pmatrix}$ with a and b rational is an algebraic torus over \mathbb{Q} , conjugated over \mathbb{R} to the maximal torus $D(\mathbb{R})$. It is clearly the Mumford-Tate group of h' .

³A kind of analytic Galois group.

1.4. Good Mumford-Tate groups. We now want to modify the definition of bad Mumford-Tate groups to obtain “good” ones. The examples of last paragraph give us a quite precise idea of the different kinds of bad Mumford-Tate groups that can appear. However, we would like to have *reductive* Mumford-Tate groups in order to have a closer analogy with the case of elliptic curves.

DEFINITION 1.4.1. Let $h \in X$. A *Mumford-Tate group* for h is a minimal reductive subgroup of $\mathrm{GL}_{2,\mathbb{Q}}$ that contains $\mathrm{BMT}(h)$.

With this new definition, a Mumford-Tate group is always reductive whereas BMT may be, for example, the group B of upper triangular matrices.

LEMMA 1.4.2. *Let $h \in X$. Then there exists a unique Mumford-Tate group for h , which we denote by $\mathrm{MT}(h)$.*

PROOF. The group $\mathrm{BMT}_{\mathbb{R}}$ contains a maximal torus of $\mathrm{GL}_{2,\mathbb{R}}$ (the image of h). We thus have three possibilities, depending on the dimension (2,3 or 4) of BMT : BMT is a maximal torus, a Borel subgroup or the whole of $\mathrm{GL}_{2,\mathbb{Q}}$. If BMT is a maximal torus or $\mathrm{GL}_{2,\mathbb{Q}}$ then MT is clearly well defined and equal to BMT . If BMT is a Borel subgroup, then the smallest reductive group that contains it is the group $\mathrm{GL}_{2,\mathbb{Q}}$. \square

We now arrive at the structure theorem for Mumford-Tate groups of geodesics.

THEOREM 1.4.3. *Let $h \in X$. The Mumford-Tate group of h is of one of the following types:*

- (1) $\mathrm{MT}(h) = \mathrm{GL}_2$, the corresponding geodesic is called *MT-generic*,
- (2) $\mathrm{MT}(h) = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ with E/\mathbb{Q} a real quadratic field, we say that the corresponding geodesic is *special* or has real multiplication by E ,
- (3) $\mathrm{MT}(h) = \mathbb{G}_{m,\mathbb{Q}}^2$, the geodesic is *rational*.

This theorem follows from the fact that all maximal tori in $\mathrm{GL}_{2,\mathbb{Q}}$ are of the form $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ with E/\mathbb{Q} étale of dimension 2. Such an algebra E is either a real quadratic field, isomorphic to \mathbb{Q}^2 , or an imaginary quadratic field. The imaginary quadratic case can not appear because the two lines F_x and F_y associated to h (see Subsection 1.2) are real.

The properties of Mumford-Tate groups have a simple interpretation in terms of dynamical properties of geodesics on the modular curve, as explained to the author by Etienne Ghys. The first case corresponds to non-closed geodesics, the second to closed geodesics not homotopic to the cusps, and the last corresponds to closed geodesics homotopic to the cusps.

1.5. Special geodesics and class field theory. Let us look more closely at the second case in Theorem 1.4.3, following Gauss’ ideas. So let $h \in X$ correspond to a special geodesic (i.e., one with real multiplication by a real quadratic field E) and let T be its Mumford-Tate group. By definition, h is a morphism into $T_{\mathbb{R}}$. The $T(\mathbb{R})$ -conjugacy class of h is reduced to $\{h\}$. The natural map $(T, \{h\}) \rightarrow (\mathrm{GL}_{2,\mathbb{Q}}, X)$ induces a map between the double coset

spaces

$$\begin{array}{ccc} T(\mathbb{Q}) \backslash (\{h\} \times T(\mathbb{A}_f) / T(\widehat{\mathbb{Z}})) & \longrightarrow & \mathrm{GL}_2(\mathbb{Q}) \backslash (X \times \mathrm{GL}_2(\mathbb{A}_f) / \mathrm{GL}_2(\widehat{\mathbb{Z}})) \\ \parallel & & \parallel \\ \mathrm{Pic}(\mathcal{O}_E) & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}) \backslash X = Y \end{array}$$

whose image gives the space of geodesics with real multiplication by E . The left term of this morphism is the ideal class group $\mathrm{Pic}(\mathcal{O}_E)$ of this real multiplication field. The bottom arrow interprets $\mathrm{Pic}(\mathcal{O}_E)$ in terms of geodesics, a well known result of Gauss. The top arrow is an adelic formulation of this result which may admit a generalization to higher rank spaces.

We want to arrive at a similar geodesic interpretation for the connected component of the idèle class group $\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}))$. Recall that if (G, H) is a pair consisting of a reductive group over \mathbb{Q} and a $G(\mathbb{R})$ -left space H , we denote by

$$\mathrm{Sh}(G, H) = \lim_{\leftarrow K} G(\mathbb{Q}) \backslash (H \times G(\mathbb{A}_f) / K)$$

where $K \subset G(\mathbb{A}_f)$ runs over all compact open subgroups. In [Del79], 2.2.3, it is shown that

$$\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})) = \mathrm{Sh}(T, \pi_0(T(\mathbb{R}))).$$

At this point, the morphism $(T, \pi_0(T(\mathbb{R}))) \rightarrow (\mathrm{GL}_2, X)$ induces a map

$$(1) \quad \mathrm{Sh}(T, \pi_0(T(\mathbb{R}))) \rightarrow \mathrm{Sh}(\mathrm{GL}_2, X).$$

Unfortunately, the archimedean component $\pi_0(T(\mathbb{R})) := T(\mathbb{R}) / T(\mathbb{R})^+$ is killed under this mapping because $X \cong \mathrm{GL}_2(\mathbb{R}) / T(\mathbb{R})$. Therefore, one replaces X by $X^\pm \cong \mathrm{GL}_2(\mathbb{R}) / T(\mathbb{R})^+$ in morphism (1) where X^\pm is the space of morphisms $h \in X$ oriented by the choice of orientations s_x, s_y on the corresponding real lines F_x, F_y . This space projects naturally onto X . The resulting map

$$\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})) = \mathrm{Sh}(T, \pi_0(T(\mathbb{R}))) \rightarrow \mathrm{Sh}(\mathrm{GL}_2, X^\pm)$$

yields the geodesic interpretation of the connected component of the idèle class group.

We are however far from a theory of real multiplication because a “natural” rational structure on the space of geodesics, analogous to the coordinates on the modular curve given by the j and \mathcal{P} functions, is missing.

