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On the Euler number of an orbifold

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Dedicated to Hans Grauert on his sixtieth birthday

This short note illustrates connections between Lothar Göttsche's results from the preceding paper and invariants for finite group actions on manifolds that have been introduced in string theory. A lecture on this was given at the MPI workshop on "Links between Geometry and Physics" at Schloß Ringberg, April 1989.

Invariants of quotient spaces. Let G be a finite group acting on a compact differentiable manifold X. Topological invariants like Betti numbers of the quotient space X/G are well-known:

$$b_i(X/G)=\dim H^i(X,\mathbf{R})^G=\frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(g^*\,|\,H^i(X,\mathbf{R}))\,.$$

The topological Euler characteristic is determined by the Euler characteristic of the fixed point sets X^g:

$$e(X/G)=\frac{1}{|G|}\sum_{g\in G}e(X^g)\,.$$

Physicists' formula. Viewed as an orbifold, X/G still carries some information on the group action. In [DHVW1, 2; V] one finds the following string-theoretic definition of the "orbifold Euler characteristic":

$$\boxed{e(X,G)=\frac{1}{|G|}\sum_{gh=hg}e(X^{\langle g,h\rangle})}\,.$$

Here summation runs over all pairs of commuting elements in $G\times G$, and $X^{\langle g,h\rangle}$ denotes the common fixed point set of g and h. The physicists are mainly interested in the case where X is a complex threefold with trivial canonical bundle and G is a finite subgroup of $SU(3)$. They point out that in some situations where X/G has a resolution of singularities $\widetilde{X/G}\xrightarrow{\pi}X/G$ with trivial canonical bundle $e(X,G)$ is just the Euler characteristic of this resolution [DHVW2; St-W].

In this paper we consider some well-known examples from algebraic geometry and check to what extent the formula

$$e(X,G)=e(\widetilde{X/G})$$

holds. We will also do this in the local situation of a matrix group $G\subset U(n)$ acting on $\mathbf{C}^n$, since in this non-compact case all the invariants considered here are meaningful as well.

Some elementary calculations. For a fixed $g\in G$ the elements commuting with g form the centralizer $C(g)$. The conjugacy class $[g]$ is a system of representatives for $G/C(g)$, so we have

$$\#C(g)\cdot\#([g])=|G|.$$

Since simultaneous conjugation of g and h by some element of G leaves $e(X^{\langle g,h\rangle})$ fixed, using the classical formula for $e(X/G)$ we can write $e(X,G)$ as a sum over the conjugacy classes of G:

$$\begin{aligned} e(X,G) &= \frac{1}{|G|}\sum_{[g]}\#([g])\sum_{h\in C(g)} e(X^{\langle g,h\rangle}) \\ &= \frac{1}{|G|}\sum_{[g]}\#([g])\cdot\#C(g)\cdot e(X^g/C(g)). \end{aligned}$$

So we get an equivalent definition which sometimes is more useful than the original one:

$$\boxed{e(X,G)=\sum_{[g]} e(X^g/C(g))}\,.$$

For a free action we immediately get $e(X,G)=e(X/G)$, and we also see that some assumption is necessary: For a cyclic group of order n acting on $\mathbf{P}^1(\mathbf{C})$ with two fixed points, the quotient is $\mathbf{P}^1(\mathbf{C})$ again, whereas $e(\mathbf{P}^1,G)=e(\mathbf{P}^1)+(n-1)\cdot 2=2n$.

