Differentiable Manifolds and Quadratic Forms

Lectures by

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Errata (§0 - §3)

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0-17	1	$\alpha_1 \xi_{11}^2 + \ldots\ldots$	
A-5	9-12	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$
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0. Quadratic forms.

Let A be an integral domain. By a lattice over A (or simply an A-lattice) we shall mean a finitely generated, free, unitary A-module. The terms base, rank, etc., will be employed in the usual fashion. Occasionally, we shall consider A itself as an A-lattice of rank 1 (the module structure being defined by the ring operations in A). For a given A-lattice, V, the module $\mathrm{Hom}_A(V,A) = V'$ is again a lattice of the same rank. We shall call V' the dual lattice of V. There is a pairing (called the Kronedker product) from $V' \times V$ into A defined by $\langle x',x\rangle = x'(x)$ for $x' \in V'$ and $x \in V$.

For our purpose, a quadratic form, $f = (f,V)$, over A may be defined to be a bilinear, symmetric pairing

$$f\colon V \times V \to A \quad ,$$

where V is an A-lattice and, in this case, we shall also say that f is defined on V. To such a form, f there corresponds a linear map $\varphi : V \to V'$, to be called the correlation associated with f, given by

$$\langle \varphi(x),y\rangle = f(x,y) \ , \tag{0.1}$$

for all x and y in V. The form, f, will be termed non-degenerate if its associated correlation, φ, is injective (i.e. a monomorphism). This property is also characterized by

(0.2) $f(x,y) = 0$ for all $y \in V \Rightarrow x = 0$.

If a base $(e_i)_{1 \leq i \leq r}$ is chosen in V , there is a dual base $(e'_i)_{1 \leq i \leq r}$ in V' characterized by the property that $\langle e'_i, e_j \rangle = \delta_{ij}$ (the Kronecker deltas). Put $\varphi(e_i) = \sum_j \alpha_{ij} e'_j$, then $\alpha_{ij} = f(e_i, e_j)$. The matrix $M = M_f = (\alpha_{ij})$ will be called the matrix of f (with respect to (e_i)), and its rank, which is independent of the base chosen, will be called the rank of f . We shall only consider non-degenerate quadratic forms; therefore the rank of a form (f,V) is the same as the rank of the lattice V . By means of a chosen base, (e_i), we may express each element of V by a column matrix, e.g., $x^t = (x_1, \ldots, x_r)$ and $y^t = (y_1, \ldots, y_r)$, where the super-script, t, denotes the transposition. Then we have

$$f(x,y) = x^t My = \sum_{i,j} \alpha_{ij}\, x_i\, y_j \qquad (\alpha_{ij} = \alpha_{ji}) .$$

If $(\tilde{e}_i)_{1 \leq i \leq r}$ is another base for V, obtained from (e_i) by applying an invertible (= non-singular) matrix, P , then the matrix, $\tilde{M}$, of f relative to the new base is clearly given by $\tilde{M} = P^t MP$. Thus the determinant of M and $\tilde{M}$ differs only by the square of a unit. For a given base in V , we define the determinant, det f , of f to be $\det M_f$. Therefore $\det f$ is, apart from a factor which is the square of a unit, independent of the choice of the base in V.

Two quadratic forms $f = (f,V)$ and $q = (q,W)$ over the same integral domain, A, are said to be <u>equivalent</u>, in symbol $f \sim g$, if there exists an isomorphism, $u: V \to W$, of V onto W such that $f(x,y) = g(u(x), u(y))$. Such an isomorphism, u, will be referred to as an <u>isometry</u> (more precisely, an (f,g)-isometry). In particular, if $V = W$ and $f = g$, we shall call u an <u>automorph</u> (of f). Suppose we are given a quadratic form g on W and a lattice, V, isomorphic to W. For each isomorphism, $u: V \to W$, there is a unique quadratic form, $f = u^* g$, defined on V, which renders u an isometry.

If bases are chosen in V and W, then f and g are expressed by matrices M_f and M_g. We have, plainly, the following lemma.

LEMMA (0.3) The quadratic forms, f and g, are equivalent if, and only if, there exists and invertible matrix, P, such that $M_f = P^t M_g P$, i.e. the matrices M_f and M_g are congruent.

For this reason it is often convenient to use the matrical language in dealing with quadratic forms; only in this case one has to be careful to observe what properties are invariant under a change of bases.

Let f_i be quadratic forms defined on A-lattices V_i, $(i = 1,2)$. A quadratic form, f, defined on the direct sum, $V_1 \oplus V_2$, is termed the <u>sum</u> of f_1 and f_2, or in symbol, $f = f_1 \dot{+} f_2$, if

$$f(x_1 \oplus x_2 , y_1 \oplus y_2) = f_1(x_1, y_1) + f_2(x_2 , y_2)$$

If a base $(e_1, \ldots, e_k; e_{k+1}, \ldots, e_s)$ is given for $V_1 \oplus V_2$, then the matrices, M, M_1 and M_2, of the forms f, f_1 and f_2 respectively are related by

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

If $f \sim f_1 \dot{+} f_2$, we shall say that f_1 (or f_2) <u>splits off</u> from f or f <u>decomposes into</u> f_1 and f_2 (under the equivalence relation $\sim$).

A quadratic form, $f = (f,V)$, will be called non-singular if its associated correlation $\mathcal{f}: V \to V'$ is an isomorphism. This means that the matrix, M, of f is invertible or, what is the same thing, det f is a unit in A. If A is a field, then the properties of being non-singular or non-degenerate coincide. If the restriction of f on a sub-lattice, V_1, of V is non-singular, we shall say that f is <u>non-singular on</u> V_1. Notice that if f_1, f_2 are non-singular, so is $f_1 \dot{+} f_2$.

LEMMA (0.4) Suppose that f is a quadratic form defined on $V_1 \oplus V_2$, which is non-singular on V_1. Then the restriction, f_1, of f on V_1 splits off from f..

In fact, with respect to a base $(e_1, \ldots, e_k ; e_{k+1}, \ldots, e_r)$ of $V_1 \oplus V_2$, the matrix, M, of f is of the form

$$M = \begin{pmatrix} M_1 & L^t \\ L & N \end{pmatrix} \qquad M_1 = M_1{}^t \text{ , } N = N^t \text{ ,}$$

where M_1 is, by assumption, invertible. Let P be the invertible matrix

$$P = \begin{pmatrix} I & -M_1{}^{-1} L^t \\ 0 & I \end{pmatrix} .$$

Then

$$P^t MP = \begin{pmatrix} M_1 & 0 \\ 0 & -LM_1{}^{-1} L^t + N \end{pmatrix} ,$$

which, in view of (0.3), proves the lemma.

A quadratic form f on V is termed <u>unary</u>, <u>binary</u>, <u>ternary</u>, ..., <u>r-ary</u>, according as the rank of V is $1, 2, 3, \ldots, r$. To assert that a quadratic form decomposes into unary forms is equivalent to saying that the matrix of the form is congruent to a diagonalized matrix.

From now on we shall assume that the integral domain, A, is furthermore a <u>local domain</u> in the sense that there exists a unique maximal ideal, $\mathfrak{m}$, in A. This assumption enables us to conclude the following trivial but useful properties of A :

(i) The set, A^* , of all the units of A coincides with $A \setminus \mathfrak{m}$.

(ii) If α, β are units and ξ, η are non-units in A , then $\alpha\beta$ and $\alpha + \xi$ are units and $\xi\eta$ and $\alpha\xi$ are non-units.

The following theorem generalizes the classical diagonalization theorem of symmetric matrices.

THEOREM (0.5) Every non-singular quadratic form, f, over a local domain, A , decomposes into unary and binary forms. If, in addition, $2 \in A^*$ then f decomposes into unary forms.

To prove (0.5), we chose an arbitrary base $(e_i)_{1 \leq i \leq r}$ for the lattice V on which f is defined, and write the matrix of f as $M = (\alpha_{ij})$. If a diagonal entry α_{ii} is a unit, then the corresponding unary form splits off in virtue of (0.4). We may thus assume that no diagonal entry is in A^* . Since det f is a unit, $\alpha_{11} \in A^*$ implies that $\alpha_{1i} \in A^*$ for some i , $1 \leq i \leq r$; we lose nothing by assuming $\alpha_{12} \in A^*$. But then the matrix

$$(0.6) \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} , \qquad \alpha_{11} \in \mathfrak{m}, \alpha_{12} = \alpha_{21} \in A^* ,$$

has determinant $\alpha_{11}\alpha_{22} - \alpha_{12}^2 \in A^*$ and hence invertible. The first assertion now follows from (0.4) and a trivial induction. To prove the second assertion we need only to show that the matrix (0.6) is congruent to one whose diagonal entries are units; for then we may apply (0.4) to decompose the corresponding binary form. Indeed the assumption $2 \in A^*$ shows that the matrix,

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is invertible, and

$$P^t \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} P = \begin{pmatrix} \alpha_{11} + 2\alpha_{12} + \alpha_{22} & * \\ * & \alpha_{11} - 2\alpha_{12} + \alpha_{22} \end{pmatrix}$$

gives the desired congruence.

Consider now a quadratic form, f, defined on V. If $a \in V$ is such that $f(a,a) = \alpha \in A^*$, then the linear map, $u_a\colon V \to V$, given by

$$u_a(x) = \frac{2f(x,a)}{\alpha} a - x$$

is an involution, since $u_a u_a = \text{identity}$. It is easy to check that u_a is an automorph leaving the element a fixed (i.e. $u_a(a) = a$). This automorph is also known as a reflection.

In the rest of this section we shall <u>assume tacitly that</u> A <u>is a local domain with a maximal ideal</u> $\mathfrak{m} \subset A$ <u>and that</u> 2 <u>is a unit in</u> A .

LEMMA (0.7) Let $f = (f,V)$ be a quadratic form over A . If the elements $x, y \in V$ are such that $f(x,x) = f(y,y) = \alpha \in A^*$, then there is an automorph u of f which interchanges x and y .

Indeed, the relation $f(a,a) \in A^*$ must be satisfied by either $a = x - y$ or $a = x + y$. In the first case $u = u_a$ is the desired automorph and in the second, $u = -u_a$ is the desired automorph.

THEOREM (0.8) Let f_1, f_2 and h be quadratic forms over A , where h is non-singular. If $f_1 \dotplus h \sim f_2 \dotplus h$ then $f_1 \sim f_2$. (Witt)

This may be regarded as the "<u>cancellation law</u>" for forms. Since h decomposes into unary forms by (0.5), we lose nothing by assuming h itself being unary in the statement of (0.8). Let the lattices on which f_1 , f_2 and h are defined be denoted by V_1, V_2 and W , respectively. Then W has rank 1; let w be its base. By assumption, there is an isometry $u: V_1 \oplus W \to V_2 \oplus W$. Then $(f_2 \dotplus h)(w,w) = h(w,w) = (f_1 \dotplus h)(w,w) = (f_2 \dotplus h)(u(w), u(w))$, which is a unit because h is non-singular. We apply (0.7) to obtain an automorph, v, of $f_2 \dot{+} h$ such that $v(w) = u(w)$ and $v(u(w)) = w$. The composite map $vu : V_1 \oplus W \to V_2 \oplus W$ is an isometry which carries w into w. Therefore the restriction of vu on V_1 gives the required isometry to conclude $f_1 \sim f_2$.

For notational convenience, we shall use $\mathbb{G}_n$ to denote the multiplicative cyclic group of order n. Thus $\mathbb{G}_n$ is actually the group $\mathbb{Z}_n$ written multiplicatively. The multiplicative group, A^*, of A has a subgroup $A^{**} = \{x^2 \mid x \in A^*\}$. We shall assume, for a moment, that the ring A satisfies the additional condition

$$(0.10) \qquad A^*/A^{**} \approx \mathbb{G}_2 .$$

Let the two cosets in A^*/A^{**} be represented by 1 and ε, $(1, \varepsilon \in A^*)$. It follows quickly from (0.5) that every quadratic form, f, over A is equivalent to one of the form

$$x_1^2 + x_2^2 + \dots + x_k^2 + \varepsilon x_{k+1}^2 + \dots + \varepsilon x_{k+m}^2 .$$

Let g be another quadratic form over A and

$$g \sim y_1^2 + y_2^2 + \dots + y_r^2 + \varepsilon y_{r+1}^2 + \dots + \varepsilon y_{r+s}^2 .$$

We may assume that $k \leq r$. From (0.8), we see that $f \sim g$ if, and only if

$$(0.11)) \qquad \varepsilon(\tilde{x}_1^2 + \dots + \tilde{x}_{m-s}^2) \sim (\tilde{y}_1^2 + \dots + \tilde{y}_{r-k}^2).$$

In particular, if $A = R$, the real field, then ε may be taken as -1. The relation (0.11) holds if, and only if, $m-s = r-k = 0$. This gives us <u>Sylvester's law of inertia</u>.

Recall that the determinant, det f, of a form, f, is determined up to the square of a unit. Hence for non-singular forms over A , det f modulo A^{**} is well defined. We shall denote by DET f the coset in A^{*}/A^{**} containing det f . Now assume that A satisfies yet another condition.

(0.12) the matrices $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are congruent over A .

Then, with the notations defined above, $f \sim g$ if, and only if, $m - s = r - k$ is even; in other words, if DET f = DET g . We have shown

THEOREM (0.13) If A satisfies the additional assumptions (0.10) and (0.12), then two quadratic forms, f and g, of the same rank over A are equivalent if, and only if DET f = DET g.

We have been working under the general assumption that A is a local domain in which 2 is a unit. Perhaps a few examples of such domains are now in order. In fact, any field of characteristic different from 2 is such a domain (with $\mathfrak{m} = \{0\}$). As examples we mention

a) $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ the fields of rational, real and complex numbers,

b) $F(p)$ the field of p-adic numbers, where p is a (rational) prime, and

c) $\mathbb{Z}_p$ the (finite) field of rational integers modulo p, where p is an odd prime.

For local domain which are not fields we have

d) $R(p)$, $Q(p)$ the ring of p-adic integers and the ring of rational p-adic integers (p odd prime). We may regard $Q(p)$ as $R(p) \cap \mathbb{Q}$ which consists of all the fractions $\frac{a}{b} \in \mathbb{Q}$ with $b \not\equiv 0 \pmod{p}$. Notice that the maximal ideal, $\mathfrak{m}$, in $R(p)$ is generated by the single element, $p \in R(p)$. Indeed $R(p)$ is a principal ideal domain in which each ideal is of the form $p^r R(p)$. Thus it is meaningful to talk about congruence modulo p^r in $R(p)$. We shall also make the convention that $R(\infty) = F(\infty) = \mathbb{R}$.

As one may expect, particular choice of A leads to more specific properties. We shall sample a few of them which we shall have occasion to use. If $A = Z_p$, the prime field of integers modulo p, (p odd) then $A^* \approx \mathbb{G}_{p-1}$ is of even order, and hence $A^*/A^{**} \approx \mathbb{G}_2$. We define the Legendre symbol $(q \mid p)$ for $q \in A^*$ by

$$(0.14) \quad \begin{cases} (q \mid p) = 1 & \text{if } q \in A^{**} \\ (q \mid p) = -1 & \text{if } q \notin A^{**} \end{cases}$$

From (0.13) we have

COROLLARY (0.15) If $A = \mathbb{Z}_p$ (p odd prime), then two quadratic forms f , g of the same rank over A are equivalent if, and only if $(\det f \mid p) = (\det g \mid p)$.

To prove (o.15) we need only check that (0.12) holds in $\mathbb{Z}_p$. Indeed, we shall prove the following lemma which clearly implies (0.12).

LEMMA (0.16) If α and β are elements of $\mathbb{Z}_p^*$, then $\alpha x^2 + \beta y^2 = 1$ is solvable in $\mathbb{Z}_p$.

For, if we denote by H the subset $\{0, 1, 2 \ldots, \frac{p-1}{2}\}$ of $\mathbb{Z}_p$, the maps $i, j : H \to \mathbb{Z}_p$, given by

$$i(x) = \alpha x^2 \quad , \quad j(y) = 1 - \beta y^2 ,$$

are both injective. Therefore their image sets have at least one element in common. This clearly proves (0.16).

We recall that a sequence $\{\alpha_i\}$ of p-adic numbers converges to a limit (in $F(p)$) if $(\alpha_{n+1} - \alpha_n) \to 0$ as $n \to \infty$, (see e.g. Van der Waerden: Algebra I, 5te Aufl. p. 255). We may also talk about convergence in lattice over $F(p)$ or $R(p)$ just as we do, say, in real vector spaces.

LEMMA (0.17) Let $f = (f,V)$ be a quadratic form over $R(p)$, not necessarily non-singular. If there exists an element $x \in V$ such that $f(x,x) \equiv c (\mathrm{mod}\ p^w)$, where $c \in R(p)^*$ and $w = 1$ or 3 according as p is odd or $p = 2$, then there is $\tilde{x} \in V$ such that $f(\tilde{x},\tilde{x}) = c$.

Put $f(x,x) - c = p^t u$, where $t \geq 1$ if p is odd and $t \geq 3$ if $p = 2$. Define

$$x_1 = x - \frac{1}{2} \cdot \frac{p^t u}{f(x,x)} \cdot x .$$

This is meaningful because $f(x,x)$ is invertible and, even in case where $p = 2$, the factor $\frac{1}{2}$ is admissible because $t \geq 1$. Then we have

$$\begin{aligned} f(x_1,x_1) - c &= \frac{1}{4} p^{2t} u^2 f(x,x) \\ &\equiv 0 \ (\mathrm{mod}\ p^{t+1}) \end{aligned}$$

in all cases. By this means we may construct a sequence $x_1, x_2, \ldots$ of elements in a sub-lattice of rank 1 in V such that $\{x_i\}$ converges to an element $\tilde{x} \in V$, and $f(\tilde{x},\tilde{x}) = c$. This proves the lemma.

Let $\mathfrak{m}$ be the maximal ideal in $R(p)$. Then $R(p)/\mathfrak{m} = \mathbb{Z}_p$, the prime field of characteristic p. Thus for each $\alpha \in R(p)^*$, α modulo p is in $\mathbb{Z}_p^*$. We have

COROLLARY (0.18) If p is an odd prime, then $R(p)^*/R(p)^{**} \approx \mathfrak{G}_2$ and if $p = 2$, $R(2)^*/R(2)^{**} \approx \mathfrak{G}_2 \cdot \mathfrak{G}_2$.

In fact, putting f to be the unary form αx^2 in (0.17) for $\alpha \in R(p)^*$, the case where p is odd follows immediately from the fact that $\mathbb{Z}_p^*/\mathbb{Z}_p^{**} \approx G_2$. For the case $p = 2$, similar argument applies and the desired result may be obtained by appealing to the properties of $R(2)$. Using (0.16), we get

COROLLARY (0.19) If α and β are elements in $R(p)^*$ and p is odd, then $\alpha x_1^2 + \beta x_2^2 = 1$ has a solution in $R(p)$.

It is of interest to study when does $\alpha x_1^2 + \beta x_2^2 = 1$ have a solution in other local rings. For fields $F(2), F(3), \ldots, F(\infty) = \mathbb{R}$, this leads to the <u>Hilbert symbol</u> $(\alpha,\beta)_p$ defined as follows: for $\alpha, \beta \in F(p)^*$, $p = 2, 3, \ldots$ (finite primes) or $p = \infty$,

$$\begin{cases} (\alpha,\beta)_p = 1 & \text{if } \alpha x_1^2 + \beta x_2^2 = 1 \text{ has a solution in } F(p), \\ (\alpha,\beta)_p = -1 & \text{otherwise.} \end{cases}$$

We list with B. W. Jones the following properties of the Hilbert symbols, (see Jones: The Arithmatic Theory of Quadratic forms, Carus Monographs No. 10, p. 27).

1. $(\alpha,\beta)_\infty = 1$ unless α and β are both negative
2. $(\alpha,\beta)_p = (\beta,\alpha)_p$
3. $(\alpha\rho^2, \beta\sigma^2)_p = (\alpha,\beta)_p$
4. $(\alpha,-\alpha)_p = 1$
5. If $\alpha = p^a\alpha_1$, $\beta = p^b\beta_1$ with $\alpha_1, \beta_1 \in R(p)^*$, then

a. $(\alpha,\beta)_p = (-1 \mid p)^{ab} (\alpha_1 \mid p)^b (\beta_1 \mid p)^a$,

if p is odd and $\neq \infty$.

b. $(\alpha,\beta)_2 = (2 \mid \alpha_1)^b (2 \mid \beta_1)^a (-1)^{(\alpha_1-1)(\beta_1-1)/4}$.

5' If p is prime to $2\alpha\beta$, $(\alpha,\beta)_p = 1$, for p finite and $\alpha,\beta \in R(p)$.

6. $(\alpha,\beta)_p (\alpha,\gamma)_p = (\alpha,\beta\gamma)_p$.

7. $(\alpha,\alpha)_p = (\alpha,-1)_p$.

8. $(\alpha\rho, \beta\rho)_p = (\alpha,\beta)_p (\rho, -\alpha\beta)_p$.

9. If β is a non-square in $F(p)$ and $c = 1$ or -1 , there is for each prime p an integer α such that $(\alpha,\beta)_p = c$.

10. If $a, b \in Q^*$,

$$\prod_p (a,b)_p = 1 \quad .$$

where the product extends over all primes including ∞ .