2. Algebraic groups and moduli spaces of non-commutative tori

2.1. The moduli space of elliptic curves. This section recalls the classical construction of the universal family of elliptic curves in terms of mixed Shimura varieties [Pin90], 10.7. For classical Shimura varieties, we will refer to [Del79]. The basic definition of mixed Shimura varieties can be found in [Pin90], 2.1, 3.1, or [Mil90][VI]. We only need in this paper the notion of pre-Shimura datum defined in the Notations Section.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ be the Deligne torus. This is an \mathbb{R} -algebraic group such that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. Let $w : \mathbb{G}_{m, \mathbb{R}} \rightarrow \mathbb{S}$ be the weight morphism given by the natural inclusion $\mathbb{R}^* \subset \mathbb{C}^*$. Let $\mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$ be the Hodge

morphism that sends z to the pair $(z, 1)$ and let $\tilde{\mu}$ the morphism that sends z to the pair $(1, z)$. We call $\tilde{\mu}$ the *anti-Hodge morphism*.

Let V be an \mathbb{R} -vector space. Recall that a representation $h : \mathbb{S}_{\mathbb{C}} \rightarrow \mathrm{GL}(V_{\mathbb{C}})$ in the \mathbb{C} -vector space $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ gives an ascending filtration, the so called *weight filtration*, $W_{\bullet} V_{\mathbb{C}}$ given by the cocharacter $h \circ w_{\mathbb{C}}$ and a descending filtration, the so called *Hodge filtration*, $F^{\bullet} V_{\mathbb{C}}$ given by the cocharacter $h \circ \mu$. We will also be interested by another descending, that we call the *anti-Hodge filtration*, $\tilde{F}^{\bullet} V_{\mathbb{C}}$ given by the cocharacter $h \circ \tilde{\mu}$. Note that if h is not defined over \mathbb{R} , then the anti-Hodge filtration is not the complex conjugate filtration of $F^{\bullet} V_{\mathbb{C}}$.

Let $\mathbb{H}^{\pm} := \{\tau \in \mathbb{C}, \tau \notin \mathbb{R}\}$ be the Poincaré double half plane. This space can be identified with the $\mathrm{GL}_2(\mathbb{R})$ -conjugacy class of the morphism of real algebraic groups $h_i : \mathbb{S} \rightarrow \mathrm{GL}_{2, \mathbb{R}}$ that maps $z = a + ib \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$ to the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. The map between this conjugacy class and $\mathbb{H}^{\pm} \subset \mathbb{P}^1(\mathbb{C})$ is given by associating to $h : \mathbb{S} \rightarrow \mathrm{GL}_{2, \mathbb{R}}$ the line $F(h) := \mathrm{Ker}(h_{\mathbb{C}} \circ \mu(z) - z \cdot \mathrm{id}) \subset \mathbb{C}^2$.

DEFINITION 2.1.1. The pair $(\mathrm{GL}_2, \mathbb{H}^{\pm})$ is called the *classical modular Shimura datum*.

Let K be the compact open subgroup $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of $\mathrm{GL}_2(\mathbb{A}_f)$. The associated Shimura variety of level K is by definition

$$\begin{aligned} \mathcal{M} = \mathrm{Sh}_K(\mathrm{GL}_{2, \mathbb{Q}}, \mathbb{H}^{\pm}) &= \mathrm{GL}_2(\mathbb{Q}) \backslash (\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f) / K) \\ &= \mathrm{GL}_2(\mathbb{Z}) \backslash \mathbb{H}^{\pm} \\ &= \mathrm{PGL}_2(\mathbb{Z}) \backslash \mathbb{H}, \end{aligned}$$

that is to say the classical modular curve.

Now we recall the construction of the universal family $\mathcal{E} \rightarrow \mathcal{M}$ of elliptic curves.

Let P be the group scheme $V \rtimes \mathrm{GL}_2$ with $V := \mathbb{G}_a^2$ the standard representation⁴ of GL_2 . Fix a rational splitting $s : \mathrm{GL}_2 \rightarrow P$ of the natural projection map $\pi : P \rightarrow \mathrm{GL}_2$, for example the map $g \mapsto (0, g) \in V \rtimes \mathrm{GL}_2$. All such splittings are conjugates under $V(\mathbb{Q}) \subset P(\mathbb{Q})$. Define $h' : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ by $h' := s_{\mathbb{C}} \circ h_{i, \mathbb{C}}$. We will denote by \mathbb{H}' the $P(\mathbb{R})$ -orbit of h' in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ for the conjugation action.

DEFINITION 2.1.2. The pair (P, \mathbb{H}') is called the *pre-Shimura datum of the universal elliptic curve*.

LEMMA 2.1.3. *The space \mathbb{H}' does not depend on the choice of the splitting s .*

PROOF. This follows from the fact that all such splittings are conjugates under $V(\mathbb{Q})$ and that \mathbb{H}' is a $P(\mathbb{R})$ -orbit. \square

Let $K^P := P(\hat{\mathbb{Z}}) \subset P(\mathbb{A}_f)$. The mixed Shimura variety associated to the data (P, \mathbb{H}') and K^P is, by definition,

$$\mathcal{E} = \mathrm{Sh}_{K^P}(P, \mathbb{H}') = P(\mathbb{Q}) \backslash (\mathbb{H}' \times P(\mathbb{A}_f) / K^P) = P(\mathbb{Z}) \backslash \mathbb{H}'.$$

⁴i.e $V(\mathbb{Z}) = \mathbb{Z}^2$

The natural projection map $\pi : P \rightarrow \mathrm{GL}_2$ induces a projection morphism of Shimura data $(P, \mathbb{H}') \rightarrow (\mathrm{GL}_2, \mathbb{H}^\pm)$ and a projection map

$$\mathcal{E} \rightarrow \mathcal{M}$$

between the corresponding mixed Shimura varieties.

Theorem 10.10 of [Pin90] shows that this map (up to the choice of finer level structures $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ and $K^P \subset P(\mathbb{A}_f)$) gives the universal family of elliptic curves.

REMARK 2.1.4. We want to stress here that the fiber of $\mathbb{H}' \rightarrow \mathbb{H}^\pm$ over some $h : \mathbb{S} \rightarrow \mathrm{GL}_{2, \mathbb{R}}$ is given by the one dimensional \mathbb{C} -vector space $\mathbb{C}^2/F(h) \cong V(\mathbb{R})$. This will be useful in the next section.

2.2. The universal family of geodesics. We will now use the same ideas to construct, for the space of geodesics, an analog of the universal family of elliptic curves.

Let $h_0^Q : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow \mathrm{GL}_{2, \mathbb{R}}^2$ be the morphism that sends the pair $(x, y) \in (\mathbb{R}^*)^2$ to the pair of matrices $((\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & y \end{smallmatrix}))$. Let $Z_Q := \mathrm{GL}_2(\mathbb{R}) \cdot h_0^Q$ be the $\mathrm{GL}_2(\mathbb{R})$ -conjugacy class of h_0^Q in $\mathrm{Hom}(\mathbb{G}_{m, \mathbb{R}}^2, \mathrm{GL}_{2, \mathbb{R}}^2)$. The multiplication map $m : \mathrm{GL}_2^2 \rightarrow \mathrm{GL}_2$ (which is not a group homomorphism) induces a natural $\mathrm{GL}_2(\mathbb{R})$ -equivariant bijection

$$\begin{array}{ccc} Z_Q & \xrightarrow{m} & X \\ (g \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g^{-1}, g \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} g^{-1}) & \mapsto & g \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g^{-1}. \end{array}$$

In other words, we have constructed an isomorphism $m : (\mathrm{GL}_2, Z_Q) \rightarrow (\mathrm{GL}_2, X)$ of pre-Shimura data.

