Localised eigenfunctions in Šeba billiards

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Abstract

We describe some new families of quasimodes for the Laplacian perturbed by the addition of a potential formally described by a Dirac delta function. As an application we find, under some additional hypotheses on the spectrum, subsequences of eigenfunctions of Šeba billiards that localise around a pair of unperturbed eigenfunctions.

1   Introduction

One of the unsolved questions in the analysis of quantum eigenfunctions concerns possible limiting distributions as the eigenvalue tends to infinity. For eigenfunctions of the Laplace operator on certain surfaces with arithmetical properties, it has been proved [1, 2] that there is only one possible limit; all sequences of eigenfunctions become uniformly distributed. On the other hand, Hassell [3] has proved the existence of chaotic billiard domains in $\mathbb{R}^2$ for which zero-density subsequences of eigenfunctions fail to equi-distribute in the limit.

We consider the Laplace operator plus potential supported at a single point. Such a potential has been variously referred to as delta-interaction potential, Fermi pseudo-potential or zero-range potential in different parts of the literature. Mathematically this operator can be constructed using the tools of self-adjoint extension theory.

We will prove our results for the case where the underlying space is a compact 2-dimensional manifold, for which the Laplace operator has eigenfunctions and eigenvalues denoted by Φ_j and E_j respectively. We perturb this operator with a delta potential supported at the point p, which will remain fixed throughout, and suppressed from notations. This perturbation can be realised by a 1-parameter family of self-adjoint operators, H_Θ, indexed by an angle Θ which controls the strength of the perturbation.

We fix a finite interval $I \subseteq \mathbb{R}$ containing at least one E_j, and define, for notational convenience,

$$\zeta_I(s, \lambda) := \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \lambda)^s}. \tag{1}$$

Let $\sigma \in [0, 1]$. We define

$$\psi(x) := \sum_{E_j \in I} \frac{\overline{\Phi_j(p)}}{E_j - \mu} \Phi_j(x) + \sigma \sum_{E_j \notin I} \left(E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right) \frac{\overline{\Phi_j(p)}}{1 + E_j^2} \Phi_j(x), \tag{2}$$

for μ a solution to

$$\zeta_I(1, \mu) = \sigma \sum_{E_j \in I} \left(E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right) \frac{|\Phi_j(p)|^2}{1 + E_j^2}. \tag{3}$$

Our main results are as follows:

Theorem 1.1. *The pair (ψ, μ) is a quasimode for H_Θ with discrepancy d, where*

$$d^2 = \frac{(1 - \sigma)^2 \zeta_I(0, \mu) + \sigma^2 \sum_{E_j \notin I} \left(1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta}(E_j - \mu) \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}}{\zeta_I(2, \mu) + \sigma^2 \sum_{E_j \notin I} \left(E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right)^2 \frac{|\phi_j(p)|^2}{(1 + E_j^2)^2}} \tag{4}$$

Furthermore, if ψ_1, ψ_2 are defined by (2) for $\mu_1 \neq \mu_2$, two solutions of (3) then

$$\langle \psi_1, \psi_2 \rangle = \sigma^2 \sum_{E_j \notin I} \left(E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}. \tag{5}$$

The construction of families of quasimodes is a key step in Hassell's proof [3], as well as the proofs of many recent results on localisation of quantum eigenfunctions [4, 5, 6, 7, 8, 9]. One reason for this is that quasimodes can often be used to approximate eigenfunctions. In general (see the introduction to section 3 for precise statements) the smaller the discrepancy, the closer quasimodes are to true eigenfunctions. For this reason it is important to know when the discrepancy can be made small. In this direction we have the following corollary to theorem 1.1:

Corollary 1.2. *Let $\sigma = 1$ and let $I = [0, T]$ where $T > E_1$. Then the discrepancy d of the quasimode ψ satisfies*

$$d \ll \frac{\mu}{\sqrt{T}}. \tag{6}$$

Let $\sigma = 0$ and I be any interval containing at least two E_j. If $\mu \in I$ then the discrepancy d of ψ satisfies

$$d \leqslant \frac{1}{\sqrt{2}} \ell(I), \tag{7}$$

where $\ell(I)$ is the length of I. If, additionally, I contains precisely two E_j, then we have

$$d \leqslant \frac{1}{2} \ell(I). \tag{8}$$

In particular, the quasimodes with $\sigma = 1$ and μ held fixed or slowly growing, can be made arbitrarily precise by choosing T as large as desired.