Properties 1, 2, 3 are obvious. For property 4, notice that $x_1 = (1 + \alpha^{-1})/2$, $x_2 = (1 - \alpha^{-1})/2$ is a solution. Property 5' can be deduced from (0.19). For property 5 we remark that, since $R(p)^*/R(p)^{**} \approx G_2$ (p odd), $(\alpha_1 \mid p) = 1$ if α_1 is a square; otherwise $(\alpha_1 \mid p) = -1$. If $p = 2$, then $(2 \mid \alpha_1) = (-1)^{(\alpha_1^2-1)/8}$. For property 10, we notice that the product is well defined, since all but finite factors are equal to 1 . We shall omit the proofs of properties 5 through 10, (cf. Jones op. cit. pp. 28-31).

Our next goal is to define the Hasse-Minkowski symbol for a quadratic form over $F(p)$ through a diagonalization of its matrix. An invariant definition of this symbol may be found in an appendix at the end of this section.

Let diag $(\alpha_1\alpha_2 \dots \alpha_r)$ denote the diagonalized matrix whose diagonal entries are $\alpha_1, \alpha_2, \dots, \alpha_r$. A quadratic form, f, is said to be diagonal if its matrix has been diagonalized, i.e. f is of the form $\alpha_1 x_1^2 + \dots + \alpha_r x_r^2$.

LEMMA (0.20) Let f and g be two diagonal forms over a field K. Then f may be carried into g through successive application of binary transformations such that at each stage the resulting form remains diagonal.

Let the matrices of f and g be, respectively, $M = \text{diag}\ (\alpha_1 \dots \alpha_r)$ and $N = \text{diag}\ (\beta_1 \dots \beta_r)$. First observe that by binary transformations we may permute α_i's among themselves. Our proof is by induction on r. The case $r = 2$ is trivial. Using theorem (0.8) and the induction hypothesis, we see that for $r > 2$ it suffices to show that diag $(\alpha_1 \dots \alpha_r)$ may be transformed into the form diag $(\beta_1\ \gamma_2 \dots \gamma_r)$ by binary transformations such that at each stage the resulting form remains diagonal. By assumption, there is a non-singular matrix $R = (\rho_{ij})$ such that $R^tMR = N$. Since $\beta_1 = \Sigma\alpha_i\rho_{i1}^2 \neq 0$ we may, by rearranging the order of α_i's (hence ρ_{ij}'s) if necessary, assume that $\alpha_1 \xi_{11}^2, \alpha_1 \quad {}_{11}^2 + \alpha_2 \xi_{21}^2, \ldots\ldots, \sum_i \alpha_i \xi_{i1}^2$ are all distinct from zero. Now, the binary transformation whose matrix is of the form

$$U = \left(\begin{array}{cc|c} \xi_{11} & -\xi_{12}\,\alpha_2 & \\ \xi_{21} & \xi_{11}\,\alpha_1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{r-2} \end{array}\right)$$

carries M into $U^tMU = \operatorname{diag}\,((\alpha_1 \xi_{11}^2 + \alpha_2 \xi_{21}^2)\,\gamma_2\,\alpha_3 \cdots \alpha_r)$ which may then be carried by binary transformations into $\operatorname{diag}\,((\alpha_1 \xi_{11}^2 + \alpha_2 \xi_{21}^2)\,\alpha_3 \cdots \alpha_r\,\gamma_2)$. Since $\alpha_1' = \alpha_1 \xi_{11}^2 + \alpha_2 \xi_{21}^2 \neq 0$. Put $\alpha_2' = \alpha_3$, $\xi_{11}' = 1$, the above construction may be repeated with α_1', α_2', ξ_{11}' in places of α_1, α_2 and ξ_{11}. The proof of (0.20) is completed.

Now let f be a non-singular quadratic form over $F(p)$. We may diagonalize f according to (0.5). Let the matrix of f be $M_f = \operatorname{diag}\,(\alpha_1 \ldots \alpha_r)$. We define

(0.21) $$c_p(f) = c_p(\alpha_1 \ldots \alpha_r) = \prod_{i<j} (\alpha_i, \alpha_j)_p \,,$$

which is either 1 or -1 . We shall show that $c_p(f)$ depends only on the form, f, but not on the particular diagonalization M_f. Indeed, if $\operatorname{diag}\,(\beta_1 \ldots \beta_r)$ gives another diagonalization of f then

LEMMA (0.22) $\quad c_p(\alpha_1 \ldots \alpha_r) = c_p(\beta_1 \ldots \beta_r)$.

For the case $r = 1$, we define $c_p(f) = 1$, and there is nothing to prove. It is easily seen that, for $r = 2$, $c_p(f) = 1$ or -1 according as $f(x,x) = 1$ has or has not as solution; this property is obviously independent of the diagonalization. For the general case, we observe that, by properties (2) and (3) of the Hilbert symbols,

$$c_p(\alpha_1 \dots \alpha_r) = c_p(\alpha_1\alpha_2)\, c_p(\alpha_3 \dots \alpha_r)(\alpha_1\alpha_2, \alpha_3\alpha_4 \cdots \alpha_r)_p .$$

Observe also that $c_p(\alpha_1 \dots \alpha_r)$ is unchanged if we permute the order of α_i's. Therefore $c_p(\alpha_1 \dots \alpha_r)$ is invariant under binary transformations provided that the resulting form is again diagonal. The lemma now follows from (0.20).

The symbol, $c(f)$, which depends only on the quadratic form, f, will be called the Hasse-Minkowski symbol.

COROLLARY (0.23) Let f, f_1 and f_2 be quadratic forms over $F(p)$ such that $f = f_1 \dotplus f_2$. Then

$$c_p(f) = c_p(f_1)\, c_p(f_2)\, (\det f_1, \det f_2)_p .$$

A quadratic form, f, over $R(p)$ can always be regarded as a form over $F(p)$, since $R(p) \subset F(p)$; thus the symbol $c_p(f)$ is also defined. One deduces easily from property (5') of the Hilbert symbol that

LEMMA (0.24) If f is a non-singular quadratic form over $R(p)$, $p \neq 2$. Then $c_p(f) = 1$.

By the same token we may consider $c_p(f)$ for

$p = 2, 3, \ldots, \infty$ if f is a quadratic form over Q, the field of rational numbers. Recall that if a, b, are rational numbers, then $(a,b)_p$, for p ranging over $2, 3, 5, \ldots \infty$, has at most finitely many values $\neq 1$ and $\prod_p (a,b)_p = 1$. Therefore we have

LEMMA (0.25) If f is a non-degenerate quadratic form over Q then

$$\prod_p c_p(f) = 1$$

where p ranges over $2, 3, \ldots, \infty$.

Appendix A.

Let B be a commutative monoid (= semi-group) in the sense of Bourbaki (Algebre CH. I), i.e., no unit is assumed. We use $\dot{+}$ to denote the monoid operation in B. Consider B as a set of elements and let $\mathcal{F}(B)$ be the free abelian group generated by the elements of B. The group operation in $\mathcal{F}(B)$ will be denoted by $+$. We may consider B as embedded in $\mathcal{F}(B)$. Let $\mathfrak{A}(B)$ be the subgroup of $\mathcal{F}(B)$ generated by the elements of the form

$$(b_1 \dot{+} b_2) - b_1 - b_2 \qquad (b_1 , b_2 \in B)$$

and let $G(B) = \mathcal{F}(B) / \mathfrak{A}(B)$. There is an obvious canonical map $j : B \to G(B)$ which is a monoid-homomorphism. With respect to the map j, $G(B)$ has the following universal property: if $h: B \to G$ is (monoid-) homomorphism of B into an abelian group, G, then there exists a group-homomorphism $g : G(B) \to G$ such that $gj = h$; that is, the diagram is commutative.

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[dr, "h"'] & G(B) \arrow[d, dashed, "g"] \\ & G \end{tikzcd}$$

EXERCISE (A.1) Show that the natural map $j : B \to G(B)$ is injective if, and only if, the cancellation law holds in B.

EXAMPLE (A.2) Let X be a topological space and let

B be the set of equivalence classes of real (resp. complex) vector bundles over X . (We do not assume X to be connected. The typical fibre may vary from (connected-) component to component.) With respect to Whitney sum, $\dot{+}$, B is a commutative monoid and the group G(B) is usually denoted by KO(X) (Resp. K(X)).

EXAMPLE (A.3) Let A be an integral domain and let $\mathcal{F}(A)$ (resp. $\mathcal{F}_0(A)$) be the set of equivalence classes of non-degenerate (resp. non-singular) quadratic forms (of various ranks) over A . Then $\mathcal{F}(A)$ (resp. $\mathcal{F}_0(A)$) is a commutative monoid with respect to the operation, $\dot{+}$, defined in §0 . Notice that the cancellation law holds in $\mathcal{F}_0(A)$ (cf. (0.8)) if A is suitably restricted. Notice also $\mathcal{F}_0(A) = \mathcal{F}(A)$ if A is a field. The group $G(\mathcal{F}(A))$ (resp. $G(\mathcal{F}_0(A))$) will be denoted by G(A) (resp. $G_0(A)$). This example will be our main concern in this Appendix.

There is an augmentation $rk : G(A) \longrightarrow \mathbf{Z}$ defined by assigning to each class of quadratic forms the rank of a representative. Let $f_1 = (f_1, V_1)$ and $f_2 = (f_2, V_2)$ be quadratic forms over A . We define the <u>product,</u> $f_1 \otimes f_2$, of f_1 and f_2 to be the quadratic form over $V_1 \otimes_A V_2$ characterized by

$$f_1 \otimes f_2\ (a_1 \otimes a_2,\ b_1 \otimes b_2) = f_1(a_1, b_1)\ f_2(a_2, b_2)$$

where $a_i, b_i \in V_i$, $i = 1,2$. This operation, $\otimes$, induces a ring structure on the group $G(A)$ and the map, rk, extends to a ring homomorphism, $rk : G(A) \to \mathbb{Z}$. This ring, $G(A)$, will be referred to as the Grothendieck-Witt ring (of quadratic forms over A).

REMARK. The groups $KO(X)$ and $K(X)$ are also augmentable. For simplicity, we assume X be connected. An augmentation $\dim : KO(X) \to \mathbb{Z}$ (resp. $\dim : K(X) \to \mathbb{Z}$) may be defined by assigning to each vector bundle the real (resp. complex) dimension of its typical fibre. The tensor product of vector bundles induces a ring structure on $KO(X)$ (resp. $K(X)$), known as the Grothendieck ring of X. We shall not discuss this structure here (cf. e.g. Atiyah-Hirzebruch, Bull. AMS 65(1959) pp. 276-281).

Returning to $G(A)$, we let 1 denote the class of quadratic forms represented by the unary form $f = x^2$. There is a ring-homomorphism $\varepsilon : \mathbb{Z} \to G(A)$ defined by $\varepsilon(1) = 1$. Clearly $rk \cdot \varepsilon$ = identity. Let $\hat{G}(A)$ denote the kernel of rk; then $\hat{G}(A)$ is an ideal in $G(A)$. The short exact sequence

$$0 \to \hat{G}(A) \to G(A) \underset{\varepsilon}{\overset{rk}{\rightleftarrows}} \mathbb{Z} \to 0$$

of rings and ring-homomorphisms splits since $rk \circ \varepsilon$ = identity on $\mathbb{Z}$. Then $G(A) = \mathbb{Z} \oplus \hat{G}(A)$.

There is another homomorphism, $DET : G_0(A) \to A^*/A^{**}$, given by the "determinant", DET, of quadratic forms. Since $\mathrm{DET}(f_1 \dotplus f_2) = \mathrm{DET}(f_1)\,\mathrm{DET}(f_2)$, this is a group-homomorphism. If A is such that (0.8) holds true, then we may embed A^*/A^{**} in $G_o(A)$ by identifying each element $a \in A^*/A^{**}$ to the class of the unary form $a\,x^2$. Observe that $A^*/A^{**} \subset G_o(A)$ generates $G_o(A)$ additively in view of (0.5) if $2 \in A^*$. If $A = F$ is a field of characteristic $\neq 2$, then $F^*/F^{**} \subset G(F) = G_o(F)$. The restriction of DET on $\hat{G}(F)$ is already an epimorphism onto F^*/F^{**}.

In fact, for $a \in F^*/F^{**}$, $\eta'(a) = (a-1) \in \hat{G}(F)$ has the property $\mathrm{DET}(a-1) = a$. Therefore we have the short exact sequence (of groups)

$$(A.4) \qquad 0 \to L(F) \to \hat{G}(F) \xrightarrow{\mathrm{DET}} F^*/F^{**} \to 1 \quad ,$$

where $L(F)$ is the kernel of DET. The map $\eta' : F^*/F^{**} \to \hat{G}(F)$ given above is, in general, not a homomorphism, hence (A.4) may not split. Notice also that image of η' in $\hat{G}(F)$ generates $\hat{G}(F)$ additively.

EXAMPLES (A.5) $\quad G(\mathbb{R}) \approx \mathbb{Z} + \mathbb{Z}$, $G(\mathbb{C}) \approx \mathbb{Z}$ and $G_o(\mathbb{Z}) \approx \mathbb{Z} + \mathbb{Z}$.

We shall only show the first assertion. If $F = \mathbb{R}$, then $F^*/F^{**} = \{1, -1\} \approx G_2$. Every quadratic form may be written (uniquely) as $\alpha^+ 1 + \alpha^- (-1)$. Therefore

$G(\mathbb{R}) \approx \mathbb{Z} + \mathbb{Z}$. The second assertion is trivial. We shall prove the third assertion later.

Next we propose to study $G(F(p))$. If p is an odd prime, then $F^*/F^{**} \approx \mathbb{G}_2 \cdot \mathbb{G}_2$ $(F = F(p))$. Indeed, from (o.18), $R(p)^*/R(p)^{**} \approx \mathbb{G}_2$; let its elements be represented by $\{1, \epsilon\}$. Then F^*/F^{**} is represented by $\{1, \epsilon, p, p\epsilon,\}$ $\approx \mathbb{G}_2 \cdot \mathbb{G}_2$. Since $(\epsilon,\epsilon)_p = 1$, $(\epsilon,p)_p = -(-1 \mid p)$ and $(p,p)_p = (-1 \mid p)$, we have

$$p \equiv 1 \pmod 4$$

$$\begin{aligned} (\varepsilon,\varepsilon)_p &= 1 \\ (\varepsilon,p)_p &= -1 \\ (p,p)_p &= -1 \end{aligned}$$

LEMMA (A.6) For $\alpha,\beta \in F(p)^*/F(p)^{**}$,

$$(\alpha,\beta)_p = 1 \iff \alpha + \beta = 1 + \alpha\beta \text{ in } G(F(p)) .$$

Indeed, $(\alpha,\beta)_p = 1$ means that $\alpha x_1{}^2 + \beta x_2{}^2 = 1$ has a solution. This is equivalent to saying that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & \alpha\beta \end{pmatrix}$$

are congruent, which clearly proves the lemma. Thus we have, in $G(F(p)) = G(F)$,

$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$
$2\epsilon = 2$	$2\epsilon = 2$
$\epsilon + p \neq 1 + \epsilon p$	$\epsilon + p \neq 1 + \epsilon p$
$2p = 2$	$2p \neq 2$

where $2 = 1 + 1$ (in $G(F)$) .

In case $p \equiv 1 \pmod 4$, we have

$$\begin{aligned} G(F) &= \mathbb{Z} \oplus \hat{G}(F) \\ &= \underset{1}{\mathbb{Z}} \oplus \underset{(\epsilon - 1)}{\mathbb{Z}_2} \oplus \underset{(p - 1)}{\mathbb{Z}_2} \oplus \underset{(\epsilon - 1)(p - 1)}{\mathbb{Z}_2} \end{aligned}$$

where the element beneath a group shows a generator of that group. The last summand, $\mathbb{Z}_2$, is $L(F)$. To be precise, we must show that $(\epsilon - 1) \neq 0$, $(p - 1) \neq 0$ and $(\epsilon - 1)(p - 1) \neq 0$; but these follow from DET $(\epsilon - 1) \neq 1$, DET $(p - 1) \neq 1$ and $\epsilon + p \neq 1 + \epsilon p$, respectively.

For the case $p \equiv 3 \pmod 4$, we have

$$\begin{aligned} G(F) &= \mathbb{Z} \oplus \hat{G}(F) \\ &= \underset{1}{\mathbb{Z}} \oplus \underset{(\epsilon-1)}{\mathbb{Z}_2} \oplus \underset{(p-1)}{\mathbb{Z}_4} . \end{aligned}$$

This can be seen as follows. Property 6 of the Hilbert symbol shows that $(p,\epsilon p)_p = (\epsilon,p)_p (p,p) = 1$. Therefore $p + \epsilon p = 1 + \epsilon$, i.e. $G(F)$ is generated additively by $(\epsilon - 1)$ and $(p - 1)$ only, and $(p - 1)$ has order 4 in $G(F)$. The element $2(p - 1)$ generates $L(F)$; therefore $L(F) \approx \mathbb{Z}_2$.

Finally, we arrive at the case where $F = F(2)$. As we know $R(2)^*/R(2)^{**} \approx G_2 \cdot G_2$; let it be represented by $\{1, \epsilon, \mathcal{f}, \epsilon\mathcal{f}\}$, $(\epsilon^2 = 1, \mathcal{f}^2 = 1)$. Then $F^*/F^{**} \approx G_2 \cdot G_2 \cdot G_2$ is represented by $\{1, \epsilon, \mathcal{f}, \epsilon\mathcal{f}, 2, 2\epsilon, 2\mathcal{f}, 2\epsilon\mathcal{f}\}$; in fact, using integers modulo 8 we have $\epsilon = 3$

$\mathcal{f} = 5$. Then

(1) $(\epsilon,\epsilon)_2 = -1$

(2) $(\epsilon,\mathcal{f})_2 = 1$

(3) $(\epsilon,2)_2 = -1$

(4) $(\mathcal{f},\mathcal{f})_2 = 1$

(5) $(\mathcal{f},2)_2 = -1$

(6) $(2,2)_2 = 1$

Putting $x_1 = \epsilon$, $x_2 = \mathcal{f}$, $x_3 = 2$, we obtain

$$x_1 + x_2 = 1 + x_1 x_2$$
$$2x_2 = 2$$
$$2x_3 = 2$$
$$x_1 + x_1 x_3 = 1 + x_3$$
$$x_1 x_2 + x_3 = 1 + x_1 x_2 x_3 \ .$$

Using an argument similar to that used before, we get

$$G(F(2)) = \underset{1}{Z} \oplus \underset{(x_2-1)}{\mathbb{Z}_2} \oplus \underset{(x_3-1)}{\mathbb{Z}_2} \oplus \underset{(x_1-1)}{\mathbb{Z}_4}$$

and $L(F(2))$ is generated by $2(x_1 -1)$. We leave the details to the reader.

We conclude this Appendix by giving an invariant definition of the Hasse-Minkowski symbol. For each finite prime, p, let $c_p' : G(F(p)) \longrightarrow L(F(p))$ be a map defined by $c_p'(f) = f - \mathrm{DET}\, f \cdot 1 - \mathrm{rk} f \cdot 1 + 1$. Identify $L(F(p))$ with $\mathbb{Z}_2$, then $c_p'(f) = 0$ or 1 . Define

$$c_p(f) = (-1)^{c_p'(f)}$$

In case $p = \infty$ (i.e. $F(p) = \mathbb{R}$) , write $f = \alpha^+ \cdot 1 + \alpha^-(-1)$;

then α^- is called the <u>index</u> of f. Define

$$c_\infty(f) = (-1)^{\alpha^-(\alpha^- - 1)/2} .$$

To see that this symbol $c_p(f)$ coincides with the Hasse-Minkowski symbol defined in §0 we argue as follows. Trivially, (write DET f.1 as DET f, etc.),

$$c'_p(f \dotplus g) = f + g - \text{DET } f . \text{ DET } g - \text{rk } f - \text{rk} g + 1 ,$$

$$c'_p(f) + c'_p(g) = f - \text{DET } f - \text{rk } f + 1 + g - \text{DET } g - \text{rk} g + 1 .$$

Therefore,

(A.7) $$c'_p(f \dotplus g) - c'_p(f) - c'_p(g) = \text{DET } f + \text{DET } g - \text{DET } f \text{ DET } g - 1 .$$

From (A.6) we have

$$(\text{DET } f, \text{DET } g)_p = (-1)^{\text{RHS}} ,$$

where RHS = right hand side of (A.7). Thus

$$(-1)^{c'_p(f \dotplus g) - c'_p(f) - c'_p(g)} = (\text{DET } f, \text{DET } g)_p$$

or

(A.8) $$c_p(f \dotplus g) = c_p(f)\, c_p(g)\, (\text{DET } f, \text{DET } g)_p$$

since this newly defined $c_p(f)$ coincides with the one defined in §0 for unary forms, f, and since every form over $F(p)$ decomposes into unary forms, the desired result now follows from (A.8) and (0.5).

COROLLARY (A.9) Let f be a quadratic form over $F(p)$, p finite. Then the equivalence class of f in $G(F(p))$ is completely determined by the rk f, DET f and $c_p(f)$.

REFERENCES

1. Jones, B. W. The Arithmetic Theory of Quadratic forms (Carus Math. Monographs, N.Y., 1950).

2. Ledermann, W., An Arithmetical Property of Quadratic Forms. Comm. Math. Helv. 33 (1959) 34-37.

3. Milnor, J. W. On simply Connected 4-manifolds Symp. Intern. de Topologia Alg. (Mexico 1958), 122-128.

4. Witt, E. Theorie der quadratischen Formen in beliebigen Körpern. Journal für Math., 176 (1937) 31-44.

1. Certain Arithmatical Properties of Quadratic Forms.

Let B be an integral domain which contains A as a sub-domain. Then B may be considered as a torsion-free module over A. If V is an A-lattice, $V \otimes_A B$ is, in the natural way, a B-lattice whose rank over B is the same as the rank of V over A. A quadratic form f defined on V extends naturally into a quadratic form $f^B = (f^B, V \otimes B)$ on $V \otimes B$. This notation will be used throughout these notes.