Let $V := \mathbb{G}_a^2$ be the standard representation⁵ of GL_2 . Let Q' be the group scheme $V^2 \rtimes \mathrm{GL}_2^2$ and Q be the group scheme $V^2 \rtimes \mathrm{GL}_2$. We will also denote by $h_0^Q : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow Q'_\mathbb{R}$ the morphism obtained by composing h_0^Q with a rational section of the natural projection $Q' \rightarrow \mathrm{GL}_2^2$. Such a section is unique up to an element of $V^2(\mathbb{Q})$. Let $Y_Q := Q(\mathbb{R}) \cdot h_0^Q \subset \mathrm{Hom}(\mathbb{G}_{m, \mathbb{R}}^2, Q'_\mathbb{R})$ be the $Q(\mathbb{R})$ -conjugacy class of h_0^Q . It does not depend on the chosen section because $V^2 \subset Q$.

DEFINITION 2.2.1. We call the pair (Q, Y_Q) the *pre-Shimura datum of the universal family of geodesics*.

The next lemma will explain this definition.

Let $\pi' : Q' \rightarrow \mathrm{GL}_2^2$ be the natural projection. This projection induces a natural map $Y_Q \rightarrow Z_Q$ that is compatible with the projection $\pi : Q \rightarrow \mathrm{GL}_2$. This yields a morphism of pre-Shimura data $\pi : (Q, Y_Q) \rightarrow (\mathrm{GL}_2, Z_Q)$. If we compose this morphism with m , we get a natural morphism of pre-Shimura data

$$(Q, Y_Q) \rightarrow (\mathrm{GL}_2, X)$$

which is in fact the quotient map by the additive group V^2 .

The Shimura fibered space $\mathrm{Sh}(Q, Y_Q) \rightarrow \mathrm{Sh}(\mathrm{GL}_2, X)$ can be considered as a universal family of geodesics because of the following lemma.

⁵i.e., $V(\mathbb{Z}) = \mathbb{Z}^2$.

LEMMA 2.2.2. *The fiber of the projection $\mathrm{Sh}_{Q(\widehat{\mathbb{Z}})}(Q, Y_Q) \rightarrow \mathrm{Sh}_{\mathrm{GL}_2(\widehat{\mathbb{Z}})}(\mathrm{GL}_2, X)$ over a point $[(V, F_x, F_y)]$ of the space of geodesics is the space $\mathbb{Z}^2 \backslash \mathbb{R}^2 / F_x \times \mathbb{Z}^2 \backslash \mathbb{R}^2 / F_y$, product of the two leaves spaces of the corresponding linear foliations on the torus $\mathbb{Z}^2 \backslash \mathbb{R}^2 = V(\mathbb{Z}) \backslash V(\mathbb{R})$.*

PROOF. We can embed Q' in $\mathrm{GL}_3^{\mathbb{Z}}$ by considering pairs of matrices of the form

$$\left(\begin{pmatrix} A_1 & v_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} A_2 & v_2 \\ 0 & 1 \end{pmatrix} \right)$$

with $A_1, A_2 \in \mathrm{GL}_2$ and $v_1, v_2 \in V$. The morphism h_0^Q is now given by the matrix $\left(\begin{pmatrix} Z_x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} Z_y & 0 \\ 0 & 1 \end{pmatrix} \right)$ with $Z_x = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, $Z_y = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$. The $Q(\mathbb{R})$ -conjugacy class Y_Q of h_0^Q is given by pairs of matrices of the form

$$(M_x, M_y) := \left(\begin{pmatrix} AZ_x A^{-1} & (I - AZ_x A^{-1})v_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} AZ_y A^{-1} & (I - AZ_y A^{-1})v_2 \\ 0 & 1 \end{pmatrix} \right)$$

with $A \in \mathrm{GL}_2(\mathbb{R})$. The projection $Y_Q \rightarrow Z_Q$ sends such a matrix to the pair $(AZ_x A^{-1}, AZ_y A^{-1})$. The action of $(v'_1, v'_2) \in V^2(\mathbb{R})$ by conjugation on a pair (M_x, M_y) as above gives

$$\left(\begin{pmatrix} AZ_x A^{-1} & (I - AZ_x A^{-1})(v_1 + v'_1) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} AZ_y A^{-1} & (I - AZ_y A^{-1})(v_2 + v'_2) \\ 0 & 1 \end{pmatrix} \right).$$

In this way, we see directly that the fibre $Y_{Q,h}$ of this projection over some $h \in Z_Q \cong X$ is a $V^2(\mathbb{R})$ -homogeneous space. The stabilizer of any point of $Y_{Q,h}$ for this $V^2(\mathbb{R})$ -action is the sum $F_y \oplus F_x \subset V(\mathbb{R}) \oplus V(\mathbb{R})$. This shows that $Y_{Q,h} \cong V(\mathbb{R})/F_y \times V(\mathbb{R})/F_x$. Since the projection is equivariant with respect to the projection $Q \rightarrow \mathrm{GL}_2$, the fibre of the projection $\mathrm{Sh}_{Q(\widehat{\mathbb{Z}})}(Q, Y_Q) \rightarrow \mathrm{Sh}_{\mathrm{GL}_2(\widehat{\mathbb{Z}})}(\mathrm{GL}_2, X)$ at the point $[\mathrm{GL}_2(\mathbb{Z}) \cdot h]$ is given by the space $\mathbb{Z}^2 \backslash \mathbb{R}^2 / F_x \times \mathbb{Z}^2 \backslash \mathbb{R}^2 / F_y$. \square

2.3. The moduli problem for non-commutative tori. In this section, we want to understand Manin's point of view of the moduli space of non-commutative tori (in terms of pseudo-lattices) in the spirit of Hodge structures. To this end, we introduce the notion of pre-lilac.

DEFINITION 2.3.1. A (rank 2) *pre-lilac*⁶ is a pair (M, F) of a free \mathbb{Z} -module of rank two and a real line $F \subset M_{\mathbb{R}}$. A *morphism of pre-lilacs* $(M_1, F_1) \rightarrow (M_2, F_2)$ is a morphism of Abelian groups $f : M_1 \rightarrow M_2$ such that $f_{\mathbb{R}}(F_1) \subset F_2$.

