# MODULES AND MORITA THEOREM FOR OPERADS

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*Dedicated to A. N. Tyurin on the occasion of his 60th birthday*

## §0. Introduction and summary.

**(0.1) Morita theory.** Let $A, B$ be two commutative rings. If their respective categories of modules are equivalent, then $A$ and $B$ are isomorphic. This is not true anymore if $A$ and/or $B$ are not assumed to be commutative. Morita theory describes the relashionship between $A$ and $B$ in this case. The fact that a ring is not determined uniquely by its category of modules has deep implications for non–commutative geometry which tends to substitute an elusive non–commutative space by the category of sheaves on it.

In this note we construct a fragment of Morita theory for operads. More precisely, let $k$ be a field. For a $k$–linear operad $\mathcal{P}$, we construct matrix operads $\mathrm{Mat}\,(n, \mathcal{P})$ and prove that their respective categories of representations (that is, algebras) are equivalent.

In order to compare the situation with the classical one, let us remind the exact statement of the Morita theorem. For $A, B$ as above, denote by $A$–Mod, $B$–Mod their respective categories of left modules. They are equivalent iff one can find a $(B, A)$–bimodule $M$ which is finitely generated and projective as $B$–module. Then the functor $A$–Mod $\to B$–Mod :

$$(0.1.1) \qquad\qquad N \mapsto M \otimes_A N$$

establishes an equivalence, and $A$ is isomorphic to an algebra of the type $e\,\mathrm{Mat}(n, B)e$ where $e$ is the idempotent defining $M$.

A $k$–linear operad $\mathcal{P}$ can be considered as an "associative ring" (or rather monoid) in the monoidal category **S**–Vect whose objects are families of representations of all the symmetric groups $\mathbf{S}_n, n \geq 0$. The plethysm monoidal product in this category (denoted $\circ$ below) is not symmetric. This adds a new dimension of non–commutativity to the situation. In particular, the notions of left and right modules become asymmetric, and in our Morita theorem we replace left modules by $\mathcal{P}$–algebras, whereas right modules remain right $\mathcal{P}$–modules in **S**–Vect. Denote the categories of these objects $\mathcal{P}$–Alg and Mod–$\mathcal{P}$ respectively.

Our argument proceeds as follows.

(i) We use a kind of relative tensor product operation $\circ_{\mathcal{P}} : \text{Mod–}\mathcal{P} \times \mathcal{P}\text{–Alg} \to$ Vect (see sec. 1.3 below) to construct an operadic version of the functor (0.1.1). Let $\mathcal{Q}$ be another $k$–linear operad. Consider an $\mathbf{N}$–graded $\mathcal{Q}$–algebra $M$ which is simultaneously a right $\mathcal{P}$–module such that both structures are compatible in the sense that will be made explicit in sec. 1.4. Then for any $A$ in $\mathcal{P}$–Alg the product $M \circ_{\mathcal{P}} A$ is in $\mathcal{Q}$–Alg.

(ii) Given an $M$ in Mod–$\mathcal{P}$, we can construct its endomorphism operad $\mathcal{Q} = \text{Op End}_{\mathcal{P}}(M)$. It consists of the part of $\bigoplus_n \text{Hom}\,(M^{\otimes n}, M)$ compatible with the grading and the action of $\mathcal{P}$. Then $M$ becomes a $\mathcal{Q}$–algebra, so that the construction of (i) provides the functor $\mathcal{P}$–Alg $\to \mathcal{Q}$–Alg.

(iii) Finally, if we take for $M$ a free module of rank $n$ in Mod–$\mathcal{P}$, its endomorphism operad is denoted $\text{Mat}\,(n, \mathcal{P})$, and the functor produced in (ii) turns out to be an equivalence of categories: see (1.8.1).

Hopefully, this construction can be extended to a fuller statement providing the necessary conditions for equivalence as well.

Moreover, the multifaceted analogies between linear operads and rings suggest several other Morita–like contexts. For example, one can ask which equivalences may exist between the categories of right modules for different operads. Notice that equivalences between the categories of modules over algebras of various operadic types can be studied in principle by passing to the universal enveloping algebras and applying Morita theory for associative algebras.

**(0.2) Plan of the paper.** The first section is devoted to the Morita theorem. We first remind the generalities on operads, operadic modules, and algebras, in order to fix notation. We proceed by describing several examples and constructions related to the operadic modules which deserve to be better known: see e. g. Theorem (1.6.3) for the construction of a Lie algebra, which provides in particular a family of canonical vector fields on any formal Frobenius manifold. Finally, we prove the operadic Morita theorem described above.

The simplest application of the classical Morita theory to supergeometry furnishes the following fact: *the category of $D$–modules on a supermanifold $M$ is equivalent to the one on its underlying manifold $M_{\text{red}}$.* (To get a more sensible statement, $\mathbf{Z}_2$–grading must be introduced; for more sophisticated generalizations see [P]). In fact, the algebra of differential operators of the $\mathbf{Z}_2$–graded exterior algebra $W := \wedge_k(V)$ of a finite dimensional vector space $V$ is isomorphic to the matrix algebra $\text{End}_k(W)$. Therefore, sheaves of algebras $\text{Diff}_M$ and $\text{Diff}_{M_{\text{red}}}$ are Morita equivalent. In the second section of this paper we discuss some operadic versions of this remark.

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## §1. Modules over operads
## and abstract Morita theory.

**(1.1) Monoidal structures on S–Vect.** We work over a fixed base field $k$ and denote by Vect the category of $k$–vector spaces. We denote by $\mathbf{S}$ the category whose objects are the standard finite sets $\langle n \rangle = \{1, 2, ..., n\}$, $n \geq 0$ and morphisms are bijections, i.e., $\mathrm{Hom}_{\mathbf{S}}(\langle n \rangle, \langle n \rangle) = \mathbf{S}_n$ is the symmetric group. By an $\mathbf{S}$–space we mean a contravariant functor $\mathbf{S} \to$ Vect, i.e., a collection $\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$ where $\mathcal{V}(n)$ is a vector space with a right $\mathbf{S}_n$-action. As well known, an $\mathbf{S}$-spave $\mathcal{V}$ defines a functor on the category of all finite sets and their bijections. More precisely, if $I$ is any set with $n$ elements, then we put

$$(1.1.1) \qquad \mathcal{V}(I) = \left( \bigoplus_{\phi:\langle n \rangle \longrightarrow I} \mathcal{V}(n) \right)_{\mathbf{S}_n}.$$

Here $\phi$ runs over all bijections. Let $\mathbf{S}$–Vect be the category of $\mathbf{S}$–spaces. It possesses three important monoidal structures. Let $\mathcal{V} = \{\mathcal{V}(n)\}$ and $\mathcal{W} = \{\mathcal{W}(n)\}$ be two $\mathbf{S}$–spaces. Set

$$(1.1.2) \qquad (\mathcal{V} \otimes \mathcal{W})(n) = \mathcal{V}(n) \otimes_k \mathcal{W}(n),$$

$$(1.1.3) \quad (\mathcal{V} \boxtimes \mathcal{W})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbf{S}_i \times \mathbf{S}_j}^{\mathbf{S}_n} (\mathcal{V}(i) \otimes_k \mathcal{W}(j)) = \bigoplus_{I \sqcup J = \langle n \rangle} \mathcal{V}(I) \otimes_k \mathcal{W}(J),$$

$$(1.1.4) \qquad \mathcal{V} \circ \mathcal{W} = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{\mathbf{S}_n} \mathcal{W}^{\boxtimes n}.$$

In other words,

$$(\mathcal{V} \circ \mathcal{W})(n) = \bigoplus_{p \geq 0} \mathcal{V}(p) \otimes_{\mathbf{S}_p} \left( \bigoplus_{I_1 \sqcup ... \sqcup I_p = \langle n \rangle} \mathcal{W}(I_1) \otimes_k ... \otimes_k \mathcal{W}(I_p) \right)$$

Here the $\mathbf{S}_p$-action on the right takes the summand corresponding to $(I_1, ..., I_p)$ to the summand corresponding to $(I_{s(1)}, ..., I_{s(p)})$, $s \in \mathbf{S}_p$. This summand will be different unless $I_\nu = I_{s(\nu)} = \emptyset$ for all $\nu$ not fixed by $s$. This means that

$$(1.1.5)$$
$$(\mathcal{V} \circ \mathcal{W})(n) = \bigoplus_{p \geq 0} \bigoplus_{a_1+...+a_p=n} \left( \mathcal{V}(p) \otimes_k \mathcal{W}(a_1) \otimes_k ... \otimes_k \mathcal{W}(a_p) \right)_{\mathrm{Aut}(O(a_1,...,a_p))}.$$

Here $O(a_1, ..., a_p) \subset \langle p \rangle$ is the set of $\nu$ such that $a_\nu = 0$ and $\mathrm{Aut}(O(a_1, ..., a_p)) \subset \mathbf{S}_p$ is the group of self-bijections of this set.

Also, for $\mathcal{V} \in \mathbf{S}$–Vect and $X \in$ Vect define the vector space

$$(1.1.6) \qquad \mathcal{V}\langle X \rangle = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{\mathbf{S}_n} X^{\otimes n}.$$

The products (1.1.3–4) are analogous to the operations of multiplication and composition of formal power series $\sum v_n x^n / n!$, and (1.1.6) is analogous to the evaluation of such series. The product (1.1.4) is known as plethysm. Notice that for a "constant" $\mathbf{S}$-space, i.e., an $\mathbf{S}$-space $\mathcal{W}$ of the form $\mathcal{W}(0) = X$, $\mathcal{W}(n) = 0, n > 0$ , the plethysm $\mathcal{V} \circ \mathcal{W}$ is again "constant", corresponding to the vector space $\mathcal{V}\langle X \rangle$.

Each of the three products makes $\mathbf{S}$–Vect into a $k$–linear monoidal category, with $\otimes$ and $\boxtimes$ giving in fact symmetric monoidal structures, whereas $\circ$ is non–symmetric. The unit objects with respect to the three structures are:

$$(1.1.7) \qquad \mathcal{C}om : \quad \mathcal{C}om(n) = k, n \geq 0,$$

$$(1.1.8) \qquad \mathbf{1} : \quad \mathbf{1}(0) = k, \mathbf{1}(n) = 0, n \neq 0,$$

$$(1.1.9) \qquad \mathbf{I} : \quad \mathbf{I}(1) = k, \mathbf{I}(n) = 0, n \neq 1.$$

Note that we have canonical identifications

$$(1.1.10) \qquad (\mathcal{V} \circ \mathcal{W})\langle X \rangle = \mathcal{V}\langle \mathcal{W}\langle X \rangle \rangle,$$

$$(1.1.11) \qquad (\mathcal{V} \boxtimes \mathcal{W}) \circ \mathcal{X} = (\mathcal{V} \circ \mathcal{X}) \boxtimes (\mathcal{W} \circ \mathcal{X}),$$

$$(1.1.12) \qquad (\mathcal{V} \boxtimes \mathcal{W})\langle X \rangle = \mathcal{V}\langle X \rangle \otimes_k \mathcal{W}\langle X \rangle,$$

which correspond to the familiar rules of dealing with power series. In addition, we have the natural morphisms of $\mathbf{S}$–spaces

$$(1.1.13) \qquad (\mathcal{V}_1 \otimes \mathcal{V}_2) \circ (\mathcal{W}_1 \otimes \mathcal{W}_2) \to (\mathcal{V}_1 \circ \mathcal{W}_1) \otimes (\mathcal{V}_2 \circ \mathcal{W}_2)$$

and of vector spaces

$$(1.1.14) \qquad (\mathcal{V}_1 \otimes \mathcal{V}_2)\langle X_1 \otimes X_2 \rangle \to \mathcal{V}_1\langle X_1 \rangle \otimes \mathcal{V}_2\langle X_2 \rangle.$$

For example, (1.1.13) is the "diagonal map" sending the summand

$$\left( \mathcal{V}_1(p) \otimes \mathcal{V}_2(p) \otimes \mathcal{W}_1(a_1) \otimes \mathcal{W}_2(a_1) \otimes ... \otimes \mathcal{W}_1(a_p) \otimes \mathcal{W}_2(a_p) \right)_{\mathrm{Aut}(O(a_1,...,a_p))} \subset$$

$$\subset \big( (\mathcal{V}_1 \otimes \mathcal{V}_2) \circ (\mathcal{W}_1 \otimes \mathcal{W}_2) \big)(n), \quad a_1 + ... + a_p = n,$$

see (1.1.5), into the tensor product of the summand

$$\left( \mathcal{V}_1(p) \otimes \mathcal{W}_1(a_1) \otimes ... \otimes \mathcal{W}_1(a_p) \right)_{\mathrm{Aut}(O(a_1,...,a_p))} \subset (\mathcal{V}_1 \circ \mathcal{W}_1)(n)$$

and of the summand

$$\left( \mathcal{V}_2(p) \otimes \mathcal{W}_2(a_1) \otimes ... \otimes \mathcal{W}_2(a_p) \right)_{\mathrm{Aut}(O(a_1,...,a_p))} \subset (\mathcal{V}_2 \circ \mathcal{W}_2)(n).$$

This map is injective, if the **S**-spaces $\mathcal{W}_1$ and $\mathcal{W}_2$ are null in degree 0. The morphism (1.1.14) is defined in a similar way: it is the degree 0 component of (1.1.13) for the "constant" **S**-spaces $\mathcal{W}_i$ corresponding to $X_i$.