A quadratic form over A will be termed integral, rational or real according as $A = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Let f be a real quadratic form whose matrix has been diagonalized according to (0.5). Let α^+ (resp. α^-) denote the number of positive (resp. negative) diagonal entries. Then the integers α^+ and α^- are independent of the diagonalization in virtue of Sylvester's law of inertia, which we have shown in §0. From what we just said, we may regard α^+ and α^- as being defined for integral and rational forms also. Since we only consider non-degenerate quadratic forms, $\alpha^+ + \alpha^- = rk(f)$, the rank of f. Define the signature, $\tau(f)$, of the form f to be the integer $\alpha^+ - \alpha^-$. Notice that α^+ and α^-, and hence the signature, are also defined for forms over $Q(p) = Q \cap R(p)$, the ring of rational p-adic integers.

A quadratic form, f, over $\mathbb{Z}$ or $R(2)$ is said to be <u>even</u> if $f(x,x) \equiv 0 \pmod 2$ for every $x \in V$, where V is the lattice on which f is defined; otherwise f is termed odd. Since

$$(1.1) \qquad f(x+y,\ x+y) = f(x,x) + f(y,y) + 2f(x,y),$$

f is even if, and only if, the diagonal entries of its matrix are all even, where an element $t \in R(2)$ is called even or odd according as $t \equiv 0 \pmod 2$ or not.

Consider a quadratic form, $f = (f,V)$, over $\mathbb{Z}$ or $R(2)$ with odd determinant (this property clearly independent of the base chosen in V). There exists an element $w \in V$, in general not unique, satisfying the relation

$$(1.2) \qquad f(x,x) \equiv f(x,w) \pmod 2$$

for each $x \in V$. For if $x = \sum_i x_i e_i$ and $w = \sum_i w_i e_i$, where $e_1, \ldots, e_r$ constitute a base of V and w_i, x_i are elements in $\mathbb{Z}$ or $R(2)$ as the case may be, then

$$f(x,x) = f(\sum x_i e_i, \sum x_i e_i)$$
$$\equiv \sum_i f(e_i,e_i)\, x_i^2 \pmod 2 .$$

On the other hand,

$$f(x,w) = \sum_{i,j} f(e_i,e_j)\, x_i w_j .$$

The relation (1.2) is equivalent to

$$f(e_i,e_i) \equiv \sum_j f(e_i,e_j)\, w_j \pmod 2 .$$

Since $\det f \not\equiv 0 \pmod 2$, we may solve for the coefficients

w_j , $j = 1, \dots r$,. Such an element, w , will be called a <u>characteristic</u> <u>element</u> of f . If $\tilde{w}$ is another characteristic element of f , then $f(x, \tilde{w} - w) \equiv 0$ (mod 2) for every $x \in V$. It follows from the assumption $\det f \not\equiv 0$ (mod 2) that

$$(1.3) \qquad \tilde{w} = w + 2z$$

for some $z \in V$. Conversely, any element, w , satisfying (1.3) is a characteristic element whenever w is one. Now

$$f(\tilde{w},\tilde{w}) = f(w,w) + 4f(w,z) + 4f(z,z) \ .$$

We conclude from (1.2) that

$$(1.4) \qquad f(w,w) \equiv f(\tilde{w},\tilde{w}) \pmod 8$$

It is scarcely necessary to remark that, in $R(2)$, mod 8 means modulo the ideal $2^3 R(2)$. Since the integers, $\mathbb{Z}$, form a sub-ring of $R(p)$ for $p = 2, 3, \dots, \infty$; we may, in particular, consider an integral form as a form over $R(2)$.

THEOREM (1.5) Let f be a quadratic form over $R(2)$ with odd determinant. Then

$$(1.6) \qquad f(w,w) - r - \det f + 1 \equiv 0 \pmod 4$$

where $r = \operatorname{rk} f$ is the rank of f .

In view of (1.4), the relation (1.6) is meaningful if $\det f$ modulo 4 is well-defined; but this is clear since $\det f$ is determined up to the square of a unit and we are working in $R(2)$. One additional remark before we proceed to prove (1.5): throughout these notes the equality sign, $=$, will also stand for the phrase "is canonically isomorphic to".

Let $f = (f,V)$ be the given quadratic form. If $r = 1$, then $V = R(2)$ and $f(x,y) = axy$, is odd, where $a = \det f$ (with respect to the canonical base). Thus $w = 1$ is a characteristic element of f and (1.6) is clearly satisfied If $r = 2$ then $V = R(2) \oplus R(2)$. If furthermore f is an even form, the matrix of f relative to the canonical base may be written as

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$$

where c is odd because $\det f$ is odd. Therefore $c^2 \equiv 1 \pmod 4$. Now, $w = 0$ is a characteristic element of f , so we have $f(w,w) - r - \det f + 1 = 0 - 2 - 4ab + c^2 + 1 \equiv 0 \pmod 4$. Thus (1.5) is proved for unary and binary even forms.

Assume that $f = f_1 \dotplus f_2$ is a decomposition of f , where f_i is defined on V_i , $i = 1,2$, and $V_1 \oplus V_2 = V$. Obviously f_i has odd determinant and $r_1 + r_2 = r$ where r_i is the rank of f_i $(i = 1,2)$. Suppose that, by induction, the theorem has been proven for quadratic forms of rank $\leq \max(r_1,r_2)$. Let w_i be a characteristic element of f_i $(i = 1,2)$. Then $w = w_1 \oplus w_2$ is a characteristic element of f . Since $\det f_i$ is odd for $i = 1,2,$ we have

$$(\det f_1 - 1)(\det f_2 - 1) = \det f_1 \det f_2 - \det f_1$$
$$- \det f_2 + 1 \equiv 0 \pmod 4 .$$

On the other hand,

$$\begin{aligned} & f(w,w) - r - \det f + 1 \\ & = f_1 \dotplus f_2(w_1 \oplus w_2, w_1 \oplus w_2) - r_1 - r_2 - \det f_1 \det f_2 + 1 \\ & = f_1 (w_1,w_1) + f_2(w_2,w_2) - r_1 - r_2 - \det f_1 \det f_2 + 1 \\ & \equiv -\det f_1 \det f_2 + \det f_1 + \det f_2 - 1 \pmod 4, \end{aligned}$$

by induction hypothesis. Therefore

$$f(w,w) - r + \det f + 1 \equiv 0 \pmod 4 .$$

To complete the proof we need only show that every quadratic form of odd determinant may be written as a sum of unary and binary even forms with odd determinants. A slight modification of the proof of (0.5) will prove the last statement and hence the theorem.

Now, for an element $a \in R(2)$, the exponent $(-1)^a$ may be defined by stipulating

$$(-1)^a = \begin{cases} 1 , & \text{if } a \text{ is even} \\ -1 , & \text{if } a \text{ is odd} . \end{cases}$$

We then define

$$\tilde{c}_2(f) = (-1)^{\frac{f(w,w) - r - \det f + 1}{4}} .$$

Recall that for a quadratic form, f, over $R(2)$ the

Hasse-Minkowski symbol $c_2(f)$ is also defined. We have

THEOREM (1.8) $\tilde{c}_2(f) = c_2(f)$.

Indeed, if f_1 and f_2 (hence $f = f_1 \dot{+} f_2$) are forms over $R(2)$ with odd determinant, then in virtue of (1.7) and property (5b) of the Hilbert symbols we have

$\tilde{c}_2\,(f) = \tilde{c}_2\,(f_1)\,\tilde{c}_2(f_2)\quad(\det f_1, \det f_2)_2$.

Comparing this with (0.23) we see that $\tilde{c}$ and c behave the same way on sum of forms. It follows that the proof of (1.8) will be completed if we can prove the statement for unary and binary even forms. With the notations used in the proof of (1.5) we see that if f is unary, $R(2) = V$, $w = 1$ and $\det f = a \in R(2)^*$ then $\tilde{c}_2(f)$ $= 1 = c_2(f)$. If f is a binary even form, $V = R(2) \oplus R(2)$, $w = 0$, then

$$c_2\,(f) = (-1)^{\frac{-1 - 4ab + c^2}{4}} = (-1)^{ab}$$

because $c^2 - 1 \equiv 0 \pmod 8$. We consider two cases:

<u>case 1</u>. $a = b = 0$. In this case the matrix of f is

$$\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

which is congruent over $F(2)$ to $\mathrm{diag}(\alpha,-\alpha)$. Therefore $c(f) = 1$. Trivially, $\tilde{c}_2(f) = 1$; hence $c_2(f) = \tilde{c}_2(f)$.

<u>case 2</u>. At least one of a and b is not zero, let it be a . We split $2a$ off from f to obtain

$$\begin{aligned} c_2(f) &= \left(2a, \frac{4ab - c^2}{2a}\right)_2 \\ &= \left(2a, \frac{4ab - c^2}{2a}\right)_2 (2a, -2a)_2 \\ &= (2a, c^2 - 4ab)_2 \end{aligned}$$

Recall that $(2|\alpha) = 1$ or -1 according as $\alpha \equiv 1, 7 \pmod 8$ or $\alpha \equiv 3, 5 \pmod 8$. It is now a routine matter to check that $\tilde{c}_2(f) = c_2(f)$ (use property (5b) of the Hilbert symbols). This proves case 2 and hence the theorem.

Now let f be a quadratic form over $Q(2) = R(2) \cap Q$. Then the signature, $\tau = \tau(f)$, is defined. We have

THEOREM (1.9) For non-singular quadratic forms, f, over $Q(2)$,

$$c_2(f)c_\infty(f) = (-1)^{\frac{f(w,w) - \tau - \det f + \operatorname{sgn} \det f}{4}}$$

Let α^+ and α^- be defined as at the beginning of this section. Then $\tau = \alpha^+ - \alpha^-$, $r = \alpha^+ + \alpha^-$ and, obviously, $\operatorname{sgn} \det f = (-1)^{\alpha^-}$. In view of Theorem (1.8), we consider

$$\frac{f(w,w) - \tau - \det f + \operatorname{sgn}\det f}{4} - \frac{f(w,w) - r - \det f + 1}{4}$$

$$= \frac{-\tau + r + \operatorname{sgn}\det f - 1}{4}$$

$$= \frac{2\alpha^- + (-1)^{\alpha^-} - 1}{4} \equiv \frac{\alpha^-(\alpha^- - 1)}{2} \pmod 2 .$$

It is easy to compute $c_\infty(f)$ directly from the definition and property (1) of the Hilbert symbols; indeed

$$c_{\infty}(f) = (-1)^{\frac{\alpha^-(\alpha^- - 1)}{2}}$$

This completes the proof of the theorem.

Since the only units in $\mathbb{Z}$ are 1 and -1, the determinant $\det f$ of an integral form f, is well-defined. An integral form is called unimodular (= non-singular over $\mathbb{Z}$) if its determinant is ± 1; in this case $\det f - \operatorname{sgn} f = 0$. Furthermore, for such forms f, $c_p(f) = 1$ if p is an odd prime, in virtue of property (5') of the Hilbert symbols. From Lemma (0.25) we deduce that $c_2(f)\, c_{\infty}(f) = 1$. This together with (1.9) proves the following

COROLLARY (1.10) If f is an integral unimodular quadratic form then

$$f(w,w) - \tau \equiv 0 \pmod 8 .$$

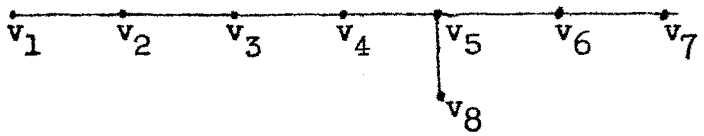
In particular, if f is, in addition, an even form, then $w = 0$ is a characteristic element. Thus we have

COROLLARY (1.11) If f is an integral, unimodular, even quadratic form then

$$\tau(f) \equiv 0 \pmod 8 .$$

We remark that this last result is the best possible in the sense that there actually exists a quadratic form satisfying the hypothesis with $\tau = 8$. Indeed, let E_8

denote the following graph (= 1-dimensional simplicial complex, cf. §4)



Construct a matrix $M = (\mu_{ij})$ with integral entries according to the following formulae

$\mu_{ii} = 2$, for $i = 1,2,\ldots,8$,

$\mu_{ij} = 1$, if the vertices v_i and v_j are joined by an edge (= 1-simplex) in E_8 ,

$\mu_{ij} = 0$, otherwise.

Thus

$$M = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & 0 & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ 0 & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix} .$$

The corresponding even quadratic form is easily seen to be unimodular with signature $\tau = 8$.

We call attention to this method of constructing integral forms from a graph. Similar constructions will be used later. Before a more general definition is given we shall refer to the quadratic form mentioned above as "the" quadratic form associated to E_8 .

2. The Quadratic form of a 4k-dimensional Manifold.

Unless otherwise specified, by a manifold we mean a connected, compact, orientable, differentiable manifold with or without boundary, together with a given orientation. Thus all manifolds under consideration are oriented. An n-dimensional manifold is also called an n-manifold.

Let M be a 4k-manifold. The homology group $H_{2k}(M;\mathbb{Z})$ is finitely generated. Therefore $V = H_{2k}(M;\mathbb{Z})/(\text{Torsion})$ is a lattice over $\mathbb{Z}$. There is a pairing,

$$S : V \times V \to \mathbb{Z} ,$$

defined by the intersection number of cycles. Precisely, if a, b are elements in $H_{2k}(M;\mathbb{Z})$, represented respectively by cycles z_1, z_2 in $Z_{2k}(M;\mathbb{Z})$, we define $\tilde{S}(a,b) = z_1 \cdot z_2$, the intersection number of z_1 and z_2. This pairing is clearly well-defined, bilinear and, since $2k$ is an even number, symmetric. Observe that $\tilde{S}(a,b)$ vanishes if a or b is a torsion element and hence it induces a quadratic form $S = (S,V)$, on V into the integers. Henceforth, this quadratic form will be written as S_M, and will be called the quadratic form of M. We shall also write $\tau(M)$ for $\tau(S_M)$, the signature of S_M.

The Poincaré duality for an n-manifold, M, may be expressed by an isomorphism

$$H_i(M;\mathbb{Z}) \approx H^{n-i}(M, \partial M;\mathbb{Z})$$

for each i $(0 \leq i \leq n)$. In particular, for $n = 4k$, $i = 2k$, we have

$$H_{2k}(M;\mathbb{Z}) \approx H^{2k}(M, \partial M,\mathbb{Z}) .$$

Denote the elements in $H^{2k}(M, \partial M;\mathbb{Z})$ which correspond to a, b, ..., in $H_{2k}(M;\mathbb{Z})$ under this isomorphism, by α, β, ..., respectively. Since

$$\alpha \cup \beta \in H^{4k}(M, \partial M;\mathbb{Z}) = \mathbb{Z} ,$$

(where the equality sign, as remarked before, means "is canonically isomorphic to"), we may consider $\alpha \cup \beta$ as an integer, which is precisely the intersection number of a and b (see e.g. [H-W], p. 156). In this sense we may regard S_M as a pairing defined on $H^{2k}(M, \partial M;\mathbb{Z})/(\text{Torsion})$ into $\mathbb{Z}$ by the cup-product.

Exercise (2.1) Show that if M is a 4k-dimensional unbounded manifold (i.e. $\partial M = \emptyset$) then S_M is unimodular (see e.g. Milnor [1]).

Our main concern, at least for a while, will be 4-manifolds. Many results stated below have higher dimensional analogues; but this does not concern us at this moment.

Let M be an unbounded 4-manifold, and let $H^*(M;\mathbb{Z}_2)$ be its cohomology ring with coefficients in $\mathbb{Z}_2$. The quadratic form of M over $\mathbb{Z}_2$ is again given by the cup-product : $S(x,y) = x \cup y$. In this case there exists a unique characteristic element $w_2 = w_2(M)$ in $H^2(M;\mathbb{Z}_2)$ satisfying $x \cup x = x \cup w_2$, for each $x \in H^2(M;\mathbb{Z}_2)$. We remark that w_2 is actually the middle Stiefel-Whitney class of M since M is oriented.

The following theorems are known. (Rohlin [2], Borel-Hirzebruch [3], see also Kervaire-Milnor [1] and [2]).

THEOREM (2.2) If M is an unbounded 4-manifold with $w_2 = o$, then $\tau(M) \equiv 0 \pmod{16}$. (Rohlin)

We remind the reader that all manifolds under consideration are supposed to be differentiable. We do not know whether (2.2) holds in general for oriented topological 4-manifolds. (Notice that the statement of (2.2) still makes sense in this case.)

Let $\pi : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ be the reduction modulo 2, i.e. the homorphism induced by the coefficient epimorphism $\mathbb{Z} \to \mathbb{Z}_2$. We have

THEOREM (2.3) If $d \in H^2(M;\mathbb{Z})$ is such that $\pi d = w_2$, then $S_M(d,d) \equiv \tau(M) \pmod{8}$, where S_M is the integral quadratic form of M. (Borel-Hirzebruch)

The assumption of (2.3) implies that d is a characteristic element of S_M, (because the cup-product commutes with the homomorphism π). Hence (2.3) follows immediately from (1.10) and (2.1). A weakened version of (2.2), i.e., $\tau(M) \equiv 0 \pmod 8$, can then be deduced from (2.3).

The short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ of coefficients induces an exact sequence

(2.4) $\quad \ldots \to H^2(M;\mathbb{Z}) \xrightarrow{2} H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2) \to \ldots.$

in homology, where the number 2 above an arrow means multiplication by 2. If M has no 2-torsion (i.e., the 2-component of the torsion subgroup of $H^*(M;\mathbb{Z})$ is zero), then (2.4) gives rise to a short exact sequence

$$0 \to H^2(M;\mathbb{Z}) \xrightarrow{2} H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2) \to 0 .$$

Therefore,

$$\begin{aligned} H^2(M;\mathbb{Z}_2) &\approx H^2(M;\mathbb{Z})/2.H^2(M;\mathbb{Z}) \\ &\approx \frac{H^2(M;\mathbb{Z})}{\text{Torsion}} \Big/ \frac{2.H^2(M;\mathbb{Z})}{\text{Torsion}} . \end{aligned}$$

Thus w_2 corresponds, under the above isomorphism, to the residue class containing the characteristic elements of S_M. We conclude that, assuming M has no 2-torsion, $w_2 = 0$ if, and only if, the quadratic form S_M is even. We may thus state

THEOREM (2.2a) If M is a 4-manifold without 2-torsion and S_M is even, then $\tau(M) \equiv 0 \pmod{16}$.

Hence the quadratic form associated to E_8 mentioned in §1 cannot occur as the quadratic form of a 4-manifold without 2-torsion. The following theorem, due to Kervaire and Milnor [2], sharpens the results of (2.2) and (2.3).

THEOREM (2.5) In the statement of (2.3) if the dual class of d is represented by a differentiably imbedded 2-sphere in M, then $S(d,d) \equiv \tau(M)$ (Mod 16).

We hope to include later a proof of this theorem and some examples.

3. An application of Rohlin's theorem, μ-invariants.

Let G be an abelian group. By a G-homology k-sphere, X, we mean an unbounded manifold whose homology groups over G are isomorphic to those of a k-sphere; that is, $H_i(X;G) \approx G$, for $i = 0, k$, and $H_i(X;G) = 0$, otherwise. Note that if $G \neq 0$, this implies that X is k-dimensional.

We are interested in a particular class, $\mathcal{X}_3$, of $\mathbb{Z}_2$-homology 3-spheres. A $\mathbb{Z}_2$-homology 3-sphere, X, is an element of $\mathcal{X}_3$ if, and only if, X bounds a 4-manifold Y(i.e. $\partial Y = X$) satisfying the conditions

(3.1) a) $H_1(Y;\mathbb{Z})$ has no 2-torsion, and

b) S_Y is an even quadratic form.

We remark that from cobordism theory, every 3-manifold bounds. Here we merely impose an additional condition, (3.1), to the manifolds they bound. For each $X \in \mathcal{X}_3$ we define

$$\mu(X) = \frac{-\tau(Y)}{16} \in \mathbb{Q}/\mathbb{Z}$$

(the number $-\tau(Y)/16$ reduced modulo 1). It is understood that Y satisfies (3.1).

THEOREM (3.2) For $X \in \partial\mathcal{Y}_3$, $\mu(X)$ is invariant under orientation-preserving diffeomorphisms.

For this reason we shall call $\mu(X)$ the μ-invariant of X. This is a special case of the μ-invariants studied by Eells and Kuiper [1]. Clearly, to prove (3.2) it suffices to show that $\mu(X)$ is independent of the choices of Y (provided that Y satisfies (3.1) and $\partial Y = X$, of course). Let Y_1 and Y_2 be two such 4-manifolds with $\partial Y_1 = \partial Y_2 = X$. Without loss of generality, we may assume that $Y_1 \setminus X$ and $Y_2 \setminus X$ have no common points. Let $-Y_2$ denote the manifold obtained from Y_2 by reversing its orientation. Then $M = Y_1 \cup (-Y_2)$ is a topological manifold, oriented in accordance with Y_1 and with $-Y_2$. In other terms, M is constructed from Y_1 and $-Y_2$ by "pasting" their boundaries together according to the identity map on X. Notice that M has, after smoothing if necessary, a differentiable structure compatible with those on Y_1 and on $-Y_2$. Thus M is a manifold in our sense. We refer the reader to Milnor [2], [3] for details about pasting and smoothing.

LEMMA (3.3) The manifold, M, obtained above has the following properties:

1) M has no 2-torsion,

2) S_M is even, and

3) $S^{\mathbb{Q}}_{Y_1} + (-S^{\mathbb{Q}}_{Y_2}) = S^{\mathbb{Q}}_M$;

where $S^{\mathbb{Q}}$ is defined as in §1 .

