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Description of algebraically constructible functions

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Abstract. The algebraically constructible functions on a real algebraic set are the sums of signs of polynomials on this set. We prove a formula giving the minimal number of polynomials needed to write generically a given algebraically constructible function as a sum of signs. We also prove a characterization of the polynomials appearing in a generic presentation of the function with the minimal number of polynomials. Both results are effective.

Introduction

Let $V \subset \mathbb{R}^N$ be a real algebraic set. (All the algebraic sets we consider here are zero sets of polynomials in some $\mathbb{R}^N$.) We will denote by $\mathscr{P}(V)$ and $\mathscr{R}(V)$ respectively the ring of polynomials on V and the ring of regular functions on V. If V is irreducible, let $\mathscr{K}(V)$ be the field of rational functions on V. *Algebraically constructible functions* on V have been defined by McCrory and Parusiński in [9], as linear combinations, with integer coefficients, of Euler characteristics of fibres of proper regular morphisms. These authors use them to study the topology of real algebraic sets: in [9] they reformulate the Akbulut–King conditions of algebraicity in dimension $\leqslant 3$, and in [10] they give new necessary conditions for dimension 4.

If P is a polynomial function on V (or, more generally, a regular function or a Nash function), we define the *sign of* P as the function $(\operatorname{sgn} P) : V \to \mathbb{Z}$ such that for all $x \in V$

$$(\operatorname{sgn} P)(x) = \begin{cases} 1 & \text{if } P(x) > 0, \\ -1 & \text{if } P(x) < 0, \\ 0 & \text{if } P(x) = 0. \end{cases}$$

Parusiński and Szafraniec on one hand ([12]), and Coste and Kurdyka on the other ([8]), have proved independently that the algebraically constructible functions on V are exactly the sums of signs of polynomials on V.

Let $\varphi : V \to \mathbb{Z}$ be an algebraically constructible function. There are clearly many ways to write φ as a sum of signs of polynomials on V. For instance, for any

$P \in \mathscr{P}(V)$, we have $\varphi = \varphi + \operatorname{sgn} P + \operatorname{sgn}(-P)$, so we can get a presentation as long as we want. We are interested here in a presentation as short as possible.

We will work *generically*, i.e. outside an algebraic subset of V of dimension strictly smaller than the dimension of V. We will write $=_{\text{gen}}$ for an equality holding generically on V.

We prove a formula giving the minimal number of polynomials (counted "with multiplicities") needed to write generically an algebraically constructible function as a sum of signs of polynomials. This formula allows us to calculate effectively this minimal number, using an induction on the dimension of the space. The proof is a transposition to the geometric case of a result for quadratic forms over spaces of orderings: the isotropy theorem. There is a similar formula for Nash constructible functions.

Then, using the same type of proofs, we give results about the polynomials appearing in a generic presentation of a given algebraically constructible function with the minimal number of polynomials. Such polynomials are said to be *represented* by the function.

The paper is organized as follows: Section 1 is devoted to a short presentation of spaces of orderings, Section 2 contains the formula for the minimal number of polynomials, and Section 3 gives a characterization of the represented polynomials.

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1 Spaces of orderings

1.1 Presentation. We present spaces of orderings in the context of the real spectrum of a field. For a complete definition we refer to [3] or [11]. Here $\mathbb{Z}/2\mathbb{Z} = \{-1, 1\}$, and we denote by $\mathscr{F}(X, \mathbb{Z}/2\mathbb{Z})$ the set of functions from a set X to $\mathbb{Z}/2\mathbb{Z}$.

Let K be a real field. The set of the orderings of K (as a field) is called the *real spectrum* of K and denoted $\operatorname{Spec}_r K$. If a is a non-zero element of K, we define the function $(\operatorname{sgn} a)$: $\operatorname{Spec}_r K \to \mathbb{Z}/2\mathbb{Z}$ which maps an ordering σ to the sign of a for σ. Denote

$$G = \{\operatorname{sgn} a \mid a \in K - \{0\}\} \subset \mathscr{F}(\operatorname{Spec}_r K, \mathbb{Z}/2\mathbb{Z}).$$

Then $(\operatorname{Spec}_r K, G)$ is a space of orderings.

A subset C of $\operatorname{Spec}_r K$ is said to be *constructible* if it is a finite union of sets of the form

$$\{\sigma \in \operatorname{Spec}_r K \mid (\operatorname{sgn} a_1)(\sigma) = 1, \ldots, (\operatorname{sgn} a_r)(\sigma) = 1\}$$

with $a_1, \ldots, a_r \in K$. We consider the *constructible topology* on $\operatorname{Spec}_r K$, that is, the topology on $\operatorname{Spec}_r K$ for which the constructible subsets of $\operatorname{Spec}_r K$ form a basis.

Consider now a (non-empty) closed subset F of $\operatorname{Spec}_r K$. The set F is a *fan* of K if

$$\forall \sigma_1, \sigma_2, \sigma_3 \in F\ \exists \sigma \in F\ \forall g \in G : g(\sigma) = g(\sigma_1)g(\sigma_2)g(\sigma_3)$$

i.e. "the product of three elements of F is still in $\operatorname{Spec}_r K$ and belongs to F", where the product of orderings means the product of signs for these orderings.

Remark 1.1. Any subset of $\operatorname{Spec}_r K$ of one or two elements is a fan, and is called a *trivial fan*. The cardinality of a finite fan is always a power of two, since a fan has a structure of a $(\mathbb{Z}/2\mathbb{Z})$-affine space, with the product as inner operation, and the natural scalar multiplication.

Let X be a (non-empty) closed subset of $\operatorname{Spec}_r K$. Then the set X is a *subspace* of $\operatorname{Spec}_r K$ if there is no fan F of K such that $X \cap F$ has exactly three elements. If $H = \{(\operatorname{sgn} a)|_X \mid a \in K\backslash\{0\}\}$, then the couple (X, H) is a space of orderings.

Example 1.2. Let X be a set with a single element, and let H be the set of the two constant functions $X \to \{1\}$ and $X \to \{-1\}$. Then (X, H) is a space of orderings, called *the atomic space* and denoted by E.

If σ is an ordering of K and B is a valuation ring of K, we say that σ and B are *compatible* if for any a in K and any b in the maximal ideal $\mathfrak{m}$ of B, the relation $0 < a < b$ for σ implies $a \in \mathfrak{m}$. Then σ induces an ordering $\bar{\sigma}$ on the residue field k of B by

$$(\operatorname{sgn} \bar{a})(\bar{\sigma}) = (\operatorname{sgn} a)(\sigma) \quad \text{for } a \in B\backslash\mathfrak{m}$$

where $\bar{a}$ denotes the class of a in k. Conversely, if $\bar{\sigma} \in \operatorname{Spec}_r k$, the orderings of K compatible with B and inducing $\bar{\sigma}$ are called *pullbacks of $\bar{\sigma}$ via B*. If X is a subspace of $\operatorname{Spec}_r k$ (respectively a fan of k), then the set of the pullbacks of the elements of X via B is a subspace of $\operatorname{Spec}_r K$ (respectively a fan of K).