This notion is equivalent to Manin's notion of pseudo-lattice with weak morphisms (see [Man]) but it is easier for us to formulate our results in terms of lilacs because of their analogy with complex structures.

From a pre-lilac (M, F) , one can construct a non-commutative algebra $\mathcal{A}(M, F)$, called the *Kronecker foliation algebra* $C^\infty(M \backslash M_{\mathbb{R}}) \rtimes F$. We will call such an algebra a *non-commutative torus* because the choice of an element e of a basis for M allows one to construct a Morita equivalent algebra $\mathcal{T}(M, F, e) = C^\infty(\mathbb{Z}.e \backslash M_{\mathbb{R}}/F) \rtimes [\mathbb{Z}.e \backslash M]$ which is an irrational rotation algebra, i.e., a non-commutative torus in the usual sense. The Morita equivalence

⁶Abbreviation for **L**ine in a **L**Atti**C**e.

$\mathcal{A}(M, F) \sim \mathcal{T}(M, F, e)$ follows from [GBVF01], corollary 12.20⁷. We are interested in the set of isomorphism classes of Kronecker foliation algebras. Let (\mathbf{Nct}) be the category of such algebras with $*$ -isomorphisms as morphisms. Let $(\mathbf{pre-Lilacs})$ be the category of pre-lilacs with isomorphisms as morphisms. The assignment $(M, F) \mapsto \mathcal{A}(M, F)$ gives a functor

$$T : \begin{array}{ccc} (\mathbf{pre-Lilacs}) & \rightarrow & (\mathbf{Nct}) \\ (M, F) & \mapsto & \mathcal{A}(M, F). \end{array}$$

For an object A of (\mathbf{Nct}) , denote by $S : HC_2(A) \rightarrow HC_0(A)$ the periodicity map in cyclic homology, by $\text{ch} : K_0(A) \rightarrow HC_2(A)$ the Chern character defined in [Lod98], and by $\text{ch}_{\mathbb{R}} : K_0(A) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow HC_2(A)$ the corresponding map of \mathbb{R} -vector spaces.

There is also a functor in the other direction

$$L : \begin{array}{ccc} (\mathbf{Nct}) & \rightarrow & (\mathbf{pre-Lilacs}) \\ A & \mapsto & (K_0(A), \text{ch}_{\mathbb{R}}^{-1}(\text{Ker}(S)) \subset K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}) \end{array}$$

The facts that ch induces an isomorphism $K_0(A) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow HC_2(A)$, and that the filtration by $\text{Ker}(S)$ on cyclic homology is real with respect to the real structure given by K -theory can be deduced from the explicit calculations of Lemma 54 of [Con85]. Notice that this functor was already present in the paper [CDS97].

LEMMA 2.3.2. *Two Kronecker foliation algebras \mathcal{A}_1 and \mathcal{A}_2 are Morita equivalent if and only if they are isomorphic as $*$ -algebras.*

PROOF. This comes from the fact that for every objects $\mathcal{A}(M, F) \in (\mathbf{Nct})$, one has $L(\mathcal{A}(M, F)) \cong (M, F)$. To prove this, we can look at the Morita equivalent algebra $\mathcal{T}(M, F, e)$, for e an element of a basis for M and use the explicit calculation of its Chern character in [Con85]. We thus obtain that for all object $\mathcal{A} \in (\mathbf{Nct})$, $T \circ L(\mathcal{A}) \cong \mathcal{A}$. To finish the proof, if the two Kronecker algebras are Morita equivalent, then Morita invariance of cyclic cohomology, K -theory and Chern character implies $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$ and this implies $\mathcal{A}_1 \cong \mathcal{A}_2$. \square

The relation with Manin's definition of pseudo-lattices associated to non-commutative tori is the following. The long exact sequence of cyclic homology gives

$$HH_2(A) \xrightarrow{I} HC_2(A) \xrightarrow{S} HC_0(A) \xrightarrow{B} HH_1(A)$$

and we know that S is surjective in this case (by explicit calculation given in [Con85]), so it gives an isomorphism $HC_2(A)/\text{Ker}(S) \xrightarrow{S} HC_0(A)$. Since the Chern character is compatible with S , the natural map $S \circ \text{ch} : K_0(A) \rightarrow HC_0(A)$ induced by $\text{ch} : K_0(A) \rightarrow HC_2(A)$ is equal to the Chern character $\text{ch} : K_0(A) \rightarrow HC_0(A)$. The pseudo-lattices that Manin considers are given by this Chern character. So the functor that associates to a pre-lilac (M, F) the pseudo-lattice $(M, M_{\mathbb{C}}/F_{\mathbb{C}})$ has a natural interpretation in cyclic homology as the association $(K_0(A), \text{ch}_{\mathbb{R}}^{-1}(\text{Ker}(S))) \rightsquigarrow (K_0(A), HC_0(A) = HC_2(A)/\text{Ker}(S))$.

The two functors T and L naturally identify the *set* of isomorphism classes of pre-lilacs and the *set* of isomorphism classes of non-commutative tori.

⁷See also [GBVF01], theorem 12.17 for a choice-of-basis free proof of this fact.

Fix a free \mathbb{Z} -module M of rank two. The projective space $\mathbb{P}(M_{\mathbb{R}})$ over $M_{\mathbb{R}}$ gives a parameter space for lines $F' \subset M_{\mathbb{R}}$. Two such lines correspond to isomorphic pre-lilacs if and only if they are exchanged by some $g \in \mathrm{GL}(M)$. So the set of isomorphism classes of pre-lilacs is $\mathrm{GL}(M) \backslash P(M_{\mathbb{R}}) \cong \mathrm{GL}_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{R})$. As said before, this is also the moduli set for non-commutative tori.

We also remark that there is a natural projection

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Z}) \backslash X &\rightarrow \mathrm{GL}_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{R}) \\ (M, F_x, F_y) &\mapsto (M, F_x) \end{aligned}$$

from the space of geodesics to the moduli space of pre-lilacs. In non-commutative geometry, the left hand side of this projection can be interpreted as the moduli space for triples

$$(\mathcal{A}_1, \mathcal{A}_2, \psi : K_0(\mathcal{A}_1) \xrightarrow{\sim} K_0(\mathcal{A}_2))$$

consisting of two Kronecker foliation algebras and an isomorphism ψ between their K_0 groups, such that if F_1 and F_2 are the real lines in K -theory constructed in this section, $\psi_{\mathbb{R}}(F_1) \oplus F_2 = K_0(\mathcal{A}_2)_{\mathbb{R}}$.