Before we proceed to prove (3.3), observe that the lemma is sufficient to conclude (3.2). For property 3) implies $\tau(M) = \tau(Y_1) - \tau(Y_2)$, and properties 1) and 2) together with Rohlin's theorem (2.2a), imply that $\tau(M) \equiv 0 \pmod{16}$; Theorem (3.2) then follows. To prove (3.3) we need

LEMMA (3.4) If X is a $\mathbb{Z}_2$-homology k-sphere then

a) $H_i(X;\mathbb{Z})$ is a torsion group of odd order for $i \neq 0, k$,

b) X is a $\mathbb{Q}$-homology sphere, and

c) X is an $R(2)$-homology sphere.

Consider the exact sequence of homology groups

(3.5) $\ldots \to H_{i+1}(X;\mathbb{Z}_2) \to H_i(X;\mathbb{Z}) \xrightarrow{2} H_i(X;\mathbb{Z}) \to \ldots .$

induced by the coefficient exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z} \to 0$. If $i \neq 0, k$, then $H_i(X;\mathbb{Z}) \xrightarrow{2} H_i(X;\mathbb{Z})$ is an epimorphism. As the groups involved are finitely generated, we conclude that $H_i(X;\mathbb{Z})$ is a torsion group of odd order. Since b) is an obvious consequence of a), it remains to prove c). Using the exact sequence $0 \to R(2) \xrightarrow{2} R(2) \to \mathbb{Z}_2 \to 0$, we may replace $\mathbb{Z}$ by $R(2)$ in (3.5) to obtain

$$\cdots \to H_{i+1}(X;\mathbb{Z}_2) \to H_i(X;R(2)) \xrightarrow{2} H_i(X;R(2))$$

$\to H_i(X;\mathbb{Z}_2) \to \cdots$. Recall that $R(2)$ is a PID (= principal ideal domain) and $H_i(X;R(2))$ is finitely generated. Hence

$$(3.6) \qquad H_i(X;R(2)) \approx \oplus \sum_{k=1}^{m} R(2)/\mathfrak{a}_k ,$$

where $\mathfrak{a}_k$'s are ideals of $R(2)$ such that $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_m \neq R(2)$ (cf. [B-a], Ch. 7, § 4, Theorem 2). The only ideals in $R(2)$ are the zero ideal and those generated by 2^j ; therefore each summand in (3.6) is of the form $R(2)$ or $R(2)/2^j.R(2) \approx \mathbb{Z}_{2^j}$. It follows that if $H_i(X;R(2)) \xrightarrow{2} H_i(X;R(2))$ is an epimorphism, then $H_i(X;R(2)) = 0$. This proves c) and hence (3.4).

Returning to (3.3), we consider X, Y_1 , Y_2 , and M as defined there and apply the Mayer-Vietoris sequence to obtain

$$(3.7) \quad \cdots \to H_i(X) \to H_i(Y_1) \oplus H_i(Y_2) \to H_i(M) \to H_{i-1}(X) \to \cdots ,$$

where the coefficient group is not specified. From (3.4) we know that X is a $\mathbb{Q}$-homology 3-sphere. Therefore $H_2(M;\mathbb{Q})$ has an injective representation.

$$(3.8) \qquad 0 \to H_2(Y_1;\mathbb{Q}) \oplus H_2(Y_2;\mathbb{Q}) \to H_2(M;\mathbb{Q}) \to 0 .$$

Taking orientations into account, it is not hard to see that $S^Q_{Y_1} \dotplus (-S^Q_{Y_2}) = S^Q_M$. This proves 3) of (3.3).

Next, we take integral coefficients in (3.7) to obtain

$$\cdots \to H_1(X;\mathbb{Z}) \to H_1(Y_1;\mathbb{Z}) \oplus H_1(Y_2;\mathbb{Z}) \to H_1(M;\mathbb{Z}) \to \tilde{H}_0(X;\mathbb{Z})$$

where $\tilde{H}_0(X;\mathbb{Z}) = 0$ is the reduced homology group. Since Y_1 and Y_2 have no 2-torsion and since $H_1(M;\mathbb{Z})$ is a torsion group (by (3.4a)), it follows that $H_1(M;\mathbb{Z})$ has no 2-component. Property 1) of (3.3) now follows from Poincaré duality (of homology groups, see e.g. [S-T] p. 245, Satz II).

Finally, we take $R(2)$ as coefficient group in (3.7) and apply (3.4) to obtain an injective representation

$$0 \to H_2(Y_1;R(2)) \oplus H_2(Y_2;R(2)) \to H_2(M;R(2)) \to 0$$

just as in (3.8). Since Tor(odd torsion, $R(2)) = 0$, then by the universal coefficient theorem, $H_2(Y_i;R(2)) = H_2(Y_i;\mathbb{Z}) \otimes R(2)$, $i = 1, 2$, and $H_2(M;R(2)) = H_2(M;\mathbb{Z}) \otimes R(2)$. Recall that $S^{R(2)}_{Y_i}$ is defined on $\dfrac{H_2(Y_i;\mathbb{Z})}{\text{Torsion}} \otimes R(2) \approx \dfrac{H_2(Y_i;\mathbb{Z}) \otimes R(2)}{\text{Torsion}}$, $i = 1,2$. Thus

$$S^{R(2)}_M = S^{R(2)}_{Y_1} \dotplus (-S^{R(2)}_{Y_2}) .$$

The last two terms are even by assumption; hence $S_M^{R(2)}$ is also even. The last statement implies that S_M itself is even, which completes the proof of (3.3).

As an example we shall apply the invariant $\mu(X)$ to lens spaces. Let the standard 3-sphere, S^3, be represented in the complex 2-space, $\mathbb{C}^2$, by $S^3 = \{(z_1,z_2) \mid z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$. Then S^3, as submanifold of $\mathbb{C}^2$, inherits a canonical orientation. Let n and q be integers, $0 < q < n$, such that $(n,q) = 1$, (i.e., n and q are co-prime). Define an operation of the group $\mathbb{Z}_n$ on S^3 by the formula

$$(3.9) \qquad \gamma \cdot (z_1,z_2) = (e^{\frac{2\pi i \gamma}{n}} \cdot z_1,\ e^{\frac{2\pi i q \gamma}{n}} \cdot z_2)$$

for each $\gamma \in \mathbb{Z}_n$. Then $\mathbb{Z}_n$ acts freely and differentiably on S^3. The quotient space (space of orbits) $S^3/\mathbb{Z}_n = L(n,q)$ is, by definition, the lens space of type (n,q), which inherits an orientation and a differentiable structure from S^3. Therefore $L(n,q)$ is a manifold in our sense. Since $L(n,q)$ has S^3 as its universal covering space,

$$\pi_1(L(n,q)) = \mathbb{Z}_n = H_1(L(n,q);\mathbb{Z}).$$

Thus the Betti numbers of $L(n,q)$ are 1,0,0,1 and there is a torsion coefficient, n, at dimension 1. If n is odd, $L(n,q)$ is a $\mathbb{Z}_2$ - homology sphere.

THEOREM (3.10) For each odd integer $n > 0$, $L(n,q) \in \mathcal{T}_3$. $\mathcal{T}_3$; siehe Seite 3.1

A proof of this theorem can be found in $\not p$. Granting this, we proceed to study the μ-invariant of such lens spaces.

LEMMA (3.11) With respect to the canonical orientations, $L(n,q) = -L(n,n-q)$.

In fact, the orientation reversing differentiable involution, $\alpha: (z_1,z_2) \to (z_1,\bar{z}_2)$, on S^3 carriès $L(n,n-q)$ onto $-L(n,q)$. Explicity, for $\gamma \in \mathbb{Z}_n$,

$$(e^{\frac{2\pi i\gamma}{n}} \cdot z_1, e^{\frac{2\pi i q\gamma}{n}} \cdot z_2) = (e^{\frac{2\pi i\gamma}{n}} \cdot z_1,$$

$$e^{\frac{-2\pi i q\gamma}{n}} \cdot z_2) = \alpha(e^{\frac{2\pi i\gamma}{n}} z_1, e^{\frac{2\pi i(n-q)\gamma}{n}} \cdot z_2).$$

Thus, to study the μ-invariants of $L(n,q) \in \mathcal{T}_3$, we may restrict our attention to the case where q takes even values. Recall that a fraction n/q can be expanded into (finite) continued fractions

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ddots - \cfrac{1}{b_s}}}$$

$$= [b_1,b_2,b_3,\ldots,b_s] \quad \text{(notation)},$$

where b_i is an integer with $|b_i| \geq 2$, $i = 1, 2, \ldots, s$. In case where n is odd and q is even, there is a unique expansion $n/q = [b_1,b_2,\ldots,b_s]$ such that

positive

each b_i is even (the proof is elementary). Denote by $p^+ = p^+(n,q)$ (resp. $p^- = p^-(n,q)$) the number of positive (resp. negative) b_i's in this unique expansion of n/q. We have

RECIPE (3.12) $\mu(L(n,q)) = \frac{p^+ - p^-}{16} \in \mathbb{Q}/\mathbb{Z}$.

A proof of this fact will be given later. As examples we have

$7/6 = [2,2,2,2,2,2]$ $\qquad p^+ = 6,\ p^- = 0$,

$7/2 = [4.2]$ $\qquad p^+ = 2,\ p^- = 0$.

Therefore,

$\mu(L(7,6)) = 3/8$,

$\mu(L(7,2)) = 1/8$,

$\mu(L(7,1)) = -\mu(L(7,6)) = -3/8 \equiv 5/8 \pmod 1$.

We recall that (J. H. C. Whitehead [1]),

THEOREM (3.13) Two lens spaces, $L(n,q)$ and $L(n,q')$, have the same homotopy type if, and only if, qq' or $-qq'$ is a quadratic residue modulo n .

Thus $L(7,1)$ and $L(7,2)$ have the same homotopy type; but they are not diffeomorphic because $\mu(L(7,1))$ differs from $\mu(L(7,2))$ and from $-\mu(L(7,2))$.

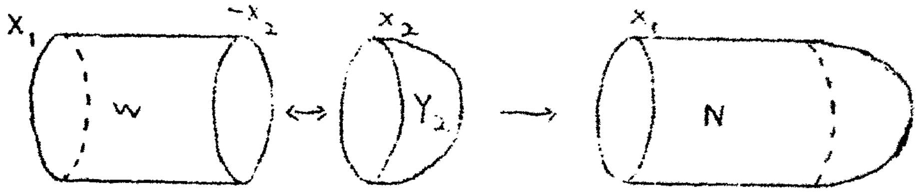
Now let X_1 and X_2 be unbounded n-manifolds. We say that X_1 and X_2 are h-cobordant (=J-equivalent) if, and only if, there exists an (n+1)-manifold, W, such that

1) $\partial W = X_1 \cup -X_2$ (disjoint union) and,

2) both X_1 and X_2 are deformation retracts of W .

THEOREM (3.14) Let X_1 and X_2 be 3-manifolds in $\mathcal{S}_3'$. If X_1 and X_2 are h-cobordant then $\mu(X_1) = \mu(X_2)$.

This follows quickly from the "pasting and smoothing" technique. Indeed, let $X_1 = \partial Y_1$ and $X_2 = \partial Y_2$, where Y_i satisfies (3.1), $i = 1,2$, and let X_1 and X_2 co-bound W (i.e., $\partial W = X_1 \cup -X_2$), which gives the cobordant relation. We paste W and Y_2 to obtain a manifold, N. Notice that we already have the right orientations on the two copies of X_2.



Since X_2 is a deformation retract of W, the manifold N is homotopically equivalent to Y_2 and hence $S_N = S_{Y_2}$. On the other hand, N satisfies (3.1) with $\partial N = X_1$. It follows that $\tau(Y_2) = \tau(N) \equiv \tau(Y_1)$ (mod 16), (cf. (3.3)). This proves (3.14).

It follows that $L(7,1)$ and $L(7,2)$ are not even h-cobordant.

We conclude this section by proving the following lemma, which shows that certain $\mathbb{Z}_2$ - homology spheres arise naturally. The higher dimensional analogues are immediate.

LEMMA (3.15) Let Y be a simply-connected, torsion-free 4-manifold with boundary, $\partial Y = X$. If $\det S_Y$ is odd, X is a $\mathbb{Z}_2$ - homology sphere.

Consider the sequence

$$H_2(Y) \xrightarrow{j_*} H_2(Y,X) \xrightarrow{\partial} H_1(X) \to 0$$

which, as a portion of the homology sequence of the pair (Y,X), is exact (integral homology being understood here). Since M has no torsion, $H^2(Y) = \mathrm{Hom}(H_2(Y),\mathbb{Z})$. We apply the Poincaré-Lefschetz duality,

$$H_i(Y,X) = H^{4-i}(Y),$$

to obtain a commutative diagram

$$\begin{tikzcd}
H_2(Y) \arrow[r] \arrow[dr, "\mathcal{G}"'] & H_2(Y,X) = H^2(Y) = \mathrm{Hom}(H_2(Y),\mathbb{Z}) \arrow[r] & H_1(X) \arrow[r] & 0 \\
& \mathrm{Hom}(H_2(Y),\mathbb{Z}) \arrow[ur, "\psi"'] & &
\end{tikzcd}$$

The homomorphism $\mathcal{G}$ may be interpreted as follows : for each a in the lattice $H_2(Y)$ there is an element $h_a \in \mathrm{Hom}(H_2(Y),\mathbb{Z})$ given by $h_a(x) = S_Y(a,x)$, then $\mathcal{G}(a) = h_a$. Obviously ψ induces an isomorphism ψ_0 from coker $\mathcal{G}$ (= cokernel of $\mathcal{G}$) onto $H_1(X)$. Since

$H_2(Y)$ and $\mathrm{Hom}(H_2(Y),\mathbb{Z})$ are lattices of the same rank, we have

(3.16) $\quad H_1(X)$ is finite $\Longleftrightarrow \det S_Y \neq 0$.

Now, coker φ is determined, up to an isomorphism, by the elementary divisors of φ , we conclude that

(3.17) $\quad \det S_Y \neq 0 \Rightarrow$ order of $H_1(X) = |\det S_Y|$.

This, in particular, implies (3.15) .

1) We introduced the class γ_3 (see p. 3-1) merely for avoiding certain detailed arguments. In fact, γ_3 is precisely the class of <u>all</u> $\mathbb{Z}_2$-homology 3-spheres. To see this we first recall a well-known fact that every 3-dimensional (orientable, differentiable) manifold is parallelizable. Therefore, according to Milnor [2] (see also M.W. Hirsch [1]), every unbounded 3-manifold bounds a (compact) simply connected π-manifold. Now let X be any $\mathbb{Z}_2$-homology 3-sphere. We shall show that $X \in \gamma_3$. Let Y be a simply connected π-manifold such that $\partial Y = X$. Then the condition (3.1a) is trivially fulfilled. Let M be the double of Y (i.e. $M = Y \cup -Y$, obtained by pasting Y and $-Y$ together along their boundaries). It is clear that S_M is unimodular and $w_2(M) = 0$ (since $w_2(Y) = 0$ and $\partial Y = X$ is a $\mathbb{Z}_2$-homology sphere); therefore the quadratic form S_M is even. On the other hand, we have

$$S_M^{R(2)} = S_Y^{R(2)} \dotplus (-S_Y^{R(2)})$$

(see p. 3-5). We conclude that S_Y is even and hence $X \in \gamma_3$. The following theorem follows easily from our discussion:

THEOREM (3.18) Let X be a $\mathbb{Z}_2$-homology 3-sphere. If X is embeddable in R^4 then $\mu(X) = 0$.

The spherical dodecahedral space is a classical example of the Poincaré manifolds (see e.g. [S -T] p.218). The μ-invariant of this space, as we shall show later, is $1/2 \neq 0$ and hence it is not embeddable in Euclidean 4-space.

2) The totality of $\mathbb{Z}_2$-homology 3-shperes, γ_3, is apparently closed under the connected sum operation; i.e., if X_1 and X_2 are elements in γ_3, then the connected sum, $X_1 \# X_2$, is again in γ_3. In fact, γ_3 endowed with the operation, $\#$, is a commutative monoid with identity. The invariant, μ, may be thought of as a map

$$\mu : \gamma_3 \to \mathbb{Q}/\mathbb{Z} .$$

One proves easily the following

THEOREM (3.19) The map, $\mu : \gamma_3 \to \mathbb{Q}/\mathbb{Z}$, is a momoid-homomorphism (with respect the operation $\#$ in γ_3 and the additive operation in $\mathbb{Q}/\mathbb{Z}$) which maps the identity into the identity.

The operation, $\#$, is compatible with the h-cobordant relation in γ_3 (see Milnor [2], Lemma 2.3) . Therefore μ is a homomorphism of the monoid of h-cobordant classes in γ_3 into $\mathbb{Q}/\mathbb{Z}$.

4. Plumbing.

A fibre bundle, η , having total space, E , projection, p , and base space, B , will be denoted by $\eta = (E,p,B)$. We shall consider mainly bundles with base space $B = S^2$. The principal $SO(2)$-bundles over S^2 are classified by $\pi_1(SO(2)) \approx \mathbb{Z}$, with the Hopf fibration corresponding to a generator in $\mathbb{Z}$, (see e.g. [S] p. 99 and p. 105; here and in the sequel we shall identify $SO(2)$ with S^1) . Therefore, if fixed orientations of $SO(2)$ and S^2 are chosen, every principal $SO(2)$-bundle over S^2 corresponds to a unique integer, known as its <u>characteristic</u> <u>number</u>. We recall briefly these notions. Let $\eta = (E,p,S^2)$ be a given principal $SO(2)$-bundle, where S^2 and $SO(2)$ are oriented. Choose a base point $x_0 \in S^2$ and a neighborhood U of x_0, homeomorphic to an open 2-disk. Then the frontier, $\dot{U}$, of U is an 1-sphere. The restricted bundles $E \mid (S^2 \setminus U)$ and $E \mid \overline{U}$ are both trivial ([S] , p. 53). Let $t : E \mid \overline{U} \to SO(2)$ be a trivialization and let $\sigma : (S^2 \setminus U) \to E$ be a (partial) cross-section. The homotopy class of $t\,\sigma \mid \dot{U} : \dot{U} \to SO(2)$ defines an element in $\pi_1(SO(2)) = \mathbb{Z}$ (given by the degree of the map $t\,\sigma \mid \dot{U}$) which depends only on the bundle η and the orientations chosen. This integer is then called the characteristic number of the bundle.

We now make the orientation convention so that the Hopf fibration has characteristic number -1 , and we shall denote this bundle by $\xi_{-1} = (X_{-1}, p_{-1}, S^2)$. The Hopf (fibre-) map, $p = p_{-1}$, may be described as follows, ([S] , p. 105). Write $S^3 = \{(z_1,z_2) \in ¢^2 \mid z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$ and represent S^2 by the complex projective line in which each point is given by an equivalence class $[z_1,z_2]$, $z_1 \cdot z_2 \neq 0$; (the equivalence relation being $(z_1,z_2) \sim \lambda(z_1,z_2)$, $\lambda \neq 0$) . We then define $p : S^3 \to S^2$ by $p(z_1,z_2) = [z_1, z_2]$. For any $[z_1,z_2] \in S^2$, we may find a $(z_1',z_2') \in S^3$ (by normalization) such that $[z_1',z_2'] = [z_1,z_2]$. The fibre $p^{-1}[z_1,z_2]$ is clearly the circle $e^{i\theta}(z_1',z_2')$, $0 \leq \theta \leq 2\pi$. It follows that the action of $\mathbb{Z}_n$ on S^3 defined in (3.9), is compatible with the fibration, p , if $q = 1$. We recall that in this case (i.e. $q = 1$), the action is given by

$$\gamma(z_1,z_2) = (e^{\frac{2\pi i \gamma}{n}} \cdot z_1, e^{\frac{2\pi i \gamma}{n}} \cdot z_2) , \quad \gamma \in \mathbb{Z}_n .$$

Passing to the quotient structure, we obtain a bundle $\xi_{-n} = (X_{-n}, p_{-n}, S^2)$, where $X_{-n} = S^3/\mathbb{Z}_n = L(n,1)$. Diagrammatically,

$$\begin{tikzcd} S^3 \arrow[rr, "f"] \arrow[dr] & & L(n,1) \arrow[dl, "p_{-n}"] \\ & S^2 & \end{tikzcd}$$

where f is an n-fold covering map. We also know that $\pi_1(L(n,1)) = \mathbb{Z}_n$. Comparing f with the classification theorem of $SO(2)$-bundles cited above, we see that f induces a homomorphism $\pi_1(SO(2)) \xrightarrow{n} \pi_1(SO(2))$, which carries the characterisitic number of ξ_{-1} to that of ξ_{-n} . This shows that the characteristic number of ξ_{-n} is indeed $-n$, and in this way ξ_m may be constructed for each $m \in \mathbb{Z}$.

