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1 Charney, J. G., and Phillips, N. A., "Numerical Integration of the Quasi-geostrophic Equations for Barotropic and Simple Baroclinic Flows," *J. Meteorol.*, **10**, No. 2, 71–99 (1953).

2 Charney, J. G., "On the Scale of Atmospheric Motions," *Geofys. Publikasjoner, Oslo*, **17**, No. 2, 17 pp. (1948).

3 This iteration method is essentially the extrapolated Liebmann method. See Frankel, S. P., "Convergence Rates of Iterative Treatments of Partial Differential Equations," *Math. Tables and Other Aids to Computation*, **4**, No. 30, 65–75 (1950); and Young, D., "Iterative Methods for Solving Partial Difference Equations of Elliptic Type" (1951). Seminar given at Numerical Analysis Seminar of Ballistic Research Laboratories, Aberdeen Proving Ground, Md. (based on Ph.D. thesis by author, Harvard University, Cambridge, Mass., 1950).

ARITHMETIC GENERA AND THE THEOREM OF RIEMANN-ROCH FOR ALGEBRAIC VARIETIES

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Introduction.—In a preceding note[1] we posed the problem whether the arithmetic genus $\Pi(V_n)$ of a (non-singular) algebraic variety[2] coincides with the Todd genus $T(V_n)$. The purpose of the present note is to prove that this is actually the case. Moreover we prove a main theorem (M) which gives a formula for the Euler-Poincaré characteristic of V_n with respect to the cohomology[3] of V_n with coefficients in the sheaf (faisceau) of local holomorphic cross-sections of any complex analytic bundle W over V_n which has the complex vector space C_q as fibre and the linear group $GL(q, C)$ as structure group. The main theorem expresses this Euler-Poincaré characteristic as a polynomial in the Chern classes of the tangential bundle of V_n and in the Chern classes of the bundle W. As special cases one gets the Todd formula $\Pi(V_n) = T(V_n)$ and the "Theorem of Riemann-Roch for arbitrary dimensions." The author wishes to extend his hearty thanks to Professors A. Borel, K. Kodaira, and D. C. Spencer, with all of whom he had many valuable discussions. The notes of Kodaira and Spencer[4] are essentially used. The author also wants to point out that the main theorem (in a slightly different formulation) was conjectured by J. P. Serre in a letter to Kodaira and Spencer. The proof of the main theorem uses the index theorem of the author[1] which involves essentially the theory of "cobordisme" due to R. Thom. Full details of the proof of the main theorem will appear elsewhere.

1. The Main Theorem and Some of Its Consequences.—Let V_n be an algebraic variety (non-singularly imbedded in some complex projective space). Let W be a complex analytic bundle over V_n with the complex vector space C_q as fibre and the linear group $GL(q, C)$ as structure group. Denote by c_i the Chern classes of the tangential bundle of V_n and by d_j the Chern classes of the bundle W. ($c_i \in H^{2i}(V_n, Z)$, $d_j \in H^{2j}(V_n, Z)$; $0 \leqq i \leqq n$, $0 \leqq j \leqq q$, $c_0 = d_0 = 1$.) We introduce the formal roots γ_i, δ_j:

$$\sum_{i=0}^{n} c_i x^i = \prod_{i=1}^{n} (1 + \gamma_i x), \qquad \sum_{j=0}^{q} d_j x^j = \prod_{j=1}^{q} (1 + \delta_j x).$$

Every formal power series which is symmetric in the γ_i as well as in the δ_j will be considered as a power series in the Chern classes c_i and d_j and hence as an element of the cohomology ring of V_n. Denoting by Q the rationals we define the operator κ_n on the cohomology ring $H^*(V_n, Q)$ as follows: For every $u \in H^*(V_n, Q)$ we take the component of topological dimension $2n$ which we consider in the unique fashion as a rational number. This number is denoted by $\kappa_n[u]$.