REMARK 2.3.3. As we saw in Section 1.5, the study of nontrivial level structures on geodesics (and thus on non-commutative tori) is also interesting because of class field theory. We will denote by $\mathrm{Irrat}(X)^{\pm} \subset X^{\pm}$ the space of pairs of irrational oriented lines in \mathbb{R}^2 . Let $N > 1$ be an integer, and let K_N be the group defined by the exact sequence $1 \rightarrow K_N \rightarrow \mathrm{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \rightarrow 1$. Then the Shimura space $\mathrm{Sh}_{K_N}(\mathrm{GL}_2, \mathrm{Irrat}(X)^{\pm})$ is the moduli space of tuples

$$(M, F_x, F_y, s_x, s_y, \phi : M \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2)$$

consisting of a free \mathbb{Z} -module M of rank 2, two irrational lines F_x, F_y in the underlying real vector space, equipped with two orientations s_x, s_y , and a level structure ϕ . Perhaps this space has a moduli interpretation in terms of tuples

$$(\mathcal{A}_1, \mathcal{A}_2, \psi : K_0(\mathcal{A}_1) \xrightarrow{\sim} K_0(\mathcal{A}_2), \phi : K_0(\mathcal{A}_1) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2).$$

The orientation on the real lines could be given by the image of the positive cone in K -theory. The author's knowledge is, however, not sharp enough to be sure of this non-commutative moduli interpretation.

REMARK 2.3.4. Polishchuk's remarkable work [Pol03] on the relation of analytic non-commutative tori with nonstandard t -structures on derived categories of coherent sheaves on usual elliptic curves seems to be promising, because it allows one to give some rationality properties to the objects we have on hand. For example, if E is an elliptic curve over \mathbb{Q} and we fix some t -structure on its associated analytic curve given by a line with real quadratic slope on $K_0(E) \otimes_{\mathbb{Z}} \mathbb{R}$, we can ask questions about the rationality of this t -structure. The Algebraic Proj construction in [Pol02] for real multiplication non-commutative tori also allows one to ask rationality questions.

2.4. The universal non-commutative torus. For at least four reasons (esthetic symmetry, class field theory, Mumford-Tate groups and the complications that appear in this paragraph), it seems to be more natural to study the moduli space for *pairs* of non-commutative tori as above (which is also the space of geodesics) rather than the moduli space of solitary non-commutative

tori. However, we still want to construct a universal non-commutative torus because it permits to understand the relation with Tate mixed Hodge structures.

Let h'_0 be the morphism of algebraic groups given by

$$h'_0 : \mathbb{G}_{m,\mathbb{R}}^2 \rightarrow \mathrm{GL}_{2,\mathbb{R}} \\ (x, y) \mapsto \begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix}.$$

ASIDE 2.4.1 (rational boundary component). If we denote by U the group of unipotent upper triangular matrices and by P_1 the group of matrices of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ in $\mathrm{GL}_{2,\mathbb{Q}}$, then the $U(\mathbb{C})$ -conjugacy class Y'_1 of $h'_{0,\mathbb{C}}$ is identified with $U(\mathbb{C}) = \mathbb{C}$. Let Y_1 be $Y'_1 \times \{\pm 1\}$. Pink calls the pair (P_1, Y_1) a *rational boundary component* of $(\mathrm{GL}_2, \mathbb{H}^\pm)$. These kind of (mixed) Shimura data appear in the toroidal compactification of Shimura varieties. They are parameter spaces for mixed Hodge structures. In our particular case, the mixed Hodge structures are extensions

$$0 \rightarrow \mathbb{Z}(1) \rightarrow M \rightarrow \mathbb{Z} \rightarrow 0$$

that can be described algebraically by 1-motives of the form $[\mathbb{Z} \rightarrow \mathbb{G}_{m,\mathbb{C}}]$ (see [Del74] section 10 for a definition of 1-motives). Level structures on these one-motives are strongly related to roots of unity in \mathbb{C} , i.e., to generators of \mathbb{Q}^{ab} . In some sense (see [Pin90], 10.15 to give a precise meaning to this affirmation), we can say that $\mathrm{Sh}(P_1, Y_1)$ is a universal family over the moduli space $\mathrm{Sh}(\mathbb{G}_{m,\mathbb{Q}}, \{\pm 1\})$ of primitive roots of unity.

We will denote by X' the $\mathrm{GL}_2(\mathbb{R})$ -conjugacy class of h'_0 . Recall that X' is the set $\{(F_{xy}, F_0)\}$ of pairs of distinct lines in \mathbb{R}^2 of respective weights 0 and xy . There is a natural projection map $X' \rightarrow \mathbb{P}^1(\mathbb{R})$ given by $(F_{xy}, F_0) \mapsto F_0$.

Recall from Section 2.1 that P is the group scheme $V \rtimes \mathrm{GL}_2$, with $V := \mathbb{G}_a^2$ the standard representation of GL_2 . As usual, we also denote by $h'_0 : \mathbb{G}_{m,\mathbb{R}}^2 \rightarrow P_{\mathbb{R}}$ the morphism obtained by composing h'_0 with a rational section of the natural projection $P \rightarrow \mathrm{GL}_2$. Such a section is unique up to an element of $V(\mathbb{Q})$. Let Y_P be the $P(\mathbb{R})$ -conjugacy class of h'_0 . It is independent of the choice of the section of $P \rightarrow \mathrm{GL}_2$ because $V \subset P$.

Let $V_P = \mathbb{Q}^3$ be the standard representation of P given by an embedding $P \hookrightarrow \mathrm{GL}_3$. For $h \in Y_P$, we let $F^0(h) := \{v \in V_{P,\mathbb{R}} \mid h(x, y) \cdot v = v\}$. On Y_P , we have the equivalence relation:

$$h \sim h' \Leftrightarrow F^0(h)V' = F^0(h')V'.$$

Let $\overline{Y_P} = Y_P / \sim$ be the corresponding quotient.

DEFINITION 2.4.2. We call the pair $(P, \overline{Y_P})$ the *pre-Shimura datum of the universal family of non-commutative tori*.

The following lemma shows us that the Shimura fibered space $\mathrm{Sh}(P, \overline{Y_P}) \rightarrow \mathrm{Sh}(\mathrm{GL}_2, \mathbb{P}^1(\mathbb{R}))$ can be considered as a *universal family of non-commutative tori*.

LEMMA 2.4.3. *The fiber of the projection*

$$\mathrm{Sh}_{P(\widehat{\mathbb{Z}})}(P, Y_P) \rightarrow \mathrm{Sh}_{\mathrm{GL}_2(\widehat{\mathbb{Z}})}(\mathrm{GL}_2, \mathbb{P}^1(\mathbb{R})) = \mathrm{PGL}_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{R})$$

over a point (M, F_0) of the space of pre-lilacs is the space $M \setminus M_{\mathbb{R}} / F_0$ of leaves of the corresponding foliation on the two-torus $M \setminus M_{\mathbb{R}}$.