Example 1.3. We will use the following construction (see [4, Ex. 2.2]). Let A be a regular local ring of dimension d with quotient field K, and let $(x_1, \dots, x_d)$ be a regular system of parameters of A. Consider the valuation ring B of the place $K = K_0 \to K_1 \cup \infty \to \cdots \to K_d \cup \infty$, where K_i is the quotient field of the ring $A_i = A/(x_1, \dots, x_i)$ and the place $K_i \to K_{i+1} \cup \infty$ corresponds to the valuation ring $A_{i(\overline{x_{i+1}})}$ of K_i. The ring B is a discrete valuation ring of rank d. It dominates A and has the same residue field k. Any ordering $\bar{\sigma}$ of k has exactly 2^d pullbacks σ in K via B, and each of them is determined by the signs given to $x_1, \dots, x_d$.

We come now to the notion of form over a space of orderings.

Definition 1.4. Let (X, G) be a space of orderings. A *form of dimension r* over X is a class of r-tuples of elements of G modulo the relation

$$(f_1, \dots, f_r) \sim (g_1, \dots, g_r) \quad \text{iff } \forall \sigma \in X : f_1(\sigma) + \cdots + f_r(\sigma) = g_1(\sigma) + \cdots + g_r(\sigma).$$

We denote by $\langle f_1, \dots, f_r \rangle$ the class of $(f_1, \dots, f_r) \in G^r$. If X' is a subspace of X, the *restriction* of the form $\rho = \langle f_1, \dots, f_r \rangle$ to X' is the form $\rho|_{X'} = \langle f_1|_{X'}, \dots, f_r|_{X'} \rangle$ over X'.

Example 1.5. If K is an ordered field, a quadratic form (in the usual sense) of dimension r over K is a symmetric matrix of dimension r with entries in K. We can diagonalize this matrix, and the diagonal matrix we get corresponds to the previous definition for the real spectrum of K. The usual definition of signature of a quadratic form coincides with the following one.

Definition 1.6. The *signature* of the form $\rho = \langle f_1, \dots, f_r \rangle$ over the space of orderings X is the function $\hat{\rho} : X \to \mathbb{Z}$ defined by $\hat{\rho}(\sigma) = f_1(\sigma) + \cdots + f_r(\sigma)$.

A form over X is *anisotropic* if there is no form over X with the same signature and a strictly smaller dimension. A form which is not anisotropic is said to be *isotropic*.

If $\rho = \langle f_1, \dots, f_r \rangle$ and $\rho' = \langle g_1, \dots, g_s \rangle$ are two forms over X and h is an element of G, we define $\rho + \rho' = \langle f_1, \dots, f_r, g_1, \dots, g_s \rangle$ and $h\rho = \langle hf_1, \dots, hf_r \rangle$. Then we have $\widehat{\rho + \rho'} = \hat{\rho} + \hat{\rho}'$ and $\widehat{h\rho} = h\hat{\rho}$.

1.2 Structure. We present now two basic operations on spaces of orderings: addition and extension.

Let (X_1, G_1) and (X_2, G_2) be two spaces of orderings. The *sum* $(Y, H) = (X_1, G_1) + (X_2, G_2)$ is defined by $Y = X_1 \sqcup X_2$ (disjoint union) and $(g_1, g_2) \in H = G_1 \times G_2$ acting as

$$(g_1, g_2)(\sigma) = \begin{cases} g_1(\sigma) & \text{if } \sigma \in X_1, \\ g_2(\sigma) & \text{if } \sigma \in X_2. \end{cases}$$

The resulting space is a space of orderings.

If now (Y, H) is a space of orderings and H' is a group of exponent two, the *extension* $(Y, H)[H']$ is the couple $(\widehat{H'} \times Y, H' \times H)$, where $\widehat{H'}$ denotes the group of homomorphisms from H' to $\mathbb{Z}/2\mathbb{Z}$ and the functions are defined by

$$(h', h)(\alpha, \sigma) = \alpha(h')h(\sigma) \quad \text{for } (\alpha, \sigma) \in \widehat{H'} \times Y \text{ and } (h', h) \in H' \times H.$$

This defines a space of orderings.

Example 1.7. If B is a discrete valuation ring of rank d of a field K, and if Y is a subspace of the real spectrum of its residue field, then the set of pullbacks of the elements of Y via B is the extension $Y[(\mathbb{Z}/2\mathbb{Z})^d]$.

These two operations are very important, as the following theorem shows ([3, IV.5.1], [11, 4.2.2]).

Theorem 1.8 (Structure theorem). *Any finite space of orderings can be built from a finite number of atomic spaces, by a finite number of additions and extensions by $\mathbb{Z}/2\mathbb{Z}$. This construction is unique up to isomorphism.*

We now explain the behaviour of forms under addition and extension of spaces of orderings. Let X_1, X_2 be spaces of orderings. It follows from the definition of the sum that a form ρ of dimension r over $X_1 + X_2$ can be seen as a couple (ρ_1, ρ_2) where ρ_l is a form of dimension r over X_l for $l = 1, 2$. The form ρ is anisotropic over $X_1 + X_2$ if and only if ρ_1 is anisotropic over X_1 or ρ_2 is anisotropic over X_2.

If now Y is a space of orderings and H' a group of exponent two, an anisotropic form ρ over $Y[H']$ can be written in a unique way as $\rho = \sum_{h' \in H'} h' \rho_{h'}$, where the $\rho_{h'}$'s are anisotropic forms over Y, and only a finite number of $\rho_{h'}$'s are different from the "empty" form $\langle\ \rangle$.

1.3 Application to algebraically constructible functions. Let $V \subset \mathbb{R}^N$ be an irreducible real algebraic set, and let φ be an algebraically constructible function on V. Then φ is in particular constructible, i.e. there exists a finite semi-algebraic partition of V such that φ is constant on each element of the partition.

Denote $\Sigma_V = \mathrm{Spec}_r \mathscr{K}(V)$. As in [7], we identify the algebraically constructible function $\varphi = \sum_{j=1}^{r} \mathrm{sgn}\, P_j$ considered generically on V, with the signature $\tilde{\varphi}$ of the form $\langle f_1, \ldots, f_r \rangle$ over Σ_V, where $f_j = \mathrm{sgn}\, P_j$ on Σ_V.

Assume φ takes the value $k \in \mathbb{Z}$ on a semi-algebraic subset S of V. Consider the constructible subset $\tilde{S}$ of Σ_V defined by the same boolean combination of equations and sign conditions as S. The set $\tilde{S}$ is well-defined (see [5]). Then, the function $\tilde{\varphi}$ takes the value k on $\tilde{S}$.

The minimal number of polynomials needed to describe φ generically is the dimension of the anisotropic form over Σ_V with signature $\tilde{\varphi}$. In the same way, a polynomial P appears in a generic presentation of φ with the minimal number of polynomials if and only if the sign of P is an entry of the anisotropic form over Σ_V with signature $\tilde{\varphi}$. So instead of studying the geometric situation, we will study forms in the algebraic context of spaces of orderings.