MAIN THEOREM (M). *Using the above notations we put*

$$\chi(V_n, W) = \sum_{i=0}^{n} (-1)^i \dim H^i(V_n, W),$$

where $H^i(V_n, W)$ is the i-dimensional cohomology group of V_n with coefficients in the sheaf (faisceau) of local holomorphic cross-sections of W. We have

$$\chi(V_n, W) = \kappa_n \left[(e^{\delta_1} + e^{\delta_2} + \ldots + e^{\delta_q}) \prod_{i=1}^{n} \frac{-\gamma_i}{e^{-\gamma_i} - 1} \right] = \kappa_n \left[e^{c_1/2} (e^{\delta_1} + e^{\delta_2} + \ldots + e^{\delta_q}) \prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh \gamma_i/2} \right].$$

Remark: $x/\sinh x$ is a power series in x^2. Since the γ_i^2 can be regarded as the formal roots of the Pontrjagin polynomial[1] of V_n, we see that $\chi(V_n, W)$ is a polynomial in c_1, the Pontrjagin classes of V_n, and the Chern classes of the bundle W.

Now let W be the trivial bundle over V_n with C_1 as fibre. The main theorem implies[5]

THEOREM 1. *The Todd genus of an algebraic variety is equal to the arithmetic genus:*

$$\Pi(V_n) = \sum_{i=0}^{n} (-1)^i g_i = T(V_n).$$

More generally, let F be an arbitrary complex line bundle (fibre C_1) over V_n. Denote by $f = c(F)$ its characteristic class ($f \in H^2(V_n, Z)$). The main theorem implies

THEOREM 2. *We have*

$$\chi(V_n, F) = \kappa_n \left[e^{f + c_1/2} \prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh \gamma_i/2} \right].$$

Hence $\chi(V_n, F)$ is a polynomial in $f + c_1/2$ and the Pontrjagin classes of V_n.

Theorem 2 can be considered as a generalization of the theorem of Riemann-Roch to arbitrary dimensions. It contains the known Riemann-Roch theorems for $n = 1, 2, 3$. In the case $n = 2$ the term $\dim H^1(V_n, F)$ has to be identified with the superabundance of F. A similar remark applies to $\dim H^1(V_n, F)$ and $\dim H^2(V_n, F)$ in the case $n = 3$. In general, the term $\dim H^n(V_n, F)$ is equal to dim $H^0(V_n, K - F)$, where K is the canonical bundle of V_n. The characteristic class of K is equal to $-c_1$. Kodaira[6] proved that $H^i(V_n, F) = 0$ for $i > 0$, if the characteristic cohomology class of the bundle $F - K$ "contains" a closed positive definite Hermitian form. In this case $F - K$ is called ample. An arbitrary divisor D of V_n represents a complex line bundle,[7] which we also denote by D. Obviously dim $|D| + 1 =$ dim $H^0(V_n, D)$. Denoting the cohomology class of D by d we get

THEOREM 2*. *If $D - K$ is ample (in the sense of Kodaira), then we have*

$$\dim |D| + 1 = \kappa_n \left[e^{d + c_1/2} \prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh \gamma_i/2} \right].$$

(For $n = 1$, this is the classical formula: dim $|D| = d - p$, where $c_1 = 2 - 2p$.)

Applying (M) to the case where W is the bundle of complex covariant p-vectors, yields[5]

THEOREM 3. *Let $h^{p,q}$ be the number of harmonic forms of type (p, q) on V_n. We have*

$$\chi^p(V_n) = \chi(V_n, W) = \sum_{q=0}^{n} (-1)^q h^{p,q} = \kappa_n \left[\left(\sum e^{-(\gamma_{i_1} + \ldots + \gamma_{i_p})} \right) \cdot \prod_{i=1}^{n} \frac{-\gamma_i}{e^{-\gamma_i} - 1} \right].$$

The sum on the right side has to be extended over all the $\binom{n}{p}$ possible combinations.