**(1.2) Operads and modules.** A $k$-linear operad $\mathcal{P}$ can be defined as a monoid object in (**S**–Vect, $\circ$). This means that $\mathcal{P}$ is an **S**–space endowed with associative multiplication (which is a morphism in **S**–Vect) and a distinguished element

$$(1.2.1) \qquad\qquad \mu_\mathcal{P} : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}, \ 1 \in \mathcal{P}(1)$$

with the usual properties. Spelling these data out with the help of (1.1.5), we get the maps

$$(1.2.2) \qquad \mu_{m_1,...,m_l} : \mathcal{P}(l) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_l) \to \mathcal{P}(m_1 + \cdots + m_l)$$

satisfying May's axioms (see [M], [S], [Gi–K]).

A right, resp. left $\mathcal{P}$–module is an **S**–space $M$ together with a multiplication morphism

$$(1.2.3) \qquad\qquad {}^M\mu : M \circ \mathcal{P} \to M, \quad \text{resp.} \quad \mu^M : \mathcal{P} \circ M \to M$$

satisfying the usual asociativity and unit requirements (cf. [Mar1-2], [Re], [BJT]). We denote by Mod–$\mathcal{P}$, resp. $\mathcal{P}$–Mod, the categories of right, resp. left $\mathcal{P}$-modules. Note that Mod–$\mathcal{P}$ is an abelian $k$-linear category, because the plethysm product $\circ$ is linear in the first argument, but $\mathcal{P}$–Mod is not even additive.

Similarly, a $\mathcal{P}$–algebra is a vector space $A$ together with a morphism of vector spaces

$$(1.2.4) \qquad \mu^A : \mathcal{P}\langle A \rangle \to A$$

satisfying the usual requirements. We denote by $\mathcal{P}$–Alg the category of $\mathcal{P}$–algebras. It is also not additive.

If $V$ is a vector space, then $\mathcal{P}\langle V \rangle$ is a $\mathcal{P}$–algebra called *the free algebra generated by $V$*.

Given an operad $\mathcal{P}$, one defines (see [Ad], §2.3) the corresponding category of operators (or PROP) $U(\mathcal{P})$ as follows. Objects of $U(\mathcal{P})$ are the same as for $\mathbf{S}$, i.e., the sets $\langle n \rangle$, $n \geq 0$, while

$$(1.2.5) \qquad \mathrm{Hom}_{U(\mathcal{P})}(\langle m \rangle, \langle n \rangle) = \bigoplus_{f:\langle m \rangle \to \langle n \rangle} \bigotimes_{i=1}^{n} \mathcal{P}(f^{-1}(i)),$$

where $f$ runs over all maps of sets $\langle m \rangle \to \langle n \rangle$ and the value of $\mathcal{P}$ on a set is defined by (1.1.1). The composition is induced by the composition in $\mathcal{P}$. The following is then obvious by construction.

**(1.2.6) Proposition.** *The category of right $\mathcal{P}$-modules is equivalent to that of contravariant functors $U(\mathcal{P}) \to \mathrm{Vect}$.*

**(1.3) Examples of operads and modules.**

**(1.3.1) The endomorphism operad.** Any vector space $V$ determines the operad $\mathrm{Op\,End}\,(V)$ with

$$\mathrm{Op\,End}\,(V)(n) = \mathrm{Hom}_k\,(V^{\otimes n}, V)$$

and the multiplication law (1.2.2) given by the composition of multilinear maps. A structure of the $\mathcal{P}$–algebra on $V$ is the same as a morphism of operads $\mathcal{P} \to \mathrm{Op\,End}(V)$.

An example of a right $\mathrm{Op\,End}(V)$–module is given by $V^{\star} = \{(V^*)^{\otimes n}\}_{n \geq 0}$. The module structure is given by taking the superposition of an $n$–linear form $V^{\otimes n} \to k$ with $n$ multilinear maps $V^{\otimes a_i} \to V$, $i = 1, ..., n$.

More generally, if $(\mathcal{C}, \otimes)$ is any $k$-linear symmetric monoidal category, any object $V \in \mathcal{C}$ gives rise to an operad $\mathrm{Op\,End}_{\mathcal{C}}(V)$ defined in a similar way.

**(1.3.2) The trivial operad.** The unit object $\mathbf{I} \in (\mathbf{S}$–$\mathrm{Vect}, \circ)$ is an operad called the trivial operad. An $\mathbf{I}$–algebra is the same as a vector space and an $\mathbf{I}$-module (left or right) is the same as an $\mathbf{S}$–space.

**(1.3.3) Free and projective right modules.** Any operad $\mathcal{P}$ is a left and a right module over itself. It follows that for any vector space $V$ the **S**–space $V \otimes \mathcal{P} = \{V \otimes_k \mathcal{P}(n)\}_{n \geq 0}$ is a right $\mathcal{P}$–module. Such modules will be called free (of finite rank, if dim $V < \infty$). A right module will be called projective, if it is a direct summand of a free module.

For example $\operatorname{Op}\operatorname{End}(V)$ is isomorphic, as a right module over itself, to the direct sum of $\dim(V)$ copies of $V^\star$, so $V^\star$ is a projective module.

**(1.3.4) The commutative operad.** The **S**–space $\mathcal{C}om$ has a natural operad structure with all the maps (1.2.2) being the canonical identifications $k \otimes ... \otimes k \to k$. A $\mathcal{C}om$–algebra is the same as a commutative algebra with unit in the usual sense. Note that $\mathcal{C}om = \operatorname{Op}\operatorname{End}(k)$.

Let $F$ be the category whose objects are the finite sets $\langle n \rangle$, $n \geq 0$ as above, but morphisms are all maps of finite sets. Thus **S** is the subcategory of $F$ formed by all objects and all isomorphisms. By an $F$–space, we mean a contravariant functor $\mathcal{V} : F \to \mathrm{V}ect$, whose value on $\langle n \rangle$ will be denoted $\mathcal{V}(n)$. Such a functor gives, in particular, an **S**–space. We denote by $F$–Vect the category of $F$–spaces.

The category $F$ should not be confused with the category $\Gamma$ of Segal [Se] whose objects are finite *pointed* sets $[n] = \{0, 1, ..., n\}$, $n \geq 0$ and morphisms are all maps taking 0 to 0.

**(1.3.5) Proposition.** *The category* $\mathrm{Mod}$–$\mathcal{C}om$ *of right* $\mathcal{C}om$–*modules is equivalent to* $F$–Vect *so that the forgetful functor* $\mathrm{Mod}$–$\mathcal{C}om \to$ **S**–Vect *is identified with the restriction functor* $F$–Vect $\to$ **S**–Vect.

*Proof:* Follows from the definition (1.2.5) of the category $U(\mathcal{P})$ and Proposition 1.2.6.

For example, if $A$ is a commutative algebra, then the morphism of operads $\mathcal{C}om \to \operatorname{Op}\operatorname{End}(A)$ together with the construction of (e) makes $A^\star$ into a right $\mathcal{C}om$-module. On the other hand, $\langle n \rangle \mapsto A^{\otimes n}$ is a covariant functor $F \to \mathrm{Vect}$ (cf. [L], Proposition 3.2 or [Pi], Sect. 1.7), and thus $\langle n \rangle \mapsto (A^*)^{\otimes n}$ is an $F$-space.

**(1.3.5) The associative operad and simplicial spaces.** Let $\mathcal{A}ss$ be the operad whose algebras are associative algebras with unit ([Gi–K]). Thus $\mathcal{A}ss(n)$ is the regular representation of $\mathbf{S}_n$. Let us describe right $\mathcal{A}ss$–modules. Denote by $\widetilde{F}$ the category whose objects are the finite sets $n \rangle$ as before and a morphism $\Phi : \langle m \rangle \to [n]$ consists, first, of a map $\phi : \langle m \rangle \to \langle n \rangle$ in the ordinary sense and, second, of a choice of a total order on each fiber $\phi^{-1}(i)$. The composition of morphisms in $\widetilde{F}$ is defined using the lexicographic ordering of the fibers of a composition. Thus we have a functor $\widetilde{F} \to F$. As before, we define a $\widetilde{F}$–space as a contravariant functor $\widetilde{F} \to \mathrm{Vect}$. The proof of the following proposition is also straightforward from (1.2.5-6).

**(1.3.6) Proposition.** *The category of right $\mathcal{A}ss$–modules is equivalent to the category of $\widetilde{F}$–spaces.*

Let us now discuss the relation between $\widetilde{F}$-spaces and simplicial spaces. Let $\Delta$ be the standard simplicial category, with objects $[n] = \{0, ..., n\}$ and monotone maps as morphisms. Thus contravariant functors $V_\bullet : \Delta \to \text{Vect}$ are simplicial vector spaces, with the value of $V_\bullet$ at $[n]$ denoted $V_n$. We will use the standard notation for the face and degeneracy operators

(1.3.7) $\qquad \partial_{n,i} : V_n \to V_{n-1}, \, 0 \leq i \leq n, \quad s_{n,i} : V_n \to V_{n+1}, \, 0 \leq i \leq n.$

On the other hand, let $\Delta_+ \subset F$ be the subcategory with the same objects $\langle n \rangle$ bu only monotone maps as morphisms. The bijections $[n] \to \langle n+1 \rangle$ (taking $i$ to $i+1$) identify $\Delta$ with the full subcategory of $\Delta_+$ on objects $\langle n \rangle, n > 0$. Thus a $\Delta_+$-space $\mathcal{V}$ is the same as, first, a simplicial space $V_\bullet$ with $V_n = \mathcal{V}(n+1)$ and, second, the datum of a vector space $V_{-1} = \mathcal{V}(0)$ together with a linear map $\partial_{0,0} : V_0 \to V_{-1}$ satisfying $\partial_{0,0} \circ \partial_{1,0} = \partial_{0,0} \circ \partial_{1,1}$. Such objects are traditionally called augmented simplicial spaces, and $\partial_{0,0}$ is called the augmentation. Note that every simplicial space can be considered as an augmented one, by taking $V_{-1} = 0$.