Loop spaces. For $g\in G$ we consider the space of paths

$$\mathscr{L}(X,g):=\{\alpha:\mathbf{R}\to X\mid \alpha(t+1)=g\alpha(t)\}.$$

G acts on the disjoint union of these spaces by $(h\alpha)(t):=h\cdot\alpha(t)$. Obviously h transforms $\mathscr{L}(X,g)$ into $\mathscr{L}(X,hgh^{-1})$. We form the quotient

$$\mathscr{L}(X,G):=\Big(\biguplus_{g}\mathscr{L}(X,g)\Big)\Big/G=\biguplus_{[g]}(\mathscr{L}(X,g)/C(g)).$$

The real numbers act on the $\mathscr{L}(X,g)$ and on $\mathscr{L}(X,G)$ by transforming $\alpha(t)$ to $\alpha(t+c)$. The fixed point set of this action is

$$\biguplus_{[g]}(X^g/C(g))\subset\mathscr{L}(X,G),$$

where X^g is embedded in $\mathscr{L}(X,g)$ as the set of constant paths. This corresponds to the inclusion of X in the ordinary loop space $\mathscr{L}(X)$ as the fixed point set of the obvious S^1-action. On each component $\mathscr{L}(X,g)$ our $\mathbf{R}$-action is in fact an action of S^1 as well because $\alpha(t+\operatorname{ord}(g))=\alpha(t)$. So we can take the Euler characteristic with respect to this action, i.e. the Euler characteristic of the fixed point set, and get the orbifold invariant $e(X,G)$.

Quotient singularities. If G is a finite subgroup of $U(n)$ acting on $\mathbf{C}^n$, then every fixed point set is contractible. Thus $e(\mathbf{C}^n,G)$ equals the number of conjugacy classes, i.e. the number of isomorphism classes of irreducible representations of G.

If in particular $G\subset SU(2)$, then the corresponding 2-dimensional quotient singularity has a minimal resolution $\widetilde{\mathbf{C}^2/G}$ by a configuration of rational (-2)-curves. This is equivalent to $\widetilde{\mathbf{C}^2/G}$ having trivial canonical bundle. If the number of exceptional curves is k, then $e(\widetilde{\mathbf{C}^2/G})=k+1$. Now the McKay correspondence states that the number of non-trivial irreducible representations of G equals this number k of exceptional curves, hence $e(\widetilde{\mathbf{C}^2/G})=k+1=e(\mathbf{C}^2,G)$.

For resolution configurations containing other than (-2)-curves and therefore having non-trivial canonical divisor the result is false: If G is a cyclic subgroup of $U(2)$ generated by

$$\begin{pmatrix} \exp\left(2\pi i\,\frac{p}{n}\right) & 0 \\ 0 & \exp\left(2\pi i\,\frac{q}{n}\right) \end{pmatrix},$$

p, q relatively prime to n, we have $e(\mathbf{C}^2,G)=n$. But now the resolution graph consists of rational curves with self-intersections $-a_i$ determined by the continued fraction $\frac{n}{r}=a_1-\cfrac{1}{a_2-\cfrac{1}{\cdots}}$, where $r\equiv p/q \bmod n$, $0<r<n$. In the case $G\subset SU(2)$ considered above we have $r=n-1$, the continued fraction has length $n-1$ with entries $a_i=2$, and the result is true. But for $p=q$ there is just one $(-n)$-curve, so $e(\widetilde{\mathbf{C}^2/G})=2$ equals $e(\mathbf{C}^2,G)=n$ only if $n=2$, i.e. $G\subset SU(2)$.

In higher dimensions the same phenomenon occurs: If $G\subset SU(n)$ is generated by a diagonal matrix $\operatorname{diag}(\zeta,\ldots,\zeta)$ for ζ a primitive n-th root of unity, then a resolution of $(\mathbf{C}^n/G)$ consists of a single $\mathbf{P}^{n-1}$ with normal bundle $\mathcal{O}(-n)$ and we have $e(\widetilde{\mathbf{C}^n/G})=n=e(\mathbf{C}^n,G)$.