We attach to every algebraic variety the polynomial

$$\chi_y(V_n) = \sum_{p=0}^{n} \chi^p(V_n) y^p$$

and get by easy calculations

THEOREM 3*. *We have*

$$\chi_y(V_n) = \kappa_n \left[\prod_{i=1}^{n} Q_y(\gamma_i) \right],$$

where

$$Q_y(x) = \frac{-x(y + 1)}{e^{-x(y+1)} - 1} - yx.$$

Theorem 3* is for $y = 0$ a restatement of Theorem 1. For $y = -1$ it gives that

$$\chi_{-1}(V_n) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}$$

is equal to the Euler-Poincaré characteristic c_n of V_n. For $y = 1$ we obtain

THEOREM 4. *We have*

$$\chi_1(V_n) = \sum_{p,q=0}^{n} (-1)^q h^{p,q} = \kappa_n \left[\prod_{i=1}^{n} \frac{\gamma_i}{\operatorname{tgh} \gamma_i} \right].$$

Theorem 4 is known even for Kähler manifolds, because the left side of the equation is equal to the index $\tau(V_n)$.[8] The right side is also equal to $\tau(V_n)$ by the index theorem of the author.[1] The main theorem admits many further applications which we do not mention in this note.

2. A Sketch of the Proof of the Main Theorem.—In section 1 we derived the Theorems 1–4 from the main theorem (M). Actually the proof of (M) goes almost in the opposite direction and involves the following steps.

2.1. Proof of Theorem 1 by Means of the Known Theorem 4.—A complex manifold M_m is called a split manifold, if the structure group $GL(m, C)$ of its tangential bundle can be reduced complex analytically to the group $\Delta(m, C)$ of all triangular matrices (all entries above the diagonal are 0). Let V_n be an arbitrary (non-singular) algebraic variety, the tangential bundle has $GL(n, C)$ as structure group. Hence we can construct the associated bundle with $GL(n, C)/\Delta(n, C)$ as fibre. We obtain in this way an algebraic variety V_m^* of dimension $m = n + {}^1/_2 n(n-1)$. The fact that V_m^* is algebraic can be proved, for example, by using the general theorem of Kodaira[9] that every complex projective bundle over an algebraic variety is an algebraic variety. It can be proved that V_m^* is a split manifold. It is almost obvious that $\Pi(V_m^*) = \Pi(V_n)$. On the other hand, it follows from Theorem 4.2 in the preceding note[1] that $T(V_m^*) = T(V_n)$. Hence it suffices to prove Theorem 1 for all *algebraic* split varieties. Now let V_m^* be an arbitrary algebraic split variety. Then we have over V_m^* an increasing sequence of bundles W_i (in the sense of section 1), where W_i has C_i as fibre and where W_m is the tangential bundle of V_m^* ($0 \leqq i \leqq m$). We denote the complex line bundle W_i/W_{i-1} by A_i ($1 \leqq i \leqq m$) and its cohomology class by a_i. We use the notation of the preceding note and obtain readily the formula[10]

$$2^m T(V_m^*) = \sum_{j=0}^{m} \sum \tau(a_{r_1}, a_{r_2}, \ldots, a_{r_j}). \tag{1}$$

The inner sum has to be extended always over all the possible $\binom{m}{j}$ combinations. By the four-term formula of Kodaira-Spencer[11] we define in the faisceau theory virtual indices χ_1. This can be done for all algebraic varieties. (It can be done not only for χ_1, but even for χ_y.) We know that $\chi_1 = \tau$ for all non-singular algebraic varieties, (Theorem 4). Since χ_1 and τ both fulfill the "functional equation" 3.3 of the preceding note,[1] we get by an induction argument on the dimension that χ_1 and τ also coincide in the virtual case.[12]

By some rather complicated calculations based on exact sequences of sheaves we can prove the following formula which corresponds to (1)

$$2^m \Pi(V_m^*) = \sum_{j=0}^{m} \sum \chi_1(A_{r_1} \circ A_{r_2} \circ \ldots \circ A_{r_j}) \tag{2}$$

Here the terms χ_1 denote virtual indices.[13] Formulae (1) and (2) imply Theorem 1 for split varieties, and hence Theorem 1 is proved in general.

