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Chern Numbers of Algebraic Surfaces

An Example

F. Hirzebruch

Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, D-5300 Bonn 3, Federal Republic of Germany

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An algebraic surface has the Chern numbers c_2 (Euler-Poincaré characteristic or Euler number) and c_1^2 (selfintersection number of a canonical divisor). For a minimal surface of general type these numbers are positive and satisfy

$$c_1^2 \leqq 3c_2 \quad \text{(see [4, 6])}. \tag{1}$$

If $\{E_i\}$ is a finite set of disjoint smooth elliptic curves on a minimal surface of general type, then $E_iE_i<0$ and

$$-\sum E_iE_i \leqq 3c_2 - c_1^2 \quad \text{(see [5, Sect. 7])}. \tag{2}$$

We shall construct a sequence X_n $(n=2,3,\ldots)$ of minimal surfaces of general type with the following properties

$$c_2(X_n) = n^7 \tag{3}$$

$$3c_2(X_n) - c_1^2(X_n) = 4n^5. \tag{4}$$

There are $4n^4$ *smooth disjoint elliptic curves on* X_n *with selfintersection number* $-n$, *showing that we have the equality sign in* (2). As a consequence of (3) and (4) we have

$$c_1^2(X_n)/c_2(X_n) < 3,$$

but

$$\lim_{n\to\infty} c_1^2(X_n)/c_2(X_n) = 3. \tag{5}$$

Holzapfel [2, 3] has made a thorough investigation of Picard modular surfaces as quotients of the complex ball. He showed that the Picard modular surfaces give examples of (2) with equality sign where the E_i correspond to the resolution of the cusps. Holzapfel observed that the Picard modular surfaces give many examples of surfaces with c_1^2/c_2 arbitrarily near to 3. But it may be of some interest to give a direct geometric construction of surfaces X_n satisfying (5). Perhaps the X_n we are going to construct are Picard modular surfaces. In any case, it should be conjectured that the universal cover of $X_n - \bigcup E_i$ is the ball. For more references to the literature see [1]. Anyhow, this note should be regarded as a continuation of [1].

1. Abelian Coverings of an Abelian Surface

Let $\zeta = e^{2\pi i/6}$ and T the elliptic curve

$$T = \mathbb{C}/\mathbb{Z}\cdot 1 + \mathbb{Z}\zeta .$$

Consider the abelian surface $T\times T$ whose points are denoted by (z, w). The curve T has complex multiplication by the algebraic integers $\alpha\in\mathbb{Q}[\zeta]$ and $w=\alpha z$ is an elliptic curve T_α in $T\times T$ isomorphic to T. We take the following curves on $T\times T$

$$T_0 : w=0$$
$$T_\infty : z=0$$
$$T_1 : w=z$$
$$T_\zeta : w=\zeta z .$$

The four curves pass through the origin of $T\times T$ and do not intersect anywhere else. It suffices to remark that $T_1 \cap T_\zeta = \{(0,0)\}$. This follows from the fact that $1-\zeta$ is a unit, namely $1-\zeta = -\zeta^2$.

Let U_n be the group of n-division points of $T\times T$,

$$U_n = \{(z,w)|(nz, nw) = (0,0)\} .$$

It has order n^4. The group U_n acts on $T\times T$ by translations, each of the four curves remaining invariant under a group of n^2 translations contained in U_n. Therefore $U_n(T_0)$, $U_n(T_\infty)$, $U_n(T_1)$, $U_n(T_\zeta)$ each consists of n^2 smooth disjoint elliptic curves. Altogether we have $4n^2$ elliptic curves. Through each of the n^4 points of U_n (considered as subset of $T\times T$) pass 4 of the $4n^2$ curves. There are no other intersections.

We introduce the divisors

$$D_0 = U_n(T_0), \quad D_\infty = U_n(T_\infty), \quad D_1 = U_n(T_1), \quad D_\zeta = U_n(T_\zeta)$$

on $T\times T$. Each such divisor is a sum of n^2 disjoint smooth curves (of multiplicity 1). In the integral homology group $H_2(T\times T, \mathbb{Z})$ each D_i ($i\in\{0,\infty,1,\zeta\}$) is divisible by n^2, since uT_i and T_i determine the same homology class if u is a translation. In fact,

$$D_i \approx n^2 T_i ,$$

where $\approx$ denotes linear equivalence. This is clear, because T_i is a direct factor of $T\times T$ and since on an elliptic curve $n^2(0)$ is linearly equivalent to the divisor consisting of the n^2 division points of order n. Let f_i be the meromorphic function on $T\times T$ whose divisor equals $D_i - n^2 T_i$. It is uniquely determined up to a constant factor. Observe that the divisor of f_i/f_j restricted to a component of D_k (where i, j, k are distinct elements of $\{0,\infty,1,\zeta\}$) is of the form $n^2(a) - n^2(b)$ where a, b are points of the component such that $a-b$ is an n-division point in the underlying group structure. Therefore $n(a)\approx n(b)$.