PROOF. The proof is essentially the same as the one of Lemma 2.2.2. The additional fact to check is that forgetting the F_{xy} part of the filtration is compatible with the quotient, which follows from the definition. \square

3. Two higher dimensional examples

3.1. Hilbert modular varieties. Let E be a totally real number field with ring of integer \mathcal{O}_E , let $I := \text{Hom}(E, \mathbb{R})$ and let $n := \text{card}(I)$. Denote by G the group scheme $\text{Res}_{\mathcal{O}_E/\mathbb{Z}} \text{GL}_2$. We then have $G_{\mathbb{R}} = \prod_{\iota: E \rightarrow \mathbb{R}} \mathbb{G}_{\mathbb{R}, \iota} \cong \prod_{\iota: E \rightarrow \mathbb{R}} \text{GL}_{2, \mathbb{R}}$.

Let $h := \prod_{\iota: E \rightarrow \mathbb{R}} h_{\iota} : \mathbb{S} \rightarrow \mathbb{G}_{\mathbb{R}}$ with $h_{\iota} : \mathbb{S} \rightarrow \text{GL}_{2, \mathbb{R}}$ the map that sends $z = a + ib \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$ to the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Let X be the $G(\mathbb{R})$ -conjugacy class of this morphism. We have an isomorphism $X \cong \prod_{\iota: E \rightarrow \mathbb{R}} \mathbb{H}^{\pm}$.

DEFINITION 3.1.1. The Shimura datum (G, X) is called the *Hilbert modular Shimura datum*.

Let $h(E)$ be the Hilbert class number of E . Let K be the compact open subgroup $G(\hat{\mathbb{Z}})$ of $G(\mathbb{A}_f)$. The associated Shimura variety is by definition

$$\begin{aligned} \mathcal{M} = \text{Sh}_K(G, X) &= G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)) / K \\ &= \text{GL}_2(\mathcal{O}_E) \backslash X \text{ if } h(E) = 1, \\ &\cong \text{GL}_2(\mathcal{O}_E) \backslash (\prod_{\iota: E \rightarrow \mathbb{R}} \mathbb{H}^{\pm}) \text{ if } h(E) > 1, \end{aligned}$$

i.e., the Hilbert modular variety. Now let B' be the subgroup of upper triangular matrices in $\text{GL}_{2, \mathcal{O}_E}$ and denote by B the group scheme $\text{Res}_{\mathcal{O}_E/\mathbb{Z}} B'$.

ASIDE 3.1.2 (rational boundary component). The group scheme B is a maximal parabolic subgroup of G and corresponds to a rational boundary component (P_1, X_1) of (G, X) as in [Pin90], 4.11. The canonical model of the associated mixed Shimura variety is a moduli space defined over \mathbb{Q} for 1-motives with additional structures.

Let $h_0^H : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow G_{\mathbb{R}}$ be the morphism given on each simple component of $G_{\mathbb{R}}$ by $h_0 : (x, y) \in (\mathbb{R}^*)^2 \mapsto \text{diag}(x, y) := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Let X_H be the $G(\mathbb{R})$ -conjugacy class of h_0^H .

DEFINITION 3.1.3. We will call the pair (G, X_H) the *pre-Shimura datum of the moduli space of Hilbert lilacs*.

The corresponding Shimura space $\text{Sh}_{G(\hat{\mathbb{Z}})}(G, X_H)$ is a moduli space for tuples (M, F_x, F_y, i) consisting of a rank $2n$ free \mathbb{Z} -module M , equipped with a decomposition

$$M_{\mathbb{R}} = F_x \oplus F_y$$

of its underlying real vector space in two n -dimensional subspaces, and with a morphism $i : E \hookrightarrow \text{End}(M_{\mathbb{Q}}, F_x, F_y)$ (i.e., a morphism compatible with the decomposition). These objects are called *Hilbert lilacs*.

EXAMPLE 3.1.4. Let $E := \mathbb{Q}(\sqrt{2})$ and $F := \mathbb{Q}(\sqrt{2}, \sqrt{3})$. For each embedding ι in $\text{Hom}(E, \mathbb{R})$, choose one embedding c_ι over it in $\text{Hom}(F, \mathbb{R})$. Such a choice is called an *RM type* for F/E . Equip $M := \mathcal{O}_F$ with the ‘‘Hodge decomposition’’

$$M_{\mathbb{R}} = F_x \oplus F_y$$

where $F_x := \bigoplus_{\iota} F_{c_\iota} \cong \mathbb{R}^2$, and F_y is the other component in the natural decomposition of $M_{\mathbb{R}}$. Then the lilac (M, F_x, F_y) gives a point in the space of Hilbert lilacs corresponding to E , and this point has Mumford-Tate group contained in $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \subset G = \text{Res}_{E/\mathbb{Q}} \text{GL}_2$. It is called a *special point* of this space.

Let h_0^{HQ} be the product morphism $\prod_{\iota: E \rightarrow \mathbb{R}} h_0^Q : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow G_{\mathbb{R}}^2$, where $h_0^Q : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow \text{GL}_{2, \mathbb{R}}^2$ is the morphism that sends the pair $(x, y) \in (\mathbb{R}^*)^2$ to the pair of matrices $\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \right)$.

Let $Z_{HQ} := G(\mathbb{R}) \cdot h_0^{HQ}$ be the $G(\mathbb{R})$ -conjugacy class of h_0^{HQ} in $\text{Hom}(\mathbb{G}_{m, \mathbb{R}}^2, G_{\mathbb{R}}^2)$. The multiplication map $m : G^2 \rightarrow G$ (which is not a group homomorphism) induces, as in Subsection 2.2, a natural isomorphism $m : (G, Z_{HQ}) \rightarrow (G, X_H)$ of pre-Shimura data.

Let $V := \text{Res}_{\mathcal{O}_E/\mathbb{Z}} \mathbb{G}_{a, E}^2$ be the standard representation⁸ of G . Let Q' be the group scheme $V^2 \rtimes G^2$ and let Q be the group scheme $V^2 \rtimes G$. We will also denote by $h_0^{HQ} : \mathbb{G}_{m, \mathbb{R}}^2 \rightarrow Q'_{\mathbb{R}}$ the morphism obtained by composition with a rational section of the natural projection $Q' \rightarrow G^2$. Such a section is unique up to an element of $V^2(\mathbb{Q})$.

Let $Y_{HQ} := Q(\mathbb{R}) \cdot h_0^{HQ} \subset \text{Hom}(\mathbb{G}_{m, \mathbb{R}}^2, Q'_{\mathbb{R}})$ be the $Q(\mathbb{R})$ -conjugacy class of h_0^{HQ} . It does not depend on the chosen section because $V^2 \subset Q$.

DEFINITION 3.1.5. We call the pair (Q, Y_{HQ}) the *pre-Shimura datum of the universal family of Hilbert lilacs*.

The next lemma will explain this definition. Let $\pi' : Q' \rightarrow G^2$ be the natural projection. This projection induces a natural map $Y_{HQ} \rightarrow Z_{HQ}$ that is compatible with the natural projection $\pi : Q \rightarrow G$. This yields a morphism of pre-Shimura data $\pi : (Q, Y_{HQ}) \rightarrow (G, Z_{HQ})$. If we compose this morphism with m , we get a natural morphism of pre-Shimura data

$$(Q, Y_{HQ}) \rightarrow (G, X_H)$$

which is in fact the quotient map by the additive group V^2 .