We are interested in ascertaining when true eigenfunctions of H_Θ have mass supported on our quasimodes. Without any assumptions on the spectrum of the Laplacian we can prove the following.

Proposition 1.3. *For any consecutive eigenvalues $E_{\mathrm{a}} < E_{\mathrm{b}} < E_{\mathrm{c}} < E_{\mathrm{d}}$ from the sequence $(E_j)_{j=1}^{\infty}$, let $I = [E_{\mathrm{b}}, E_{\mathrm{c}}]$ and take $\sigma = 0$. Choose μ so that $\mu \in I$. Then there is an eigenfunction ϕ of H_{Θ} with eigenvalue in the interval $(E_{\mathrm{a}}, E_{\mathrm{d}})$ such that*

$$|\langle \phi, \psi \rangle| \geqslant \frac{\|\psi\|}{\sqrt{3}} \left(1 - \frac{(E_{\mathrm{c}} - E_{\mathrm{b}})^2}{4 \min\{E_{\mathrm{d}} - E_{\mathrm{c}}, E_{\mathrm{b}} - E_{\mathrm{a}}\}^2} \right)^{1/2}. \tag{9}$$

Proposition 1.3 is most interesting when the sequence of eigenfunctions Φ_j do not equidistribute. (For example if they are solutions to a PDE which is subject to separation of variables; see below.) Then, by considering an infinite subset of the spectrum $\{E_j\}$ along which the right-hand side of (9) is bounded away from zero, proposition 1.3 proves the existence of a sequence of eigenfunctions of H_{Θ} which fail to equi-distribute. Such a subset of $\{E_j\}$ does exist since the mean level spacing is constant.

Clearly the best that proposition 1.3 can achieve is to prove that a sequence of quasi-modes has an overlap of up to $1/\sqrt{3}$ with a subsequence of true eigenfunctions. In order to prove that a sequence of quasimodes converges *fully* towards a sequence of eigenvalues of H_{Θ} we need to make some assumptions on the spectrum of the Laplacian. Sufficient conditions for this and a precise statement of the result (theorem 4.4) are given in section 4.

The history of the study of the spectral properties of differential operators perturbed by the addition of a delta scatterer goes back at least to [10], in which a one-dimensional lattice of delta interactions was used to model an electron moving in a crystal lattice. A comprehensive historical review is given in the appendix to the book [11].

Part of our interest in the subject comes from the Šeba billiard which was introduced in [12]. In this work, a hard-walled rectangular billiard with a potential supported at a single point was considered. In terms of classical dynamics, the motion is integrable, since only a zero-measure set of trajectories meet the point at which the potential is supported. However, diffraction effects are introduced when one considers the quantum spectrum of the corresponding Schrödinger operator.

Šeba billiards have become important since the observation [13, 14] that the quantum spectral statistics belong to a new universality class, different from the classes from random matrix theory conjecturally associated to chaotic dynamical systems [15, 16] or the statistics of a Poisson process conjecturally associated to fully-integrable dynamical systems [17]. It is now known that general integrable systems perturbed by the addition of such a localised scatterer also belong to the same universality class [18], as do quantum Neumann star graphs [19, 20]. Characteristic features of the spectral statistics of this universality class are an exponential decay of large level spacings, together with level repulsion.

Several analytical studies of these spectral statistics have been made [18, 21, 22, 23, 24, 25]. Typically, a key feature of these arguments is the assumption of Poissonian behaviour for the eigenvalues of the billiard table without scatterer, a conjectured consequence of the integrable dynamics (the Berry-Tabor conjecture) [17].

In the final section of this article we apply theorem 4.4 to the original Šeba billiard. Our final result is a proof that there exists a subsequence of eigenfunctions of the Šeba billiard that become localised on a pair of consecutive eigenfunctions of the unperturbed billiard, if the spectrum of the unperturbed billiard satisfies an assumption which is consistent with the Berry-Tabor conjecture.

