

WEAK FROBENIUS MANIFOLDS

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Abstract. We establish a new universal relation between the Lie bracket and \circ -multiplication of tangent fields on any Frobenius (super)manifold. We use this identity in order to introduce the notion of “weak Frobenius manifold” which does not involve metric as part of structure. As another application, we show that the powers of an Euler field generate (a half of) the Virasoro algebra on an arbitrary, not necessarily semi-simple, Frobenius supermanifold.

0. Introduction. B. Dubrovin introduced and thoroughly studied in [D] the notion of Frobenius manifold. By definition, it is a structure (M, g, \circ) where M is a manifold, \circ is an associative, commutative and \mathcal{O}_M -bilinear multiplication on the tangent sheaf \mathcal{T}_M , and g is a flat metric on M , invariant with respect to \circ . The main axiom is the local existence of a function Φ (Frobenius potential) such that the structure constants of \circ in the basis ∂_a of flat local fields are given by the tensor of third derivatives $A_{ab}{}^c = \Phi_{ab}{}^c$ with one index raised with the help of g .

We start with establishing a new universal identity (1) between the \circ -multiplication and the Lie bracket. It follows formally from the Poisson (or Leibniz) identity, but is strictly weaker, and the algebra of tangent fields on a Frobenius manifold is never Poisson. For further comments see section 5. We show that this identity encodes an essential part of the potentiality property, at least in the semisimple case.

We then use it in order to introduce in section 3 “weak Frobenius manifolds” that is, Frobenius manifolds without a fixed flat metric. We explain the relation of this notion to Dubrovin’s notion of twisted Frobenius manifolds ([D], Appendix B.) The importance of weak Frobenius manifolds is related to the fact that in the constructions of K. Saito and Barannikov–Kontsevich the metric is the part of the structure that comes last, and (at least in the theory of singularities) requires considerable additional work.

Finally, in section 6 we extend the construction of the Virasoro algebra from the Euler field, previously known only in the semisimple case, to the general situation.

As a general reference on the basics of the theory of Frobenius manifolds we use [M] (summarized in [MM].) In particular, our manifolds are supermanifolds (say, in the complex analytic category). The multiplication \circ is called semisimple, if locally (\mathcal{T}_M, \circ) is isomorphic to \mathcal{O}_M^n with componentwise multiplication. The basic idempotent vector fields are then denoted e_i .

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1. Definition. An F -manifold is a pair (M, \circ) , where M is a (super)manifold and \circ is an associative supercommutative \mathcal{O}_M -bilinear multiplication $\mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{T}_M$ satisfying the following identity: for any (local) vector fields X, Y, Z, W we have

$$\begin{aligned} & [X \circ Y, Z \circ W] - [X \circ Y, Z] \circ W - (-1)^{(X+Y)Z} Z \circ [X \circ Y, W] \\ & - X \circ [Y, Z \circ W] + X \circ [Y, Z] \circ W + (-1)^{YZ} X \circ Z \circ [Y, W] \\ & - (-1)^{XY} Y \circ [X, Z \circ W] + (-1)^{XY} Y \circ [X, Z] \circ W + (-1)^{X(Y+Z)} Y \circ Z \circ [X, W] = 0 \quad (1) \end{aligned}$$

Here and in section 5 we write, say, $(-1)^{(X+Y)Z}$ as a shorthand for $(-1)^{(\tilde{X}+\tilde{Y})\tilde{Z}}$, where \tilde{X} is the parity of X .

1.1. Remarks. (i) The left hand side of (1) is \mathcal{O}_M -polylinear in X, Y, Z, W . In other words, it is a tensor. This can be checked by a completely straightforward, although lengthy, calculation.