The Shimura fibered space $\text{Sh}(Q, Y_{HQ}) \rightarrow \text{Sh}(G, X_H)$ can be considered as a universal family of Hilbert lilacs because of the following lemma.

LEMMA 3.1.6. *The fiber of the projection $\text{Sh}_{Q(\widehat{\mathbb{Z}})}(Q, Y_Q) \rightarrow \text{Sh}_{G(\widehat{\mathbb{Z}})}(G, X)$ over a point $[(V, F_x, F_y, i)]$ of the space of Hilbert lilacs is the space $V \setminus V_{\mathbb{R}}/F_x \times V \setminus V_{\mathbb{R}}/F_y$, product of the two leaves spaces of the corresponding linear foliations on the torus $V \setminus V_{\mathbb{R}}$.*

PROOF. The proof is essentially the same as in Lemma 2.2.2. □

⁸i.e., $V(\mathbb{Z}) = \mathcal{O}_E^2$.

REMARK 3.1.7. The centralizer $C_{G(\mathbb{R})}(h_0^H)$ of our basis morphism is isomorphic to the maximal torus $T(\mathbb{R})$ of $G(\mathbb{R})$. We define the Mumford-Tate group of some $h \in X_H$ as in Definition 1.4.1 as a reductive envelope, defined by André, Kahn and O'Sullivan, [AKO02]. Such a reductive envelope is well defined up to the centralizer of the enveloped group. We will now change a little bit our description of X_H in order to have a well defined Mumford-Tate group for all points in this (and other) spaces. These groups could also be of some interest in the study of dynamical properties of the corresponding foliations, as suggested to the author by Yves André and Etienne Ghys.

Let D be the basic maximal torus of G , i.e., $D := \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m^2$. Let $\mathbb{D} := D_{\mathbb{R}}$ be the corresponding real algebraic group, and let $h_{0,\mathcal{R}}^H : \mathbb{D} \rightarrow G_{\mathbb{R}}$ be the natural inclusion. Let \mathcal{R}_H be the $G(\mathbb{R})$ -conjugacy class of $h_{0,\mathcal{R}}^H$. The inclusion⁹ $\mathbb{G}_{m,\mathbb{R}}^2 \subset \mathbb{D}$ induces a $G(\mathbb{R})$ -equivariant bijection $\mathcal{R}_H \rightarrow X_H$, i.e., an isomorphism of pre-Shimura data

$$(G, \mathcal{R}_H) \rightarrow (G, X_H).$$

DEFINITION 3.1.8. Let $h \in \mathcal{R}_H$. We define the *bad Mumford-Tate group* of h to be the smallest \mathbb{Q} -algebraic subgroup $\text{BMT}(h) \subset G$ such that $\text{im}(h) \subset \text{BMT}(h)(\mathbb{R})$. A *Mumford-Tate group* for h is a minimal reductive subgroup of G that contains $\text{BMT}(h)$.

LEMMA 3.1.9. *Let $h \in \mathcal{R}_H$. There exists a unique Mumford-Tate group for h . It will be denoted by $\text{MT}(h)$.*

PROOF. The bad Mumford-Tate group of h contains a maximal torus T of G (because the image of h is a maximal torus over \mathbb{R}). We know from [SGA64], XIX, 2.8 (or other classical references on algebraic groups) that the centralizer $C_G(T)$ of T in G is T itself, because G is a reductive group. We have an inclusion of centralizers $C_G(\text{BMT}(h)) \subset C_G(T) = T$, and we already know that $T \subset \text{BMT}(h)$. So we have $C_G(\text{BMT}(h)) \subset \text{BMT}(h)$, and the fact that a reductive envelope of a group is well defined up to the centralizer of the group implies that, in this case, the Mumford-Tate group is well-defined. \square

REMARK 3.1.10. This result gives a motivation to study the pair (G, \mathcal{R}_H) itself. This pair is called the *shore datum of the Hilbert modular datum* (G, X) . Another good reason to use a bigger basis group \mathbb{D} for morphisms is given by the need, in Manin's real multiplication program, to take into account archimedean places in class field theory of totally real fields. This has been investigated in Section 1.5 in the case $E = \mathbb{Q}$ and in [Pau04] for other quite general examples.

3.2. The moduli space of Abelian surfaces. We first recall the construction of the moduli space of Abelian surfaces and then construct one of its irrational boundaries. As we will see, this gives a non-totally degenerate example, that is in some sense more general than the case of Hilbert moduli spaces.

Let $V = \mathbb{Z}^4$ and $\psi : V \times V \rightarrow \mathbb{Z}$ be the standard symplectic form given by the matrix $J := \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ with $I_2 \in M_2(\mathbb{Z})$ the identity matrix. Let G

⁹Induced by the rational inclusion $\mathbb{G}_{m,\mathbb{Q}}^2 \subset D$.

be the corresponding group scheme of symplectic similitudes, whose points in a \mathbb{Z} -algebra A are given by $G(A) = \mathrm{GSp}_{4,\mathbb{Z}}(A) := \{M \in \mathrm{M}_4(A) \mid \exists \alpha(M) \in A^* \text{ with } MJM^{-1} = \alpha(M) \cdot J\}$.

Let $h_i^S : \mathbb{S} \rightarrow G_{\mathbb{R}}$ be the morphism that maps $z = a + ib \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$ to the matrix

$$\begin{pmatrix} aI_2 & bI_2 \\ -bI_2 & aI_2 \end{pmatrix},$$

and denote by \mathcal{S}^{\pm} the $G(\mathbb{R})$ -conjugacy class of h_i , that is usually called the two dimensional Siegel space.

DEFINITION 3.2.1. The datum $(\mathrm{GSp}_4, \mathcal{S}^{\pm})$ is called the *Siegel Shimura datum*.

Let K be the compact open subgroup $G(\hat{\mathbb{Z}})$ of $G(\mathbb{A}_f)$. The associated Shimura variety is by definition

$$\begin{aligned} \mathrm{Sh}_K(\mathrm{GSp}_4, \mathcal{S}^{\pm}) &= G(\mathbb{Q}) \backslash (\mathcal{S}^{\pm} \times G(\mathbb{A}_f) / K) \\ &= \mathrm{GSp}_4(\mathbb{Z}) \backslash \mathcal{S}^{\pm}, \end{aligned}$$

i.e., the Siegel modular variety, which is the moduli space of principally polarized Abelian surfaces.