**(1.3.8) Proposition.** *(a) There is an embedding of categories $\Delta_+ \subset F$, identical on objects, so for an $\widetilde{F}$-space $\mathcal{V}$ the collection of $V_n = \mathcal{V}(n+1)$, $n \geq -1$, forms an augmented simplicial space.*

*(b) If $V_\bullet = \{V_n\}_{n \geq -1}$ is an augmented simplicial space, then the collection of $V_{n+1} \otimes_k k[\mathbf{S}_{n+1}]$ forms an $\widetilde{F}$–space.*

For example, if $A$ is an associative algebra with 1, then $A^\star$ is a right $\mathcal{A}ss$-module. On the other hand, setting $V_n = (A^*)^{\otimes(n+1)}$ we get an augmented simplicial space with $\partial_{n,i}$ given by contractions with 1 and $s_{n,i}$ by inserting the map $A^* \to A^* \otimes_k A^*$ dual to the multiplication.

*Proof of (1.3.8):* (a) If $\phi : \langle m \rangle \to \langle n \rangle$ is a monotone map, we take on each $\phi^{-1}(i)$ the total order induced from $\langle m \rangle$. This gives a morphism $\widetilde{\phi}$ of $\widetilde{F}$. If $\phi$ and $\psi$ are composable monotone maps, then one sees readily that $\widetilde{\phi\psi} = \widetilde{\phi}\widetilde{\psi}$.

(b) This is a consequence of the following property of $\widetilde{F}$. Let $\Phi = (\phi, \gamma)$ be any morphism of $\widetilde{F}$, so $\phi : \langle m \rangle \to \langle n \rangle$ is a map of sets and $\gamma$ is a system of total orders on the $\phi^{-1}(i)$. Then there is a unique permutation $s \in \mathbf{S}_m$ such that $\Phi = \widetilde{\phi}s$ where $\widetilde{\phi}$ is as in the proof of (a) and $s$ is considered as a morphism of $\widetilde{F}$ in an obvious way (its fibers are singletons so do not need ordering).

**(1.3.9) The stable curves operad and its modules.** For a finite set $I$ we denone by $\overline{M}_{g,I}$ the Deligne-Mumford stack classifying stable curves of genus $g$ with marked points $(x_i)_{i \in I}$ labeled by $I$ cf. [Kn]. For any injective map $\phi : I \to J$

of finite sets and ang $g$ such that $\overline{M}_{g,I} \neq \emptyset$ there is a natural morphism of stacks $\overline{M}_{g,J} \to \overline{M}_{g,I}$ called the stable forgetting (cf. [Man2], p. 93).

We first consider the case $I = [n] = \{0, 1, ..., n\}$. The point $x_0$ on a curve from $\overline{M}_{g,[n]}$ will be called the root point. The group $\mathbf{S}_n$ acts upon $\overline{M}_{g,[n]}$ by renumbering all points except for $x_0$. Moreover, we have the morphisms

$$(1.3.10) \qquad \overline{M}_{g,[l]} \times \overline{M}_{0,[m_1]} \times \ldots \overline{M}_{0,[m_l]} \to \overline{M}_{g,[m_1+\cdots+m_l]}$$

gluing the root point of the universal curve parametrized by $\overline{M}_{0,[m_i]}$ to the $i$–th labeled point of the universal curve parametrized by $\overline{M}_{g,[l]}$, $i = 1, \ldots, l$.

If $g = 0$, we get an operad $\overline{M}$ in the monoidal category of smooth projective manifolds with $\overline{M}(n) = \overline{M}_{0,n+1}$, $n \geq 2$ and $\overline{M}(1) = \{\mathrm{pt}\}$, $\overline{M}(0) = \emptyset$ The compositions (1.2.2) not involving $\overline{M}(1)$ are given by (1.3.10) while the unique element of $\overline{M}(1)$ is the unit. To produce a $k$–linear operad we put $H_*\overline{M}(n) := H_*(\overline{M}_{0,n+1}, k)$ and define the structure maps via pushforward of the geometric maps.

Algebras over $H_*\overline{M}$ (more precisely, cyclic algebras, cf. below) are called Cohomological Field Theories. Formal completion of such an algebra $H$ at zero has a natural structure of a formal Frobenius manifold. The theory of Gromov–Witten invariants produces such a structure on the cohomology of any smooth projective algebraic manifold. For all of this see e.g. [Man2].

We now construct a family of right modules over $\overline{M}$. Let $S$ be a finite set (possibly empty) and let $g$ be such that $\overline{M}_{g,S} \neq \emptyset$. For any $n$ consider the $\mathbf{S}_n$-action on $\overline{M}_{g,S\sqcup\langle n\rangle}$ given by renumbering points with labels in $\langle n\rangle$. We have then the morphisms

$$(1.3.11) \qquad \overline{M}_{g,S\sqcup\langle l\rangle} \times \overline{M}_{0,[m_1]} \times \ldots \overline{M}_{0,[m_l]} \to \overline{M}_{g,S\sqcup\langle m_1+\cdots+m_l\rangle},$$

defined in a way completely analogous to (1.3.10). These morphisms define on the $\mathbf{S}$-stack $\overline{\mathcal{M}}_g^S$, $\overline{\mathcal{M}}_g^S(n) = \overline{M}_{g,S\sqcup\langle n\rangle}$, the structure of right $\overline{M}$-module. Again, in order to pass to the $k$–linear situation, it suffices to apply any homology theory with coefficients in $k$. One can even consider Chow groups because the Künneth formula holds for the left hand side of (1.3.11) in the Chow theory.

Actually, both sides of (1.3.11) admit compatible morphisms onto $\overline{M}_{g,S}$ : at the left hand side, project onto $\overline{M}_{g,S\sqcup\langle l\rangle}$, forget $x_1, \ldots, x_l$ and stabilize, at the right hand side forget $x_1, \ldots, x_{m_1+\cdots+m_l}$ and stabilize. For a stable $S$-pointed curve $(C, (x_s))$ of genus $g$ let $\widehat{C}_{(x_s)}^l$ be the fiber of $\overline{M}_{g,S\sqcup\langle l\rangle} \to \overline{M}_{g,S}$ over the point represented by $(C, (x_s))$. This is in fact an algebraic variety (not just a stack). Now, the above discussion leads to the following result.

**(1.3.12) Proposition.** *(a) For any stable $S$-pointed curve $(C, (x_s))$ the collection of the $\mathbf{S}_l$-varieties $\widehat{C}^l_{(x_s)}$ forms a right module over the operad $\overline{M}$.*

*(b) If $\phi : T \to S$ is an injective map and $(D, (y_t))$ is the stable $T$-pointed curve obtained from $(C, (x_s))$ by stable forgetting, then we have natural morphisms of $\mathbf{S}_l$-varieties $\widehat{C}^l_{(x_s)} \to \widehat{D}^l_{(y_t)}$ which form a morphism of right $\overline{M}$-modules.*

For example, if $S = \emptyset$ and $C$ is smooth, then $\widehat{C}^l$ is the Beilinson–Ginzburg–Fulton–MacPherson "resolution of diagonals" of $C^l$, see [BG]. The construction of [BG] is actually applicable to any smooth curve (stable or not) and the $\overline{M}$-module structure in this case can also be constructed directly, using Proposition 3.8 of *loc. cit.* One can also compare with [Mar2] which essentially deals with a real version of $\widehat{C}^l$ (for $C$ a circle).

As before, one can produce from each geometric module in Proposition 1.3.12, a $k$–linear homology module. Such modules for different choices of $(C, (x_s))$ form a constructible sheaf over $\overline{M}_{g,S}$, smooth along the natural stratification by the type of the dual graph.

The last remark concerns enlarged symmetry of $\overline{M}_{g,l+1}$. In fact, the whole $\mathbf{S}_{l+1}$ acts upon this space rather than its subgroup $\mathbf{S}_l$. The axiomatization of this symmetry leads to the notion of the *cyclic operad* introduced and studied in [Ge–K]. Algebras and modules over a cyclic operad may also admit a cyclic structure, and we may ask for a cyclic version of Morita theory. We leave this question for future research.

**(1.4) Relative plethysm.** We will now review the relative plethysm (or circle-over construction) for modules over an operad ([Re]).

Let $\mathcal{P}$ be an operad, $M$ a right $\mathcal{P}$–module and $N$ a left $\mathcal{P}$–module. Their relative plethysm $M \circ_{\mathcal{P}} N \in \mathbf{S}$–Vect is defined as the cokernel of (the difference of) the two morphisms

$$(1.4.1) \qquad \partial_0, \partial_1 : M \circ \mathcal{P} \circ N \xrightarrow{\quad\longrightarrow\quad} M \circ N, \quad \partial_0 = {}^M\mu \circ \mathrm{Id}_N, \ \partial_1 = \mathrm{Id}_M \circ \mu^N.$$

This construction is similar to the usual tensor product of a right and a left module over an algebra and in fact can be performed for "modules" over any monoid object in any monoidal category (provided the cokernels exist). Similarly, let $A$ be a $\mathcal{P}$–algebra. The relative evaluation $M \circ_{\mathcal{P}} \langle A \rangle \in$ Vect is, by definition, the cokernel of the two morphisms

$$(1.4.2) \qquad \partial_0, \partial_1 : (M \circ \mathcal{P})\langle A \rangle \xrightarrow{\quad\longrightarrow\quad} M\langle A \rangle, \quad \partial_0 = {}^M\mu\langle \mathrm{Id}_A \rangle, \ \partial_1 = \mathrm{Id}_M\langle \mu_A \rangle.$$

The following canonical identifications are proved by mimicking the standard arguments for modules over an algebra:

$$(1.4.3) \qquad\qquad \mathcal{P} \circ_{\mathcal{P}} N = N, \quad M \circ_{\mathcal{P}} \mathcal{P} = \mathcal{P}, \quad \mathcal{P} \circ_{\mathcal{P}} \langle A \rangle = A.$$

**(1.4.4) Example.** Let $V$ be a finite–dimensional $k$–vector space and $\mathcal{P} = \mathrm{Op\,End}(V)$, $A = V$, $M = V^\star$, see (1.3.1). Then we claim that

$$V^\star \circ_{\mathrm{Op\,End}(V)} \langle V \rangle = k.$$

Indeed, we have a morphism of vector spaces

(1.4.5) $$\widetilde{\phi}: \quad V^\star \langle V \rangle = \bigoplus_{n \geq 0} (V^*)^{\otimes n} \otimes_{S_n} V^{\otimes n} \quad \rightarrow \quad k,$$

which on the $n$–th summand is induced by the $n$–th tensor power of the canonical pairing. Clearly $\widetilde{\phi}$ descends to a surjective morphism $\phi : V^\star \circ_{\mathrm{Op\,End}(V)} \langle V \rangle \to k$. Let us prove that $\phi$ is injective. For this, notice first that the image in $V^\star \circ_{\mathrm{Op\,End}(V)} \langle V \rangle$ of the $n$th summand in $V^\star \langle V \rangle$ factors through

$$\left( (V^*)^{\otimes n} \otimes_{\mathrm{End}(V)^{\otimes n}} V^{\otimes n} \right)_{S_n} = k.$$

In other words, we have morphisms $\psi_n : k \to V^\star \circ_{\mathrm{Op\,End}(V)} \langle V \rangle$ such that $\sum_{n \geq 0} \psi_n$ is surjective. Let us now prove that the images of the $\psi_n$ coincide with each other. For this, choose an identification $V \to k^d$, $d = \dim V$ and use this to make $V$ into a commutative associative algebra (the direct sum of $d$ copies of $k$). We get an element $m \in \mathrm{Op\,End}(V)(2)$ giving this algebra structure. Now, using the $\mathrm{Op\,End}(V)$–linearity conditions for $m$ and several copies of 1, we identify the $\mathrm{Im}(\psi_n)$ with each other.