Thus f_i/f_j restricted to a component of D_k becomes an n^{th} power of a meromorphic function.

Using the f_i we construct the algebraic surface X_n.

The function field of X_n will be the function field of $T\times T$ with $(f_1/f_0)^{1/n}$, $(f_\infty/f_0)^{1/n}$, $(f_\zeta/f_0)^{1/n}$ adjoined to it. This already defines the birational equivalence class of X_n. To get a smooth model we blow up the n^4 points of $T\times T$ belonging to U_n. We get an algebraic surface Y_n with Euler number n^4 and n^4 exceptional curves L_j ($j\in U_n$) arising from blowing up. The strict transforms of the D_i are smooth curves $\tilde{D}_i$ in Y_n which do not intersect each other and $(f_1/f_0)^{1/n}$, $(f_\infty/f_0)^{1/n}$, $(f_\zeta/f_0)^{1/n}$ determine a smooth covering X_n of Y_n of degree n^3 ramified above the $\tilde{D}_i$ with ramification index n. Since the branching locus consists of elliptic curves which have Euler number 0, the Hurwitz method for calculating Euler numbers gives

$$c_2(X_n)=n^3c_2(Y_n)=n^3\cdot n^4=n^7. \tag{6}$$

Let $\pi: X_n\to Y_n$ be the covering map. Then

$$\pi^*\tilde{D}_k=n\bar{D}_k \quad (k\in\{0,\infty,1,\zeta\})$$

and $\bar{D}_k$ consists of n^4 disjoint elliptic curves. Over each of the n^2 components of $\tilde{D}_k$ lie n^2 components of $\bar{D}_k$. This follows from the fact mentioned above that f_i/f_j, restricted to a component of $\tilde{D}_k$, is the n^{th} power of a meromorphic function. Each component of $\bar{D}_k$ has selfintersection number $-n$. To check this observe first that $D_kD_k=0$ and $\tilde{D}_k\tilde{D}_k=-n^4$, since D_k passes through the n^4 points of $T\times T$ which were blown up. Then we have

$$\begin{aligned}\pi^*\tilde{D}_k\cdot\pi^*\tilde{D}_k&=n^3(\tilde{D}_k\cdot\tilde{D}_k)=-n^7=(n\bar{D}_k)\cdot(n\bar{D}_k)\\ \bar{D}_k\bar{D}_k&=-n^5 \quad (k\in\{0,\infty,1,\zeta\}).\end{aligned} \tag{7}$$

Thus each of the n^4 components of $\bar{D}_k$ has selfintersection number $-n$.

For each exceptional curve L_j of Y_n (where $j\in U_n$) the lifted curve π^*L_j is a smooth curve $\bar{L}_j$ in X_n with Euler number

$$e(\bar{L}_j)=-2n^3+4n^2. \tag{8}$$

This follows from Hurwitz's formula since the map $\bar{L}_j\to L_j$ is of degree n^3 with ramification over 4 points of L_j with ramification index n everywhere. We have

$$\bar{L}_j\bar{L}_j=-n^3. \tag{9}$$

The surface Y_n has a unique effective canonical divisor K, namely $K=\sum L_j$ ($j\in U_n$). Thus

$$\bar{K}=\sum_{j\in U_n}\bar{L}_j+(n-1)\sum_{i\in\{0,1,\infty,\zeta\}}\bar{D}_i \tag{10}$$

is an effective canonical divisor of X_n. For $n\geqq 2$, the divisor $\bar{K}$ does not contain any rational curves. So X_n is free of exceptional curves, because an exceptional curve is contained in every effective canonical divisor. *Hence the surface X_n is minimal for $n\geqq 2$.*

For the calculation of $c_1^2(X_n)=\bar{K}\bar{K}$ we have to use that

$$\bar{L}_j\bar{D}_i=n^2 \quad (j\in U_n, i\in\{0,\infty,1,\zeta\}) \tag{11}$$

which follows from the fact that over a ramification point on L_j of $\bar{L}_j \to L_j$ lie n^2 points on $\bar{L}_j$. Using (7), (9), and (11) we get from (10)

$$\begin{aligned} c_1^2(X_n) &= \sum \bar{L}_j\bar{L}_j + 2(n-1)\sum \bar{L}_j\bar{D}_i + (n-1)^2 \sum \bar{D}_i\bar{D}_i \\ &= n^4(-n^3) + 2(n-1)\cdot 4n^4 \cdot n^2 + (n-1)^2 \cdot 4(-n^5) \\ c_1^2(X_n) &= 3n^7 - 4n^5 . \end{aligned} \tag{12}$$

This number is positive for $n \geqq 2$ and thus X_n is of general type.

We have proved all statements mentioned in the introduction. The $4n^4$ elliptic curves of selfintersection number $-n$ come from the four elliptic curves T_0, T_∞, T_1, T_ζ.