This result is a rigorous derivation of a formal argument first proposed in [26] and mirrors a related result proved for quantum graphs with a star-shaped connectivity [27]. These so-called quantum star graph can be considered as a singular perturbation of a disconnected set of one-dimensional bonds, each supporting a wave-function. In [27] the existence of subsequences of eigenfunctions that become localised on a pair of bonds was proved. This is exactly analogous to the localisation onto a pair of unperturbed billiard eigenfunctions in theorem 4.4. In both [27] and theorem 4.4 the main idea of the proof is to show localisation in an eigenfunction with eigenvalue lying between two closely-spaced eigenvalues of the unperturbed problem.

2 Realisation of the perturbed operator

Let $\mathcal{M}$ be a compact 2-dimensional Riemannian manifold, possibly with piecewise-smooth boundary, and let Δ be a self-adjoint Laplacian on $\mathcal{M}$.

The realisation of the operator formally defined by

$$H = -\Delta + c\delta(x - p), \tag{10}$$

where $p \in \mathcal{M}$ and δ is the Dirac delta function, using the theory of self-adjoint extensions is given in many places in the literature. We refer the reader to [28, 29] for the details. Here we recapitulate only that which is necessary to fix notations. We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and inner product of $L^2(\mathcal{M})$.

Since $\mathcal{M}$ is compact, $-\Delta$ has a complete basis of eigenfunctions, Φ_j, with corresponding eigenvalues E_j which we write in non-decreasing order.

We will remove from the list of eigenvalues any E_j for which $\Phi_j(p) = 0$. Such eigenfunctions are not affected by a delta-scatterer at p, and so it is convenient to exclude them from the spectrum. This further allows us to assume that the spectrum $\{E_j\}$ is simple, without losing generality.

To see this, consider an eigenspace of dimension $r > 1$ spanned by the eigenfunctions $\{\tilde{\phi}_1, \ldots \tilde{\phi}_r\}$. Then the vectors $(\tilde{\phi}_1(p), \ldots, \tilde{\phi}_r(p))^{\mathrm{T}}$ and $(R, 0, \ldots, 0)^{\mathrm{T}}$ in $\mathbb{C}^r$, where

$$|R|^2 = \sum_{i=1}^{r} |\tilde{\phi}_i(p)|^2, \tag{11}$$

have identical norm. This means that we can find a unitary $r \times r$ matrix mapping the first vector to the second. Multiplying U by the vector of eigenfunctions $(\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^{\mathrm{T}}$ leads to a new basis for the eigenspace, in which all but the first eigenfunction vanishes at the point p, and the corresponding eigenvalue is counted with multiplicity one.

The resulting spectrum is therefore ordered so that

$$E_1 < E_2 < E_3 \cdots \tag{12}$$

We will frequently use Weyl's law with remainder estimate [30]:

$$N(E) := \sum_{E_j \leqslant E} |\Phi_j(p)|^2 = \frac{E}{4\pi} + \mathrm{O}(E^{1/2}), \tag{13}$$

4

where the implied constant[1] may depend on the position of the point $p \in \mathcal{M}$.

Define

$$g_{\pm i}(x) := \sum_{j=1}^{\infty} \frac{\Phi_j(x)\overline{\Phi_j(p)}}{E_j \mp i}. \tag{14}$$

Then $g_{\pm i} \in L^2(\mathcal{M})$ and in fact they are the Green functions for the resolvent of $-\Delta$ at the imaginary energies $\pm i$, satisfying

$$\langle f, g_{\pm i} \rangle = (-\Delta \pm i)^{-1} f(p). \tag{15}$$

In particular,

$$\langle \Phi_j, g_{\pm i} \rangle = ((-\Delta \pm i)^{-1}\Phi_j)(p)$$
$$= \frac{\Phi_j(p)}{E_j \pm i}, \tag{16}$$

which will be useful to know later.