**(1.5) Bimodules and functors between categories of operadic algebras.** Let $\mathcal{P}, \mathcal{Q}$ be two operads. A $(\mathcal{Q}, \mathcal{P})$–bimodule ([Mar1] [Re]) is, by definition, a space $M \in \mathbf{S}$–Vect together with a right $\mathcal{P}$–module structure and a left $\mathcal{Q}$–module structure on $M$, which commute, i.e., give rise to a well-defined map

(1.5.1) $$\mathcal{Q} \circ M \circ \mathcal{P} \to M.$$

**(1.5.2) Examples.** (a) Any operad $\mathcal{P}$ is a $(\mathcal{P}, \mathcal{P})$-bimodule.

(b) Let $V, W$ be two vector spaces. Define the $\mathbf{S}$–space $\mathrm{Op\,Hom}(V, W)$ by

$$\mathrm{Op\,Hom}(V, W)(n) = \mathrm{Hom}_k(V^{\otimes n}, W).$$

Then $\mathrm{Op\,Hom}(V, W)$ is an $(\mathrm{Op\,End}(W), \mathrm{Op\,End}(V))$–bimodule.

Similarly, if $(\mathcal{C}, \otimes)$ is any symmetric monoidal category and $V, W$ are objects of $\mathcal{C}$, we define an $(\mathrm{Op\,End}_{\mathcal{C}}(W), \mathrm{Op\,End}_{\mathcal{C}}(V))$–bimodule $\mathrm{Op\,Hom}_{\mathcal{C}}(V, W)$

Let $L$ be a left $\mathcal{Q}$–module, $N$ a right $\mathcal{P}$–module and $M$, as before, a $(\mathcal{Q}, \mathcal{P})$–bimodule. Then $L \circ_{\mathcal{Q}} M$ is naturally a right $\mathcal{P}$-module, $M \circ_{\mathcal{P}} N$ is naturally a left $\mathcal{Q}$–module and we have a canonical associativity isomoprhism

$$(1.5.3) \qquad (L \circ_{\mathcal{Q}} M) \circ_{\mathcal{P}} N \simeq L \circ_{\mathcal{Q}} (M \circ_{\mathcal{P}} N),$$

which allows us to write more complicated iterated relative plethysm without parentheses. Similarly, if $A$ is a $\mathcal{P}$–algebra, we have a canonical isomoprhism

$$(L \circ_{\mathcal{Q}} M) \circ_{\mathcal{P}} \langle A \rangle \simeq L \circ_{\mathcal{Q}} \langle M \circ_{\mathcal{P}} \langle A \rangle \rangle.$$

As a consequence, note a formula for the relative evaluation on a free algebra:

$$(1.5.4) \qquad M \circ_{\mathcal{Q}} \langle \mathcal{Q} \langle A \rangle \rangle \simeq M \langle A \rangle$$

(take $\mathcal{P} = \mathbf{I}$ to be the trivial operad).

**(1.5.5) Examples.** (a) Let $V$ be a finite–dimensional vector space. Then $V \otimes \mathcal{C}om = \{V\}_{n \geq 0}$ is naturally an $(\operatorname{Op} \operatorname{End}(V), \mathcal{C}om)$–bimodule. The arguments used in Example 1.4.4 establish an identification

$$V^{\star} \circ_{\operatorname{Op} \operatorname{End}(V)} (V \otimes \mathcal{C}om) \simeq \mathcal{C}om.$$

(b) Let $V, W, X$ be finite–dimensional $k$–vector spaces. Then we have an identification

$$\operatorname{Op} \operatorname{Hom}(W, X) \circ_{\operatorname{Op} \operatorname{End}(W)} \operatorname{Op} \operatorname{Hom}(V, W) \simeq \operatorname{Op} \operatorname{Hom}(V, X)$$

(as bimodules). This follows easily from the above example and Example 1.4.4, because $\operatorname{Op} \operatorname{Hom}(W, X)$ is isomorphic, as a right $\operatorname{Op} \operatorname{End}(W)$–module, to the direct sum of $\dim X$ copies of $W^{\star}$, and $\operatorname{Op} \operatorname{End}(W)$ is isomorphic, as a right module over itself, to the direct sum of $\dim W$ copies of $W^{\star}$.

The importance of $(\mathcal{Q}, \mathcal{P})$–bimodules for us is that any such bimodule $M$ defines a functor $f_M : \mathcal{P}\text{–Alg} \to \mathcal{Q}\text{–Alg}$,

$$(1.5.6) \qquad f_M(A) = M \circ_{\mathcal{P}} \langle A \rangle.$$

The action of $\mathcal{Q}$ on $f_M(A)$ is transferred from the action on $M$. Relative plethysm of bimodules corresponds to the composition of functors:

$$(1.5.7) \qquad f_{N \circ_{\mathcal{Q}} M} = f_N \circ f_M,$$

as it follows from (1.5.4).

**(1.5.8) Example.** Let $\mathcal{P} = \mathbf{I}$ be the trivial operad, so that $\mathcal{P}\text{–Alg} = \operatorname{Vect}$. An $(\mathbf{I}, \mathbf{I})$–bimodule $M$ is just an $\mathbf{S}$–space. The functor $f_M : \operatorname{Vect} \to \operatorname{Vect}$ is then

$$V \mapsto M \langle V \rangle = \bigoplus_{n \geq 0} M(n) \otimes_{\mathbf{S}_n} V^{\otimes n}.$$

Functors of this kind are called analytic in [J].

The following corollary of Example 1.5.5(b) is the first instance of the general Morita–type theorem to be proved later.

**(1.5.9) Theorem.** *Let $V, W$ be any two finite–dimensional $k$–vector spaces. Then the functor $f_{\mathrm{Op\,Hom}\,(V,W)}$ induces an equivalence between the categories of algebras over $\mathrm{Op\,End}\,(V)$ and over $\mathrm{Op\,End}\,(W)$. In particular, all these categories are equivalent to the category of commutative algebras, as $\mathcal{C}om = \mathrm{Op\,End}\,(k)$.*

In fact, the functor $f_{\mathrm{Op\,Hom}(V,W)}$ can be described in a more explicit way, using the tensor product of modules over a ring.

**(1.5.10) Proposition.** *Let $V, W$ be as before and $A$ be an $\mathrm{Op\,End}(V)$–algebra. Then, as a vector space,*

$$\mathrm{Op\,Hom}\,(V,W) \circ_{\mathrm{Op\,End}\,(V)} \langle A \rangle \simeq \mathrm{Hom}_k(V,W) \otimes_{\mathrm{End}\,(V)} A,$$

*where we view $A$ as a left module over the ring $\mathrm{End}(V) = \mathrm{Op\,End}(V)(1)$.*

**Proof.** As we pointed out already, $\mathrm{Op\,Hom}(V,W)$ is isomorphic, as a right $\mathrm{Op\,End}(V)$–module, to a direct sum of $\dim W$ copies of $V^\star$. Notice also that both sides of the proposed equality are additive in $W$. Thus, it is enough to treat the particular case $W = k$, so that $\mathrm{Op\,Hom}(V,W) = V^\star$. Next, since the functor

$$f_{\mathrm{Op\,Hom}\,(k,V)} : \mathrm{Op\,End}\,(k)\,\mathrm{–Alg} = \mathcal{C}om\,\mathrm{–Alg} \to \mathrm{Op\,End}\,(V)\,\mathrm{–Alg}$$

is an equivalence, we can assume that $A$ lies in the image of this functor. However, $\mathrm{Op\,Hom}(k,V)$ is isomorphic to $V \otimes_k \mathcal{C}om$ as a right $\mathcal{C}om$–module. Therefore, for a commutative algebra $B$ we have

$$f_{\mathrm{Op\,Hom}(k,V)}(B) = (V \otimes_k \mathcal{C}om) \circ_{\mathcal{C}om} \langle B \rangle = V \otimes_k B.$$

Now, the left and the right hand sides of the proposed equality are, respectively

$$V^\star \circ_{\mathrm{Op\,End}(V)} \langle V \otimes_k B \rangle, \quad V^* \otimes_{\mathrm{End}(V)} (V \otimes_k B).$$

But both these spaces are canonically identified with $B$: the first one in virtue of Theorem 1.5.9 and Example 1.5.5, the second one by elementary linear algebra. Proposition is proved.

**(1.6) Right modules as a tensor category: the role of the product $\boxtimes$.** Consider the symmetric monoidal structure $\boxtimes$ on $\mathbf{S}$–Vect, see (1.1.3). The identification (1.1.12) shows that any vector space $X$ gives rise to a tensor functor ("evaluation at $X$")

(1.6.1) $$\mathrm{Ev}_X : (\mathbf{S}\mathrm{–Vect}, \boxtimes) \to (\mathrm{Vect}, \otimes), \quad \mathcal{V} \mapsto \mathcal{V}\langle X \rangle.$$

In particular, taking $X = k$, we get a vector space

(1.6.2) $$\underline{\mathcal{V}} := \mathcal{V}\langle k \rangle = \bigoplus \mathcal{V}(n)_{\mathbf{S}_n}$$

which depends on $\mathcal{V}$ is a multiplicative (with respect to $\boxtimes$) way. The first part of the following proposition was observed in [F].

**(1.6.3) Proposition.** *Let $\mathcal{P}$ be any operad. Then the operation $\boxtimes$ makes* Mod–
$\mathcal{P}$, *the category of right $\mathcal{P}$–modules, into a symmetric monoidal abelian category.
For any vector space $X$ the functor* $\mathrm{Ev}_X$ *is a tensor functor on* $(\mathrm{Mod}\text{–}\mathcal{P}, \boxtimes)$, *i.e.,
it takes $\boxtimes$ into $\otimes$.*

**Proof.** Follows from (1.1.12).

Let now $M$ be a right $\mathcal{P}$–module. We define the operad $\mathrm{Op\,End}_{\mathcal{P}}(M)$ as the
endomorphism operad of $M$ as an object of the symmetric monoidal category Mod–
$\mathcal{P}$:

$$(1.6.4) \qquad \mathrm{Op\,End}_{\mathcal{P}}(M)(n) = \mathrm{Hom}_{\mathrm{Mod}-\mathcal{P}}(M^{\boxtimes n}, M).$$

Similarly, for two right $\mathcal{P}$–modules $M, N$ we define an $(\mathrm{Op\,End}_{\mathcal{P}}(N), \mathrm{Op\,End}_{\mathcal{P}}(M))$–
bimodule $\mathrm{Op\,Hom}_{\mathcal{P}}(M, N)$ by

$$(1.6.5) \qquad \mathrm{Op\,Hom}_{\mathcal{P}}(M, N)(n) = \mathrm{Hom}_{\mathrm{Mod}-\mathcal{P}}(M^{\boxtimes n}, N).$$

**(1.6.6) Proposition.** *We have an isomorphism of operads $u : \mathcal{P} \to \mathrm{Op\,End}_{\mathcal{P}}(\mathcal{P})$.*

**Proof.** The construction of $u$ is straightforward: the composition $\mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$
gives, upon restriction to the $n$th summand in (1.1.4), a $\mathbf{S}_n$–equivariant morphism
of right $\mathcal{P}$–modules $\mathcal{P}(n) \otimes_k \mathcal{P}^{\boxtimes n} \to \mathcal{P}$, and the associativity of $\mu$ implies that $u$ is
a morphism of operads. To see that it is injective, look at the action of $u(p)$, $p \in$
$\mathcal{P}(n)$, on $1 \otimes \cdots \otimes 1 \in \mathcal{P}(1)^{\otimes n}$. To check the surjectivity, consider any element
$\varphi = (\varphi_{l_1,\ldots,l_n}) \in \mathrm{Op\,End}_{\mathcal{P}}(\mathcal{P})(n)$. Put $p = \varphi_{1,\ldots,1}(1 \otimes \cdots \otimes 1) \in \mathcal{P}(n) \subset \mathcal{P}^{\boxtimes n}(n)$.
Then $\varphi = u(p)$. In fact, for $q_i \in \mathcal{P}(l_i)$ we have

$$\varphi_{l_1,\ldots,l_n}(q_1 \otimes \cdots \otimes q_n) = \varphi_{l_1,\ldots,l_n}(1(q_1) \otimes \cdots \otimes 1(q_n))$$

$$= (\varphi_{1,\ldots,1}(1 \otimes \cdots \otimes 1))(q_1, \ldots, q_n) = p(q_1, \ldots, q_n).$$

Proposition is proved.