2 Number of polynomials

Let $V \subset \mathbb{R}^N$ be a real algebraic set, and let $\varphi : V \to \mathbb{Z}$ be an algebraically constructible function. We want to calculate the minimal number $N(\varphi)$ of polynomials needed to write φ generically as a sum of signs of polynomials, i.e.

$$N(\varphi) = \min\left\{ r \in \mathbb{N} \mid \exists P_1, \ldots, P_r \in \mathscr{P}(V) : \varphi =_{\mathrm{gen}} \sum_{j=1}^{r} \mathrm{sgn}\, P_j \text{ on } V \right\}.$$

This means that if the same polynomial appears several times in the presentation, we will count it at each appearance. So we count the minimal number of polynomials "with multiplicities".

We will denote by $M(\varphi)$ the maximal generic value of the absolute value of φ. Since each polynomial in the presentation contributes the value 1 or -1, we have

$$N(\varphi) \geqslant M(\varphi).$$

If $V_1, \dots, V_r$ are the irreducible components of V, then we have

$$N(\varphi) = \max\{N(\varphi|_{V_1}), \dots, N(\varphi|_{V_r})\}.$$

Example 2.1. If $\dim V = 1$, we have $N(\varphi) = M(\varphi)$ for any algebraically constructible function φ. Indeed, by the previous remark, we may assume that V is irreducible. We can write generically φ as the sum of $M(\varphi)$ constructible functions, each of them taking generically the values 1 and -1. The space Σ_V is a so-called SAP-space, since its stability index is $s(\Sigma_V) = \dim V = 1$ (cf. [3, III.3.4], [11, 3.3]). Using [3, III.3.2] or [11, 3.3.1], we get that any constructible function on V with generic values in $\{1, -1\}$ is generically the sign of a polynomial.

If $\dim V \geqslant 2$, this equality no longer holds for a general algebraically constructible function. For instance, consider $\varphi : \mathbb{R}^2 \to \mathbb{Z}$ defined by

$$\varphi(x, y) = \begin{cases} 2 & \text{if } x \geqslant 0 \text{ and } y \geqslant 0, \\ -2 & \text{else.} \end{cases}$$

Then $M(\varphi) = 2$ and $N(\varphi)$ is even. We have $\varphi =_{\text{gen}} \operatorname{sgn} x + \operatorname{sgn} y + \operatorname{sgn}(xy) - 1$, so $N(\varphi) \leqslant 4$. If φ was generically the sum of the signs of two polynomials, these polynomials should be generically positive on the first quadrant and generically negative outside. Such polynomials do not exist, so $N(\varphi) = 4 > M(\varphi)$.

To make the presentation clear we start by introducing the algebraic tools we use to estimate $N(\varphi)$. Then we will present the formula, and prove it. Finally we will extend it to the Nash case.

2.1 Algebraic tools. From now on we denote $\mathbb{Z}/2\mathbb{Z} = \{1, a\}$ and $\widehat{\mathbb{Z}/2\mathbb{Z}} = \{1, \alpha\}$, that is, α is the identity.

Lemma 2.2. *Let X be a space of orderings, and let $\varphi : X \to \mathbb{Z}$ be the signature of a form over X. We denote by $N(X, \varphi)$ the dimension of the anisotropic form over X with signature φ.*

If X is a sum $X = X_1 + X_2$, then

$$N(X, \varphi) = \max\{N(X_1, \varphi|_{X_1}), N(X_2, \varphi|_{X_2})\}.$$

If X is an extension $X = Y[\mathbb{Z}/2\mathbb{Z}]$, we define two functions ψ', ψ'' on Y by $\psi'(\sigma) = \frac{1}{2}(\varphi(1, \sigma) + \varphi(\alpha, \sigma))$ and $\psi''(\sigma) = \frac{1}{2}(\varphi(1, \sigma) - \varphi(\alpha, \sigma))$. Then ψ' and ψ'' are signatures of forms over Y and

$$N(X, \varphi) = N(Y, \psi') + N(Y, \psi'').$$

Proof. Let ρ be the anisotropic form over X such that $\hat{\rho} = \varphi$.

Assume first $X = X_1 + X_2$. Write $\rho = (\rho_1, \rho_2)$ where ρ_l is a form over X_l for $l = 1, 2$. Then $N(X, \varphi) = \dim \rho = \dim \rho_1 = \dim \rho_2$, and either ρ_1 or ρ_2 is anisotropic. As ρ_l represents $\varphi|_{X_l}$, we have $N(X_l, \varphi|_{X_l}) \leqslant N(X, \varphi)$, with equality if ρ_l is anisotropic. This proves the first point of the lemma.

Assume now that X is the extension $X = Y[\mathbb{Z}/2\mathbb{Z}]$. We can write $\rho = \rho_1 + a\rho_a$ where ρ_1 and ρ_a are anisotropic forms over Y. Then $\varphi(1, \sigma) = \hat{\rho}_1(\sigma) + \hat{\rho}_a(\sigma)$ and $\varphi(\alpha, \sigma) = \hat{\rho}_1(\sigma) - \hat{\rho}_a(\sigma)$ for $\sigma \in Y$, so $\psi' = \hat{\rho}_1$ and $\psi'' = \hat{\rho}_a$. We get

$$N(X, \varphi) = \dim \rho = \dim \rho_1 + \dim \rho_a = N(Y, \psi') + N(X, \psi''). \qquad \square$$

The formula of the next paragraph is a geometric version of the following result ([3, IV.6.4], [11, 4.3.1]):

Theorem 2.3 (Isotropy theorem). *Let X be a space of orderings, and ρ an anisotropic form over X. Then there exists a finite subspace Y of X such that $\rho|_Y$ is still anisotropic.*

2.2 The geometric result. Let $V \subset \mathbb{R}^N$ be a real algebraic set, and let $\varphi : V \to \mathbb{Z}$ be an algebraically constructible function. We use the notion of walls of the function φ. This notion has been used by Acquistapace, Andradas, Broglia and Vélez to study basicness ([2]) and separation ([1]) of semi-algebraic sets, and by the author to characterize algebraically constructible functions ([7]). We recall the definition.

If $S \subset V$ is a semi-algebraic set, we denote by S^* its regularized version

$$S^* = \mathrm{Int}(\mathrm{Adh}(\mathrm{Int}(S) \cap \mathrm{Reg}(V))).$$

A *wall* of φ is an irreducible component, of codimension one in V, of the Zariski closure of the Euclidean boundary of a $(\varphi^{-1}(m))^*$.

By the remark at the beginning of the section, we can work independently on each irreducible component of V. So from now on, we assume that V is irreducible.

If $\overline{V}$ is a one-point compactification of V, we can extend φ to a function $\bar{\varphi}$ on $\overline{V}$, by giving any value at the additional point. Then $\bar{\varphi}$ is algebraically constructible on $\overline{V}$, and we have $N(\bar{\varphi}) = N(\varphi)$. So from now on, we also assume that V is compact.