(ii) Introduce the expression measuring the deviation of the structure $(\mathcal{T}_M, \circ, [,])$ from that of Poisson algebra on (\mathcal{T}_M, \circ) :

$$P_X(Z, W) := [X, Z \circ W] - [X, Z] \circ W - (-1)^{XZ} Z \circ [X, W]. \quad (2)$$

Then (1) is equivalent to the following requirement:

$$P_{X \circ Y}(Z, W) = X \circ P_Y(Z, W) + (-1)^{XY} Y \circ P_X(Z, W). \quad (3)$$

2. Theorem. a). Let (M, g, A) be a Frobenius manifold with multiplication \circ . Then (M, \circ) is an F -manifold.

b). Let (M, \circ) be a pure even F -manifold, whose multiplication law is semisimple on an open dense subset. Assume that it admits an invariant flat metric g defining the cubic tensor A as in [M], I.(1.2). Then (M, g, A) is Frobenius manifold.

Proof. a). Since the left hand side of (1) is a tensor, it suffices to check that it vanishes on quadruples of flat fields $(X, Y, Z, W) = (\partial_a, \partial_b, \partial_c, \partial_d)$. Flat fields (super)commute so that only five summands of nine survive in (1). Denoting the structure constants $A_{ab}{}^c$ as in [M], I.(1.4) and calculating the coefficient of ∂_f in (1), we can represent it as a sum of five summands, for which we introduce special notation in order to explain the pattern of cancellation:

$$\begin{aligned} & \sum_e A_{ab}{}^e \partial_e A_{cd}{}^f - (-1)^{(a+b)(c+d)} \sum_e A_{cd}{}^e \partial_e A_{ab}{}^f = \alpha_1 + \beta_1, \\ & (-1)^{(a+b)c} \sum_e \partial_c A_{ab}{}^e A_{ed}{}^f = \alpha_2, \quad (-1)^{(a+b+c)d} \sum_e \partial_d A_{ab}{}^e A_{ec}{}^f = \gamma_1, \end{aligned}$$

$$-(-1)^{a(b+c+d)} \sum_e \partial_b A_{cd}^e A_{ea}^f = \gamma_2, \quad -(-1)^{(c+d)b} \sum_e \partial_a A_{cd}^e A_{eb}^f = \beta_2.$$

Here we write, say, $(-1)^{(a+b)c}$ as a shorthand for $(-1)^{(\widetilde{x}_a + \widetilde{x}_b)\widetilde{x}_c}$.

Use potentiality in order to interchange the subscripts e, c in α_1 (a sign emerges). After this we see that

$$\alpha_1 + \alpha_2 = (-1)^{(a+b)c} \partial_c \left(\sum_e A_{ab}^e A_{ed}^f \right).$$

Similarly, permuting a and e in β_1 we find

$$\beta_1 + \beta_2 = -(-1)^{(c+d)b} \partial_a \left(\sum_e A_{cd}^e A_{eb}^f \right).$$

Now rewrite γ_1 permuting a, d , and γ_2 permuting b, c . Calculating finally $\beta_1 + \beta_2 + \gamma_1 + \gamma_2$ we see that it cancels with $\alpha_1 + \alpha_2$ due to the associativity relations [M], I.(1.5).

b). Clearly, (M, g, A) is an associative pre-Frobenius manifold in the sense of [M], Definition I.1.3, so that it only remains to check its potentiality in the domain of semisimplicity. To this end we will use the Theorem I.3.3 of [M].

Let (e_i) be the basic idempotent local vector fields. Applying (3) to $X = Y = e_i$ we get $P_{e_i} = 2e_i \circ P_{e_i}$ so that $P_{e_i} = 0$. Applying then (2) to $(X, Z, W) = (e_i, e_j, e_j)$, $i \neq j$, we see that $[e_i, e_j] = 0$. This is the first condition of the Theorem I.3.3. The second one expresses invariance and flatness of the metric in canonical coordinates, which we have already postulated.

2.1. Corollary (of the proof). *Semisimple F -manifolds are exactly those manifolds (M, \circ) which everywhere locally admit a basis of pairwise commuting \circ -idempotent vector fields, or, which is the same, Dubrovin's canonical coordinates.*

In fact, we have already deduced from (1) that that e_i pairwise commute. Conversely, if they commute, (1) holds for any quadruple of idempotents.