Let

$$\mathcal{D}_p := \{f \in \mathrm{Dom}(\Delta) : \langle f, \delta_p \rangle = 0\}, \tag{17}$$

and define the operator H_0 with domain $\mathcal{D}_p$ by

$$H_0 : f \mapsto -\Delta f. \tag{18}$$

H_0 is a symmetric, but not self-adjoint operator. In fact its deficiency subspaces are spanned by $g_{\pm i}$.

It follows from the von Neumann theory [31] that,

$$\mathrm{Dom}(H_0^*) = \mathcal{D}_p \oplus \mathrm{span}\{g_i, g_{-i}\}. \tag{19}$$

Since the deficiency indices are equal, H_0 possesses self-adjoint extensions, constructed as follows.

First of all, note that we can write for $\psi \in \mathrm{Dom}(H_0^*)$,

$$\psi = \hat{\psi} + a_+(\psi)g_i + a_-(\psi)g_{-i}, \tag{20}$$

where $\hat{\psi} \in \mathcal{D}_p$ and $a_\pm(\psi) \in \mathbb{C}$. In fact we have

$$H_0^*\psi = H_0\hat{\psi} + i a_+(\psi)g_i - i a_-(\psi)g_{-i}. \tag{21}$$

Since the deficiency indices of H_0 are both equal to 1, there is a 1-parameter family of self-adjoint extensions, H_Θ, $0 < \Theta \leqslant 2\pi$ with

$$\mathrm{Dom}(H_\Theta) = \{\psi \in \mathrm{Dom}(H_0^*) : a_-(\psi) = -e^{i\Theta}a_+(\psi)\}. \tag{22}$$

We take the self-adjoint operator H_Θ to be the realisation of the formal operator (10).

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[1]The notations $f = \mathrm{O}(g)$ and $f \ll g$ both mean that there exists a positive constant C (the "implied constant") such that $f \leqslant Cg$.

3 Quasimodes

3.1 Definitions and basic properties

Let H be a self-adjoint operator in a Hilbert space, without continuous spectrum.

Definition 3.1. *A quasimode of H with discrepancy d is a pair $(\psi, \mu) \in \mathrm{Dom}(H) \times \mathbb{R}$ such that*

$$\|(H - \mu)\psi\| \leqslant d\|\psi\|. \tag{23}$$

We are interested in the situation when the quasi-eigenvalue μ and quasi-eigenfunction ψ approximate true eigenvalues λ_j and eigenfunctions ϕ_j of H. In this direction, the following classical results apply (see, e.g. [32, 9])

For a quasimode with discrepancy d, the interval $[\mu - d, \mu + d]$ contains at least one eigenvalue of H.

If we consider instead, the interval $[\mu - M, \mu + M]$ where $M > 0$, then

$$\sum_{\lambda_j \notin [\mu - M, \mu + M]} |\langle \psi, \phi_j \rangle|^2 \leqslant \frac{d^2}{M^2} \|\psi\|^2. \tag{24}$$

In particular, if ψ is normalised, and the interval $[\mu - M, \mu + M]$ contains only a single eigenvalue with eigenfunction ϕ, then there is a phase $\chi \in [0, 2\pi)$ such that

$$\|\phi - \mathrm{e}^{\mathrm{i}\chi}\psi\| \leqslant \frac{2d}{M}. \tag{25}$$

These results will be the main tools by which we relate the quasimodes constructed in the next subsection to the eigenfunctions and eigenvalues of H_Θ.

3.2 Quasimodes of delta perturbations

Let $I \subseteq \mathbb{R}$ be a finite interval containing at least one point E_j of the spectrum of $-\Delta$. Let $\sigma \in [0, 1]$. We will associate to the interval I a family of quasimodes parametrised by σ.

We first define

$$\psi_{\sigma, I, z} := \sum_{E_j \in I} \frac{\overline{\Phi_j(p)}}{E_j - z} \Phi_j + \frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}} \mathcal{P}_{I^{\mathrm{c}}} \left(g_{\mathrm{i}} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}} \right), \tag{26}$$

where $\mathcal{P}_S$ is the spectral projection operator onto the set S,

$$\mathcal{P}_S f := \sum_{E_j \in S} \langle f, \Phi_j \rangle \Phi_j, \tag{27}$$

and I^{c} is the complement to I. We have the following:

Lemma 3.2. *For $z \neq E_j$ for any $E_j \in I$, the function $\psi_{\sigma, I, z}$ satisfies*

$$\|\psi_{\sigma, I, z}\|^2 = \zeta_I(2, z) + \sigma^2 \sum_{E_j \notin I} \left(E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}, \tag{28}$$

with the second term being bounded by a constant independent of I, z and $\sigma \in [0, 1]$.