Now, consider $\pi : V' \to V$, a sequence of blowings-up with smooth centers, such that V' is non-singular, and that the walls in V' of the algebraically constructible function $\varphi \circ \pi$ are non-singular with normal crossings intersections. (By this we mean that there is a family of polynomials $P_1, \ldots, P_s \in \mathscr{P}(V')$ describing $((\varphi \circ \pi)^{-1}(k))^*$, for $k \in \mathbb{Z}$, such that all the P_j's are at normal crossings in V'.) We have $N(\varphi \circ \pi) = N(\varphi)$. So we consider from now on this non-singular situation.

Remark 2.4. Since V is non-singular, φ is generically constant on each of the connected components of the complement of the union of the walls.

Indeed, let C be such a connected component. Denote by Y the union of the Euclidean boundaries of the $(\varphi^{-1}(m))^*$ for $m \in \mathbb{Z}$, and let $\varphi^* = \sum_{m \in \mathbb{Z}} m \mathbf{1}_{(\varphi^{-1}(m))^*}$. The function φ^* is constant on each connected component of $V \setminus Y$. If $x \in C$, the codimension of the germ Y_x in V_x is at least two, so $(V \setminus Y)_x$ is connected and φ^* is constant on $(V \setminus Y)_x$. Since φ and φ^* are generically equal on V, the function φ is generically constant on a neighbourhood of x.

Fix now $x \in C$. We define a function $f : C \to \{0, 1\}$ by $f(y) = 1$ if the generic value of φ near y is the same as near x, and $f(y) = 0$ else. Then f is continuous, so it is constant and equal to 0, and φ is generically constant on C.

Let W be a wall of φ. We consider the algebraically constructible function $\partial_W \varphi$ on W defined in [7]. We only recall here the generic definition in our non-singular case.

Let x be a point of W, such that x belongs to no other wall of φ. The function φ is generically constant near x on each of the two sides of W. Then, $\partial_W \varphi(x)$ is the average of the generic values of φ on each of these two sides.

We define another function on W. Let t be a polynomial on V which is an uniformizer of the regular local ring $\mathscr{R}(V)_{\mathscr{I}(W)}$. Then we set

$$\partial_W^t \varphi = \partial_W(\varphi \cdot \operatorname{sgn} t).$$

The function $\partial_W^t \varphi$ is algebraically constructible on W, and is generically equal to the half of the difference of the generic values of φ on the two sides of W. The definition of $\partial_W \varphi$ and $\partial_W^t \varphi$ can be compared to the definition of the shadow and countershadow of a semi-algebraic set on a wall in [1].

Remark 2.5. If $t' \in \mathscr{P}(V)$ is another uniformizer of $\mathscr{R}(V)_{\mathscr{I}(W)}$, then the functions $\partial_W^t \varphi$ and $\partial_W^{t'} \varphi$ are a priori different, but $N(\partial_W^t \varphi) = N(\partial_W^{t'} \varphi)$. Indeed, if $\partial_W^t \varphi =_{\text{gen}} \sum_{j=1}^r \operatorname{sgn} P_j$ on W, then $\partial_W^{t'} \varphi =_{\text{gen}} \sum_{j=1}^r \operatorname{sgn}(t \cdot t' \cdot P_j)$ on W. This allows us to talk about $N(\partial_W^t \varphi)$ without making precise the chosen uniformizer t.

Theorem 2.6. *Let $V \subset \mathbb{R}^N$ be an irreducible real algebraic set which is compact and non-singular. Let $\varphi : V \to \mathbb{Z}$ be an algebraically constructible function whose walls are non-singular with normal crossings intersections. Then*

$$N(\varphi) = \max\left\{M(\varphi), \max_{W \text{ wall of } \varphi} (N(\partial_W \varphi) + N(\partial_W^t \varphi))\right\}.$$

This theorem reduces the problem of calculating $N(\varphi)$ to a finite number of similar problems in lower dimension. By induction on the dimension, it is sufficient to calculate this number of polynomials in dimension one. By Example 2.1, in dimension one we have $N(\varphi) = M(\varphi)$, and we can calculate $N(\varphi)$ in an effective way for any dimension of V.

Remark 2.7. If the generic values of φ are contained in an interval $[\delta - k, \delta + k]$ with $\delta \in \mathbb{Z}$ and $k \in \mathbb{N}$, then the generic values of $\partial_W \varphi$ are in $[\delta - k, \delta + k]$, and the generic values of $\partial_W^t \varphi$ are in $[-k, k]$. As $M(\varphi) \leqslant k + |\delta|$, we get by induction on the dimension that

$$N(\varphi) \leqslant 2^{\dim V - 1} k + |\delta|.$$

This bound was already in [7], where its sharpness is proved if k is even. (For odd values of k, this bound can be improved a bit to get a sharp one, see [7].)

2.3 Proof of Theorem 2.6. We have seen that $N(\varphi) \geqslant M(\varphi)$. Let W be a wall of φ, and let t be an uniformizer of $\mathscr{R}(V)_{\mathscr{I}(W)}$ which is a polynomial on V. Consider the space of orderings $X = \Sigma_W[\mathbb{Z}/2\mathbb{Z}]$. The residue field of $\mathscr{R}(V)_{\mathscr{I}(W)}$ is $\mathscr{K}(W)$, and we embed X in Σ_V via this ring $\mathscr{R}(V)_{\mathscr{I}(W)}$, using t, as in Example 1.7. We have clearly $N(\varphi) \geqslant N(X, \tilde{\varphi}|_X)$ with the notations of Lemma 2.2, and by the second point of this lemma we get

$$N(X, \tilde{\varphi}|_X) = N(\Sigma_W, \widetilde{\partial_W \varphi}) + N(\Sigma_W, \widetilde{\partial_W^t \varphi}) = N(\partial_W \varphi) + N(\partial_W^t \varphi).$$

This proves the inequality $\geqslant$.

To prove the other inequality, we consider the anisotropic form ρ over Σ_V such that $\hat{\rho} = \tilde{\varphi}$. We want to calculate $N(\varphi) = \dim \rho$. By the isotropy Theorem 2.3, there exists a finite subspace X' of Σ_V such that $\rho|_{X'}$ is still anisotropic. We choose X' of minimal cardinality for this property. Then X' cannot be a sum $X_1' + X_2'$, since then either $\rho|_{X_1'}$ or $\rho|_{X_2'}$ would be anisotropic, which would contradict the minimality of the cardinality of X'. Thus, it follows from the structure Theorem 1.8 that X' is the atomic space or an extension.

If X' is the atomic space, $X' = \{\sigma\}$, then $N(\varphi) = |\tilde{\varphi}(\sigma)| \leqslant M(\varphi)$.