3. Definition. *A weak Frobenius manifold is an F -manifold (M, \circ) such that in a neighborhood U of any point there exists a flat invariant metric g making (U, g, \circ) Frobenius manifold. We will call such metrics compatible (with the given F -structure.)*

Thus, a weak Frobenius manifold is a Frobenius manifold without a fixed metric.

Any semisimple F -manifold is automatically weak Frobenius. This follows from the results of [M], Chapter II, §3, which reduce the construction of compatible metrics to the solution to Schlesinger's equations. We do not know whether there exist non-semisimple F -manifolds which are not weak Frobenius.

4. Sheaves of compatible metrics and Euler fields. Let (M, \circ) be a weak Frobenius manifold. Then compatible metrics on M form a sheaf \mathcal{M}_M . Assume now that M admits an identity e . Then there is an embedding $\mathcal{M}_M \hookrightarrow \Omega_M^1$ which sends each metric g to the respective coidentity ε_g defined by $i_X \varepsilon_g = g(e, X)$. In fact, knowing ε_g we can reconstruct $g : g(X, Y) = i_{X \circ Y}(\varepsilon_g)$. We will call ε_g compatible 1-forms.

We will now impose an additional restriction and denote by \mathcal{F}_M the sheaf of those compatible metrics for which e is flat. We will call such metrics *admissible*. If $g \in \mathcal{F}_M$, then ε_g is closed and g -flat: see [M], I.(2.4). It is important to understand the structure of this sheaf of sets. Again, the situation is rather transparent on the tame semisimple part of M . We will state and prove Dubrovin's theorem which provides a neat local description of admissible metrics *with fixed rotation coefficients* γ_{ij} considered as functions on the common definition domain of metrics.

This result should be compared with the Theorem II.3.4.3 of [M] which depicts the set of metrics *with fixed v_{ij} at a point* where

$$v_{ij} = \frac{1}{2} (u^j - u^i) \frac{\eta_{ij}}{\eta_j}, \quad \gamma_{ij} = \frac{1}{2} \frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}} = \frac{1}{u^j - u^i} \sqrt{\frac{\eta_j}{\eta_i}} v_{ij} \quad (4)$$

so that both statements refer to the closely related coordinates on the space of metrics.

4.1. Theorem. *a). Let $g = \sum_i \eta_i (du^i)^2$, $\tilde{g} = \sum_i \tilde{\eta}_i (du^i)^2$ be two \circ -invariant metrics in a simply connected domain of canonical coordinates u^1, \dots, u^n in M . Then there exist exactly 2^n vector fields ∂ in this domain such that*

$$\tilde{g}(X, Y) = g(\partial \circ X, \partial \circ Y) \quad (5)$$

for any X, Y . These fields are \circ -invertible and differ only by the signs of their e_i -components.

b). Assume that g is admissible. Then \tilde{g} defined by (5) is admissible and has the same rotation coefficients $\tilde{\gamma}_{ij} = \gamma_{ij}$ iff ∂ is g -flat.

Proof. a). Put $\partial = \sum_i D_i e_i$. Then (5) is equivalent to

$$D_i^2 = \frac{\tilde{\eta}_i}{\eta_i}. \quad (6)$$

This proves the first statement.

b). Choose a solution (D_i) of (6). Let ∇_i denote the Levi-Civita covariant derivative in the direction e_i with respect to the metric g . Using [M], I.(3.10), we see that

$$\nabla_i \left(\sum_j D_j e_j \right) =$$

$$\sum_{j \neq i} \left(e_i D_j + \frac{1}{2} D_j \frac{\eta_{ij}}{\eta_j} - \frac{1}{2} D_i \frac{\eta_{ij}}{\eta_j} \right) e_j + \left(e_i D_i + \frac{1}{2} \sum_j D_j \frac{\eta_{ij}}{\eta_i} \right) e_i. \quad (7)$$