**(1.7) A Lie algebra associated to an operad.** Let us recall the main ideas
of the Tannaka–Krein duality using [De] as a reference. If $(\mathcal{C}, \otimes)$ is a symmetric
monoidal $k$-linear category and $\Phi : \mathcal{C} \to \mathrm{Vect}$ is a tensor functor, then one can form
the group $\mathrm{Aut}(\Phi)$ of all $\otimes$-automorphisms of $\Phi$. If $\mathcal{C}$ satisfies additional properties
listed in [De] (i.e., $\mathcal{C}$ is Tannakian, in the terminology of *loc. cit.*), then $\mathrm{Aut}(\Phi)$
naturally becomes (the group of $k$-points of) a group scheme over $k$ and, moreover,
$\mathcal{C}$ becomes identified with the category of regular representations of this group
scheme. Working at the infinitesimal level and weakening the conditions we have
the following definition.

**(1.7.1) Definition.** *Let $(\mathcal{C}, \otimes)$ be a k–linear symmetric monoidal category, $\Phi : \mathcal{C} \to Vect$ a tensor functor. A derivation of $\Phi$ is a natural transformation $L : \Phi \to \Phi$ such that for any two objects $A, B \in \mathcal{C}$ the morphism $L_{A \otimes B} : \Phi(A \otimes B) \to \Phi(A \otimes B)$ coincides, after the identification $\Phi(A \otimes B) \simeq \Phi(A) \otimes \Phi(B)$, with $L_A \otimes 1_{\Phi(B)} + 1_{\Phi(A)} \otimes L_B$.*

All the derivations of $\Phi$ form, obviously, a Lie algebra which we denote $\mathrm{Der}(\Phi)$.

We now fix an operad $\mathcal{P}$ and specialize the above to the symmetric monoidal category $(\mathrm{Mod} - \mathcal{P}, \boxtimes)$ and the tensor functor $\Phi$ given by $\Phi(M) = \underline{M} = M\langle k\rangle$, see Proposition 1.6.3. This category is not Tannakian because it lacks dual objects, and $\Phi$ is not a fiber functor because it is not faithful. Nevertheless, Definition 1.7.1 is applicable and gives some Lie algebra depending only on $\mathcal{P}$. We start with a down–to–earth description of this algebra.

For $p \in \mathcal{P}(l)$, $q \in \mathcal{P}(n)$ and $1 \le i \le l$ set

$$(1.7.2) \qquad p \circ_i q = p(1, ..., 1, q, 1, ..., 1) \in \mathcal{P}(l + n - 1),$$

where $q$ on the right hand side is at the $i$th position, cf. [Mar1]. In a similar way, if $M$ is a right $\mathcal{P}$–module, $m \in M(l)$ and $q \in \mathcal{P}(n)$, we define $m \circ_i q \in M(l + n - 1)$, $1 \le i \le l$.

**(1.7.3) Theorem.** *(a) Let $\mathcal{P}$ be an operad and set for $p \in \mathcal{P}(a+1)$, $q \in \mathcal{P}(b+1)$:*

$$p \circ q = \sum_{i=1}^{a+1} p \circ_i q \quad \in \quad \mathcal{P}(a + b + 1).$$

*Then the operation $\circ$ induces on the vector space $\underline{\mathcal{P}}$ (see (1.6.2)) the structure of a pre-Lie algebra in the sense of [Ger] [CL], i.e., it satisfies the following property:*

$$(p \circ q) \circ r - p \circ (q \circ r) = (p \circ r) \circ q - p \circ (r \circ q).$$

*In particular, the operation $[p, q] = p \circ q - q \circ p$ makes $\underline{\mathcal{P}}$ into a Lie algebra which we denote $\mathcal{L}(\mathcal{P})$.*

*(b) For every right $\mathcal{P}$–module $M$ the operation*

$$(m, q) \mapsto mq = \sum_{i=1}^{a+1} m \circ_i q \in M(a + b + 1), \quad m \in M(a + 1), q \in \mathcal{P}(b + 1),$$

*induces on $\underline{M}$ the structure of a graded right module over the pre-Lie algebra $\underline{\mathcal{P}}$ and hence of a graded right module over the Lie algebra $\mathcal{L}(\mathcal{P})$, i.e., this operation satisfies the relation of a pre-Lie algebra product except that the first variable belongs to $\underline{M}$.*

*If $M_1, M_2$ are two right $\mathcal{P}$–modules, then $\underline{M_1 \boxtimes M_2}$, is identified with $\underline{M_1} \otimes \underline{M_2}$ as a $\mathcal{L}(\mathcal{P})$–module.*

*(c) For any finite–dimensional $\mathcal{P}$–algebra $A$ the Lie algebra $\mathcal{L}(\mathcal{P})$ acts on (the affine space underlying) $A$ by polynomial vector fields.*

**Proof.** (a) First of all, $\underline{\mathcal{P}}$ is the quotient of $\bigoplus \mathcal{P}(n)$ by the subspace $J$ spanned by elements of the form $p(1 - s)$, $p \in \mathcal{P}(a + 1), s \in \mathbf{S}_{a+1}$. Using the functional notation, one easily checks that $J$ is the two–sided ideal with respect to the product $\circ$. Moreover, we have, $p \circ_i q \equiv p \circ_j q \bmod J$.

The fact that $\circ$ satisfies the identity of a pre-Lie algebra, follows directly from axioms for the $\circ_i$, given in [Mar1]. Compare with [Ger], where a part of these axioms (not involving the symmetric group action) is axiomatized under the name of a "pre-Lie system" (§5) and the pre-Lie algebra identity is derived (§6).

(b) This statement is checked similarly, and we omit the details.

(c) Notice that $A^\star = \{(A^*)^{\otimes n}\}_{n \geq 0}$ is naturally a right $\mathcal{P}$–module, and $\underline{A^\star} = S(A^*)$ is the algebra of polynomial functions on (the affine space underlying) $A$. So $S(A^*)$ becomes a $\mathcal{L}(\mathcal{P})$-module, and to prove our assertion we need only to check that $\mathcal{L}(\mathcal{P})$ acts by algebra derivations. But this follows from the compatibility with tensor products.

**(1.7.4) Remark.** The proof above shows that the direct sum $\Lambda(\mathcal{P}) = \bigoplus \mathcal{P}(a+1)$ also forms a graded (pre-)Lie algebra, of which $\mathcal{L}(\mathcal{P})$ is a quotient. The Gerstenhaber bracket on the Hochschild complex of an associative algebra $A$, see [Ger], is a particular case of the construction of $\Lambda(\mathcal{P})$ but for the operad Op $\mathrm{End}(A[-1])$ in the category of graded vector spaces.

**(1.7.5) Theorem.** *The Lie algebra $\mathcal{L}(\mathcal{P})$ is identified with $\mathrm{Der}(\Phi)$ for the tensor functor described above.*

**Proof.** Part (b) of Theorem 1.6.3 means that we have a morphism of Lie algebras $u : \mathcal{L}(\mathcal{P}) \to \mathrm{Der}(\Phi)$. To see that $u$ is injective it is enough to consider an action of $p \in \mathcal{P}(a + 1)$ on $\underline{\mathcal{P}}$ where $\mathcal{P}$ is considered as a module over itself. It remains to prove surjectivity, i.e., that any derivation $D : \Phi \to \Phi$ comes from some element of $\mathcal{L}(\mathcal{P})$. To see this, consider the $D$–action on $\underline{\mathcal{P}}$ and let $q = D(1)$ be the image of $1 \in \mathcal{P}(1)$. We claim that $D = u(q)$. Indeed, let $M$ be an arbitrary right $\mathcal{P}$–module and $m \in M(a + 1)$. Then the associativity of the right $\mathcal{P}$–action on $M$ can be expressed as follows. Consider the natural morphism

$$(1.7.6) \qquad\qquad M(a + 1) \otimes_k \mathcal{P}^{\boxtimes(a+1)} \to M$$

given by restricting the action $M \circ \mathcal{P} \to M$, see (1.1.4). Let us view the source of this morphism as a right $\mathcal{P}$–module by considering $M(n)$ as just the vector space of multiplicities of the $\mathcal{P}$–module $\mathcal{P}^{\boxtimes(a+1)}$. Then (1.7.6) is a morphism of modules.

Now, because $D$ is a derivation, applying $D$ to $m \otimes 1 \otimes ... \otimes 1$ in the source of (1.7.6) we get

$$\sum_{i=1}^{a+1} m \otimes 1 \otimes ... \otimes 1 \otimes q \otimes 1 \otimes ... \otimes 1$$

with $q$ at the $i$–th place in the $i$–th term. Since (1.7.6) is a morphism of modules, we find that the $D$–action on $m$ is given by the formula in (1.7.3)(b). Theorem is proved.

**(1.7.7) Examples.** (a) Let $V$ be a finite-dimensional vector space and $\mathcal{P} = \mathrm{Op\,End}(V)$. Then

$$\underline{\mathcal{P}} = \bigoplus_{n \geq 0} S^n(V^*) \otimes V$$

can be naturally identified with $\mathrm{Der}\,k[V^*]$, the space of polynomial vector fields on $V$. Moreover, the bracket $[p, q]$ is identified with the usual Lie bracket of vector fields, so $\mathcal{L}(\mathrm{Op\,End}(V)) \simeq \mathrm{Der}\,k[V^*]$ as a Lie algebra, the action (1.7.3)(c) being the tautological one.

The fact that the Lie algebra of vector fields on $V$ comes in fact from a pre-Lie algebra, was pointed out in [Ku].

(b) Taking $V = k$ in (a), we get $\mathcal{P} = \mathcal{C}om$. Thus, $\mathcal{L}(\mathcal{C}om) = \mathrm{Der}\,k[x]$ is Lie algebra of polynomial vector fields on the line. The commutation relations for the canonical generators $e_n \in \mathcal{C}om(n)$ is directly found to be

$$[e_m, e_n] = (m - n)e_{m+n-1}.$$

In other words, $e_n$ corresponds to the vector field $x^n(d/dx)$.

Further, take $\mathcal{P} = \mathcal{A}ss$ to be the associative operad. Then $\mathcal{P}(n)_{\mathbf{S}_n} = k$ and we find that $\mathcal{L}(\mathcal{A}ss)$ is isomorphic to $\mathcal{L}(\mathcal{C}om)$, i.e., to $\mathrm{Der}\,k[x]$. As a corollary, we get part (a) of the following fact.