If X' is an extension, we write $X' = Y'[(\mathbb{Z}/2\mathbb{Z})^r]$, where Y' is not an extension. There is a valuation ring B of $\mathscr{K}(V)$ such that Y' is a subspace of the real spectrum of the residue field k of B, and such that X' is a subspace of the pullback of Y' in Σ_V via B. As V is compact, we have $\mathscr{R}(V) \subset B$. Let $\mathfrak{p}$ be the intersection with $\mathscr{R}(V)$ of the maximal ideal of B. It is a prime ideal of $\mathscr{R}(V)$. Denote by Z the zero set of $\mathfrak{p}$ in V: this is an irreducible algebraic set, and by construction $\mathscr{K}(Z)$ is a subfield of k. Let Y be the subspace of Σ_Z generated by the restrictions to $\mathscr{K}(Z)$ of the elements of Y'. Note that if $\sigma \in X'$ is a pullback of the element $\gamma \in Y'$, and $\tau \in Y$ is the restriction of γ, then τ is a specialization of σ in $\operatorname{Spec}_r \mathscr{R}(V)$. Indeed, if $f \in \mathscr{R}(V)$ is such that $(\operatorname{sgn} f)(\tau) = 1$, then $(\operatorname{sgn} f)(\gamma) = 1$, and so $(\operatorname{sgn} f)(\sigma) = 1$.

We claim that at least one wall of φ contains Z. (Note that we do not claim that two orderings in X' have a common specialization on a wall.) For, otherwise, each element τ in Y would be in the constructible subset $\tilde{C}$ of $\operatorname{Spec}_r \mathscr{P}(V)$ for some connected component C of the complement of the union of the walls. But then, all the elements of X' specializing to τ would be in $\tilde{C}$ too. Since φ is generically constant on C by Remark 2.4, the value of $\tilde{\varphi}$ would be the same on the 2^r elements of X' which are pullbacks of the same element of Y', and the restriction of ρ to the subspace $X'' = \{(1, \dots, 1)\} \times Y'$ of $X' = (\widehat{\mathbb{Z}/2\mathbb{Z}})^r \times Y'$ would be anisotropic. Indeed, the residue form $(\rho|_{X'})_{(1,\dots,1)}$ would be anisotropic of dimension $\dim \rho$ over Y', and the image of this form via the isomorphism between Y' and X'' is $\rho|_{X''}$. This way we would get again a contradiction with the minimality of the cardinality of X'.

Denote by $W_1, \dots, W_{d'}$ the walls containing Z, and by d the codimension of Z in V. As the walls have normal crossings intersections, we have $d \geqslant d'$. Let $P_1, \dots, P_r$ be polynomials on V describing the $(\varphi^{-1}(m))^*$ for $m \in \mathbb{Z}$. Since by assumption all the P_j's are at normal crossings, there is a regular system of parameters $(x_1, \dots, x_d)$ of $\mathscr{R}(V)_{\mathfrak{p}}$ such that each P_j is a monomial in $\mathscr{R}(V)_{\mathfrak{p}}$ for this system, i.e. $P_j = u_j x_1^{m_{1,j}} \dots x_d^{m_{d,j}}$ with u_j a unit of $\mathscr{R}(V)_{\mathfrak{p}}$, and $m_{1,j}, \dots, m_{d,j}$ some integers. For $i = 1, \dots, d'$, at least one of the P_j's vanishes on W_i. So we may assume that $\{x_i = 0\}$ corresponds to the wall W_i for $i = 1, \dots, d'$.

Consider now the discrete valuation ring C dominating $\mathscr{R}(V)_{\mathfrak{p}}$ with the same residue field, as explained in Example 1.3, using the parameters $(x_1, \dots, x_d)$. Let X be the pullback of Y via C. In particular, X is the pullback of a subspace X_1 of Σ_{W_1} via $\mathscr{R}(V)_{\mathscr{I}(W_1)}$. We will prove that $\rho|_X$ is anisotropic.

We consider the mapping $\theta : X' \to X$, which maps an element σ' of X', pullback of $\tau' \in Y'$, to the element σ of X, which is a pullback of the restriction of τ' in Y and satisfies $(\operatorname{sgn} x_i)(\sigma) = (\operatorname{sgn} x_i)(\sigma')$ for $i = 1, \dots, d$. We want to prove that θ is a morphism of spaces of orderings (cf. [11, 2.1]).

Let $f : X \to \{-1, 1\}$ be the restriction to X of the sign of an element of $\mathscr{K}(V)$. We have to prove that $g = f \circ \theta : X' \to \{-1, 1\}$ is the restriction to X' of the sign of an element of $\mathscr{K}(V)$. If this were not the case, by [3, IV.7.2.a)], there would be a four-element fan F' of X' such that g is positive on exactly an odd number of elements of F'. Denote $F' = \{\sigma_1', \sigma_2', \sigma_3', \sigma_4'\}$ with, say, g positive on $\{\sigma_1', \sigma_2', \sigma_3'\}$ and negative on $\{\sigma_4'\}$. Let $\sigma_l = \theta(\sigma_l')$ for $l = 1, 2, 3, 4$ and $F = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. Then f would be positive on $\{\sigma_1, \sigma_2, \sigma_3\}$ and negative on $\{\sigma_4\}$.

We prove that F is a four-element fan of X. Denote by τ_l' the element of Y' induced by σ_l' and by τ_l the restriction of τ_l' in Y, for $l = 1, 2, 3, 4$. As F' is a fan, we have $\tau_4' = \tau_1'\tau_2'\tau_3'$ hence $\tau_4 = \tau_1\tau_2\tau_3$, and if $i \in \{1, \dots, d\}$ then

$$(\operatorname{sgn} x_i)(\sigma_4) = (\operatorname{sgn} x_i)(\sigma_4') = \prod_{l=1}^{3} (\operatorname{sgn} x_i)(\sigma_l') = \prod_{l=1}^{3} (\operatorname{sgn} x_i)(\sigma_l).$$

This proves that $\sigma_4 = \sigma_1\sigma_2\sigma_3$, and F is a fan. The possible cardinalities for F are 1, 2 and 4. If the cardinality of F is not four, the value of f implies $\sigma_1 = \sigma_2 = \sigma_3 \neq \sigma_4$. But on the other hand $\sigma_4 = \sigma_1^3 = \sigma_1$, a contradiction. Therefore F is a four-element fan and we get that f is positive on exactly three elements of F, which is not possible by [3, III.3.8]. We conclude that g is the restriction to X' of the sign of an element of $\mathscr{K}(V)$, and that $\theta : X' \to X$ is a morphism of spaces of orderings.

Remark that the value of φ at an element of Σ_V inducing an ordering on $\mathscr{K}(Z)$, is determined by this induced ordering, and by the signs given to $x_1, \dots, x_d$. So for any $\sigma' \in X'$ we have $\tilde{\varphi}(\theta(\sigma')) = \tilde{\varphi}(\sigma')$.

Let $\langle f_1, \dots, f_s \rangle$ be a form over X, of signature $\tilde{\varphi}|_X$. Then $\tilde{\varphi}|_{X'} = (\tilde{\varphi} \circ \theta)|_{X'} = \sum_{j=1}^{s} f_j \circ \theta$, where $f_j \circ \theta$ is the restriction to X' of the sign of an element of $\mathscr{K}(V)$ since θ is a morphism of spaces of orderings. We conclude that $s \geqslant N(\varphi)$, and that $\rho|_X$ is anisotropic.