Using (6) and [M], I.(3.13), we can rewrite the first sum in (7) as

$$\sum_{j \neq i} \left(\sqrt{\frac{\tilde{\eta}_i}{\eta_j}} (\tilde{\gamma}_{ij} - \gamma_{ij}) \right) e_j \quad (8)$$

and the remaining terms as

$$\left(\frac{1}{2} \frac{\tilde{\eta}_{ii}}{\sqrt{\eta_i \tilde{\eta}_i}} + \sum_{j \neq i} \sqrt{\frac{\tilde{\eta}_j}{\eta_i}} \gamma_{ij} \right) e_i. \quad (9)$$

If we replace in (9) γ_{ij} by $\tilde{\gamma}_{ij}$, the resulting expression will vanish for admissible \tilde{g} because $(\sum_j e_j) \tilde{\eta}_i = 0$ (see [M], Proposition I.3.3). Subtracting this zero from (9) we finally find

$$\nabla_i \left(\sum_j D_j e_j \right) = \sum_{j \neq i} \sqrt{\frac{\tilde{\eta}_i}{\eta_j}} (\tilde{\gamma}_{ij} - \gamma_{ij}) e_j - \sum_{j \neq i} \sqrt{\frac{\tilde{\eta}_j}{\eta_i}} (\tilde{\gamma}_{ij} - \gamma_{ij}) e_i. \quad (10)$$

Hence admissibility of \tilde{g} and coincidence of the rotation coefficients imply the g -flatness of ∂ , and vice versa. This proves the second statement of the theorem.

If one does not assume semi-simplicity, a part of the preceding theorem still holds true. It is, too, due to Dubrovin.

4.2. Theorem. *Let g be an admissible metric and ∂ a g -flat even invertible vector field. Then \tilde{g} defined by (5) is admissible.*

Proof. Put $\tilde{x}^a := \sum_b g^{ab} \partial_b \partial \Phi = \partial \Phi^a$, where $(\partial_a = \partial / \partial x^a)$ is a local basis of flat vector fields and Φ is a local Frobenius potential. Then we have

$$\frac{\partial \tilde{x}^a}{\partial x^b} = \partial \Phi_b^a.$$

Hence the respective Jacobian is the matrix of the multiplication $\partial \circ$ in the basis (∂_a) . Since the latter is invertible, (\tilde{x}^a) form a local coordinate system. For the dual basis of vector fields $\tilde{\partial}_a$ we have

$$\partial \circ \tilde{\partial}_a = \sum_b \tilde{\partial}_a(x^b) \partial_b \circ \partial = \sum_{b,c,d} \tilde{\partial}_a(x^b) g^{cd} \partial_b \partial_d \partial \Phi \partial_c = \sum_{b,c} \tilde{\partial}_a(x^b) \partial_b(\tilde{x}^c) \partial_c = \partial_a. \quad (11)$$

Hence \tilde{g} has the same coefficients in the basis $(\tilde{\partial}_a)$ as g in the basis (∂_a) and is flat. Clearly, it is also \circ -invariant. It remains to show that the pre-Frobenius structure (M, \circ, \tilde{g}) is potential. It is easy to see that any local function $\tilde{\Phi}$ satisfying the equations

$$\tilde{\partial}_a \tilde{\partial}_b \tilde{\Phi} = \partial_a \partial_b \Phi \quad (12)$$

for all a, b can serve as a local potential defining the same multiplication \circ . To prove its existence, we check the integrability condition:

$$\begin{aligned} \tilde{\partial}_a \partial_b \partial_c \Phi &= \sum_d \tilde{\partial}_a(x^d) \partial_d \partial_b \partial_c \Phi = \sum_d \tilde{\partial}_a(x^d) g(\partial_d, \partial_b \circ \partial_c) \\ &= g(\tilde{\partial}_a, \partial_b \circ \partial_c) = g(\partial^{-1} \circ \partial_a, \partial_b \circ \partial_c) = (-1)^{ab} \tilde{\partial}_b \partial_a \partial_c \Phi. \end{aligned} \quad (13)$$

The same reasoning as in the end of the proof of [M], Theorem I.1.5 then shows the existence of $\tilde{\Phi}$. The identity e remains flat because \tilde{g} -flat fields form the sheaf $\partial^{-1} \mathcal{T}_M^f$ and $e = \partial^{-1} \circ \partial$. This finishes the proof.