Kummer surfaces. The quotient of an abelian surface (two-dimensional complex torus) X by the involution $\tau: x\mapsto -x$ has 16 singularities corresponding to the 16 fixed points of τ. Each singularity can be resolved by a single (-2)-curve. This minimal resolution $\widetilde{X/\langle\tau\rangle}$ is called the Kummer surface of X. It is a K3-surface with Euler characteristic 24. On the other hand $e(X,\langle\tau\rangle)=\frac{1}{2}(e(X)+3\cdot e(X^\tau))=\frac{1}{2}(0+3\cdot 16)=24$.

A Calabi-Yau manifold. This is a corresponding example in dimension three. If C is the elliptic curve with complex multiplication of order 3, the cyclic group $G=\langle\varrho\rangle$

of order 3 operates also on $X=C\times C\times C$ with 27 fixed points. As described above, each of the corresponding singularities is resolved by a $\mathbf{P}^2$, and we get

$$e(X,G)=\tfrac{1}{3}(e(X)+8\cdot e(X^g))=\tfrac{1}{3}(0+8\cdot 27)=72,$$
$$e(\widetilde{X/G})=e(X/G)-27+27\cdot e(\mathbf{P}^2)=\tfrac{1}{3}(e(X)+2\cdot e(X^g))+54=72.$$

These global results are not too surprising if one has the local results for quotient singularities, since $e(X,G)=e(X_1,G)+e(X_2,G)$ for reasonable disjoint unions $X=X_1\cup X_2$ of G-invariant subsets for which the Euler characteristic is defined.

Göttsche's formula [G1, 2]. One important class of examples consists in the symmetric powers $S^{(n)}$ of a smooth (complex-)algebraic surface S. The symmetric power is a quotient of the cartesian power S^n by the obvious action of the symmetric group $\mathscr{S}_n$. Algebraic Geometry provides a canonical resolution

$$\mathrm{Hilb}^n(S)=:S^{[n]}\xrightarrow{f}S^{(n)}$$

by the Hilbert scheme of finite subschemes of length n. The action leaves the canonical divisor of S^n invariant, so it descends to a canonical divisor on $S^{(n)}$. This divisor is not affected by the resolution, i.e. $f^*\mathscr{K}_{S^{(n)}}=\mathscr{K}_{S^{[n]}}$. If in particular S has trivial canonical divisor then so does $S^{[n]}$, but we will see that $e(S^{[n]})=e(S^n,\mathscr{S}_n)$ holds in general.

In his Diplom thesis Göttsche computed the Betti numbers of $S^{[n]}$ for an algebraic surface S. His main result is

$$\sum_{n=0}^{\infty}\tilde{P}(S^{[n]},z)\cdot t^n=\exp\left(\sum_{k=1}^{\infty}\frac{t^k}{k}\cdot\frac{\tilde{P}(S,z^k)}{1-z^{2k}t^k}\right)$$

where $\tilde{P}(X,z)$ denotes the modified Poincaré polynomial $\tilde{P}(X,z)=P(X,-z)$ $=\sum(-1)^i b_i(X)z^i$. For the Euler characteristic $e(X)=\tilde{P}(X,1)$ this simplifies to

$$\begin{aligned}\sum_{n=0}^{\infty}e(S^{[n]})\cdot t^n&=\exp\left(e(S)\sum_{i=1}^{\infty}\frac{1}{i}\frac{t^i}{1-t^i}\right)\\&=\exp\left(e(S)\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{\infty}t^{ik}\right)\\&=\exp\left(e(S)\sum_{k=1}^{\infty}-\log(1-t^k)\right)\\&=\prod_{k=1}^{\infty}(1-t^k)^{-e(S)}.\end{aligned}$$