We consider now the space X_1. The functions $\partial_{W_1}\varphi$ and $\partial^t_{W_1}\varphi$ correspond to the functions ψ' and ψ'' of Lemma 2.2, and we have

$$N(\varphi) = N(X, \tilde{\varphi}|_X) = N(X_1, \widetilde{\partial_{W_1}\varphi}|_{X_1}) + N(X_1, \widetilde{\partial^t_{W_1}\varphi}|_{X_1}) \leqslant N(\partial_{W_1}\varphi) + N(\partial^t_{W_1}\varphi).$$

The proof is complete.

2.4 The Nash case. Recall that a Nash function on $\mathbb{R}^N$ is a function which is both analytic and semi-algebraic, and a Nash subset of $\mathbb{R}^N$ is a zero set of Nash functions.

Let $V \subset \mathbb{R}^N$ be a Nash set. In [9] McCrory and Parusiński introduced *Nash constructible functions* on V: their definition is similar to that of algebraically constructible functions, but now the fibres are restricted to connected components of algebraic sets. More precisely, $\varphi : V \to \mathbb{Z}$ is Nash constructible if for $i = 1, \dots, r$, there are an integer m_i, a regular proper morphism f_i from an algebraic set Z_i to V, and a connected component T_i of Z_i such that

$$\varphi(x) = \sum_{i=1}^{r} m_i \chi(f_i^{-1}(x) \cap T_i) \quad \text{for } x \in V.$$

(Here χ denotes the Euler characteristic.) In particular, algebraically constructible functions are Nash constructible.

If $\varphi : V \to \mathbb{Z}$ is a constructible function, we define a *Nash wall* of φ as a Nash-irreducible component, of codimension one in V, of the Nash closure of the Euclidian boundary of a $(\varphi^{-1}(k))^*$. In [6] we proved that if V is compact and non-singular, and if the Nash walls of φ are non-singular with normal crossings intersections, then φ is generically Nash constructible on V if and only if φ is generically a sum of signs of Nash functions on V. If V is compact, but these regularity assumptions do not hold, this is not true: in this case, Nash constructible functions coincide with sums of signs of semi-algebraic arc-analytic functions.

Assume that V is compact and non-singular, and that the Nash walls of φ are non-singular with normal crossings intersections. In this case, for a Nash wall W, the functions $\partial_W\varphi$ and $\partial^t_W\varphi$ also are generically sums of signs of Nash functions. We transpose Theorem 2.6 to this case:

Proposition 2.8. *Let $V \subset \mathbb{R}^N$ be a compact Nash set which is Nash-irreducible and non-singular. Let $\varphi : V \to \mathbb{Z}$ be a Nash constructible function whose Nash walls are non-singular with normal crossings intersections. Denote by $N_N(\varphi)$ the minimal number of Nash functions (counted with multiplicities) needed to write generically φ as a sum of signs of Nash functions. Then*

$$N_N(\varphi) = \max\Big\{M(\varphi), \max_{W \text{ Nash wall of } \varphi} (N_N(\partial_W\varphi) + N_N(\partial^t_W\varphi))\Big\}.$$

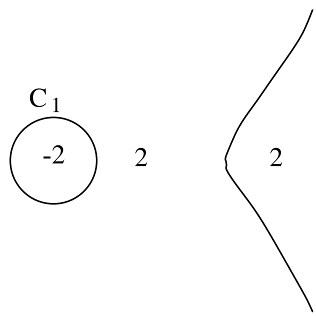
Proof. We repeat the proof of Theorem 2.6, working with the ring $\mathcal{N}(V)$ of Nash functions on V instead of the ring of rational functions.

The ring $\mathcal{N}(V)$ is noetherian by [5, Theorem 8.7.18]. If $\mathfrak{m}$ is a maximal ideal of $\mathcal{N}(V)$, then $\mathfrak{m}$ is the ideal of Nash functions vanishing at a point in M by [5, Corollary 8.6.3] and $\mathcal{N}(V)_{\mathfrak{m}}$ is a regular local ring by [5, Proposition 8.7.15]. So $\mathcal{N}(V)_{\mathfrak{p}}$ is a regular local ring for any prime ideal $\mathfrak{p}$ of $\mathcal{N}(V)$, i.e. the ring $\mathcal{N}(V)$ is regular. This allows us to build the discrete valuation ring denoted by C in the proof of Theorem 2.6. □

As in the algebraic case, we can determine $N_N(\varphi)$ by induction on the dimension, since in dimension one we have $N_N(\varphi) = M(\varphi)$.

Remark 2.9. The formulas given in Theorem 2.6 and Proposition 2.8 are very similar. However $N(\varphi)$ and $N_N(\varphi)$ are not the same in general, even if $V \subset \mathbb{R}^N$ is a compact real algebraic set which is Nash-irreducible and non-singular, and if φ is an algebraically constructible function on V with non-singular and normal crossings walls.

For instance, let $V = \mathbb{RP}^2$ with the coordinates $(x_0 : x_1 : x_2)$, and let C be the cubic of V with the equation $x_0 x_2^2 = x_1(x_1^2 - x_0^2)$. We define an algebraically constructible function φ on V in the following way:



There is only one (algebraic) wall: the cubic C, and we have $N(\partial_W \varphi) = N(\partial_W^t \varphi) = 2$, so $N(\varphi) = 4$. There is also only one Nash wall: the connected component C_1 of C, and we have $N_N(\partial_W \varphi) = 0$ and $N_N(\partial_W^t \varphi) = 2$. So $N_N(\varphi) = 2 \neq N(\varphi)$.

3 Represented polynomials

3.1 Algebraic tools.

Definition 3.1 ([3], III.1.18). Let (X, G) be a space of orderings and ρ a form of dimension d over X. We define

$$D_X(\rho) = \{g \in G \mid \exists g_2, \dots, g_d \in G : \rho = \langle g, g_2, \dots, g_d \rangle \text{ over } X\}.$$

An element of $D_X(\rho)$ is said to be *represented by ρ over X*.

Example 3.2. If ρ is isotropic, then it is clear that $D_X(\rho) = G$. Actually, these two conditions are equivalent ([11, 2.2.6(3)]).

The following lemma explains the behaviour of $D_X(\rho)$ under additions and extensions.

Lemma 3.3. *Let X be a space of orderings and ρ a form over X.*
If $X = X_1 + X_2$ and $\rho = (\rho_1, \rho_2)$ where ρ_j is a form over X_j for $j = 1, 2$, then

$$D_X(\rho) = D_{X_1}(\rho_1) \times D_{X_2}(\rho_2).$$

If $X = Y[H]$ and ρ is anisotropic, write $\rho = \sum_{h \in H} h\rho_h$. Then

$$D_X(\rho) = \bigsqcup_{h \in H} hD_Y(\rho_h).$$

Proof. The first point is clear, the second one is given by [3, IV.2.12.b)]. $\square$

Remark 3.4. Let X be a space of orderings, and let $\rho_1, \dots, \rho_n$ be forms over X.
Assume that $X = X_1 + X_2$ and denote $\rho_i = (\rho_{i,1}, \rho_{i,2})$. Then $\bigcap_{i=1}^n D_X(\rho_i) = \varnothing$ if and only if $\bigcap_{i=1}^n D_{X_1}(\rho_{i,1}) = \varnothing$ or $\bigcap_{i=1}^n D_{X_2}(\rho_{i,2}) = \varnothing$.
Assume that $X = Y[H]$ and that the ρ_i's are anisotropic. Write $\rho_i = \sum_{h \in H} h\rho_{i,h}$. Then $\bigcap_{i=1}^n D_X(\rho) = \varnothing$ if and only if for any $h \in H$, we have $\bigcap_{i=1}^n D_Y(\rho_{i,h}) = \varnothing$.
These two conditions follow easily from the previous lemma.