2. A Surface with $c_1^2 = 3c_2$

Consider the surface X_n constructed in Sect. 1. Adjoining to the function field of X_n the element $(f_0)^{1/n}$ [or equivalently $(f_1)^{1/n}$, $(f_\infty)^{1/n}$, $(f_\zeta)^{1/n}$], defines an n-fold covering of X_n ramified along the curves $\bar{L}_j$. We obtain a smooth surface W_n. By Hurwitz's method

$$c_2(W_n) = nc_2(X_n) - (n-1)\sum e(\bar{L}_j)$$

and by (6) and (8)

$$\begin{aligned} c_2(W_n) &= n^8 - (n-1)n^4(-2n^3 + 4n^2), \\ c_2(W_n) &= n^6(3n^2 - 6n + 4). \end{aligned} \tag{13}$$

The divisor $\bar{K} + \frac{n-1}{n}\sum \bar{L}_j$ gives, when lifted to W_n, a canonical divisor of W_n. Thus

$$\begin{aligned} c_1^2(W_n) &= n\left(\bar{K} + \frac{n-1}{n}\sum \bar{L}_j\right)^2 \\ &= n\left(\bar{K}^2 + 2\frac{n-1}{n}\sum \bar{K}\bar{L}_j + \frac{(n-1)^2}{n^2}\sum \bar{L}_j^2\right). \end{aligned} \tag{14}$$

By (9)–(11) we have

$$\bar{K}\bar{L}_j = 3n^3 - 4n^2 \tag{15}$$

and by (14), (12), (15), and (9)

$$\begin{aligned} c_1^2(W_n) &= n(3n^7 - 4n^5 + 2(n-1)n^3(3n^3 - 4n^2) - (n-1)^2 n^5) \\ c_1^2(W_n) &= n^6(8n^2 - 12n + 3). \end{aligned} \tag{16}$$

Hence by (13) and (16)

$$3c_2(W_n) - c_1^2(W_n) = n^6(n-3)^2 . \tag{17}$$

Thus W_3 is a surface (minimal, of general type) with $c_1^2 = 3c_2$.

We observe that

$$\lim_{n\to\infty} \frac{c_1^2(W_n)}{c_2(W_n)} = \frac{8}{3}.$$

3. Elliptic Functions

Consider an elliptic curve $\mathbb{C}/\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$. The Weierstrass σ-function can be used to describe elliptic functions with given zeros and poles: If a_i $(i=1,\dots,m)$ and b_i $(i=1,\dots,m)$ are complex numbers satisfying

$$\sum a_i=\sum b_i \bmod \mathbb{Z}\omega_1+\mathbb{Z}\omega_2,$$

then

$$\prod_{i=1}^{n}(\sigma-a_i)\Big/\prod_{i=1}^{n}(\sigma-b_i)$$

is an elliptic function with zeros and poles in a_i and b_i, respectively. Its divisor equals

$$\sum(a_i)-\sum(b_i),$$

where now addition and subtraction are in the group of divisors of the curve. In particular, we can define the elliptic function

$$F_n(z)=\prod_{a\in U_n}\sigma(z-a)/\sigma(z)^{n^2}=\text{const}\cdot\sigma(nz)/\sigma(z)^{n^2},$$

where U_n is the group of n-division points of the curve.

We now take the lattice $\mathbb{Z}+\mathbb{Z}\zeta$ of Sect. 1, and use also the notation of Sect. 1. Then we can write the functions $f_0, f_\infty, f_1, f_\zeta$ as follows

$$f_0=F_n(w),\quad f_\infty=F_n(z)$$
$$f_1=F_n(w-z),\quad f_\zeta=F_n(w-\zeta z).$$

Thus the function field of the algebraic surface X_n is the field of meromorphic functions of z and w elliptic in both variables (with respect to the lattice $\mathbb{Z}+\mathbb{Z}\zeta$) extended by adjoining

$$(F_n(z)/F_n(w))^{1/n},\quad (F_n(w-z)/F_n(w))^{1/n},\quad (F_n(w-\zeta z)/F_n(w))^{1/n}.$$

The function field of W_n is obtained by adjoining

$$F_n(z)^{1/n},\quad F_n(w)^{1/n},\quad F_n(w-z)^{1/n},\quad F_n(w-\zeta z)^{1/n}.$$

Remark. I came to the constructions of Sect. 1, Sect. 2 by trying to replace line arrangements (see [1]) by other interesting arrangements of curves on some surface. Then I realized how close the constructions of Sect. 1, Sect. 2 are to the paper of Livné (Harvard thesis, see [1]). He takes the elliptic modular surface $E(n)$ of level n and uses the n^2 sections of the elliptic fibration $E(n)\to X(n)$ as locus of ramification. I write $T\times T$ in different ways as direct product and thus as a trivial fibration and use its n^2 sections of n-division points as locus of ramification, applying several such ramifications simultaneously.

The methods of Sect. 1, Sect. 2 lead to interesting surfaces also for other elliptic curves with complex multiplication. The special fact occuring for the curve T is that the four elliptic curves T_0, T_∞, T_1, T_ζ meet only in the origin of $T\times T$.

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