**(1.7.8) Theorem.** *(a) Let $A$ be a finite–dimensional associative algebra. Then the Lie algebra $\mathrm{Der}\,k[x]$ acts by polynomial vector fields on $A$. Explicitly, $e_n = x^n(d/dx)$ acts by the vector field*

$$E_n = \left( X^n, \frac{\partial}{\partial X} \right).$$

*Here $X^n$ is the $A$–valued polynomial function on $A$ which raises every element to the $n$–th power, $\frac{\partial}{\partial X}$ is the canonical $A^*$–valued vector field on $A$ (corresponding to $\mathrm{Id} \in A^* \otimes_k A$) and $(-, -)$ is the structure pairing between $A$ and $A^*$.*

*(b) Let $A^\times \subset A$ be the set of invertible elements. Let $\mathrm{Der}\,k[x, x^{-1}]$ be the Witt algebra of regular vector fields on $k^\times = k - \{0\}$, with the basis $e_n = x^n(d/dx), n \in \mathbf{Z}$.*

*Then the operators $E_n$ defined similarly to (a), give an action of $\operatorname{Der} k[x, x^{-1}]$ by regular vector fields on $A^\times$.*

This fact for $A = \operatorname{Mat}(r, \mathbf{C})$ was noticed by N. Wallach ([W]). The proof of part (b) in general is left to the reader.

As we saw in Proposition 1.3.8, any augmented simplicial vector space gives rise to a right $\mathcal{A}ss$–module and thus, by the above, it defines a representation of $\operatorname{Der} k[x]$. Let us describe this action explicitly. Let $V_\bullet$ be an augmented simplicial vector space, with face and degeneracy operators $\partial_{n,i}, s_{n,i}$, see (1.3.7). Let $C(V_\bullet) = \bigoplus_n V_n$ be the graded vector space associated to $V$. Define the linear operators $L_p : C(V_\bullet) \to C(V_\bullet)$ of degree $p-1$, $p \geq 0$, as follows. On the summand $V_n \subset C(V_\bullet)$ we set

$$L_0 = \sum_{i=0}^{n} \partial_{n,i}, \quad L_1 = (n+1) \cdot \operatorname{Id}, \quad L_p = \sum_{i=0}^{n} s_{n-p+1,i} \ldots s_{n+1,i} s_{n,i}, \ \ p \geq 1.$$

**(1.7.9) Theorem.** *The operators $L_p$ satisfy the commutation relations $[L_p, L_q] = (p - q)L_{p+q-1}$ and thus make $C(V_\bullet)$ into a $\operatorname{Der} k[x]$–module.*

The proof can be deduced directly from the standard simplicial identities between faces and degenerations.

**(1.7.10) Example.** Here we describe the Lie algebra $\mathcal{L}(H_* \overline{M})$ for the operad $H_* \overline{M}$ introduced in (1.3.9). Consider a stable tree with $n + 1$ tails (flags which are not halves of the edges). Stability means that there are at least three flags at each vertex. Any numbering of flags by $[n]$ determines an irreducible closed submanifold of $\overline{M}_{0,[n]}$ whose open dense subset parametrizes curves of genus zero of the respective combinatorial type. The image of the homology class of this submanifold in $\mathcal{L}(H_* \overline{M})$ depends only on the isomorphism class of the respective rooted tree where root is the flag labeled by 0. For any such (isomorphism class) $\tau$ denote by $L(\tau)$ the corresponding element of the Lie algebra. Then we have:

(i) $\mathcal{L}(H_* \overline{M})$ is spanned by all $L(\tau)$. Besides the general $\mathbf{Z}$–grading (see (1.6.2)), it has an additional $\mathbf{Z}$–grading by algebraic dimension of $L(\tau)$ which is $n - 2$ less the number of edges of $\tau$.

(ii) The bracket is defined by

(1.7.11) $$[L(\sigma), L(\tau)] = \sum_i L(\sigma \circ_i \tau) - \sum_j L(\tau \circ_j \sigma).$$

Here $i$ (resp. $j$) runs over non–root flags of $\sigma$ (resp. of $\tau$), and $\sigma \circ_i \tau$ denotes the tree obtained from $\sigma$ by gluing its flag $i$ to the root of $\tau$.

(iii) The elements $L(\tau)$ are lineraly independent. This follows from the fact that all linear relations between $L(\tau)$ can be obtained by symmetrizing the linear relations between the boundary homology classes described in [Man2], sec. III.4.7, and this symmetrization produces zero.

All of this in the final count follows from Keel's description of the Chow and homology groups of $\overline{M}_{0,[n]}$ ([Ke]).

For a general $H_*\overline{M}$–algebra $H$ there is nothing to add to the description of vector fields given in the proof of (1.7.3) (c). Let us therefore consider the case of the cyclic algebra $H$, for example, quantum cohomology. In this case $H$ comes with a richer structure, namely it is endowed with a non–degenerate symmetric pairing $g$ (Poincaré form), and the algebra structure geometrically appears in the guise of Gromov–Witten $\mathbf{S}_{n+1}$–equivariant maps

$$I_{n+1} : \, H^{\otimes(n+1)} \to H^*(\overline{M}_{0,[n]}, k).$$

The operadic action needed to define $L(\tau)$

$$\mu_n : \, H_*(\overline{M}_{0,[n]}) \otimes H^{\otimes n} \to H$$

is obtained from $I_{n+1}$ by partial dualization with the help of $g$.

**(1.8) The role of the product $\otimes$.** Consider now the monoidal structure $\otimes$ on $\mathbf{S}$–Vect, see (1.1.2).

If $\mathcal{P}_1, \mathcal{P}_2$ are operads, then the distributivity maps (1.1.13) make $\mathcal{P}_1 \otimes \mathcal{P}_2$ into an operad. If $A_i$ is a $\mathcal{P}_i$–algebra, $i = 1, 2$, then $A_1 \otimes_k A_2$ is a $\mathcal{P}_1 \otimes \mathcal{P}_2$–algebra via (1.1.14). Similarly, if $M_i$ is a left (resp. right) $\mathcal{P}_i$–module, i=1,2, then $M_1 \otimes M_2$ is a left (resp. right) $\mathcal{P}_1 \otimes \mathcal{P}_2$–module.

Let $\mathcal{P}$ be an operad, $M$ be a right $\mathcal{P}$–module and $V$ be a vector space. Then we have the right $\mathcal{P}$–module $V \otimes M$, see (1.3.3). Notice that if $N, W$ are another right $\mathcal{P}$–module and a vector space, then we have an identification

(1.8.1) $$(V \otimes M) \boxtimes (W \otimes N) = (V \otimes_k W) \otimes (M \boxtimes N),$$

which implies the following.

**(1.8.2) Proposition.** *We have an isomorphism of operads*

$$\mathrm{Op\,End}_{\mathcal{P}}(V \otimes M) \simeq \mathrm{Op\,End}(V) \otimes \mathrm{Op\,End}_{\mathcal{P}}(M)$$

*and an isomorphism of bimodules*

$$\mathrm{Op\,Hom}_{\mathcal{P}}(V \otimes M, W \otimes N) \simeq \mathrm{Op\,Hom}(V, W) \otimes \mathrm{Op\,Hom}_{\mathcal{P}}(M, N).$$

**(1.8.3) Definition.** *The d by d matrix operad over $\mathcal{P}$ is defined by*

$$\mathrm{Mat}\,(d, \mathcal{P}) = \mathcal{P} \otimes \mathrm{Op}\,\mathrm{End}\,(k^d) = \mathrm{Op}\,\mathrm{End}_{\mathcal{P}}(\mathcal{P}^{\oplus d})$$

*(the last equality being a consequence of (1.8.2) and Proposition 1.6.6).*

Next, let $\mathcal{P}_i, i = 1, 2$ be operads, $M_i$ be a right $\mathcal{P}_i$–module and $N_i$ be a left $\mathcal{P}_i$–module. Then the distributivity maps (1.1.12) give rise to a morphism of **S**–modules

$$(1.8.4) \qquad (M_1 \otimes M_2) \circ_{\mathcal{P}_1 \otimes \mathcal{P}_2} (N_1 \otimes N_2) \to (M_1 \circ_{\mathcal{P}_1} N_1) \otimes (M_2 \circ_{\mathcal{P}_2} N_2).$$

In general this need not be an isomophrism. We will be interested in the case when $\mathcal{P}$ is a fixed operad, $M_1 = N_1 = \mathcal{P}_1 = \mathcal{P}$, while $\mathcal{P}_2 = \mathrm{Op}\,\mathrm{End}(V)$, $\dim V < \infty$, $M_2 = V^\star$ and $N_2 = V \otimes \mathcal{C}om$. Notice that in this case $M_2 \circ_{\mathcal{P}_2} N_2 \simeq \mathcal{C}om$, see Example 1.5.5(a). Let us denote $V \otimes \mathcal{C}om$ by $\widetilde{V}$. The specialization of the morphism (1.8.4) to our case is then:

$$(1.8.5) \qquad u : (\mathcal{P} \otimes V^\star) \circ_{\mathcal{P} \otimes \mathrm{Op}\,\mathrm{End}(V)} (\mathcal{P} \otimes \widetilde{V}) \to \mathcal{P} \otimes (V^\star \circ_{\mathrm{Op}\,\mathrm{End}(V)} \widetilde{V}) = \mathcal{P}.$$

**(1.8.6) Proposition.** *The morphism $u$ in (1.8.5) is an isomorphism of* **S**–*modules.*

**Proof.** Let $\dim(V) = d$. It is enough to show that $u^{\oplus d}$, the direct sum of $d$ copies of $u$, is an isomorphism. On the other hand, $(V^\star)^{\oplus d}$ is identified, as a right $\mathrm{Op}\,\mathrm{End}(V)$–module, with $\mathrm{Op}\,\mathrm{End}(V)$ itself and hence $(\mathcal{P} \otimes V^\star)^{\oplus d}$ is identified, as a $\mathcal{P} \otimes \mathrm{Op}\,\mathrm{End}(V)$–module, with $\mathcal{P} \otimes \mathrm{Op}\,\mathrm{End}(V)$ itself. It follows that $u^{\oplus d}$ is identified with the morphism

$$\mathcal{P} \otimes \mathrm{Op}\,\mathrm{End}(V) \circ_{\mathcal{P} \otimes \mathrm{Op}\,\mathrm{End}(V)} (\mathcal{P} \otimes \widetilde{V}) \to \mathcal{P} \otimes \left( \mathrm{Op}\,\mathrm{End}(V) \circ_{\mathrm{Op}\,\mathrm{End}(V)} \widetilde{V} \right)$$

which is just the identity map $\mathcal{P} \otimes \widetilde{V} \to \mathcal{P} \otimes \widetilde{V}$. Thus $u^{\oplus d}$ is an isomorphism and hence $u$ is an isomorphism.

**(1.9) Morita equivalence.** Consider an operad $\mathcal{P}$ and a $\mathcal{P}$–module $M$. Put $\mathcal{Q} = \mathrm{Op}\,\mathrm{End}_{\mathcal{P}}(M)$. Then $M$ is a $(\mathcal{Q}, \mathcal{P})$–bimodule, so that the construction of (1.4) gives us a functor $f_M : \mathcal{P}$–Alg$\to \mathcal{Q}$–Alg.

**(1.9.1) Theorem.** *If $M$ is free as $\mathcal{P}$–module, that is, isomorphic to $\mathcal{P}^{\oplus d}$, then $f_M$ is an equivalence of categories.*

**Proof.** As above, put $V = k^d$ so that $M = \mathcal{P} \otimes_k V = \mathcal{P} \otimes \widetilde{V}$. Let $M^* = \operatorname{Op}\operatorname{Hom}_{\mathcal{P}}(M, \mathcal{P}) = \mathcal{P} \otimes V^*$, the last identification following from Proposition 1.8.2. Then $M^*$ is a $(\mathcal{P}, \mathcal{Q})$–bimodule and we have the functor $f_{M^*} : \mathcal{Q}\text{–Alg} \to \mathcal{P}\text{–Alg}$. We claim that $f_M$ and $f_{M^*}$ are mutually inverse.