The aim of this section is to derive a geometric version of the following result ([3, IV.6.1.b)], [11, 4.3.2]) in the frame of algebraically constructible functions:

Theorem 3.5 (Local-global principle). *Let $\rho_1, \dots, \rho_n$ be forms over a space of orderings X. If $\bigcap_{i=1}^n D_X(\rho) = \varnothing$, then there exists a finite subspace Y of X such that $\bigcap_{i=1}^n D_Y(\rho|_Y) = \varnothing$.*

3.2 The geometric result. Let $V \subset \mathbb{R}^N$ be an irreducible real algebraic set. If φ is an algebraically constructible function on V, we copy the definition given in Part 3.1 for forms. We say that a polynomial $P \in \mathscr{P}(V)$ is *represented by φ on V* if there exist $P_2, \dots, P_{N(\varphi)} \in \mathscr{P}(V)$ such that φ is generically equal to $\operatorname{sgn} P + \sum_{j=2}^{N(\varphi)} \operatorname{sgn} P_j$ on V. In this case, we will say also that $\operatorname{sgn} P$ is represented by φ. We denote by $D_V(\varphi)$ the set of the polynomials represented by φ on V.
So, if ρ is the anisotropic form over Σ_V representing $\tilde{\varphi}$ and P a polynomial on V, then P belongs to $D_V(\varphi)$ if and only if the sign of P (on Σ_V) belongs to $D_{\Sigma_V}(\rho)$.

Theorem 3.6. *Let $V \subset \mathbb{R}^N$ be an irreducible real algebraic set which is compact and non-singular. Let $\varphi_1, \dots, \varphi_n$ be algebraically constructible functions on V such that all* *the walls of the φ_i's are non-singular with normal crossings intersections. We assume that none of the φ_i's is generically equal to zero.*

Then $\bigcap_{i=1}^n D_V(\varphi_i) = \emptyset$ if and only if

- *there exist $i, i' \in \{1, \dots, n\}$, and $S \subset V$ semi-algebraic of dimension* $\dim V$, *such that $M(\varphi_i) = N(\varphi_i)$, $M(\varphi_{i'}) = N(\varphi_{i'})$, and $\varphi_i|_S = -M(\varphi_i)$, $\varphi_{i'}|_S = M(\varphi_{i'})$,*

or

- *there is a wall W of one of the φ_i's such that, if we fix a polynomial uniformizer t of $\mathcal{R}(V)_{\mathcal{I}(W)}$ and if we denote $J_W = \{i \in \{1, \dots, n\} \mid N(\varphi_i) = N(\partial_W \varphi_i) + N(\partial_W^t \varphi_i)\}$, then $\bigcap_{i \in J_W} D_W(\partial_W \varphi_i) = \emptyset$ and $\bigcap_{i \in J_W} D_W(\partial_W^t \varphi_i) = \emptyset$.*

Remark 3.7. The condition $\bigcap_{i \in J_W} D_W(\partial_W^t \varphi_i) = \emptyset$ is independent of the chosen t. Indeed, let t' be another polynomial uniformizer of $\mathcal{R}(V)_{\mathcal{I}(W)}$. If a polynomial P belongs to $D_W(\partial_W^t \varphi_i)$, then $t \cdot t' \cdot P$ belongs to $D_W(\partial_W^{t'} \varphi_i)$.

Remark 3.8. If there is $i \in \{1, \dots, n\}$ such that $\varphi_i =_{\text{gen}} 0$ on V, then $D_V(\varphi_i) = \emptyset$, so $\bigcap_{i=1}^n D_V(\varphi_i) = \emptyset$.

As Theorem 2.6, Theorem 3.6 reduces a problem in dimension $\dim V$ to a finite number of similar problems in lower dimension. By induction on the dimension, we have to solve similar problems in dimension 1. In this case, only the first condition of Theorem 3.6 remains, and for any function φ_i we have $N(\varphi_i) = M(\varphi_i)$, so we can check easily if $\bigcap_{i=1}^n D_V(\varphi_i) = \emptyset$.

Remark 3.9. Again, we can transpose Theorem 3.6 from the algebraic case to the Nash case, by working with the ring of Nash functions instead of the ring of polynomials. We get the same results with Nash walls instead of walls.

3.3 Proof of Theorem 3.6. We denote $G = \{(\operatorname{sgn} P) : \Sigma_V \to \mathbb{Z}/2\mathbb{Z} \mid P \in \mathcal{P}(V) \backslash \{0\}\}$. For $i = 1, \dots, n$, let ρ_i be the anisotropic form over Σ_V representing φ_i.

We assume first that there is a polynomial $P \in \bigcap_{i=1}^n D_V(\varphi_i)$. Then, if i satisfies $M(\varphi_i) = N(\varphi_i)$, the sign of P and the sign of φ_i must be generically equal on the semi-algebraic set $\{|\varphi_i| = M(\varphi_i)\}$. So the first point of the theorem is not possible.

Consider a wall W of one of the φ_i's, and a uniformizer t of $\mathcal{R}(V)_{\mathcal{I}(W)}$. We embed the space $X = \Sigma_W[\mathbb{Z}/2\mathbb{Z}]$ into Σ_V via this ring, so that for any $\sigma \in \Sigma_W$ we have $(\operatorname{sgn} t)(1, \sigma) = 1$. If $i \in J_W$, then $\rho_i|_X$ is anisotropic. For such an i, write $\rho_i|_X = \rho_i' + a\rho_i''$, where ρ_i' and ρ_i'' are anisotropic forms over Σ_W representing $\partial_W \varphi$ and $\partial_W^t \varphi$ respectively. As $(\operatorname{sgn} P)|_X$ belongs to $\bigcap_{i \in J_W} D_X(\rho_i)$, we get that $\bigcap_{i \in J_W} D_{\Sigma_W}(\rho_i') \neq \emptyset$ or $\bigcap_{i \in J_W} D_{\Sigma_W}(\rho_i'') \neq \emptyset$ by Remark 3.4. In terms of algebraically constructible functions, this means that $\bigcap_{i \in J_W} D_W(\partial_W \varphi_i) \neq \emptyset$ or $\bigcap_{i \in J_W} D_W(\partial_W^t \varphi_i) \neq \emptyset$. The first implication is proved.

Conversely, assume that $\bigcap_{i=1}^n D_V(\varphi_i) = \emptyset$, so $\bigcap_{i=1}^n D_{\Sigma_V}(\rho_i) = \emptyset$. By Theorem 3.5, there is a finite subspace X' of Σ_V such that $\bigcap_{i=1}^n D_{X'}(\rho_i|_{X'}) = \emptyset$. We choose

X' of minimal cardinality for this property. Then, by Remark 3.4, the space X' is not a sum. According to the structure theorem, X' is the atomic space or an extension.