Compare these formulae to those obtained for symmetric powers by Macdonald [M; Z], for example:

$$\sum_{n=0}^{\infty}e(S^{(n)})\cdot t^n=(1-t)^{-e(S)}.$$

Verification of $e(S^{[n]})=e(S^n,\mathscr{S}_n)$ for symmetric powers of algebraic surfaces. Let $\mathscr{M}(n)$ denote the set of all series $(\alpha)=(\alpha_1,\alpha_2,\dots)$ of nonnegative integers with $\sum_i i\alpha_i=n$, and $\mathscr{M}:=\bigcup\mathscr{M}(n)$. The conjugacy class of a permutation $\sigma\in\mathscr{S}_n$ is determined by its type $(\alpha)=(\alpha_1,\alpha_2,\dots)\in\mathcal{M}(n)$ where α_i denotes the number of i-cycles in σ. Its fixed point set in S^n consists of all n-tuples $(x_1,\dots,x_n)$ with $x_{v_1}=\dots=x_{v_i}$ for any i-cycle $(v_1\dots v_i)$ in σ and is therefore isomorphic to $\prod_i S^{\alpha_i}$. Any element τ in the centralizer $C(\sigma)$ permutes the cycles of σ respecting their length, i.e. it induces permutations π_i of α_i elements. Thus $C(\sigma)$ maps onto $\prod_i \mathcal{S}_{\alpha_i}$, the kernel acting trivially on $\prod_i S^{\alpha_i}$. Therefore $(S^n)^\sigma/C(\sigma)=\prod_i S^{(\alpha_i)}$ is a product of symmetric powers. We can compute $e(S^n,\mathcal{S}_n)$ using the formulae of Macdonald and Göttsche:

$$\begin{aligned}\sum_{n=0}^{\infty} e(S^n,\mathcal{S}_n)\cdot t^n &= \sum_{n=0}^{\infty}\sum_{[\sigma]\subset\mathcal{S}_n} e((S^n)^\sigma/C(\sigma))\cdot t^n\\ &= \sum_{n=0}^{\infty}\sum_{(\alpha)\in\mathcal{M}(n)}\left(\prod_i e(S^{(\alpha_i)})\right)\cdot t^n\\ &= \sum_{(\alpha)\in\mathcal{M}}\prod_{i\geq 1}(e(S^{(\alpha_i)})\cdot t^{i\alpha_i})\\ &= \prod_{i=1}^{\infty}\sum_{\alpha_i=0}^{\infty}(e(S^{(\alpha_i)})\cdot t^{i\alpha_i})\\ &= \prod_{i=1}^{\infty}\frac{1}{(1-t^i)^{e(S)}}\\ &= \sum_{n=0}^{\infty} e(S^{[n]})\cdot t^n.\end{aligned}$$

Graeme Segal's interpretation (Equivariant K-theory). Equivariant K-theory of (X,G) and ordinary K-theory of the fixed point sets are related by an isomorphism of complex vector spaces [S]

$$K_G(X)\otimes\mathbf{C}\xrightarrow{\sim}\bigoplus_{[g]} K(X^g/C(g))\otimes\mathbf{C}.$$

The image of an equivariant vector bundle E on X is defined as follows: On $E|_{X^g}$ the element g still acts, leaving the base points fixed. Thus $E|_{X^g}$ splits into a direct sum of vector bundles consisting of the eigenspaces of g in every fibre. We put the corresponding eigenvalue in the second factor and get an element in $K(X^g)\otimes\mathbf{C}$. Now as $C(g)$ still acts on X^g, we can take the invariants and get something in $K(X^g)^{C(g)}\otimes\mathbf{C}=K(X^g/C(g))\otimes\mathbf{C}$. The same also holds for $K^1_G(X)$, and by the standard fact that the Euler characteristic of the complex $K^*(X)$ equals the topological Euler characteristic we can deduce

$$\begin{aligned}e(K^*_G(X)\otimes\mathbf{C}) &= \dim_{\mathbf{C}} K^0_G(X)\otimes\mathbf{C}-\dim_{\mathbf{C}} K^1_G(X)\otimes\mathbf{C}\\ &= \sum_{[g]} e(X^g/C(g))\\ &= e(X,G).\end{aligned}$$

However, since the isomorphism does not commute with Adams operations, we cannot say anything about the single Betti numbers.

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