Let D^2 denote the standard 2-disk and let $SO(2)$ operate on D^2 in the obvious fashion. We may consider the 2-disk bundle $\eta_n = (Y_n, p_n, S^2)$ associated with ξ_n , (where $p_n : Y_n \to S^2$ is the obvious extension of $p_n : X_n \to S^2$) . Then Y_n is a 4-manifold having $X_n = -L(n,1)$ as boundary. The quadratic form of Y_n can be determined as follows. First observe that $H_2(Y_n) = H_2(S^2) = \mathbb{Z}$, where the zero cross-section, $\nu : S^2 \to Y_n$, represents the "positive" generator $g \in H_2(Y_n)$. To determine S_Y it suffices to find $S_Y(g,g) = g \circ g$, the self-intersection number of the class g . Take another cross-section $\sigma : S^2 \to Y_n$, also representing g . Since $\eta_n \mid (S^2 \setminus x_0)$ is a trivial bundle, we may assume that $\sigma(x) \neq \nu(x)$, if $x \neq x_0$ in S^2 . Now let U be a neighborhood of x_0 as before and let $u : Y_n \widetilde{U} \to D^2$ be a trivialization. The degree of the map $u\sigma \mid (\bar{U} \setminus x_0) : U \setminus x_0 \to D^2 \setminus 0$ is, by definition, the intersection number of $\sigma|\bar{U}$ and $\nu|\bar{U}$ and, since σ and ν do not intersect elsewhere, this is also the intersection number of σ and ν (which is $g \circ g$) . Identifying $\pi_1(\dot{U})$ with $\pi_1(U \setminus x_0)$ and $\pi_1(SO(2))$ with $\pi_1(D^2 \setminus 0)$, we see that $g \circ g = S_Y(g,g)$ is exactly the characteristic number, n , of the bundle ξ_n .

The manifolds X_n and Y_n , $n \in \mathbb{Z}$, are the basic objects for "plumbing" which we shall soon define.

REMARK (4.1) Consider the bundle $\xi_m = (X_m, p_m, S^2)$. Let W be a submanifold of S^2 such that the restricted bundle $\xi_m \mid W = (X_W, p_m, S^2)$ is trivial, where $X_W = p_m^{-1}(W)$ and W is oriented in accordance with S^2 . Let (x) and (y) be given coordinates in W and S^1 (the typical fibre) respectively, compatible with the orientations. Then (x,y) may be chosen as coordinates in X_W so that the product structure and orientation of X_W are preserved. Such coordinates in X_W will be called <u>admissible</u>.

A point in D^2 may be expressed, in polar coordinates by a pair (r,x), where $0 \leq r \leq 1$ and $x \in R/\mathbb{Z} = R_1$, the real numbers modulo 1. If $r = 1$, we write $(1,x) = x$; thus a point in S^1 is given by $x \in R_1$. Let U be a 2-disk in S^2 and let $V = Cl(S^2 \setminus U)$, which is also a 2-disk. Then both $\xi_m | U$ and $\xi_m | V$ are trivial. Let (ρ, ξ, η) and (r, x, y) be admissible coordinates in X_U and X_V, $(x, y, \xi, \eta \in R_1)$. The boundaries ∂X_U and ∂X_V (both homeomorphic to $S^1 \times S^1$) have admissible coordinates (ξ, η) and (x, y) respectively. The fact that the characteristic number of ξ_m is m shows that if $f : \partial X_V \to \partial X_U$ is a map defined by $f(x,y) = (\xi, \eta) = (-x, y-mx)$, then the adjunction space $X_V \cup_f X_U$ is precisely X_m. The map, f, may also be expressed by the matrix

$$(4.2) \qquad \begin{pmatrix} -1 & 0 \\ -m & 1 \end{pmatrix} .$$

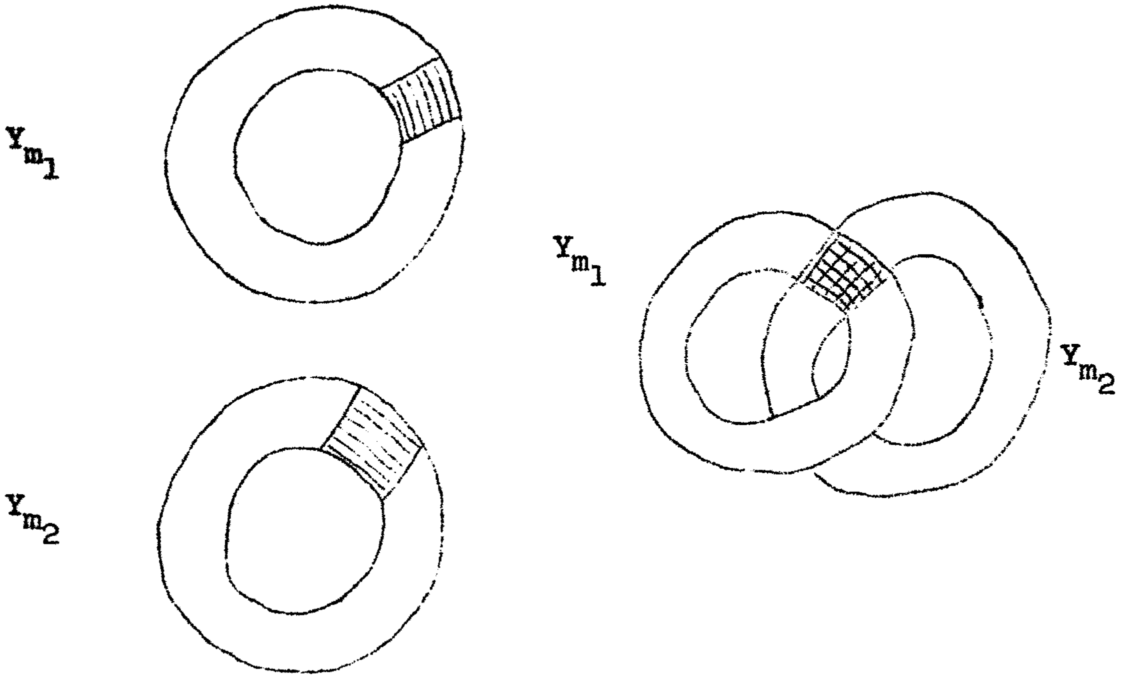
By a <u>graph</u> we shall mean a finite one-dimensional simplicial complex. A contractible graph will be termed a <u>tree</u>. Thus trees are connected. If two vertices v_i and v_j of a tree are joined by an edge (= 1-simplex we shall denote this (unique) edge by e_{ij}. Let $\mathfrak{M}$ be a set. A tree, T, is said to be <u>weighted by the elements of</u> $\mathfrak{M}$ if to each vertex, v_i, of T an element, m_i (to be called the <u>weight</u> of v_i), of $\mathfrak{M}$ is attached. If, in particular, $\mathfrak{M} = \mathbb{Z}$, we shall simply say that T is <u>weighted</u> (by integers). A weighted tree will be denoted by $T = (T, m_i)_{1 \leq i \leq s}$; its weighted vertices, by $(v_1, m_1), \ldots, (v_s, m_s)$. The adjective "weighted" will often be dropped if no confusion is likely to arise.

A plumbing operator, $P = P_k$, assigns to each tree $T = (T, m_i)$, weighted by the elements of $\pi_{2n-1}(SO(2n))$, a 4k-dimensional manifold $P(T)$. Here we restrict ourselves to the case $k = 1$ and we identify $\pi_1(SO(2))$ with $\mathbb{Z}$. We hope to expound the more general cases later.

1) If the given tree, T, contains a single weighted vertex (v,m), we define $P(T)$ to be Y_m, the total space of the bundle η_m defined above.

2) Let T be a tree consisting of two vertices, (v_1,m_1), (v_2,m_2) and an edge e_{12}. For each (v_i,m_i) we take a copy of $\eta_{m_i} = (Y_{m_i}, p_{m_i}, S^2)$, $i = 1, 2$. Choose a 2-disk, D_{ij}, in the base space of η_{m_i} and let $p_{m_i}^{-1}(D_{ij}) = Y_{ij}$. Since $\eta_{m_i} | D_{ij}$ is trivial, there is a homeomorphism $u_{ij} : D^2 \times D^2 \to Y_{ij}$ whose first component gives base coordinates and the second, the fibre coordinates. Let $t : D^2 \times D^2 \to D^2 \times D^2$ be the reflection defined by $t(x,y) = (y,x)$. Then there is a homeomorphism $f_{ji} : Y_{ji} \to Y_{ij}$ given by $f_{ji} = u_{ij}\, t u_{ji}^{-1}$. In our case, we have $f_{21} : Y_{21} \to Y_{12}$. Notice that $Y_{ij} \subset Y_{m_i}$ $(i,j = 1,2)$ we may paste

figure 1



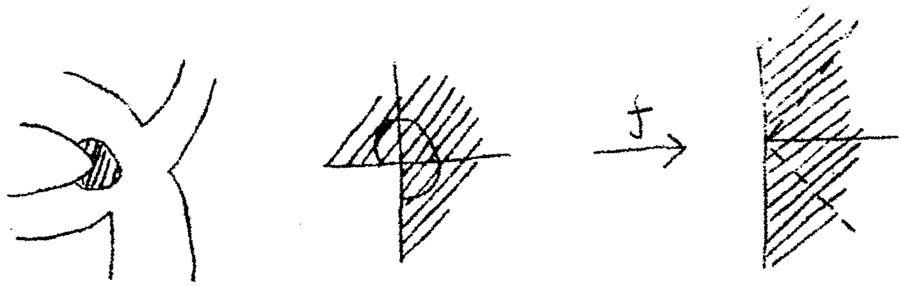
Y_{m_2} and Y_{m_1} together along Y_{21} and Y_{12} by means of f_{21} to obtain a topological 4-manifold $\widetilde{\mathcal{P}}(T)$. Observe that f_{ji} fails to be a diffeomorphism only along the "corner" $u_{ji}(\partial D^2 \times \partial D^2)$, which is a submanifold. According to Milnor [2], we may smooth $\widetilde{\mathcal{P}}(T)$ by straightening the corner to obtain a (differentiable) manifold, $\mathcal{P}(T)$, and the smoothing is essentially unique. However, the definition of $\widetilde{\mathcal{P}}(T)$ already involves arbitrary choices. It is fairly clear that $\mathcal{P}(T)$ is independent (up to a diffeomorphism) of the choice of u_{ij} , provided it is admissible. A theorem of R. Thom (Milnor [5]) assures, in particular, that $P(T)$ is independent of the choices of D_{ij} .

3) The general pattern of plumbing is now clear. Let $T = (T, m_i)_{1 \leq i \leq s}$ be a tree. For each (v_i, m_i) we take a copy of η_{m_i} . Suppose that v_i is connected to $v_j, v_k, \ldots$ by edges $e_{ij}, e_{ik}, \ldots$ in T . We take, in the base space of η_{m_i} , 2-disks $D_{ij}, D_{ik}, \ldots$ which are pairwise disjoint and construct homeomorphisms $f_{ji}, f_{ki}, \ldots$ as in 2). We then paste Y_{m_i} and Y_{m_j}, etc., together according to f_{ji}, etc., and finally we smooth the resulting topological manifold to obtain $P(T)$.

While the smoothing process has been described loosely we feel that it may be helpful to give an intuitive picture about straightening the corners (see Milnor [2]). Let R_+ denote the positive half-line of R . A point, x , at the corner, i.e., $x \in u_{ji}(\partial D^2 \times \partial D^2) = u_{ij}(\partial D^2 \times \partial D^2)$, has a neighbourhood which looks like $(\partial D^2 \times \partial D^2) \times (R_+ \times R \cup R \times R_+)$. The second factor may be "straightened" by the transformation

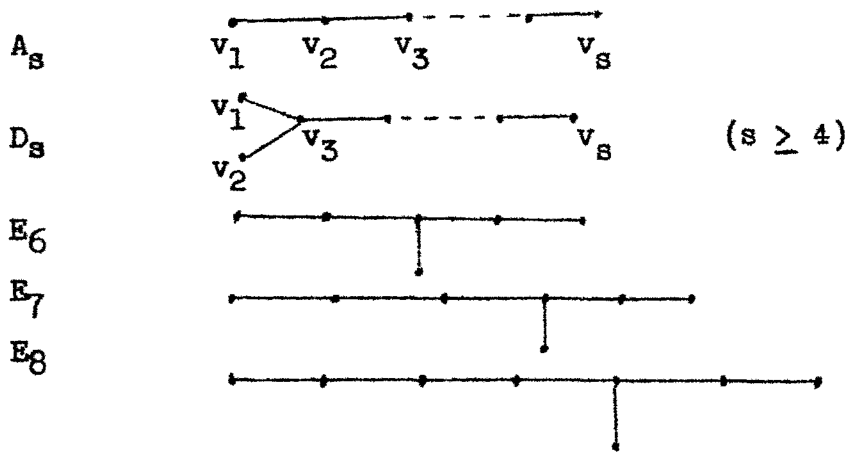
$$(r\cos\theta\ ,\ r\sin\theta) \xrightarrow{f} (r\cos\frac{2\theta+\pi}{3},\ r\sin\frac{2\theta+\pi}{3})$$

which is defferentiable except at $r = 0$.



Since Y_m has the homotopy type of a 2-sphere, it follows that, for each tree $T = (T, m_i)_{1 \leq i \leq s}$, the 4-manifold $P(T)$ has the same homotopy type as the one point union of 2-spheres. Thus $P(T)$ is simply connected and $H_2(P(T); \mathbb{Z})$ is a lattice of rank s. To a weighted tree $T = (T, m_i)_{1 \leq i \leq s}$ we may associate a quadratic form of rank s whose matrix $M = (\mu_{ij})$ is given in the same fashion as that described in p. 1-9 except $\mu_{ii} = m_i$ instead of $\mu_{ii} = 2$. Thus the quadratic form given in p. 1-9 is the form associated with (E_8, m_i) where $m_i = 2$ for $i = 1, \ldots, 8$. It is then clear that the quadratic form of $P(T)$ is the same as the quadratic form associated with T. Observe that the plumbing operator, P, is defined on all (finite) graphs; but in case the graph is not a tree, the resulting manifold will not be simply connected. Plumbing operation may be generalized by allowing other bundles to play the role of η_m.

As examples we mention the following trees which arise in the classification theory of simple Lie algebras (see e.g. Séminaire Sophus Lie, 1e année, exp. 13). These are the only trees, when weighted by 2, whose quadratic forms are positive definite.



In the following table, each vertex of the tree, T, is weighted by 2 :

T	$\mu(\partial P(T))$	$\pi_1(\partial P(T)) = F_T$	$H_1(\partial P(T);\mathbb{Z})$
A_s	$s/16$ (s odd)	C'_{s+1}	$\mathbb{Z}_{s+1}$
D_s		D''_{s-2}	$\mathbb{Z}_2 + \mathbb{Z}_2$, s even $\mathbb{Z}_4$, s odd .
E_6	6/16	T'	$\mathbb{Z}_3$
E_7		W'	$\mathbb{Z}_2$
E_8	8/16	P'	0

The groups, F_T , are the only finite subgroups of S^3 (= unit quaterions). This can be seen as follows. The finite subgroups of $SO(3)$ are known to be C_n, D_n', T, W and P (the cyclic, the dihedral, the tetrahedral, the octahedral and pentagondodecahedral groups). Regarding S^3 as a 2-fold covering group of $SO(3)$, one deduces easily that the only finite subgroups of S^3 are the cyclic groups, C_s' , and the binary groups D_s'' , T', W' and P' , obtained by lifting the groups D_s' , T, W and P of $SO(3)$.

The fundamental groups $\pi_1(\partial P(T))$ and the homology groups $H_1(\partial P(T))$ can be computed directly by finding out explicitly the generators and relations; however these groups may be obtained much easier from a more general result due to von Randow (v. Randow : Thesis, Bonn, in preparation) which we shall describe briefly. A 3-manifold will be termed a <u>Seifert manifold</u> if it can be fibered, with exceptional fibres, in S^1 over S^2 (Seifert [1]). To such a manifold, X , Seifert associated a system of integers

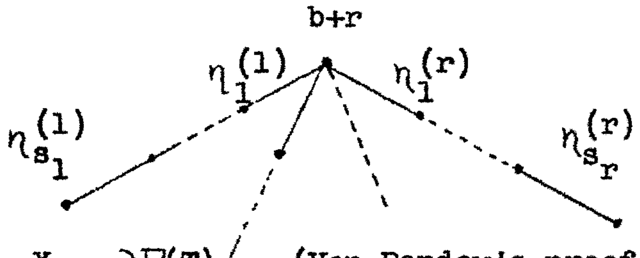
$$(b; (\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r)) ,$$

known as Seifert invariants, where r = number of exceptional fibres and the integers α_i,β_i satisfying the relations $0 < \beta_i < \alpha_i$ and $(\beta_i,\alpha_i) = 1$, $(i = 1,\ldots,r)$.

Expand $\alpha_i/(\alpha_i - \beta_i)$ into continued fraction :

$$\frac{\alpha_i}{\alpha_i-\beta_i} = [\eta_1^{(i)}, \ldots, \eta_{s_i}^{(i)}] , \quad |\eta_j^{(i)}| \geq 2 ,$$

and let T be the star-shaped tree (weighted by $\eta_j^{(i)}$'s) :



Then $X = \partial P(T)$. (Von Randow's proof of this fact is based on a special construction of lens spaces via plumbing, which we shall study later in this section. Notice that lens spaces are special Seifert manifolds.) It is known that for any finite subgroup, F, of S^3 the coset space, S^3/F, is a Seifert manifold. Indeed, each manifold, $\partial P(T)$, listed in our table is diffeomorphic to S^3/F_T. For instance, the spherical dodecahedral space, S^3/P', has Seifert invariants $(-1; (5,1), (3,1), (2,1))$. Here $r = 3$ and

$b + r = 2$

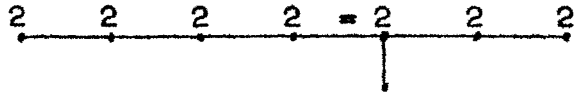
$5/(5-1) = 5/4 = [2,2,2,2]$,

$3/(3-1) = 3/2 = [2,2]$,

$2/(2-1) = 2/1 = 2$.

From what we discussed above, we get a star-shaped tree

b + r



but this is E_8 with each vertex weighted by 2 . In other words $\partial\mathcal{P}(E_8)$ is the sperical dodecahedral space if E_8 is weighted by 2 ; therefore $\pi_1(\partial\mathcal{P}(E_8)) = P'$ and $H_1(\partial\mathcal{P}(E_8)) = 0$. The other lines in the table may be justified in a similar fashion.

Notice that when $T = A_s$ (s odd), $T = E_6$ or $T = E_8$, the manifold $\partial\mathcal{P}(T)$ is a $\mathbb{Z}_2$-homology sphere and consequently the μ-invariant is defined (p. 3-13). The values of $\mu(\mathcal{P}(T))$ are obvious, since the quadratic form associated with each tree T listed above is positive definite and the quadratic form of $\mathcal{P}(T)$ is the same as that of T . In particular the μ-invariant of the spherical dodecahedral space is $8/16 = 1/2$ as we claimed toward the end of §3 .

As we have seen that in many cases we are interested in the manifold $\partial\mathcal{P}(T)$ rather than $\mathcal{P}(T)$ itself. The assignment, $\partial\mathcal{P} : T \to \partial\mathcal{P}(T)$, will also be called <u>plumbing</u>; it may be defined directly without the help of the operator $\mathcal{P}$. We describe this in three steps:

1') If $T = (v,m)$, we define $\partial P(T) = \xi_m$.

2') Let T be a tree consisting of (v_1,m_1), (v_2,m_2) and e_{12}. Take D_{12} and D_{21} in the base spaces as we did in 2). Let $X_{ij} = p_{m_i}^{-1}(D_{ij})$ and let $u_{ij} : D^2 \times S^1 \to X_{ij}$ be an admissible coordinate function $(i,j = 1,2)$. Then ∂X_{ij} is equivalent to $S^1 \times S^1$. Define $f_{21} : \partial X_{21} \to \partial X_{12}$ by putting $f = u_1 t u_2^{-1}$, where t is defined as in 2). Let $X'_{ij} = p_{m_i}^{-1}(S^2 \setminus \operatorname{Int} D_{ij})$. Then $\partial X'_{ij} = \partial X_{ij}$ $(i,j = 1,2)$. Finally we put

$$\partial \tilde{P}(T) = X'_{21} \cup_{f_{21}} X'_{12} ,$$

and then smooth it to obtain $\partial P(T)$.

3') The general procedure is now clear.