Let $h_0^S : \mathbb{G}_{m,\mathbb{C}}^2 \rightarrow G_{\mathbb{C}}$ be the morphism that associates to $(x, y) \in (\mathbb{C}^*)^2$ the matrix

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & x & 0 & 0 \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & y \end{pmatrix},$$

with $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2i}$. Let $\mathcal{R}_{\mathcal{S}} := G(\mathbb{R}) \cdot h_0^S$ be the $G(\mathbb{R})$ -conjugacy class of h_0^S . The centralizer of h_0^S is a (non-split) maximal torus of $G(\mathbb{R})$ (the subgroup $T(\mathbb{R})$ of $\mathbb{C}^* \times (\mathbb{R}^*)^2$ given by $z\bar{z} = xy$), which implies that $\mathcal{R}_{\mathcal{S}} \cong \mathrm{GSp}_4(\mathbb{R})/T(\mathbb{R})$.

The Shimura space $\mathrm{Sh}_{G(\hat{\mathbb{Z}})}(G, \mathcal{R}_{\mathcal{S}})$ is a moduli space for tuples

$$(V, F_x, F_y, \Pi, F, \psi)$$

where V is a free \mathbb{Z} -module of rank 4, F_x, F_y are two distinct real lines and Π is a real plane in $V_{\mathbb{R}} = F_x \oplus F_y \oplus \Pi$. Moreover, F is a complex line in $\Pi_{\mathbb{C}} = F \oplus \bar{F}$, and $\psi : V \times V \rightarrow \mathbb{Z}$ is a symplectic form that respects the decomposition

$$V_{\mathbb{C}} = (F_x \oplus F) \bigoplus (F_y \oplus \bar{F}).$$

This new kind of linear algebra object is quite strange and does not seem to have an easy non-commutative geometric interpretation because it mixes usual complex structures with foliations on tori. Such objects appear, however, quite often in number theory.

EXAMPLE 3.2.2. Let $K := \mathbb{Q}[x]/(x^4 - 2)$. We have a decomposition $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{C}_z$, and the \mathbb{R} -algebra \mathbb{C}_z decomposes over \mathbb{C} as a product of two copies of \mathbb{C} . Fixing a nice alternating form on K gives us exactly a point in $\mathrm{Sh}_{G(\hat{\mathbb{Z}})}(G, \mathcal{R}_{\mathcal{S}})$. This point is called a *special point* or a *quadratic multiplication point*. We showed in [Pau04] that the counting of points of this type in the

Shimura space involves interesting number theoretical information, as in the case of geodesics studied in Section 1.5.

4. Some open problems

Here are some open problems¹⁰ in our work.

- Find a higher dimensional and/or algebraic analog of non-commutative 2-tori adapted to number theoretical purposes.
- Understand, in the case of geodesics, the relation of our work with Manin’s quantum theta functions, and Stark numbers.
- More generally, find an adelic formulation of Stark’s conjectures for quadratic fields over totally real fields.
- Define a well-behaved and not ad hoc notion of level structure on non-commutative tori (resp., on Polishchuk’s t-structures).
- Find the good higher dimensional analogs of Polishchuk’s t-structures on categories of coherent sheaves of Abelian varieties.
- Study moduli spaces for stability conditions on these categories.
- Clarify, if it exists, the relationship with Darmon’s work on Stark’s conjecture for real quadratic fields.

References

- [AKO02] Yves André, Bruno Kahn, and Peter O’Sullivan. Nilpotence, radicaux et structures monoïdales. *Rend. Sem. Mat. Univ. Padova*, 108:107–291, 2002.
- [BC95] J.-B. Bost and A. Connes. Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Math. (N.S.)*, 1(3):411–457, 1995.
- [BL01] Armand Borel and Ji Lizhen. *Compactifications of locally symmetric spaces*. IAS/Michigan, 2001. Prepublication, june 2001.
- [Car98] Pierre Cartier. La folle journée, de Grothendieck à Connes et Kontsevich: évolution des notions d’espace et de symétrie. In *Les relations entre les mathématiques et la physique théorique*, pages 23–42. Inst. Hautes Études Sci., Bures, 1998.
- [CDS97] Alain Connes, Michael R. Douglas, and Albert Schwarz. Noncommutative geometry and matrix theory: compactification on tori. *arXiv*, (<http://fr.arXiv.org/abs/hep-th/9711162>), 1997.
- [CM95] A. Connes and H. Moscovici. The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5(2):174–243, 1995.
- [Con80] Alain Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B*, 290(13):A599–A604, 1980.
- [Con85] Alain Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360, 1985.
- [Con99] Alain Connes. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. *Selecta Math. (N.S.)*, 5(1):29–106, 1999.
- [Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.
- [Del79] Pierre Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.

¹⁰Some of them are collected from the literature.

- [DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*. Springer-Verlag, Berlin, 1982. Philosophical Studies Series in Philosophy, 20.
- [GBVF01] José M. Gracia-Bondía, Joseph C. Várilly, and Héctor Figueroa. *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston Inc., Boston, MA, 2001.
- [Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.
- [Man] Yuri I. Manin. Real Multiplication and noncommutative geometry.
- [Mar04] Matilde Marcolli. Lectures on Arithmetic Noncommutative Geometry. *arXiv*, (math.QA/0409520), 2004.
- [Mil83] J. S. Milne. The action of an automorphism of \mathbf{C} on a Shimura variety and its special points. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 239–265. Birkhäuser Boston, Boston, MA, 1983.
- [Mil90] J. S. Milne. Canonical models of (mixed) Shimura varieties and automorphic vector bundles. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, pages 283–414. Academic Press, Boston, MA, 1990.
- [MM01] Yuri I. Manin and Matilde Marcolli. Holography principle and arithmetic of algebraic curves. *Adv. Theor. Math. Phys.*, 5(3):617–650, 2001.
- [MM02] Yuri I. Manin and Matilde Marcolli. Continued fractions, modular symbols, and noncommutative geometry. *Selecta Math. (N.S.)*, 8(3):475–521, 2002.
- [Pau04] F Paugam. Quelques bords irrationnels de variétés de shimura. *Universitaetsverlag Goettingen, Mathematisches Institut, Seminars*, pages 1–12, 2004.
- [Pin90] Richard Pink. *Arithmetical compactification of mixed Shimura varieties*, volume 209 of *Bonner Mathematische Schriften [Bonn Mathematical Publications]*. Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.
- [Pol02] Alexander Polishchuk. Noncommutative 2-tori with real multiplication as noncommutative projective varieties. *arXiv*, (<http://fr.arXiv.org/abs/math.AG/0212306>), 2002.
- [Pol03] Alexander Polishchuk. Classification of holomorphic vector bundles on noncommutative two-tori. *arXiv*, (<http://fr.arXiv.org/abs/math.QA/0308136>), 2003.
- [SGA64] *Schémas en groupes. III: Structure des schémas en groupes réductifs*. Springer-Verlag, Berlin, 1962/1964.
- [SW99] Nathan Seiberg and Edward Witten. String theory and noncommutative geometry. *J. High Energy Phys.*, (9):Paper 32, 93 pp. (electronic), 1999.

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