Dubrovin calls the passage from g to \tilde{g} *the Legendre-type transformation*. In the Appendix B of [D] he also constructs a different type of transformations which he calls *inversion*.

What Dubrovin calls *a twisted Frobenius manifold* in our language is a weak Frobenius manifold, endowed with local admissible metrics connected by the Legendre-type transformations on the overlaps of their definition domains.

We now turn to Euler fields. Let again (M, \circ) be a weak Frobenius manifold. An even vector field E on M is called *a weak Euler field of (constant) weight d_0* if $\text{Lie}_E(\circ) = d_0 \circ$ that is, for all local vector fields X, Y we have

$$P_E(X, Y) = [E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = d_0 X \circ Y. \quad (14)$$

This is the same as [M], I.(2.6). If (M, \circ) admits an identity e , we get formally from (14) that $[e, E] = d_0 e$. Clearly, local weak Euler fields form a sheaf of vector spaces \mathcal{E}_X , and weight is a linear function on this sheaf. If (M, \circ) comes from a Frobenius manifold with flat identity e , then any Euler vector field on the latter is a weak Euler field, and e itself is a (weak) Euler field of weight zero. The latter statement follows by combining [M], Proposition I.2.2.2 and I.(2.3).

4.3. Proposition. *Commutator of any two (local) weak Euler fields is a weak Euler field of weight zero.*

Proof. We start with the following general identity: for any local vector fields X, Y, Z, W we have

$$\begin{aligned} P_{[X, Y]}(Z, W) &= \\ [X, P_Y(Z, W)] - (-1)^{XY} P_Y([X, Z], W) - (-1)^{X(Y+Z)} P_Y(Z, [X, W]) \end{aligned}$$

$$-(-1)^{XY}[Y, P_X(Z, W)] + P_X([Y, Z], W) + (-1)^{YZ}P_X(Z, [Y, W]). \quad (15)$$

In order to check this, replace the seven terms in (15) by their expressions from (2), and then rewrite the resulting three terms in the left hand side using the Jacobi identity. All the twenty four summands will cancel.

Now apply (15) to the two weak Euler fields $X = E_1, Y = E_2$. The right hand side will turn to zero. This proves our statement.

4.4. Example (Sh. Katz). The (formal) Frobenius manifold corresponding to the quantum cohomology of a projective algebraic manifold V admits at least two different Euler fields (besides e), if $h^{pq}(V) \neq 0$ for some $p \neq q$. To write them down explicitly, choose a basis (∂_a) of $H = H^*(V, \mathbf{C})$ considered as the space of flat vector fields, and let (x^a) be the dual flat coordinates vanishing at zero. Let $\partial_a \in H^{p_a, q_a}(V)$. Put $-K_V = \sum_{p_b+q_b=2} r^b \partial_b$. Then

$$E_1 := \sum_a (1 - p_a) x^a \partial_a + \sum_b r^b \partial_b,$$

$$E_2 := \sum_a (1 - q_a) x^a \partial_a + \sum_b r^b \partial_b$$

are Euler.

Let now g be an admissible metric on (M, \circ) . Weak Euler fields which are conformal with respect to g form a subsheaf of linear spaces in \mathcal{E}_M endowed with a linear function D , conformal weight: see [M], I.(2.5). A direct calculation shows that the commutator of such fields is conformal of conformal weight zero. One can also say what happens to the weights (and the full spectrum) of E when one replaces g by another metric as in (5).

4.5. Proposition. *Let (M, \circ, g) be a Frobenius manifold with flat identity e and an Euler field E , $[e, E] = d_0 e$, $\text{Lie}_E(g) = Dg$, (d_a) = the spectrum of $-\text{ad } E$ on flat vector fields. Assume that ∂ is an invertible flat field such that $[\partial, E] = d\partial$. Then E is an Euler field on (M, \tilde{g}, \circ) , $\text{Lie}_E(\tilde{g}) = (D + 2d_0 - 2d)\tilde{g}$, and the spectrum of $-\text{ad } E$ on $\partial^{-1}T_M^f$ is $(d_a + d_0 - d)$.*

We leave the straightforward proof to the reader.