First case. $X' = E$. Denote by σ the element of X'. There is $i \in \{1, \dots, n\}$ such that $\rho_i|_{X'}$ is anisotropic (else $\bigcap_{i=1}^n D_{X'}(\rho_i|_{X'})$ would be $\{X' \to \{1\}, X' \to \{-1\}\}$). For such an i, we have $|\hat{\rho}_i(\sigma)| = \dim \rho_i = N(\varphi_i) = M(\varphi_i)$, and $D_{X'}(\rho_i|_{X'})$ only contains the function $\sigma \mapsto \operatorname{sgn}(\tilde{\varphi}_i(\sigma))$. We deduce from this the first point of the theorem.

Second case. $X' = Y'[(\mathbb{Z}/2\mathbb{Z})^r]$ where Y' is not an extension. We copy the construction of the proof of Theorem 2.6: let B be a valuation ring of $\mathscr{K}(V)$ such that Y' is a subspace of the real spectrum of the residue field k of B, and X' is a subspace of the pullback of Y' via B. As before we denote by $\mathfrak{p}$ the restriction of the maximal ideal of B to $\mathscr{R}(V)$, by Z the zero set of $\mathfrak{p}$ in V, and by Y the subspace of Σ_Z generated by the restrictions to $\mathscr{K}(Z)$ of the elements of Y'.

We claim that the set Z is contained in at least one wall of one of the φ_i's. Otherwise, as in the proof of Theorem 2.6, the value of φ_i would be the same on the 2^r pullbacks in X' of the same element of Y', for $i = 1, \dots, n$. Consider $X'' = \{(1, \dots, 1)\} \times Y'$, and denote $J' = \{i \in \{1, \dots, n\} \,|\, \rho_i|_{X'} \text{ anisotropic}\}$. Then, for $i \in J'$, we would have $D_{X'}(\rho_i|_{X'}) = D_{Y'}((\rho_i|_{X'})_{(1,\dots,1)})$. As $\bigcap_{i \in J'} D_{X'}(\rho_i|_{X'}) = \varnothing$, we would get $\bigcap_{i \in J'} D_{Y'}((\rho_i|_{X'})_{(1,\dots,1)}) = \varnothing$, and by the isomorphism between X'' and Y' we would have $\bigcap_{i \in J'} D_{X''}(\rho_i|_{X''}) = \varnothing$. This would contradict the minimality of the cardinality of X'.

Let $W_1, \dots, W_{d'}$ be the walls of the φ_i's containing Z. We repeat the construction of the proof of Theorem 2.6. We get a space of orderings X and a morphism of spaces of orderings $\theta : X' \to X$. We prove that $\bigcap_{i=1}^n D_X(\rho_i|_X) = \varnothing$.

Else, there would be an element $f \in G$ such that for every $i \in \{1, \dots, n\}$, there exist $g_{i,2}, \dots, g_{i,r_i} \in G$ with $r_i = \dim \rho_i$ and $\rho_{i|X} = \langle f, g_{i,2}, \dots, g_{i,r_i} \rangle_{|X}$. For every $\sigma' \in X'$, we would have $\tilde{\varphi}_i(\sigma') = (\tilde{\varphi}_i \circ \theta)(\sigma') = (f \circ \theta)(\sigma') + \sum_{j=2}^{r_i} (g_{i,j} \circ \theta)(\sigma')$. As θ is a morphism of spaces of orderings, $f \circ \theta$ and all the $g_{i,j} \circ \theta$ would be restrictions to X' of elements of G, and $f \circ \theta$ would be in $\bigcap_{i=1}^n D_{X'}(\rho_i|_{X'})$. We would get a contradiction. So we have $\bigcap_{i=1}^n D_X(\rho_i|_X) = \varnothing$.

We can write $X = X_1[\mathbb{Z}/2\mathbb{Z}]$ where X_1 is a subspace of Σ_{W_1}. Denote $J = \{i \in \{1, \dots, n\} \,|\, \rho_i|_X \text{ anisotropic}\}$. We have $\bigcap_{i \in J} D_X(\rho_i|_X) = \varnothing$. If $i \in J$, the restriction $\rho_i|_{\Sigma_{W_1}[\mathbb{Z}/2\mathbb{Z}]}$ is a fortiori anisotropic, so $i \in J_{W_1}$ and $\bigcap_{i \in J_{W_1}} D_X(\rho_i|_X) = \varnothing$. By Remark 3.4, we get $\bigcap_{i \in J_{W_1}} D_{X_1}((\rho_i|_X)_1) = \varnothing$ and $\bigcap_{i \in J_{W_1}} D_{X_1}((\rho_i|_X)_a) = \varnothing$. In terms of algebraically constructible functions, this means that $\bigcap_{i \in J_{W_1}} D_{W_1}(\partial_{W_1} \varphi_i) = \varnothing$ and $\bigcap_{i \in J_{W_1}} D_{W_1}(\partial^t_{W_1} \varphi_i) = \varnothing$. The theorem is proved.

3.4 Recognizing represented polynomials. If φ is an algebraically constructible function on an irreducible real algebraic set $V \subset \mathbb{R}^N$, and if P is a polynomial on V, we can ask if P is represented by φ. We give the following answer, using Theorem 2.6 and the fact that P is represented by φ if and only if $N(\varphi - \operatorname{sgn} P) = N(\varphi) - 1$. Note that, if W is an hypersurface, then $\partial_W \operatorname{sgn} P$ and $\partial^t_W \operatorname{sgn} P$ are generically signs of polynomials.

Corollary 3.10. *Let $V \subset \mathbb{R}^N$ be an irreducible real algebraic set which is compact and non-singular. Let $\varphi : V \to \mathbb{Z}$ be an algebraically constructible function, and let $P \in \mathscr{P}(V) \setminus \{0\}$. We assume that the walls of φ and $\operatorname{sgn} P$ are non-singular with normal crossings intersections, and that φ is not generically equal to zero.*

Then P is represented by φ if an only if

- *if $M(\varphi) = N(\varphi)$, then P has generically the same sign as φ on the semi-algebraic set where $|\varphi| = M(\varphi)$,*

and

- *for any wall W of φ such that $N(\varphi) = N(\partial_W \varphi) + N(\partial^t_W \varphi)$,*
  - *if W is a wall of $\operatorname{sgn} P$, then $\partial^t_W \operatorname{sgn} P$ is represented by $\partial^t_W \varphi$,*
  - *if W is not a wall of $\operatorname{sgn} P$, then $\partial_W \operatorname{sgn} P$ is represented by $\partial_W \varphi$,*

and

- *for any wall of $\operatorname{sgn} P$ which is not a wall of φ, we have $N(\varphi) > N(\partial_W \varphi)$.*

Remark 3.11. In dimension 1, only the first condition remains: P is represented by φ if and only if $\operatorname{sgn} P = \operatorname{sgn} \varphi$ on $\{|\varphi| = M(\varphi)\}$ except at a finite number of points. As before, using induction on the dimension, we can reduce to this case.

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