REMARK (4.3) The map, f_{21}, may be expressed in terms of the u_{ij}-coordinates by a matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{4.4}$$

In view of remark (4.1), $\partial P(T)$ of 2') may be obtained from X'_{12} by attaching a copy of D^2 to its boundary, $\partial X'_{12}$, according to a map f_{21} given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -m & 1 \end{pmatrix} = \begin{pmatrix} -m & 1 \\ -1 & 0 \end{pmatrix} . \tag{4.5}$$

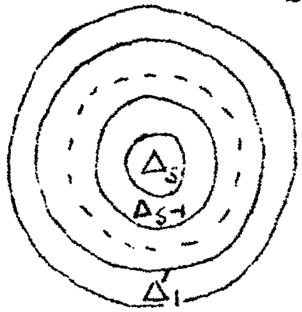
Consider the weighted tree $(A_s, m_i)_{1 \leq i \leq s}$:

$$\underset{\bullet}{m_1} \!-\!\!-\!\!-\!\! \underset{\bullet}{m_2} \!-\!\!-\!\!-\!\! \underset{\bullet}{m_3} \; - - - - - \; \underset{\bullet}{m_{s-1}} \!-\!\!-\!\!-\!\! \underset{\bullet}{m_s} \quad ,$$

where m_i's are integers. We propose to study $\partial P(A_s)$. Take a copy of ξ_{m_i} for each (v_i, m_i) in A_s and let the base space of ξ_{m_i} be denoted by S_i^2 $(i = 1, \ldots, s)$. In each S_i^2 (represented by extended complex numbers) take $D_{i,i+1}$ and $D_{i,i-1}$ centered at 0 and ∞ respectively. Let $\Delta_i =$ $S_i^2 \setminus D_{i,i+1} \setminus D_{i,i-1}$ $(i = 2, \ldots, s-1)$, $\Delta_1 = S_1^2 \setminus D_{12}$ and $\Delta_s = S_s^2 \setminus D_{s-1,s}$. We may consider Δ_1 as obtained from the space $\Delta'_1 = D^2 \setminus D_{12}$ by collapsing ∂D^2 to a point ∞ . Let

$$L = \infty \times S^1 \cup_{f_1} \Delta'_1 \times S^1 \cup_{f_2} \Delta_2 \times S^1 \cup_{f_3} \cdots \cup_{f_s} \Delta_s \times S^1 \quad ,$$

where $f_i : \partial D_{i,i-1} \times S^1 \to \partial D_{i-1,i} \times S^1$ $(f_1 : \partial D_{21} \times S^1 \to S^1)$



Parametrize $\partial D_{ij} \times S^1 = S^1 \times S^1$ by a pair of numbers (x,y) in $R/\mathbb{Z}$. From remarks (4.1) and (4.3) it is easily seen that L is homeomorphic to $\partial P(A_s)$ if the maps, f_i, are given by the matrices:

$$f_j : \quad \begin{pmatrix} -m_j & 1 \\ -1 & 0 \end{pmatrix} \qquad (j = 2,\ldots,s)$$

$$f_1 : \quad \begin{pmatrix} -m_1 & 1 \\ 0 & 0 \end{pmatrix} \quad .$$

Furthermore, in this case, L (after smoothing) is diffeomorphic to $\partial P(A_s)$. It is also easy to see that $L \approx S^1 \cup_f D^2 \times S^1$, where $f : \partial D^2 \times S^1 \to S^1$ is described by a matrix

$$(4.6) \quad \begin{pmatrix} n & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -m_1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -m_2 & 1 \\ -1 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} -m_s & 1 \\ -1 & 0 \end{pmatrix} .$$

Observe that

$$(4.6a) \quad \begin{pmatrix} -m_1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -m_2 & 1 \\ -1 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} -m_s & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} n & q \\ -q' & p \end{pmatrix} ,$$

where the integers n and q are the same as those in (4.6). Notice also that each factor in the left-hand side has determinant $= +1$, hence the determinant of the matrice $\begin{pmatrix} n & q \\ -q' & p \end{pmatrix}$ in the right-hand side is $+1$. This shows

$$(4.7) \qquad np + qq' = 1 ,$$

and it follows that n and q are co-primes.

Recall that a well-known description of the lens space $L(n,q)$ is given by

$$L(n,q) = S^1 \cup_g D^2 \times S^1 ,$$

where $g : \partial D^2 \times S^1 = S^1 \times S^1 \longrightarrow S^1$ maps a point (x,y) into $nx + qy$ $(x,y \in \mathbb{R}/\mathbb{Z})$. In other words, the matrix of g is exactly

$$\begin{bmatrix} n & q \\ 0 & 0 \end{bmatrix}$$

We have

THEOREM (4.8) Let $A_s = (A_s, m_i)_{1 \leq i \leq s}$. Then $\partial P(A_s) = L(n,q)$, where n and q are defined as in (4.6).

If we reverse the order of the tree (A_s, m_i) and plumb according to the reversed tree we get $L(n,q')$, where q' is given in (A.6a). From (4.7) it follows that $qq' = 1 \pmod n$. Our result is therefore compatible with the classical theorem concerning the lens spaces (see e.g. [S-T] p. 215 Satz II).

For given co-primes n and q, $0 \leq q \leq n$, we can find a tree $(A_s, m_i)_{1 \leq i \leq s}$ such that $\partial P(A_s) = L(n,q)$. To see this we let $\lambda_0 = n$, $\lambda_1 = q$, and use Euclidean algorithm to get

$$\begin{array}{llll}
 & \lambda_0 = a_1\lambda_1 - \lambda_2 , & 0 \leq \lambda_2 < \lambda_1 , & a_1 > 1 , \\
 & \lambda_1 = a_2\lambda_2 - \lambda_3 , & 0 \leq \lambda_3 < \lambda_2 , & a_2 > 1 , \\
(4.9) & \vdots & \vdots & \vdots \\
 & \lambda_{s-1} = a_s\lambda_s - \lambda_{s+1} , & \lambda_{s+1} = 0, \quad \lambda_s = 1 , & a_s > 1 .
\end{array}$$

This is equivalent to saying that n/q can be expanded into a continued fraction $[a_1,a_2,\ldots,a_s]$, i.e.

$$n/q = a_1 - \cfrac{1}{a_2 - \ddots - \cfrac{1}{a_s}}$$

It is clear from (4.9) that

$$\begin{pmatrix}\lambda_s & \lambda_{s+1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_s & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}a_{s-1} & 1\\ -1 & 0\end{pmatrix}\cdots\cdots\begin{pmatrix}a_1 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}n & q\\ * & *\end{pmatrix}.$$

Since $\begin{pmatrix}\lambda_s & \lambda_{s+1}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is the identity matrix, we deduce that

$$\begin{pmatrix}a_s & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}a_{s-1} & 1\\ -1 & 0\end{pmatrix}\cdots\cdots\begin{pmatrix}a_1 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}n & q\\ 0 & 0\end{pmatrix}$$

Comparing this with (4.6) we conclude that if $A_s = (A_s,m_i)_{1\le i\le s}$ is a tree with weights $m_i = -a_{s-i+1}$ $(1\le i\le s)$, then $\partial P(A_s) = L(n,q)$.

The lens space, $L(n,q)$ obtained in the preceding construction bounds a 4-manifold, $P(A_s)$, whose quadratic form is the same as that of (A_s,m_i) with $m_i = -a_{s-i+1}$. As the integers a_i may not be even, this quadratic form does not lend itself to the computation of $\mu(L(n,q))$ in case the latter is defined (cf. p. 3-1). This situation can be remedied. According to §3, the μ-invariant, $\mu(L(n,q))$, is only defined for odd n and in this case we may, without loss of generality, assume that q is even. For such n and q, (4.9) may be modified to yield:

$$(4.9a)\quad \begin{cases} \lambda_0 = n\ ,\ \lambda_1 = q \\ \lambda_0 = b_1\lambda_1 - \lambda_2\ , \quad |\lambda_2|<|\lambda_1| \quad |b_1| > 0\ , \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ \lambda_{s-1} = b_s\lambda_s - \lambda_{s+1}\ ,\ \lambda_{s+1} = 0, |\lambda_s| = 1\ , \quad |b_s| > 0\ , \end{cases}$$

where each b_i is even. This, by the way, proves an assertion announced in p. 3-7 . Let now $A_s = (A_s,n_i)_{1\leq i\leq s}$ have weights $n_i = -b_{s-i+1}$. It is easy to show that $\partial P(A_s) = L(n,q)$ in spite of the fact that λ_s may take the value -1 . In this construction, $L(n,q)$ bounds a 4-manifold, $Y = P(A_s)$, whose quadratic form, S_Y , is even. This proves (3.10). With respect to an obvious base $(e_i)_{1\leq i\leq s}$ in $H_2(Y;\mathbb{Z})$, S_Y is represented by the matrix

$$\begin{pmatrix} -b_1 & 1 & & 0 \\ 1 & -b_2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -b_s \end{pmatrix} .$$

Let M denote the matrix

$$(4.10)\qquad M = \begin{pmatrix} b_1 & 1 & & 0 \\ 1 & b_2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & b_s \end{pmatrix} .$$

Then, relative to the base $((-1)^{i+1} e_i)_{1 \leq i \leq s}$ in $H_2(Y;\mathbb{Z})$, S_Y is given by $-M$. To prove the recipe (3.12), it remains to show that $\tau(M) = p^+ - p^-$, where p^+ (resp. p^-) is the number of positive (resp. negative) diagonal entries of M. In fact, we shall prove the following lemma.

LEMMA (4.11) Let M be a matrix of the form described in (4.10) with $|b_i| > 1$. Then $\tau(M) = p^+ - p^-$.

It can be shown by an easy induction that M is congruent over $\mathbb{R}$ to a diagonal matrix diag $(c_1,\dots,c_s)$, where

$$(4.12) \qquad c_i = b_i - \cfrac{1}{b_{i+1} - \cdots - \cfrac{1}{b_s}} = [b_{i+1},\dots,b_s] .$$

If $|b_i| > 1$, then $\operatorname{sgn} c_i = \operatorname{sgn} b_i$. This obviously proves (4.11), and hence (3.12).

We end this section by a digression. Let n be an odd integer and let q be an integer prime to n. Suppose that $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t}$ is the canonical decomposition of n (into prime factors). The Jacobi symbol, $(q|n)$, may be defined through the Legendre symbol, $(q|p_i)$, by the formula

$$(q|n) = (q|p_1)^{\beta 1} \dots (q|p_t)^{\beta t} .$$

It is clear that, if q and q' are integers such that qq' is a quadratic residue modulo n (i.e. $qq' = x^2$ has a solution in $\mathbb{Z}_n$), then

(4.13) $$(q|n) = (q'|n) .$$

From Theorem (3.13), we see that $(q|n)$ is an orientation-preserving homotopy type invariant on the category of lens spaces $L(n,q)$ with odd n . Notice that the h-cobordant invariant $\mu L(n,q)$ is also defined for such spaces. We have

(4.14) $$(q|n) = (-1)^{4\mu L(n,q) + (n-1)/4} ;$$

that is the Jacobi symbol, $(q|n)$, for a fixed n is completely determined by $\mu L(n,q)$.

To prove (4.14) we first observe that, by using the identities

(4.15) $$(-1|n) = (-1)^{(n-1)/2}$$

and $\mu L(n,q) = - \mu L(n,n-q)$, we can restrict ourselves to the case where q is even. Expand n/q into continued fraction : $n = [b_1, b_2, \ldots, b_s]$ according to (4.9a), i.e., $|b_i| > 1$ and b_i even for $i = 1, 2, \ldots, s$. We have shown that $L(n,q)$ bounds a 4-manifold $P(A_s) = Y$ whose quadratic form, S_Y , may be described as follows. Let $S = -S_Y$; then the matrix of S is given by (4.10) . In other words,

$$(4.16) \qquad S = \sum_{i=1}^{s} b_i\, x_i^{\,2} + 2 \sum_{i=1}^{s} x_i\, x_{i+1}$$

$$\sim \sum_{i=1}^{s} c_i\, y_i^{\,2}$$

where the rational numbers, c_i , are given by (4.12). Therefore $\mu L(n,q) = -\tau(S_Y)/16 = \tau(S)/16 = (p^+ - p^-)/16$ (this number being understood to be reduced modulo 1) . We now compute the Hasse-Minkowski symbol $c_p(S)$ for all odd primes p .

<u>Case I</u> . $p \nmid n$.

From (3.17) we see that $\det S = \pm n$. Our assumption then implies that $p \nmid \det S$ and hence $c_p(S) = 1$, by (0.24) .

<u>Case II</u> . $p \mid n$.

In this case p is a prime factor of n , say $p = p_1$ and $n = p_1^{\beta_1} \cdots p_t^{\beta_t}$. Since $(q,n) = 1$, we get $p \nmid q$. Write (4.16) in the form

$$S = \frac{n}{q} y_1^{\,2} \dotplus \sum_{i=2}^{s} c_i y_i^{\,2} = S_1 \dotplus S_2 ,$$

where $S_1 = \frac{n}{q}\, y_1^{\,2} = c_1\, y_1^{\,2}$ and $S_2 = \sum_{i=2}^{s} c_i\, y_i^{\,2}$.

It follows that $\det S_2 = \pm q$ and, since $p \nmid q$, $c_p(S_2) = 1$.

Now S_1 is an unary form $c_p(S_1) = 1$. Therefore

$$c_p(S) = c_p(S_1)\, c_p\,(S_2)(\det S_1, \det S_2)_p =$$

$$\left(\frac{n}{q},\ q \cdot \operatorname{sgn} \det S_2\right)_p .$$

Since $p = p_i$ and $(n,q) = 1$, we may write $n/q = p^{\beta_i} \alpha_i$ and apply the Property 5) of Hilbert symbol (see p. 0-14) to obtain

$$\left(\frac{n}{q}, q \cdot \operatorname{sgn} \det S_2\right) = (q \cdot \operatorname{sgn} \det S_2 \,|\, p)^{\beta_i}$$
$$= (q|p)^{\beta_i} (\operatorname{sgn} \det S_2 |p)^{\beta_i} .$$

This, together with the conclusion from Case I, shows that

$$\prod_{\text{odd prime}} c_p(S) = \prod_{i=1}^{t} (q|p_i)^{\beta_i} (\operatorname{sgn} \det S_2 | p_i)^{\beta_i}$$

$$= (q|n)(\operatorname{sgn} \det S_2 |n) .$$

We may assume that $n/q > 0$; therefore $\operatorname{sgn} \det S_2 = \operatorname{sgn} \det S$. Applying Lemma (0.25) we get

(4.17) $\quad c_2(S)c_\infty(S) = (q|n)(\operatorname{sgn} \det S|n)$.

Since S is even, Theorem (1.9) gives

(4.18) $\quad c_2(S)\, c_\infty(S) = (-1)^{(\tau + \det S - \operatorname{sgn} \det S)/4}$.

combining (4.15), (4.17) and (4.18) we obtain (4.14) .

The lecturer is indebted to T.W.S. Cassels for a very helpful letter containing essentially the preceding proof. See also a forthcoming paper of Cassels in the Comm. Math. Helv..

5. "Riemann Surfaces" of Complex Dimension 2.

Let $M = M^{(n)}$ be a complex manifold of complex dimension n, not necessarily compact. (Complex dimensionality will be expressed by a super-script in parentheses.) A complex analytic subset of M is said to be of <u>maximal dimension</u> if it has complex co-dimension 1. We recall that such a subset, N, is given locally by the set of zeros of a holomorphic function. More precisely, if p is a point in N then there is an open neighborhood, U, of p in M and a holomorphic function, f, defined on U (not identically zero on any connected-component of U) such that a point q of U belongs to $N \cap U$ if, and only if, $f(q) = 0$. The point $p \in N$ is termed <u>regular</u> if the above mentioned function, f, may be chosen as a coordinate function about p; otherwise p is <u>singular</u>.

Let $\{f_i\}_{i \in I}$ be an indexed family of meromorphic function in which each function, f_i, is defined on an open subset, U_i, of M, $(i \in I)$ such that

a) $\{U_i\}_{i \in I}$ forms an open covering of M,

b) f_i is not identically zero on any connected-component of U_i, and

c) f_i/f_j is holomorphic and $\neq 0$ in $U_i \cap U_j$.

As we know, such a family, $\{f_i\}$, of functions defines a divisor, D, of M. We shall express this situation in symbol by $D \sim \{f_i\}_{i \in I}$. In virtue of the condition c), the locus of zeros and poles of a divisor is well-defined. This locus, together with multiplicities of its irreducible components, will often be identified with the divisor itself. If, in particular, each function in the family, $\{f_i\}$, is holomorphic, the corresponding divisor, $D \sim \{f_i\}$, is termed <u>non-negative</u>. A non-negative divisor, D, is called <u>positive</u> if its locus of zeros, N_D, is non-empty; in such case, N_D is a complex analytic subset of maximal dimension. Perhaps the simplest non-negative divisor is one given by a single, globally defined holomorphic function, i.e., $D \sim \{f\}$. For such a divisor, we shall simply write $D = (f)$ and $N_D = |f = 0|$.

To each divisor, $D \sim \{f_i\}_{i \in I}$, there is an associated holomorphic complex line bundle, $[D]$, of M given by the transition functions

$$f_{ij} = f_i/f_j \quad \text{in } U_i \cap U_j .$$

The <u>characteristic class</u>, $c_1(D)$, of D is then defined to be the Chern class, $c_1([D])$, of the bundle $[D]$. If $D = (f)$, then $[D]$ is trivial and hence $c_1(D) = 0$.

Let D be a positive divisor of M and let N_D be its locus of zeros. Each irreducible component of N_D has a fundamental homology class in the sense of Borel-Haefliger [1] . Therefore D , considered as N_D together with its multiplicities, represents a homology class, $h(D)$, in $\mathcal{H}_{2n-2}(M;\mathbb{Z})$, where the script H denotes homology with closed supports. We shall reserve H to denote the ordinary homology. Borel and Haefliger proved that

LEMMA (5.1) If D is a positive divisor then the homology class $h(D)$ in $\mathcal{H}_{2n-2}(M;\mathbb{Z})$ is dual to the characteristic class $c_1(D)$ in $H^2(M;\mathbb{Z})$.

Let V and W be $\mathbb{Z}$ modules. A bilinear form

$$g: V \times W \to \mathbb{Z}$$

may be defined by means of a correlation $\mathcal{G} : V \to W^*$ (cf. p. 0-1), where W^* is the $\mathbb{Z}$-dual of W . Let us denote by $\overline{\mathcal{H}}_i$, $\overline{H}_i$ and $\overline{H}^i$ the appropriate homology and cohomology groups modulo torsion, e.g. $\overline{\mathcal{H}}_i(M;\mathbb{Z}) = \mathcal{H}_i(M;\mathbb{Z})/(\text{torsion})$. For a compact manifold, $M = M^m$, the $\mathbb{Z}$-modules $\overline{H}_{m-1}(M;\mathbb{Z}) = V$, $\overline{H}_1(M;\mathbb{Z}) = W$ are lattices and $W^* = \overline{H}^1(M;\mathbb{Z})$. The Poincaré duality map

$H_{m-1}(M;\mathbb{Z}) \to H^1(M;\mathbb{Z})$ induces a homomorphism $\varphi : V \to W^*$ which, taken as a correlation, defines a bilinear form

$$S : \overline{H}_{m-1}(M;\mathbb{Z}) \times \overline{H}_1(M;\mathbb{Z}) \to \mathbb{Z} .$$

The integer $S(x,y)$ is precisely the intersection number of the homology classes x and y. This notion may be generalized to non-compact manifolds. In fact, if M^m is non-compact, then the $\mathbb{Z}$-modules $\overline{\mathcal{H}}_{m-1}(M;\mathbb{Z}) = V$ and $\overline{H}_1(M;\mathbb{Z})$ are torsion-free and $\overline{H}^1(M;\mathbb{Z}) \subseteq W^*$. The Poincaré duality map $\mathcal{H}_{m-1}(M;\mathbb{Z}) \to H^1(M;\mathbb{Z})$ again induces a correlation $\varphi : V \to W^*$ and hence defines a bilinear form

$$S : \overline{\mathcal{H}}_{m-1}(M;\mathbb{Z}) \times \overline{H}_1(M;\mathbb{Z}) \to \mathbb{Z} .$$

In this fashion the intersection number of a class $x \in \overline{\mathcal{H}}_{m-1}(M;\mathbb{Z})$ and a class $y \in \overline{H}_1(M;\mathbb{Z})$ is defined, i.e. $x \circ y = S(x,y) = y \circ x$. The intersection of cycles can then be defined by means of the homology classes they represent.

Let $D = (f)$ be a divisor of $M^{(n)}$. According to (5.1), $\varphi(h(D)) = c_1(D) = 0$; this proves

COROLLARY (5.2) If $D = (f)$, then the intersection number $h(D) \circ x$ vanishes for each $x \in H_2(M;\mathbb{Z})$.

(Notice that if x is a torsion class, $h(D) \circ x$ is by definition zero.) If we consider D as a cycle in M, then for any cycle, y, with compact support in M, $D \circ y = 0$.

Now, let p be a point in the complex manifold $M = M^{(n)}$. The complex line elements at p in M form a complex projective space $CP^{(n-1)}$. "Replacing" the point p of M by a copy of $CP^{(n-1)}$ we obtain a new complex manifold $\sigma_p M$; i.e., $\sigma_p M = (M \setminus p) \cup CP^{(n-1)}$. The assignment, $M \to \sigma_p M$, is known as σ-process, blowing-up operation or a quadratic transformation. To make the term "replacing" precise, we take a coordinate neighborhood, U, of p in M with local coordinates $z_1, \ldots, z_n$ centered at p (i.e. $p = (0)$). Let $CP^{(n-1)}$ be represented by homogeneous coordinates $w_1, \ldots, w_n$. There is an obvious map, φ, from $U \setminus (0)$ onto $CP^{(n-1)}$. The graph, Γ, of φ in $U \times CP^{(n-1)}$ together with $K_p = (0) \times CP^{(n-1)}$ forms a non-singular analytic subset, N, of $U \times CP^{(n-1)}$. Indeed, if $q = (w_1, \ldots, w_n) \in CP^{(n-1)}$ is such that $w_i \neq 0$, then $(0) \times q$ has a coordinate neighborhood in $U \times CP^{(n-1)}$ with local coordinates $z_1, \ldots, z_n, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$. With respect to these coordinates a point $r = (z_1, \ldots, z_n, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$ belongs to N if, and only if,

$$z_j = z_i w_j \qquad (1 \leq j \leq n, \ j \neq i). \tag{5.3}$$

Thus $w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n$ constitute a system of local coordinates in N about $(0) \times q$. We then identify $U \setminus (0)$ with $\Gamma = N \setminus K_p$ to obtain $\sigma_p M$. It can be shown that $\sigma_p M$ does not depend on the choice of the neighborhood, U, used to define it and depends only on M and the point p.

There is a projection $\pi_p : \sigma_p M \to M$ which maps K_p into p and maps $\sigma_p M \setminus K_p$ homeomorphically onto $M \setminus p$.

We now restrict ourselves to the case where $\dim_{\mathbb{C}} = 2$. Let $p \in M$, having local coordinates z_1, z_2 (centered at p) defined on a neighborhood, U, of p in M. In the complex manifold $\sigma_p M$, these coordinates are replaced by two systems of local coordinates, u, v and $\tilde{u}, \tilde{v}$, such that

$$(5.4) \qquad \begin{cases} z_1 = u \\ z_2 = uv \end{cases} \qquad (z_1 \neq 0), \qquad \begin{cases} z_1 = \tilde{u}\tilde{v} \\ z_2 = \tilde{v} \end{cases} \qquad (z_2 \neq 0),$$

as one can see from (5.3). The subspace K_p of $\sigma_p M$ is obviously given by the equations $u = 0$ and $\tilde{v} = 0$. It is also clear that topologically K_p is a 2-sphere embedded in $\sigma_p(M)$ and hence it represents a cycle with compact support in $\sigma_p(M)$.