To prove this, it suffices to construct the isomorphisms

$$(1.9.2) \qquad\qquad M^* \circ_{\mathcal{Q}} M \cong \mathcal{P} \quad \text{(as } (\mathcal{P}, \mathcal{P})\text{–bimodules).}$$

$$(1.9.3) \qquad\qquad M \circ_{\mathcal{P}} M^* \cong \mathcal{Q} \quad \text{(as } (\mathcal{Q}, \mathcal{Q})\text{–bimodules).}$$

Now, (1.9.2) is the content of Proposition 1.8.6. As for (1.9.3), we have, first of all, a morphism

$$(1.9.4) \qquad \psi : M \circ_{\mathcal{P}} M^* = (\mathcal{P} \otimes \widetilde{V}) \circ_{\mathcal{P} \otimes \mathcal{C}om} (\mathcal{P} \otimes V^{\star}) \to \mathcal{P} \otimes (V \circ_{\mathcal{C}om} V^{\star}),$$

a particular case of (1.8.4). Note that the natural morphism of $\operatorname{Op}\operatorname{End}(V)$–bimodules

$$\phi : \widetilde{V} \circ_{\mathcal{C}om} V^{\star} \to \operatorname{Op}\operatorname{End}(V)$$

is an isomorphism. This is because $\widetilde{V}$ is a free right $\mathcal{C}om$–module, so $\phi$ is isomorphic to the direct sum of $d = \dim(V)$ copies of the isomorphism $\mathcal{C}om \circ_{\mathcal{C}om} V^{\star} \to V^{\star}$. Thus (1.9.4) can be regarded as a morphism

$$\psi : M \circ_{\mathcal{P}} M^* \to \mathcal{P} \otimes \operatorname{Op}\operatorname{End}(V) = \mathcal{Q}.$$

To see that this is an isomorphism, it is enough to notice that $\psi$ is isomorphic (as a morphism of **S**–modules) to the direct sum of $d$ copies of the morphism

$$\mathcal{P} \circ_{\mathcal{P}} (\mathcal{P} \otimes V^{\star}) \to \mathcal{P} \otimes V^{\star},$$

which is an isomorphism. Theorem 1.9.1 is proved.

**(1.10) The super–version.** Most of the above constructions can be performed in any $k$–linear abelian symmetric monoidal category $\mathcal{C}$, instead of Vect. We will be particularly interested in the category SVect of super-vector spaces. Recall that objects of SVect are $\mathbf{Z}/2$–graded $k$–vector spaces, the tensor product is the usual graded one and the symmetry isomorphism $V \otimes W \to W \otimes V$ involves the Koszul sign.

If $\mathcal{P}$ is an operad in SVect, then $\mathcal{P}$–algebras (in SVect) will be called $\mathcal{P}$–superalgebras and their category will be denoted $\mathcal{P}$–SAlg.

All the statements given above for $k$–linear operads generalize to SVect without difficulty. In particular, for any super–vector space $V$ we have the endomorphism operad $\operatorname{Op}\operatorname{End}(V)$ in SVect, for any two super–vector spaces $V, W$ we have a bimodule $\operatorname{Op}\operatorname{Hom}(V, W)$ and so on.

We will need a superversion of the Morita theorem and of a more precise statement underlying it. Let $\mathcal{P}$ be any operad in SVect, $M$ any right $\mathcal{P}$–module, $\mathcal{Q} = \operatorname{Op}\operatorname{End}_{\mathcal{P}}(M)$ and $M^* = \operatorname{Op}\operatorname{Hom}_{\mathcal{P}}(M, \mathcal{P})$. Then, as before, we have a morphism

$$(1.10.1) \qquad\qquad M^* \circ_{\mathcal{Q}} M \to \mathcal{P}$$

of $(\mathcal{P}, \mathcal{P})$-bimodules and a morphism

$$(1.10.2) \qquad\qquad M \circ_{\mathcal{P}} M^* \to \mathcal{Q}$$

of $(\mathcal{Q}, \mathcal{Q})$-bimodules.

**(1.10.3) Theorem.** *If $M$ is free, i.e., isomorphic to $\mathcal{P} \otimes_k V$ where $V$ is a finite–dimensional supervector space, then the morphisms (1.10.1–2) are isomorphisms and hence the functor $f_M$ is an equivalence between $\mathcal{P}$–SAlg and $\mathcal{Q}$-SAlg.*

The proof is obtained by performing the same steps as for Theorem 1.9.1, with easy modifications to accomodate the $\mathbf{Z}/2$-grading.

## §2. Differential algebras.

**(2.1) Operadic approach to nonlinear differential equations.** Let $X$ be a smooth complex algebraic variety of dimension $p$. Denote by $\mathcal{D}(n) = \mathcal{D}_X(n)$ the sheaf of $n$–linear multidifferential operators

$$(u_1, ..., u_n) \mapsto L(u_1, ..., u_n), \quad u_i \in \mathcal{O}_X$$

acting on regular functions on $X$. If $(x_1, ..., x_p)$ is a local coordinate system on $X$, and $\partial^I = \partial_{x_1}^{i_1}...\partial_{x_p}^{i_p}$, $I = (i_1, ..., i_p)$, is the iterated partial derivative corresponding to a multiindex $I$, then a local section $L$ of $\mathcal{D}(n)$ acts on functions as follows

$$L(u_1, ..., u_n) = \sum_{I(1),...,I(n)} f_{I(1),...,I(n)}(x)(\partial^{I(1)}u_1)\cdot...\cdot(\partial^{I(n)}u_n), \quad f_{I(1),...,I(n)}(x) \in \mathcal{O}_X.$$

Thus $\mathcal{D}(1)$ is the usual sheaf of rings of linear differential operators on $X$. The collection $(\mathcal{D}_X(n))_{n \geq 0}$ forms a sheaf of operads on $X$, with the composition given by superposition of multilinear differential operators. As with any operad, $\mathcal{D}_X(n)$ is endowed with a left $\mathcal{D}_X(1)$–module (in particular, a $\mathcal{O}_X$-module) structure and with $n$ commuting right $\mathcal{D}_X(1)$–module (in particular, $\mathcal{O}_X$–module) structures.

As is well known, the study of sheaves of modules over the sheaf of rings $\mathcal{D}_X(1)$ provides a fruitful algebraic approach to the theory of linear differential equations. The introduction of the sheaf of operads $\mathcal{D}_X$ and its sheaves of algebras provides a similar algebraic language for systems of nonlinear differential equations. More precisely, let us give the following definition.

**(2.1.1) Definition.** *A system of differential equations on $X$ (polynomial in the derivatives) is a sheaf $\mathcal{A}$ of $\mathcal{D}_X$–algebras which is locally given by finitely many generators and relations. A solution to a system of equations given by $\mathcal{A}$ is a morphism of $\mathcal{D}_X$–algebras $\mathcal{A} \to \mathcal{O}_X$ (note that $\mathcal{O}_X$ is a $\mathcal{D}_X$–algebra by definition).*

To understand how this is related with the intuitive concept of a system of differential equations, consider the case of one unknown function $u$ and a system of differential equations

$$(2.1.2) \qquad P_\nu\left(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \; ...\right) = 0$$

where $P_\nu$ is a polynomial in $u$ and its derivatives with coefficients in $\mathcal{O}_X$. We polarize each homogeneous component of $P_\nu$ to a multilinear differential operator thus writing each equation from (2.1.2) as

$$(2.1.3) \qquad \sum_n L_{\nu,n}(u, ..., u) = 0.$$

where each $L_{\nu,n}$ is an $n$–ary multilinear differential operator, i.e., a section of $\mathcal{D}_X(n)$. Then we form a $\mathcal{D}_X$–algebra $\mathcal{A}$ on one generator $U$ subject only to the relations (2.1.3). A $\mathcal{D}_X$–algebra homomorphism $\mathcal{A} \to \mathcal{O}_X$ is clearly the same as a solution to (2.1.2), as $U$ should go to some function $u \in \mathcal{O}_X$ so that the defining relations of $U$ in $\mathcal{A}$ are satisfied in $\mathcal{O}_X$. Similarly, a system of equations on $n$ unknown functions is translated into an algebra with $n$ generators.

The next proposition can be checked by straightforward manipulations in local coordinates.

**(2.1.4) Proposition.** *(a) The filtration $F_\bullet \mathcal{D}_X(n)$ by total number of derivatives is compatible with the operad structure (i.e., makes $\mathcal{D}_X$ into an operad in the category of filtered quasicoherent sheaves of $\mathcal{O}_X$-modules). In particular, $F_0 \mathcal{D}_X$ is a sheaf of operads which can be identified with $\mathcal{C}om \otimes_{\mathbf{C}} \mathcal{O}_X$, and $\mathrm{gr}_\bullet^F \mathcal{D}_X$ is an operad in the category of graded sheaves.*

*(b) A $\mathcal{D}_X$-algebra is the same as a commutative $\mathcal{O}_X$-algebra (via $\mathcal{C}om \otimes_{\mathbf{C}} \mathcal{O}_X = F_0 \mathcal{D}_X \subset \mathcal{D}_X$) which is made into a left $\mathcal{D}_X(1)$-module such that vector fields in $\mathcal{D}_X(1)$ act by algebra derivations.*

Part (b) shows that our approach is in fact identical to the classical approach of "differential algebra" of Ritt [Ri]. However, the operadic point of view seems to present several advantages. Let us explain, for example, the analog of the fundamental fact that the algebra $\mathrm{gr}_\bullet^F \mathcal{D}_X(1)$ is commutative and can be identified with the algebra of functions on $T^*X$. Recall that an algebra $A$ is commutative if and only if the multiplication $A \otimes A \to A$ is an algebra morphism, i.e., $A$ is an algebra in the category of algebras.

**(2.1.5) Proposition.** *(a) Each sheaf $\mathrm{gr}_\bullet^F \mathcal{D}_X(n)$ has a natural structure of a commutative algebra so that $\mathrm{gr}_\bullet^F \mathcal{D}_X$ becomes an operad in the category of algebras.*

*(b) The spectrum of the algebra $\mathrm{gr}_\bullet^F \mathcal{D}_X(n)$ is identified with $(T_X^*)^{\oplus n}$.*

*(c) The operad structure on $\mathrm{gr}_\bullet^F \mathcal{D}_X$ is induced by a structure of the cooperad (in the category of algebraic varieties) on the collection of $(T_X^*)^{\oplus n}$ given by the maps*

$$(2.1.6) \qquad \nu_{a_1,\ldots,a_n} : (T_X^*)^{\oplus(a_1+\ldots+a_n)} \to (T_X^*)^{\oplus n} \times (T_X^*)^{\oplus a_1} \times \ldots \times (T_X^*)^{\oplus a_n},$$

$$(\xi_1, \ldots, \xi_{a_1+\ldots+a_n}) \mapsto$$

$$\left( \left( \sum_{i=1}^{a_1} \xi_i, \sum_{i=a_1+1}^{a_1+a_2} \xi_i, \ldots, \sum_{i=a_1+\ldots+a_{n-1}+1}^{a_1+\ldots+a_n} \xi_i \right), (\xi_1, \ldots, \xi_{a_1}), \ldots, (\xi_{a_1+\ldots+a_{n-1}+1}, \ldots, \xi_{a_1+\ldots+a_n}) \right).$$

Again, the proof is straightforward.

**(2.1.7) Remarks.** (a) Note that the construction (2.1.6) can be defined in fact for any abelian group $A$ (instead of $T_X^*$), making $(A^{\oplus n})_{n \geq 0}$ into a cooperad.

(b) Proposition 2.1.5(c) suggests a natural algebraic approach to microlocalization of nonlinear equations: instead of an open cone in $T_X^*$ in linear theory, the microlocalization should be done with respect to a sub-cooperad in $\{(T_X^*)^{\oplus n}\}$.