5. Relation to Poisson structure. Consider an abstract structure $(A, \circ, [,])$ where \circ , resp. $[,]$ induce on the \mathbf{Z}_2 -graded additive group A the structure of supercommutative, resp. Lie, ring. Assume that these operations satisfy the relation (1), or equivalently, (3). Then we will call $(A, \circ, [,])$, or simply A , an F -algebra. In particular, vector fields on a Frobenius manifold form a sheaf of F -algebras.

Every Poisson algebra is an F -algebra. Conversely, let A be an F -algebra, and

$$B := \{X \in A \mid P_X \equiv 0\}. \quad (16)$$

5.1. Proposition. *a). B is closed with respect to \circ and $[\cdot, \cdot]$ and hence forms a Poisson subalgebra. If A contains identity e , then $e \in B$.*

b). If A is the algebra of vector fields on a split semisimple Frobenius manifold, B is spanned by the basic idempotent fields e_i over constants. In particular, the Lie bracket in B is trivial.

Proof. a). Assume that $P_X = P_Y = 0$. We have $P_{X \circ Y} = 0$ in view of (3). Putting $X = Y = e$ in (3), we get $P_e = 0$. Finally, $P_{[X, Y]} = 0$ follows from (15).

b). Writing X, Y, Z in the basis e_i with indeterminate coefficients, one easily checks that if $P_X(Y, Z) = 0$ for all Y, Z , then the coefficients of X are constant.

6. Theorem. *Let E be an Euler field on a Frobenius manifold with identity e such that $[e, E] = d_0 e$. Then for all $m, n \geq 0$*

$$[E^{\circ n}, E^{\circ m}] = d_0(m - n) E^{\circ m + n - 1}. \quad (17)$$

Proof. We will prove slightly more. Let X be an even vector field on an arbitrary F -manifold with identity e . Since in view of (3) the map $X \mapsto P_X$ is a \circ -derivation, we have $P_{X^{\circ n}} = nX^{\circ n-1} \circ P_X$. Moreover, from (2) we have

$$P_{X^{\circ n}}(e, e) = -[X^{\circ n}, e].$$

Hence

$$[X^{\circ n}, e] = nX^{\circ n-1} \circ [X, e]. \quad (18)$$

Let us *assume* now that X satisfies the following identities: for all $n \geq 1$

$$[X^{\circ n}, X] = (1 - n) X^{\circ n} \circ [e, X]. \quad (19)$$

Then we assert that for all $m, n \geq 0$

$$[X^{\circ n}, X^{\circ m}] = (m - n) X^{\circ n + m - 1} \circ [e, X]. \quad (20)$$

In fact, the cases when m or n is ≤ 1 are covered by (18), (19). The general case can be treated by induction. We have

$$\begin{aligned} [X^{\circ n}, X^{\circ m}] &= P_{X^{\circ n}}(X^{\circ m-1}, X) + [X^{\circ n}, X^{\circ m-1}] \circ X + [X^{\circ n}, X] \circ X^{\circ m-1} = \\ &= nX^{\circ n-1} \circ ([X, X^{\circ m}] - [X, X^{\circ m-1}] \circ X) + [X^{\circ n}, X^{\circ m-1}] \circ X + [X^{\circ n}, X] \circ X^{\circ m-1} \\ &= (m - n) X^{\circ n + m - 1} \circ [e, X]. \end{aligned}$$

It remains to notice that since $[e, E] = d_0 e$, E satisfies (19) in view of the general identity [M], I.(2.12).

6.1. Remark. In the semisimple case the meaning of (19) is transparent: writing $X = \sum X_i e_i$, we must have $e_i X_j = 0$ for $i \neq j$.

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