LEMMA (5.5) The self-intersection number $K_p \circ K_p$ is -1.

Indeed, without changing of notations, z_1 may be considered as a function defined on $\pi_p^{-1}(U)$. (Strictly speaking, z_1 is a function defined on U only. When we consider z_1 as a function on $\pi_p^{-1}(U)$ we actually mean the function which is expressed in (u,v)-chart by u and in $(\tilde{u},\tilde{v})$-chart by $\tilde{u}\tilde{v}$ (see (5.4)). Similar remarks apply to (5.7).) The divisor $|z_1 = 0|$ in $\pi_p^{-1}(U)$ consists of K_p and $|\tilde{u} = 0|$; in other words $|z_1 = 0| = K_p + |\tilde{u} = 0|$. Since the intersection number $K_p \circ |\tilde{u} = 0|$ is obviously $+1$. Applying (5.2) to the open manifold $\pi_p^{-1}(U)$ we get

$$0 = K_p \circ |z_1 = 0| = K_p \circ K_p + K_p \circ |\tilde{u} = 0| .$$

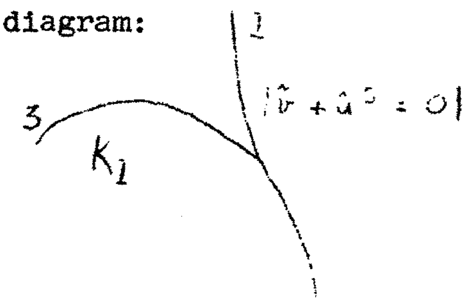
This clearly proves the lemma.

REMARK (5.6) If we ignore the complex analytic structure on M and consider $M = M^4$ as a C^∞-manifold, then $\sigma_p(M)$ is C^∞-equivalent to the connected sum $M \# (-Cp^{(2)})$.

The blowing-up operation may be iterated. Let p_1 be a point in M and consider $\sigma_{p_1} M$. Pick a point p_2 in $K_{p_1} = \pi_p^{-1}(p)$. We may construct a complex manifold $\sigma_{p_2} \sigma_{p_1} M = \sigma_{p_1 p_2} M$. Denote K_{p_i} by K_i , then K_1 and K_2 intersect at exactly one point. There is a projection, $\pi_{p_1p_2} : \sigma_{p_1p_2}M \to M$, which maps $K_1 \cup K_2$ into p. Take a point p_3 in $\pi^{-1}_{p_1p_2} = K_1 \cup K_2$ and construct $\sigma_{p_3}(\sigma_{p_1p_2}M) = \sigma_{p_1p_2p_3}M$, etc. Finally we arrive at a complex manifold $\tilde{M} = \sigma_{p_1 \cdots p_s}M$ and $\tilde{\pi} = \tilde{M} \to M$ with $\tilde{\pi}^{-1}(p_1) = K_1 \cup \cdots\cdot \cup K_s$. Notice that K_i and K_j $(i \neq j)$ are either disjoint or intersecting at one point regularly. Thus the "spherical space", $\tilde{\pi}^{-1}(p_1)$, is dual to a tree, T, in the sense that each K_i corresponds to a vertex, v_i, of T and K_i intersects K_j $(i \neq j)$ if, and only if, v_i and v_j are joint in T by an edge e_{ij}. We may weight the tree T so that v_i is weighted by the self-intersection number of K_i in $\tilde{M}$. The weighted tree so obtained is called <u>the dual weighted tree</u> of the spherical space $\tilde{\pi}^{-1}(p_1)$. The self-intersection number of K_i can easily be computed by applying (5.2) to the open manifold $\tilde{\pi}^{-1}(U)$, where U is a coordinate neighborhood of p_1.

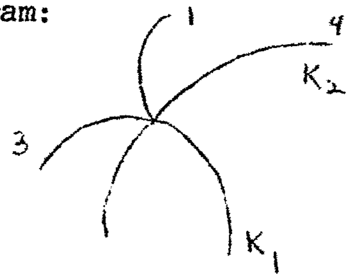
We are now ready to investigate how the resolution of singularities in the "Riemann surface" of an algebraic function by blowing-up process is related to the plumbing operation studied in §4. We believe that an example will be sufficient to clarify the situation.

EXAMPLE (5.7) Let $M = \mathbb{C}^2$ and let $f = \sqrt{z_1^{\,3} + z_2^{\,4}}$ be defined on M. The origin $p_1 = (0,0)$ is the only non-uniformizable singularity of f . Let $w = z_1^3 + z_2^4$; then the locus $|w = 0|$ is a complex analytic subset along which the branching of f occurs. Blow-up M at the point p_1 and in $\sigma_{p_1}M$ consider the local coordinates u,v and $\tilde{u},\tilde{v}$ given by (5.4) . w may be considered as a function defined on $\sigma_{p_1}M$ (see the proof of (5.5)). The divisor $|w = 0|$ in $\sigma_{p_1}M$ is expressed by $|\tilde{v}^3(\tilde{v} + \tilde{u}^3) = 0|$ and $|u^3(1+u^4v) = 0|$. The last two loci coincide except at one point. In fact, $|\tilde{v} = 0|$ and $|u = 0|$ give the 2-sphere $K_1 = K_{p_1}$ in $\sigma_{p_1}M$ and $|(1+u^4v) = 0| = |(\tilde{v} + \tilde{u}^3) = 0|$. We represent this divisor by a diagram:

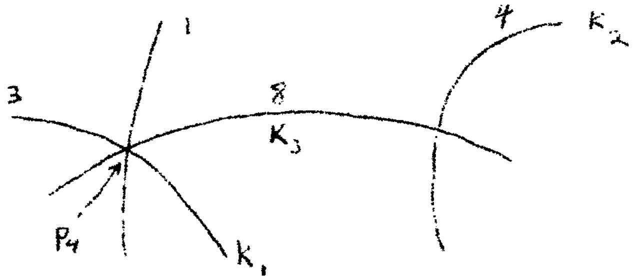


where the integers, 1 and 3 , expresses the multiplicity of the irreducible components indicated. Now the non-uniformizable singularity of $\sqrt{w} = f$ (consider as function on $\sigma_{p_1}M$) is at the point p_2 where $(\tilde{u},\tilde{v}) = 0$. Since this point does not appear in the (u,v)-chart, we may consider the restriction of w in $(\tilde{u},\tilde{v})$-chart i.e. we consider only $\tilde{v}^3(\tilde{v}+\tilde{u}^3)$. Blow-up $\sigma_{p_1}M$ at the point p_2 and in $\sigma_{p_1p_2}M$ we take local coordinates u_1,v_1

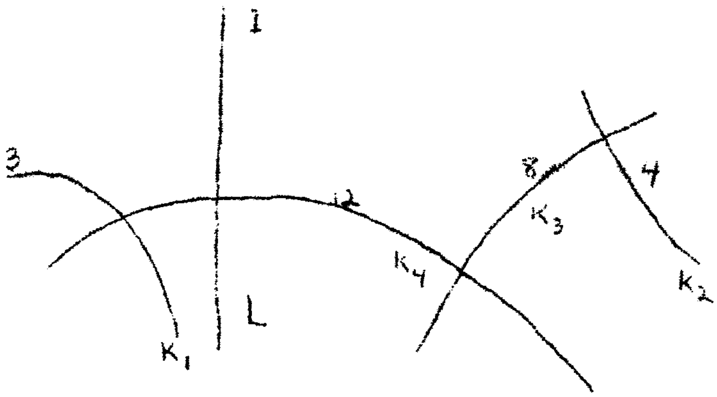
satisfying $\tilde{u} = u_1$, $\check{v} = u_1 v_1$ (for reasons similar to that given above, the other chart may be discarded). In this chart $|w = 0|$ is given by $|u_1{}^4\ v_1{}^3(v_1 + u_1{}^2) = 0|$ or in diagram:



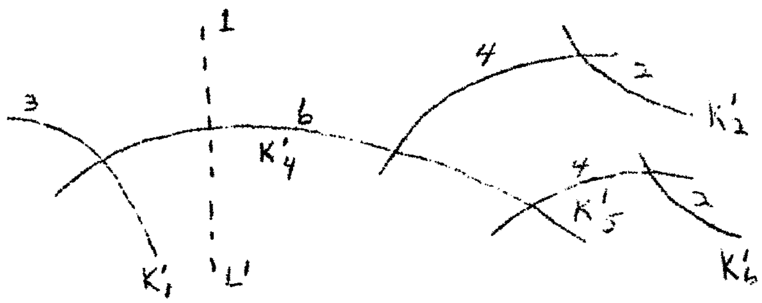
Next, let p_3 be the point where $(u_1, v_1) = (0,0)$. Blow-up at p_3, $(u_1 = u_2,\ v_1 = u_2 v_2\ ;\ u_1 = \tilde{u}_2\tilde{v}_2,\ v_1 = \tilde{v}_2)$, we see $|w = 0|$ is given by $|u_2^8 v_2^3\ (v_2 + u_2) = 0|$ and $|\tilde{u}_2{}^4\tilde{v}_2{}^8(1 + \tilde{u}_2{}^2\tilde{v}_2) = 0|$.



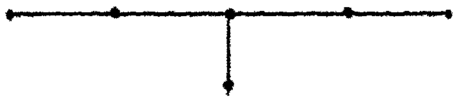
Finally we blow-up at p_4 where $(u_2, v_2) = (0,0)$ to obtain the manifold $\tilde{M} = \sigma_{p_1 p_2 p_3 p_4} M$ in which the divisor $|w = 0|$ is expressed by $|u_3{}^{12} v_3{}^3\ (v_3 + 1) = 0|$ and $|\tilde{u}_3{}^8\tilde{v}_3{}^{12}(1 + \tilde{u}_3) = 0|$. In diagram we have:



Consider the function $f = \sqrt{w}$ in $\tilde{M}$; it is now uniformizable. Branching of f only occurs along K_1 and $L = |v_3 + 1 = 0|$ where the multiplicities are odd. In the Riemann surface (of complex dimension 2) of $f = \sqrt{w}$, K_4 is lifted into a 2-sphere, K_4' , with two branches, K_1' and L' sticking out (K_4' covers K_4 twice, K_1' covers K_1 once and L' covers L once). Thus the divisor $D = |f = 0|$ of the Riemann surface can be expressed by the diagram



where K_3' and K_5' cover K_3 ; K_2' and K_6' cover K_2 . All loci except L' are 2-spheres. The dual diagram of the union of 2-spheres, $\cup K'_i$, is precisely E_6 (with vertices re-named):



We now compute the self-intersection numbers, $K_i' \circ K_i'$ $(i = 1,\ldots,6)$, by means of (5.2). First notice that if K_i' and K_j' are not disjoint then they intersect at one point regularly and hence $K_i' \circ K_j' = +1$. We have, for example,

$$0 = K_1' \circ D = 6 + 3K_1' \circ K_1' \quad ,$$

$$0 = K_4' \circ D = 3 + 1 + 4 + 4 + 6K_4' \circ K_4' \ .$$

Therefore $K_1' \circ K_1' = K_4' \circ K_4' = -2$. The reader will have no trouble to show that $K_i' \circ K_i' = -2$ for each $i = 1, 2,\ldots,6$. The dual weighted tree of the union of 2-spheres, $K = \cup K_i'$, is E_6 weighted by -2. Let us denote this tree by $(E_6;-2)$; E_6 weighted by $+2$ is then denoted by $(E_6;2)$. It is now clear that if V is a tubular neighborhood of K in the Riemann surface of (the modified) f, then V is diffeomorphic to $P(E_6;-2)$. The last space is by no means new; indeed, up to orientation it is the same as the space $P(E_6;2)$ which we studied in §4. In particular, $\partial V \approx -(S^3/T')$.

We now return to the function $f = \sqrt{z_1^3 + z_2^4}$ on $\mathbb{C}^2$. Let ψ be the projection from the Riemann surface, R_f, of f onto $\mathbb{C}^2$. Then $\psi^{-1}(0) = q$ is a singular point of R_f. Let B be the unit ball in $\mathbb{C}^2$, then $\psi^{-1}(B)$ is a neighborhood of q. The boundary of this neighborhood is obviously diffeomorphic to ∂V.

EXERCISE (5.8) Let $M = \mathbb{C}^2$. Show that for $f = \sqrt{z_1(z_1^2 + z_2^3)}$, $\sqrt{z_1^3 + z_2^5}$, $\sqrt{z_1(z_2^2 + z_1^n)}$ $(n \geq 2)$, $\sqrt{z_1^2 - z_2^n}$ $(n \geq 2)$ the dual weighted trees obtained from the preceding process are, respectively, E_7, E_8, D_{n+1}, A_{n-1} (here every tree is weighted by -2).

In general, resolving a singularity of an algebraic function, f, defined on a complex manifold $M^{(2)}$ amounts to replacing the corresponding singular point in the "Riemann surface" of f by a finite number of curves, $K_1, K_2, \ldots, K_n$, $(\dim_{\mathbb{R}} K_i = 2$, for each $i)$, of various genera such that for any pair i, j $(i \neq j)$, the curves K_i and K_j are either disjoint or intersect at one point regularly. Thus the intersection relations give rise to a dual weighted graph, each vertex corresponds to a curve ; furthermore this dual graph is connected. The quadratic form, S, associated with the dual graph (in the same fashion as that associated with a tree) may be represented by the intersection matrix $M_S = (\alpha_{ij})$, where $\alpha_{ij} = K_i \circ K_j$. We have

THEOREM (5.9) The quadratic form, S, defined above is negative definite.

The proof we are about to give is due to D. Mumford (Institut des Hautes Études Scientifiques, Publications Mathématiques, No. 9, p. 6). Let the multiplicity of K_i be denoted by m_i, which is a positive integer for each $i = 1, 2, \ldots, n$. Define a quadratic form

$$S' \sim \sum_{i,j} \alpha'_{ij} x_i x_j ,$$

where $\alpha'_{ij} = m_i m_j \alpha_{ij} = m_i K_i \circ m_j K_j$. It is clear that S is negative definite if, and only if, S' is so. Notice that

a) $\alpha'_{ij} \geq 0$ if $i \neq j$.

In virtue of (5.2) we have

b) $\sum_i \alpha'_{ij} = \sum_i (m_i K_i \circ m_j K_j) \leq 0$, for each $i = 1, \ldots, n$, and

c) $\sum_i \alpha'_{ij} < 0$ for some j.

Conditions a) and b) imply that S' is negative semi-definite. To obtain the definiteness we write S' in the form

$$\sum_{ij} \alpha'_{ij} x_i x_j = \sum_j \Big(\sum_i \alpha'_{ij} \Big) x_j^2 - \sum_{i<j} \alpha'_{ij} (x_i - x_j)^2$$

and put $S' = 0$. Condition c) shows that $x_j = 0$ for some j . This together with the connectedness of the dual graph proves that $x_1 = x_2 = \dots = x_n = 0$, i.e., S' is definite. The proof of (5.9) is thus completed.

6. A Theorem of Kervaire and Milnor

We devote this section to prove Theorem (2.5) which we stated (without proof) in §2 . This Theorem, as well as the proof we are about to see, is due to Kervaire and Milnor [2].

Let M be an unbounded (differentiable, oriented, compact) 4-manifold. Let $\pi : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ be the reduction modulo 2 and let $d \in H^2(M;\mathbb{Z})$ be such that $\pi d = w_2(M)$. Under Poincaré duality, d corresponds to an element $b \in H_2(M;\mathbb{Z})$. Theorem (2.5) states that

THEOREM (2.5) If b can be represented by a differentiable embedding, $f : S^2 \to M$, then the self-intersection number, $b \circ b = (d \cup d)[M]$, is congruent to $\tau(M)$ modulo 16.

First let us consider the case where $b \circ b = -1$. Then $f(S^2)$ in M has normal bundle equivalent to the bundle η_{-1} defined in §4. In other words $f(S^2)$ has in M a tubular neighborhood, A , whose boundary, ∂A , considered as the normal S^1-bundle of $f(S^2)$ is the Hopf fibration $S^3 \to S^2$. Put $M_1 = (M \setminus \mathrm{Int}\ A) \cup e^4$, where e^4 is a 4-cell whose boundary $\partial e^4 = S^3$, is attached to $\partial A = S^3$ by the identity map. Then

$$M = M_1 \# (-CP^{(2)}) .$$

Clearly $w_2(M_1) = 0$ and hence $\tau(M_1) \equiv 0 \pmod{16}$ by (2.2). Since $\tau(M) = \tau(M_1) + \tau(-CP^{(2)}) = \tau(M_1) - 1$, our theorem is verified for this particular case.

By reversing the orientation of M if necessary, we may suppose that $b \circ b = s \geq 0$. Let $P_1, P_2, \ldots, P_{s+1}$ be $s + 1$ copies of $-CP^{(2)}$ and let

$$M' = M \# P_1 \# P_2 \# \cdots \# P_{s+1} .$$

Using the natural isomorphism

$$j : H_2(M;\mathbb{Z}) \oplus H_2(P_1;\mathbb{Z}) \oplus \cdots \oplus H_2(P_{s+1};\mathbb{Z}) \to H_2(M';\mathbb{Z}) ,$$

let

$$c = j(b \oplus g_1 \oplus \cdots \oplus g_{s+1}) ,$$

where g_i denotes a generator of $H_2(P_i;\mathbb{Z})$. We then have

$$c \circ c = b \circ b + \sum_{i=1}^{s+1} g_i \circ g_i = -1 .$$

Using the hypothesis that b can be represented by a differentiable embedding of S^2 in M, it follows easily that c can be represented by a differentiably embedded 2-sphere. Applying the special case we just proved we get

$$\tau(M') \equiv -1 \pmod{16} .$$

Since $\tau(M') = \tau(M) - (s+1)$, we deduce that

$$b \circ b = s \equiv (M) \pmod{16} .$$

EXAMPLE (6.1) Let $M = S^2 \times S^2$ and let $\alpha, \beta \in H^2(M;\mathbb{Z})$ be the standard generators. Then $H^2(M;\mathbb{Z}) = \mathbb{Z}\cdot\alpha \oplus \mathbb{Z}\cdot\beta$ and with respect to this basis the quadratic form, S_M, of M is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

An element in $H^2(M;\mathbb{Z})$ is of the form $m\alpha + n\beta$. Its reduction modulo 2, $\pi(M\alpha + n\beta)$, is equal to $w_2(M)$ if, and only if, m and n are both even. Assume this to be the case and let $m = 2m'$, $n = 2n'$. Using the relation

$$(2m'\alpha + 2n'\beta)^2 = 8m'n' \alpha \cup \beta ,$$

it follows that the dual homology class of $m\alpha + n\beta$ can be represented by a differentiably embedded 2-sphere only if $8m'n'$ is divisible by 16 or $m'n' \equiv 0 \pmod 2$. In particular $2\alpha + 2\beta$ is not representable by a differentiably embedded 2-sphere in M.

EXAMPLE (6.2) Let $M = CP^{(2)}$ and let $g \in H^2(M;\mathbb{Z})$ be the standard generator. The dual homology class of g is obviously represented by a differentiably embedded 2-sphere; however the dual class of $3g$, for example, is not. Indeed, an element, ng, of $H^2(M;\mathbb{Z})$ satisfies the relation $\pi(ng) = w_2(M)$ if, and only if, n is odd.

Consider an element $(2k + 1)g$ in $H^2(M;\mathbb{Z})$. If its dual homology class is representable by a differentiably embedded 2-sphere then $(2k + 1)^2 \equiv 1 \pmod{16}$; i.e. $k \equiv 0,3 \pmod 4$.

On the other hand, the dual homology class of $3g$ is represented by a combinatorially embedded 2-sphere in $M = CP^{(2)}$. To see this we let z_0, z_1, z_2 be the homogeneous coordinates in $CP^{(2)}$ and let $P = P(z_0,z_1,z_2)$ be a homogeneous polynomial in three variables of (total) degree n. This polynomial defines a divisor of $CP^{(2)}$ in the obvious fashion (e.g., in the chart where $z_0 \neq 0$ we consider the function $P(1, {}^{z}1/z_0, {}^{z}2/z_0)$), and the homology class of this divisor is represented by the algebraic curve $P = 0$. If $Q = Q(z_0,z_1,z_2)$ is another homogeneous polynomial of the same degree, n, then P/Q is a meromorphic function, globally defined on $CP^{(2)}$; therefore, as a divisor it gives the zero homology class. In other words, the curves $P = 0$ and $Q = 0$ represent the same homology class of $CP^{(2)}$. Since the dual class of g is represented by the complex projective line $z_0 = 0$, the homology class represented by the curve $P = 0$ for any homogeneous polynomial, p, in z_0,z_1,z_2 of degree 3 is dual to $3g$. In particular we let $P = z_1{}^2 z_0 - z_2{}^3$.

The curve $P = 0$ has only one singularity which is a cusp at $(0,0,1)$. Using the formula

$$\text{genus} = \frac{(n-1)(n-2)}{2} - \text{local items} ,$$

we see that this curve has genus zero. It follows that the curve $P = 0$ considered as a (real) surface in the 4-manifold $CP^{(2)}$ is a 2-sphere (smooth except at a cusp) which is apparently combinatorially embedded.

REMARK (6.3) Kervaire and Milnor proved that in the manifolds $S^2 \times S^2$ and $CP^{(2)}$, any 2-dimensional homology class can be represented by a combinatorially embedded 2-sphere.