2.2. Proof of Theorem 2.—The virtual Π and the virtual T both fulfill the "functional equation" 4.3 of the preceding note. Hence the usual induction argument already used for χ_1 in 2.1 proves that Π and T also coincide in the virtual case. This proves Theorem 2 (i.e., the main theorem for $q = 1$). Moreover, the main theorem follows immediately for all cases in which the bundle W admits the triangular group $\Delta(q, C)$ as structure group.

2.3. Proof of the Main Theorem.—We use the notation of the first section. We construct over V_n the bundle V_m^* with fibre $GL(q, C)/\Delta(q, C)$ which is associated with W. The manifold V_m^* is, according to Kodaira's theorem,[9] algebraic ($m = n + {}^1/_2q(q - 1)$). We lift the bundle W up to V_m^* and call the lifted bundle W^*. The bundle W^* splits, i.e., admits $\Delta(q, C)$ as structure group. Hence Theorem 2 implies that $\chi(V_m^*, W^*)$ is given correctly by the formula of the main theorem. A spectral sequence argument shows that $\chi(V_m^*, W^*) = \chi(V_n, W)$. On the other hand, we can prove by a slight generalization of the Theorem 4.2 of the preceding note[1] that the formula of the main theorem gives for $\chi(V_m^*, W^*)$ and $\chi(V_n, W)$ the same values. This concludes the proof of the main theorem.

Remark.—The formulae (1) and (2) are special cases of more general formulae valid for χ_y with y regarded as indeterminate. The "functional equations" for $\Pi = \chi_0 = T$ and for $\chi_1 = \tau$ are special cases of a "functional equation" valid for χ_y.

[1] These PROCEEDINGS, **39**, 951–956 (1953). We use the notations of the preceding note.

[2] Complex dimensions are indicated by a subscript.

[3] All the cohomology groups occurring in this note have finite dimensions. See Cartan, H., and Serre, J. P., *Compt. rend. acad. sci., Paris*, **237**, 128–130 (1953), and Kodaira, K., these PROCEEDINGS, **39**, 865–868 (1953).

[4] Kodaira, K., and Spencer, D. C., these PROCEEDINGS, **39**, 641–649, 868–877 (1953). K. Kodaira, *loc. cit.* in ref. 3.

[5] Here we use the theorem of Dolbeault (*Compt. rend. acad. sci., Paris*, **236**, 175–177 (1953)) which states that $H^q(V_n, W^{(p)}) \cong H^{p,q}$, where $W^{(p)}$ is the bundle of covariant p-vectors and where $H^{p,q}$ is the linear space of all harmonic forms of type (p, q) on V_n.

[6] Kodaira, K., "On a Differential-Geometric Method in the Theory of Analytic Stacks," these PROCEEDINGS, **39**, 1268–1273 (1953).

[7] Conversely every complex line bundle can be represented by a divisor. See Kodaira, K., and Spencer, D. C., *loc. cit.* in ref. 4, p. 874.

[8] Hodge, W. V. D., *Proc. Int. Congr. Math.*, **I**, 182–192 (1952), and *Proc. Lond. Math. Soc.* (*3*), **1**, 104–117 (1951).

[9] Kodaira, K., not yet published.

[10] The sum on the right side starts for $j = 0$ with $\tau(V_n^*)$. The formula (1) was used by the author to prove Theorem 4.1 of the preceding note (*loc. cit.* in ref. 1). See also Hirzebruch, F., "The Index of an Oriented Manifold and the Todd Genus of an Almost Complex Manifold," Notes, Princeton University, 1953 (mimeographed).

[11] Kodaira, K., and Spencer, D. C., "On a Theorem of Lefschetz and the Lemma of Enriques-Severi-Zariski," these PROCEEDINGS, **39**, 1273–1278 (1953), formula (14).

[12] The fact that χ_1 fulfills the functional equation can be obtained from the four-term formula by some calculations.

[13] The small circle $\circ$ denotes "virtual intersection."