**(2.2) $\mathcal{D}_X$ as an endomorphism operad.** We can view the operad $\mathcal{D}_X$, though not formally as particular case, but as a close analog of the construction of the endomorphism operad $\mathrm{Op\,End}\,(V)$ from (1.2.6): we take for $V$ the space (sheaf) $\mathcal{O}_X$ of functions on $X$ and consider only local endomorphisms. All the abstract constructions of §1 can be performed in this situation and have interesting meaning.

First of all, the role of the dual space $V^*$ is played by $\omega_X$, the sheaf of volume forms on $X$, and the pairing between $V$ and $V^*$ is replaced by the Serre duality. The analog of the fact that $\mathrm{Op\,End}\,(V)(n) = (V^*)^{\otimes n} \otimes V$ is given by the following coordinate-free definition of $\mathcal{D}_X(n)$ in terms of local cohomology sheaves, generalizing Sato's definition of linear differential operators [SKK].

**(2.2.1) Proposition.** *Consider the $(n+1)$-fold Cartesian product $X^{n+1}$ whose factors will be labeled by integers $0, 1, ..., n$ and let $p_i : X^{n+1} \to X$, $i = 0, ..., n$, be the projections. Let also $\Delta \subset X^{n+1}$ be the image of the diagonal embedding of $X$. Then we have a natural identification*

$$\mathcal{D}_X(n) = \underline{H}_\Delta^{n\dim(X)}\left(\bigotimes_{i=1}^n p_i^*\omega_X\right)$$

**Proof.** We separate the question in two. First, let $f : X \to Y$ be any morphism of smooth algebraic varieties. One has then the sheaf

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$$

of $(\mathcal{D}_X(1), f^{-1}\mathcal{D}_Y(1))$-bimodules on $X$, see, e.g., [Bo] IV, §4.2 or [Ka] p. 24. Its sections can be seen as linear differential operators $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, see [Ka], *loc. cit.* Our proposition is then implied by the next two facts.

**(2.2.2) Lemma.** *If $\Gamma(f) \subset X \times Y$ is the graph of $f$ and $p_Y : X \times Y \to Y$ is the projection, then $D_{X\to Y}$ is naturally identified with $\underline{H}_{\Gamma(f)}^{\dim(Y)}(p_Y^*\omega_Y)$.*

**(2.2.3) Lemma.** *If $Y = X^n$ and $f : X \to X^n$ is the diagonal embedding, then $D_{X\to X^n} = \mathcal{D}_X(n)$.*

Lemma 2.2.2 is fairly classical, see, e.g., [Ka], p.24 or (for a similar fact about pseudodifferential operators) [SKK], p. 329.

To see Lemma 2.2.3, notice that for $n$ functions $u_1, ..., u_n \in \Gamma(U, \mathcal{O}_X)$, we have the function $u_1 \otimes ... \otimes u_n \in \Gamma(U^n, \mathcal{O}_{X^n})$, which then gives a section of $f^{-1}\mathcal{O}_{X^n}$ over $U$, still denoted $u_1 \otimes ... \otimes u_n$. Now, applying linear differential operators

$f^{-1}\mathcal{O}_{X^n} \to \mathcal{O}_X$ (i.e., sections of $\mathcal{D}_{X \to X^n}$) to sections of the form $u_1 \otimes \dots \otimes u_n$, we get precisely all the $n$-linear differential operators $(\mathcal{O}_X)^n \to \mathcal{O}_X$, as one can easily see in local coordinates. Lemma 2.2.3 and Proposition 2.2.1 are proved.

As we are almost in the endomorphism operad situation, let us further compare it to the framework of (1.3.1). Writing $\sim$ to denote the analogous objects, we have $\mathcal{P} \sim \mathcal{D}_X$, $V \sim \mathcal{O}_X$. Next, the analog of the module $V^\star$ from (1.3.1) looks as follows. Consider $X^n$, with projections $p_i : X^n \to X$, $i = 1, \dots, n$ and the diagonal $\Delta \subset X^n$, so that $\Delta$ is identified with $X$. Let

$$(2.2.4) \qquad \Omega(n) = \underline{H}_\Delta^{(n-1)\dim(X)}\left( \bigotimes_{i=1}^n p_i^* \omega_X \right) = \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X(n-1).$$

**(2.2.5) Proposition.** *The collection $\Omega = \{\Omega(n)\}_{n \geq 1}$ is naturally a right $\mathcal{D}_X$-module in the sense of (1.2).*

The proof is straightforward.

Proposition 2.2.5 generalizes the fact that $\omega_X$ is a right module over the algebra $\mathcal{D}_X(1)$.

**(2.3) $\mathcal{D}$-algebras on supermanifolds.** Let now $X$ be a supermanifold of dimension $(p|q)$, see [Man1]. Ch. 4. Thus $X$ consists of an ordinary smooth complex algebraic variety $X_{\mathrm{red}}$ of dimension $p$ and a sheaf $\mathcal{O}_X$ of $\mathbf{Z}/2$-graded supercommutative algebras which is locally isomorphic to $\mathcal{O}_{X_{\mathrm{red}}} \otimes_{\mathbf{C}} \Lambda[\xi_1, \dots, \xi_q]$. Here $\Lambda[\xi_1, \dots, \xi_q]$ is the exterior algebra on the generators $\xi_i$. By $\omega_X$ we will understand the sheaf of volume forms in the sense of Berezin, see *loc. cit.* All the constructions and statements of (2.1)–(2.2) can be immediately generalized to the super case, giving a sheaf of operads (in SVect) $\mathcal{D}_X$ on $X$ of which $\mathcal{O}_X$ is a sheaf of superalgebras.

In particular, consider the case $p = 0$, i.e., $X = \mathbf{C}^{(0|q)} = \mathrm{Spec}\,\Lambda$, where $\Lambda = \Lambda[\xi_1, \dots, \xi_q]$. Note that $\Lambda$ is finite–dimensional super–vector space over $\mathbf{C}$, of dimension $(2^{q-1}|2^{q-1})$ and $X_{\mathrm{red}}$ is in this case just a point, so a sheaf on it is just a vector space.

**(2.3.1) Proposition.** *(a) For $X = \mathbf{C}^{(0|q)}$ the sheaf of operads $\mathcal{D}_X$ on $X_{\mathrm{red}} = \{\mathrm{pt}\}$ is $\mathrm{Op\,End}(\Lambda)$, the endomorphism operad of $\Lambda$ considered as a super-vector space.*

*(b) If $X = X_{\mathrm{red}} \times \mathbf{C}^{(0|q)}$ is a split supermanifold, then $\mathcal{D}_X \simeq \mathcal{D}_{X_{\mathrm{red}}} \otimes_{\mathbf{C}} \mathrm{Op\,End}(\Lambda)$ is isomorphic to a matrix operad over $\mathcal{D}_{X_{\mathrm{red}}}$.*

**Proof.** Part (b) follows from (a). To see (a), notice first that $\mathcal{D}$-action on $\mathcal{O}$, i.e., on $\Lambda$, gives an operad morphism $\mathcal{D} \mapsto \mathrm{Op\,End}(\Lambda)$. Let $\xi^I$, $I = (1 \leq i_1 < \dots < i_m \leq q)$, be the monomial basis in $\Lambda$. Let $F_{I(1),\dots,I(n),J} \in \mathrm{Hom}_{\mathbf{C}}(\Lambda^{\otimes n}, \Lambda)$ be the

"matrix unit" which takes $\xi^{I(1)} \otimes ... \otimes \xi^{I(n)}$ to $\xi^J$ and all the other tensor products of monomials to 0. Such operators form a basis in $\operatorname{Hom}_{\mathbf{C}}(\Lambda^{\otimes n}, \Lambda)$. On the other hand, the space $\mathcal{D}(n)$ of all $n$–linear differential operators on $\Lambda$ has a basis formed by operators

$$L_{I(1),...,I(n),J}, \quad u \mapsto \xi^J(\partial^{I(1)}u)...(\partial^{I(n)}u).$$

Consider the filtration of $\Lambda$ by the powers of the maximal ideal $(\xi_1, ..., \xi_q)$ and the induced filtration on $\operatorname{Hom}_{\mathbf{C}}(\Lambda^{\otimes n}, \Lambda)$. Modulo this filtration, the action of $L_{I(1),...,I(n),J}$ is given by $F_{I(1),...,I(n),J}$, whence the statement.

Theorem 1.10.3 is now formally applicable in the situation of (2.3.1)(b) but we prefer to make a slightly more general statement. Let $\mathcal{D}_X$–SAlg be the category of sheaves of $\mathcal{D}_X$–superalgebras, see (1.10). Note that even if $X$ happens to be purely even, then a $\mathcal{D}_X$-superalgebra is still required to be $\mathbf{Z}/2$-graded.

**(2.3.2) Theorem.** *The categories $\mathcal{D}_X$–SAlg and $\mathcal{D}_{X_{\mathrm{red}}}$–SAlg are equivalent.*

This is an operadic extension of the result of I. Penkov [P] on $\mathcal{D}_X(1)$-modules.

**Proof.** Consider the natural embedding of the supermanifolds $i : X_{\mathrm{red}} \to X$. On the level of underlying spaces this is the identity and the source and target of $i$ differ only by the sheaves of rings: $\mathcal{O}_{X_{\mathrm{red}}}$ as opposed to $\mathcal{O}_X$. Accordingly, both $\mathcal{D}_X$ and $\mathcal{D}_{X_{\mathrm{red}}}$ are sheaves of operads in SVect on the same underlying space $X_{\mathrm{red}}$. This means that we can apply the formalism of Op Hom bimodules of Section 1, if we understand them in a sheaf–theoretic sense. Let

$$\mathcal{D}_{\to} = \{D_{\to}(n)\} = \underline{\operatorname{Op Hom}}(\mathcal{O}_X, \mathcal{O}_{X_{\mathrm{red}}}).$$

In other words, we set $\mathcal{D}_{\to}(n)$ to be the sheaf of $n$–linear differential operators $\mathcal{O}_X \times \cdots \times \mathcal{O}_X \to \mathcal{O}_{X_{\mathrm{red}}}$ in the obvious sense. Clearly $\mathcal{D}_{\to}$ is a $(\mathcal{D}_{X_{\mathrm{red}}}, \mathcal{D}_X)$–bimodule. Similarly, let

$$\mathcal{D}_{\leftarrow} = \{D_{\leftarrow}(n)\} = \underline{\operatorname{Op Hom}}(\mathcal{O}_{X_{\mathrm{red}}}, \mathcal{O}_X)$$

be the collection of sheaves of $n$–linear differential operators $\mathcal{O}_{X_{\mathrm{red}}} \times ... \times \mathcal{O}_{X_{\mathrm{red}}} \to \mathcal{O}_X$. This is a $(\mathcal{D}_X, \mathcal{D}_{X_{\mathrm{red}}})$–bimodule. The relative plethysm with these bimodules defines functors from $\mathcal{D}_X$–SAlg to $\mathcal{D}_{X_{\mathrm{red}}}$–SAlg and back. We claim that these functors are mutually inverse. More precisely, we have natural morphisms of sheaves of operads (in SVect)

(2.3.3) $$\mathcal{D}_{\leftarrow} \circ_{\mathcal{D}_{X_{\mathrm{red}}}} \mathcal{D}_{\to} \to \mathcal{D}_{X_{\mathrm{red}}}, \quad \mathcal{D}_{\to} \circ_{\mathcal{D}_X} \mathcal{D}_{\leftarrow} \to \mathcal{D}_X,$$

obtained by sheafification of (1.10.1–2). In virtue of (1.5.7), it is enough to show that the morphisms (2.3.3) are isomorphisms. To verify this, we can work locally, over an affine open set $U \subset X_{\mathrm{red}}$. In this case the supermanifold $(U, \mathcal{O}_X|_U)$ is split, so Proposition 2.3.1(b) together with Theorem 1.10.3 imply that our morphisms are isomorphisms after restruction on $U$. This proves our theorem.

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