7. Integral Unimodular Quadratic Forms *

Let $f = (f,V)$ be a non-degenerate integral quadratic form of rank n. The set of non-negative integers, $\{|f(x,x)|\}_{x\in V}$ has a minimum, which we shall denote by $\min f$. In case $\min f = 0$, we say that f is a zero form or, equivalently, f represents zero.

If a base $(e_i)_{1\le i\le n}$ is chosen in the lattice, V, the quadratic form, f, may be expressed by

$$f = \sum_{i,j} a_{ij}x_ix_j \; . \tag{7.1}$$

Recall that we can always choose the base (e_i) of V so that the expression (7.1) of f is Hermite-reduced. The last term can be defined inductively as follows:

1) If $f = (f,V)$ is of rank 1, then $f = a_{11}x_1^2$ is Hermite reduced.

2) For quadratic form of rank $n > 1$, the expression $f = \sum a_{ij}x_ix_j$ is Hermite-reduced if

   a) $|a_{11}| \geq 2\,|a_{1j}|$ for $j > 1$,

   b) $|a_{11}| = \min f$,

   c) $a_{11}f = (\sum a_{1i}x_i)^2 + f_1(x_2,\dots,x_n)$,

where $f_1(x_2, \dots, x_n)$ is a Hermite-reduced form of rank $n-1$.

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* The content of this section is based on an article of J - P. Serre (see Serre [1]) as much as on the original lectures.

We have

THEOREM (7.2) If $f = \sum a_{ij}x_ix_j$ is Hermite-reduced and if f is not a zero form, then

$$|a_{11}| \leq 4/3^{(n-1)/2} |\det f|^{1/n} .$$

COROLLARY (7.3) Let $f = \sum a_{ij}x_ix_j$ be an integral unimodular quadratic form of rank ≤ 5. If f is not a zero form then it is equivalent to either $\sum y_i^2$ or $-\sum y_i^2$.

This corollary shows in particular that any indefinite unimodular integral quadratic form of rank ≤ 5 represents zero. For quadratic forms of rank ≥ 5 this is true for any indefinite form. Precisely, we have

THEOREM (7.4) Every non-degenerate indefinite integral quadratic form of rank ≥ 5 represents zero (Meyer).

Proofs of these statements can be found in the useful monograph of B. W. Jones (Carus Monographs No. 10; see Theorem 23, Corollary 23 and Corollary 27d there).

We now turn to study the Grothendieck-Witt ring $G_o(\mathbb{Z})$ which we defined earlier in Appendix A. An integral unimodular quadratic form, f, represents an element (also denoted by f) in the monoid $\mathcal{F}_o(\mathbb{Z})$ (see page A - 2). The elements in $\mathcal{F}_o(\mathbb{Z})$ represented by the forms x^2, $-x^2$ and xy will be denoted respectively by

1, -1 and u . The element in $G_o(\mathbb{Z})$ represented by $f \in \mathfrak{F}_o(\mathbb{Z})$ will be denoted by $\bar{f}$. This notation differs slightly from that used in Appendix A.

LEMMA (7.5) Every unimodular indefinite odd quadratic form of rank ≥ 2 can be decomposed into the form $1 \dotplus (-1) \dotplus g$, where g is a non-singular form .

To prove this lemma we let $f = (f,V)$ be a given unimodular indefinite odd form. By (7.3) and (7.4), f is a zero form. Let $x \in V$ be an indivisible element satisfying $f(x,x) = 0$. Since f is non-singular (unimodular), the correlation, $\mathcal{G}$, of f maps x into an indivisible element $\mathcal{G}(x)$ in V' . Therefore there exists an element $y \in V$ such that $\langle \mathcal{G}(x), y \rangle = f(x,y) = 1$. We may furthermore choose y so that $f(y,y)$ is odd. For if $f(y,y)$ is even, we choose an element $z \in V$ satisfying $f(z,z) = 1 \bmod 2$ (this is possible because f is odd) and replace y by $y' = z + [1 - f(x,y)]y$. Then $f(y',y')$ is odd and $f(x,y') = 1$. Now assume $f(y,y)$ is odd and let $f(y,y) = 2m + 1$. The elements $e_1 = y - mx$, $e_2 = y - (m+1)x$ are indivisible in V for $f(e_1,e_1) = 1$ and $f(e_2,e_2) = -1$. Our lemma now follows from (0.4).

THEOREM (7.6) If f is a unimodular indefinite odd quadratic form then f is equivalent to $\alpha^+ . 1 \dotplus \alpha^-(-1)$.

Indeed, Lemma (7.5) shows that f is equivalent to $1 \dotplus (-1) \dotplus g$. Since one of $1 \dotplus g$ and $(-1 \dotplus g$ has to be indefinite, (7.6) follows from a simple induction.

COROLLARY (7.7) Two unimodular indefinite odd quadratic forms are equivalent if they have the same rank and same index.

We are now ready to show that $G_o(\mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$ as we announced in (A.5). In fact we shall show that

THEOREM (7.8) $G_o(\mathbb{Z}) = \mathbb{Z}.\overline{1} \oplus \mathbb{Z}.(\overline{-1})$.

For each $f \in \mathcal{F}_o(\mathbb{Z})$, either $1 \dotplus f$ or $(-1) \dotplus f$ is odd and indefinite. Using (7.6) we conclude that $\overline{f} = \alpha^+.\overline{1} + \alpha^-.(\overline{-1})$ in $G_o(\mathbb{Z})$ and this shows that $\overline{1}$ and $\overline{-1}$ generates $G_o(\mathbb{Z})$. Since the homomorphism $(\alpha^+,\alpha^-): G_o(\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}$ maps these generators into the free generators of $\mathbb{Z} \oplus \mathbb{Z}$, it follows that $G_o(\mathbb{Z}) = \mathbb{Z}.\overline{1} \oplus \mathbb{Z}.(\overline{-1})$.

REMARK (7.9) Corollary (1.10) , which states that $f(\omega,\omega) - \tau(f) \equiv 0 \pmod 8$ for any integral unimodular form f , now follows easily from (7.8) . The map $h : \mathcal{F}_o(\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ defined by $h(f) = \dfrac{f(\omega,\omega) - \tau}{8}$ (reduced modulo 1) is a monoid-homomorphism which induces a homomorphism $\overline{h} = G_o(\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. Corollary (1.10) is equivalent to saying that $\overline{h}$ vanishes indentically on $G_o(\mathbb{Z})$; but this is clear since $\overline{h}$ vanishes on the generators $\overline{1}$ and $\overline{-1}$ of $G_o(\mathbb{Z})$.

The structure of integral unimodular indefinite odd forms is completely determined by Theorem (7.6). The structure of unimodular indefinite even forms can also be determined (see Serre [1]). In particular two such forms are equivalent if, and only if, they have the same rank and same index. Combining this with (7.7), we have

THEOREM (7.10) Two integral unimodular indefinite quadratic forms are equivalent if, and only if, they have the same rank and same index.

As for definite forms, our knowledge is meagre. We have, for example,

THEOREM (7.11) Two integral unimodular quadratic forms of the same rank ≤ 8 are equivalent if they have the same signature and type (i.e., even or odd).

REMARK (7.12) The last theorem ceased to be true for forms of rank > 8. For example, let f be the quadratic form associated with E_8 as defined in P. 1-9 and let $f' = f \dotplus 1$. Then f' and $g = 1 \dotplus 1 \dotplus \cdots \dotplus 1$ (9 times) are of the same rank, same signature and same type. If (7.11) were true for this case, f and g would be equivalent. This is impossible for $f'(x,x) = 1$ has 2 solutions by $g(x,x) = 1$ has 18 solutions.

8. More about Quadratic Forms.

Let f be a quadratic form over a ring A. For an element $a \in A$ we shall denote by af the quadratic form form defined by $af(x,y) = a.f(x,y)$.

So far we have considered exclusively quadratic forms which are non-degenerate. This is not an essential restriction; for if $f = (f,V)$ is any quadratic form, the kernel, Rad (V), of the correlation $\varphi : V \to V'$ of f is a submodule of V and f induces a non-degenerate quadratic form, $\widetilde{f}$, on $V/\mathrm{Rad}(V)$. We shall use $\widetilde{\det}\, f$ (resp. $\widetilde{\mathrm{DET}}\, f$) to denote $\det \widetilde{f}$ (resp. $\mathrm{DET}\, \widetilde{f}$, cf. p. 0-10) and agree that if f is totally isotropic (i.e. $f(x,y) = 0$ for all x and y in V) then $\widetilde{\mathrm{DET}}\, f = 1$.

THEOREM (8.1) Every non-degenerate quadratic form, f, over $R(p)$ may be decomposed into

$$f = f_0 \dotplus pf_1 \dotplus p^2 f_2 \dotplus \dots \dotplus p^k f_k \,, \tag{8.2}$$

where f_i $(0 \leq i \leq k)$ is a non-singular quadratic form of rank $r_i \geq 0$.

COROLLARY (8.3) Every non-degenerate quadratic form over $R(p)$ $(p \neq 2)$ decomposes into unary forms.

The corollary follows immediately from (8.1) and (0.5). To prove (8.1) we choose a basis in the lattice, V, on which f is defined, and express f by its matrix

$$M_f = (\alpha_{ij}) \quad , \quad (i,j = 1, \ldots, r).$$

If every α_{ij} is divisible by $p \in R(p)$ we shall say that M_f is divisible by p and, if this is the case, put $M_f = p^t M'$, where $M' = (\alpha'_{ij})$ is not divisible by p. Notice that M' may not be invertible.

a). If not all the diagonal entries of M' are divisible by p, then some of them are units. Applying (0.4), we may split off these entries.

b). If all the diagonal entries of M' are divisible by p, then there is at least one entry off the diagonal which is a unit and we lose nothing by assuming this entry be α'_{12}; but then the minor

$$\begin{pmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{pmatrix}$$

is non-singular and hence splits off.

By repeated application of a) and b) we get $f = p^t(f_t \dotplus g)$, where f_t is non-singular and the matrix, N, of g is divisible by p. We write $N = p^s N'$ such that $p^t M'$ and repeat the preceding argument to obtain $f = p^t f_t \dotplus p^{s+t}(f_{s+t} \dotplus g')$, where f_{s+t} is nonsingular and g' is divisible by p . Our theorem now follows readily from an obvious induction.

Let r_i $(0 \leq i \leq k)$ be the rank of f_i in (8.2). Then r_i's are uniquely determined by the equivalence class of f . Indeed, the cokernel of the correlation $\mathcal{f}: V \to V'$ is isomorphic to $\sum R(p)/\mathfrak{a}_j$, where $\mathfrak{a}_1 \subset \dots \subset \mathfrak{a}_m \neq R(p)$ are ideals uniquely defined by φ, hence by f (cf. [B - Alg] CH. VII. §4. Th. 2). In $R(p)$ the only ideals are of the form $p^t R(p)$. The number of times $R(p)/p^t R(p)$ appears in $\sum R(p)/\mathfrak{a}_j$ is clearly equal to r_i . This gives an invariant definition of the r_i's .

Each f_i in (8.2) is a non-singular quadratic form. Therefore $\det f_i$ is a unit. We define

$$\gamma_i = \begin{cases} 1 , & \text{if } \det f_i \in R(p)^{**} , \\ -1 , & \text{if } \det f_i \notin R(p)^{**} . \end{cases}$$

In other words, if we identify $R(p)^*/R(p)^{**}$ with $\{1, -1\}$ then $\gamma_i = \mathrm{DET}\, f_i$. As we shall see that the γ_i's depend only on the equivalence class of f ; they are called the <u>Minkowski invariants</u> of f .

We now seek for an invariant definition of these γ_i's. If this is done, the invariance of γ_i is automatically established. Let $f = (f,V)$. We may extend f to a quadratic form over $V \otimes F(p)$. The extended f, formerly denoted by $f^{F(p)}$, will also be denoted by f for simplicity. Consider $V \subset V \otimes F(p)$ and let V^+ be the "dual" of V in $V \otimes F(p)$, i.e., $V^+ = \{x \in V \otimes F(p) \mid f(x,y) \in R(p) \text{ for every } y \in V\}$. Then $U = V^+/V$ is a finite abelian p-group. The quadratic form, f, induces a bilinear pairing

$$t : U \times U \to F(p)/R(p) .$$

Filter the group U by

$$\{0\} = U_0 \subset U_1 \subset \cdots \subset U ,$$

where $U_i = \{x \in U \mid p^i x = 0\}$. It is clear that every element in the quotient group $W_i = U_i/U_{i-1}$ is of order p. We may consider W_i as a vector space over $\mathbb{Z}_p$. It is also easy to see that the pairing, t, defined above induces a bilinear pairing

$$f_i' : W_i \times W_i \to \frac{1}{p^i} R(p) \Big/ \frac{1}{p^{i-1}} R(p) \approx \mathbb{Z}_p :$$

Therefore f_i' is a quadratic form over $\mathbb{Z}_p$ and $\widetilde{\mathrm{DET}}\, f_i' \in \mathbb{Z}_p^*/\mathbb{Z}_p^{**}$ is defined. Identifying $\mathbb{Z}_p^*/\mathbb{Z}_p^{**}$ with $G_2 = \{1, -1\}$, we see that $\widetilde{\mathrm{DET}}\, f_i' = \pm 1$. We claim that $\gamma_i = \widetilde{\mathrm{DET}}\, f_i'$ $(i = 1, \ldots, k)$. To show this we let $f_i = (f_i, V_i)$ in (8.2). Then $V = \oplus \sum_{i=0}^{k} V_i$ and $V^+ = \sum \frac{1}{p^i} V_i \subset V \otimes F(p)$. We compare the definitions of γ_i and $\widetilde{\mathrm{DET}}\, f_i'$ and apply (0.17) to conclude that $\gamma_i = \widetilde{\mathrm{DET}}\, f_i'$ for $i = 1,\ldots,k$. The invariance of γ_0 now follows as a consequence.

The following lemma is not hard to prove (see Jones, p. 91).

LEMMA (8.4) Let f and g be non-singular quadratic forms over $R(p)$. For $p \neq 2$, f and g are equivalent if, and only if, they are equivalent over $F(p)$. For $p = 2$, f and g are equivalent if, and only if, they have the same type and are equivalent over $F(2)$.

THEOREM (8.5) Two non-singular quadratic forms over $R(p)$ $(p \neq 2)$ are equivalent if they have the same rank and same DET.

könnte mit Hensel Lemma auf Restklassenkörper zurückgeführt werden!

Let f and g be two non-singular quadratic forms over $R(p)$ such that $\mathrm{rk}\, f = \mathrm{rk}\, g$ and $\mathrm{DET}\, f = \mathrm{DET}\, g$. Since $p \neq 2$, $c_p(f) = c_p(g) = 1$ in virtue of (0.24). It follows from (A.9) that f and g, when considered as quadratic forms over $F(p)$, are equivalent in $G(F(p))$ (i.e., in the notation of §7, $\bar{f} = \bar{g}$). We then apply (A.1) and (0.8) to conclude that f and g are equivalent over $F(p)$. Our theorem now follows from (8.4).

COROLLARY (8.6) For non-degenerate quadratic forms over $R(p)$ $(p \neq 2)$, the integers r_i, γ_i $(i = 0, 1, \ldots, k)$ constitute a complete set of invariants for the equivalence classes.

Returning now to integral quadratic forms. Two such forms, f and g , are said to be equivalent over $R(p)$ if $f^{R(p)} \sim g^{R(p)}$. We say that f is <u>semi-equivalent</u> to g or f and g have the same <u>genus</u> if f is equivalent to g over $R(p)$ for $p = 2, 3, \ldots \infty$. We have

THEOREM (8.7) Two integral even quadratic forms, f and g , with odd determinants are semi-equivalent if they are equivalent over $R(p)$ for all $p \neq 2$ (including $p = \infty$) .

In virtue of (8.4), we need only to show that f and g are equivalent over $F(2)$. Notice that $f \sim g$ over $R(\infty)$ implies $\operatorname{rk} f = \operatorname{rk} g$. We then apply (0.25) to obtain $c_2(f) = c_2(g)$, and hence $\tilde{c}_2(f) = \tilde{c}_2(g)$. Since f and g are even forms, the last equality shows that $\det f \equiv \det g \pmod 8$; that is $\mathrm{DET}\, f = \mathrm{DET}\, g$ (cf. (0.17) and the proof of (0.18)). It follows from (A.9), (A.1) and (0.8) that $f \sim g$ over $F(2)$ and this proves (8.7).

THEOREM (8.8) The rank, index and type (even or odd) form a complete system of invariants of the genus of an integral unimodular quadratic form.

This is a consequence of (0.24), (8.4) and (8.5).

Let $f = (f,V)$ be an integral quadratic form. The invariants γ_i and r_i of $f^{R(p)}$ will be denoted by

$\gamma_i^{(p)}$ and $r_i^{(p)}$, which are invariants of f. Consider $V \subset V \otimes Q$ and let $V^+ = \{\{x \in V \otimes Q \mid f(x,y) \in \mathbb{Z}$ for all $y \in V\}$. Define $U = V^+/V$ as before. The quadratic form, f, then induces a bilinear pairing

$$t : U \times U \to Q/\mathbb{Z} .$$

For any odd prime, p, let $U_i^{(p)} = \{u \in U \mid p^i u = 0\}$ and put $W_i^{(p)} = U_i^{(p)} \big/ U_{i-1}^{(p)}$. We then obtain a bilinear pairing

$$W_i^{(p)} \times W_i^{(p)} \to \mathbb{Z}_p .$$

It is clear that $\gamma_i^{(p)}$, $r_i^{(p)}$ $(i \geq 1)$ of f can be defined by means of this pairing.

Let $Y = Y^{4k}$ be a (compact, differentiable, oriented) 4k-manifold with boundary $\partial Y = X$. For simplicity we assume that

1) Y has no torsion,

2) Y is acyclic in dimensions $\neq 0$ or $2k$,

3) Y is a rational homology sphere.

(These assumptions may be weakened.) Consider

$$\begin{array}{ccccccc}
H_{2k}(Y) & \to & H_{2k}(Y,X) & \to & H_{2k-1}(X) & \to & 0 \\
& \searrow & \| & \nearrow & & & \\
& & H^{2k}(Y) & & & & \\
& & \| & & & & \\
& & \mathrm{Hom}(H_{2k}(Y),\mathbb{Z}) & & & &
\end{array}$$

Since $H_{2k-1}(X)$ is a finite torsion group, if we put $V = H_{2k}(Y)$ then $\mathrm{Hom}(H_{2k}(Y)\mathbb{Z}) = H_{2k}(Y,X)$ is essentially V^{+} and $H_{2k}(X) = U$. The quadratic form, S_Y, of Y induces a bilinear pairing

$$L : U \times U \to Q/\mathbb{Z},$$

which, as informed reader may have already observed, gives the linking numbers in X. Thus L, and consequently $\gamma_i^{(p)}$ and $r_i^{(p)}$ $(i \geq 1$, p odd), are invariants of the oriented homotopy type of $X = \partial Y$. From (8.7) we conclude that if Y_1 and Y_2 are 4k-manifolds with boundaries satisfying 1) - 3) and if the quadratic forms S_1 and S_2 of Y_1 and Y_2 are even, with odd determinants, and are equivalent over $R(\infty) = \mathbb{R}$ then S_1 and S_2 have the same genus provided ∂Y_1 and ∂Y_2 have the same oriented homotopy type.

References

Books and Monographs

[H] Hirzebruch, F. Neue Topologische Methoden in der Algebraischen Geometrie (Berlin, 1956-1962)

[H-W] Hilton, P. J. and Wylie, S. Homology Theory (Cambridge, 1960)

[S-T] Seifert, H. and Threlfall, W. Lehrbuch der Topologie (Leipzig, 1934)

[B-Alg] Bourbaki, N. Algèbre.

Papers and Notes

Borel, A. and Hirzebruch, F.

[1] Characteristic classes and homogeneous spaces I Am. J. Math. 80(1958), 459-538.

[2] -----II. ibid 81(1959), 315-382.

[3] -----III. ibid 82(1960), 491-504.

Hirzebruch, F.

[1] Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei Komplexen Veränderlichen. Math. Ann. Bd 126, S. 1-22 (1953).

Kervaire, M. A. and Milnor, J. W.

[1] Bernoulli number, homotopy groups, and a theorem of Rohlin. proc. Intern. Congress of Math, (Edinbergh, 1958), 454-458.

[2] On 2-spheres in 4-manifolds. Proc. Nat. Acad. of Sci., U. S. A. 47(1961), 1651-1657.

Milnor, J. W.

[1] On simply connected 4-manifold. Symposium Intern. de Topologia Algebraica (Mexico, 1958), 122-128.

[2] Differentiable manifolds which are homotopy spheres. (Princeton 1959).

[3] A procedure for Killing homotopy groups of differentiable manifolds. Proc. of Symposia in Pure Math. A.M.S. Vol. III.

[4] Lectures on characteristic classes. (Princeton 1957)

[5] Differentiable structures. [Princeton 1960).

Rohlin, V. A.

[1] Classification of mappings of S^{n+3} onto S^n (Russian) Dokl. Akad. Nauk SSSR. 81(1951), 19-22.

[2] A new result in the theory of 4-dimensional manifolds (Russian) ibid 84(1952) 221-224.

Seifert, H.

[1] Topologie dreidemensionaler gefaserter Räume. Acta Math. 60(1932), 147-238.

Serre, J-P.

[1] Formes bilinéares symetriques entières à discriminant ± 1. H. Cartan Séminaire 14^e année Exp. 14.

Whitehead, J. H. C.

[1] On incidence matrices, nuclei and homotopy type. Ann. Math. 42(1941), 1197-1237.