Proof. We have

$$\|\psi_{\sigma,I,z}\|^2 = \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - z)^2} + \frac{\sigma^2}{|1 - \mathrm{e}^{\mathrm{i}\Theta}|^2} \left\| \mathcal{P}_{I^c}(g_{\mathrm{i}} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}}) \right\|^2. \tag{29}$$

By (16) we get

$$\begin{aligned}
\langle \Phi_j, g_{\mathrm{i}} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}} \rangle &= \left(\frac{1}{E_j + \mathrm{i}} - \frac{\mathrm{e}^{-\mathrm{i}\Theta}}{E_j - \mathrm{i}} \right) \Phi_j(p) \\
&= \frac{E_j(1 - \mathrm{e}^{-\mathrm{i}\Theta}) - \mathrm{i}(1 + \mathrm{e}^{-\mathrm{i}\Theta})}{1 + E_j^2} \Phi_j(p) \\
&= (1 - \mathrm{e}^{-\mathrm{i}\Theta}) \left(\frac{E_j}{1 + E_j^2} - \frac{\sin\Theta}{1 - \cos\Theta} \frac{1}{1 + E_j^2} \right) \Phi_j(p), \tag{30}
\end{aligned}$$

using

$$\mathrm{i} \frac{1 + \mathrm{e}^{\mathrm{i}\Theta}}{1 - \mathrm{e}^{\mathrm{i}\Theta}} = \frac{-\sin\Theta}{1 - \cos\Theta}. \tag{31}$$

By Parseval's identity

$$\frac{1}{|1 - \mathrm{e}^{\mathrm{i}\Theta}|^2} \left\| \mathcal{P}_{I^c}(g_{\mathrm{i}} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}}) \right\|^2 = \sum_{E_j \notin I} \left(E_j - \frac{\sin\Theta}{1 - \cos\Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}. \tag{32}$$

Finally, to show that the right-hand side of (32) is finite and does not depend on I, we observe that it is bounded by

$$\sum_{j=1}^{\infty} \left(E_j - \frac{\sin\Theta}{1 - \cos\Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2} = \int_0^{\infty} \left(\frac{1}{1 + t^2} \left(t - \frac{\sin\Theta}{1 - \cos\Theta} \right) \right)^2 \mathrm{d}N(t), \tag{33}$$

writing the sum as a Riemann-Stieltjes integral. The spectral counting function $N(t)$ was defined in (13). Integrating by parts, we get

$$\sum_{E_j \notin I} \left(E_j - \frac{\sin\Theta}{1 - \cos\Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2} \leqslant - \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{1 + t^2} \left(t - \frac{\sin\Theta}{1 - \cos\Theta} \right) \right)^2 N(t)\, \mathrm{d}t. \tag{34}$$

Since $N(t) \ll t$ by Weyl's law, we see that the integral in (34) is a finite constant. $\qquad\square$

Let $\mu = \mu(\sigma, I)$ be a solution to

$$\sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{E_j - \mu} = \frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}} \mathcal{P}_I(g_{\mathrm{i}} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}})(p). \tag{35}$$

Then the pair $(\psi_{\sigma,I}, \mu)$ is a quasimode for H_Θ, where $\psi_{\sigma,I} \coloneqq \psi_{\sigma,I,\mu}$. This follows from the following proposition:

Proposition 3.3. *The function $\psi_{\sigma,I}$ belongs to $\mathrm{Dom}(H_\Theta)$ and satisfies*

$$\|(H_\Theta - \mu)\psi_{\sigma,I}\|^2 = (1 - \sigma)^2 \zeta_I(0, \mu) + \sigma^2 \sum_{E_j \notin I} \left(1 + E_j \mu + \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu) \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}. \tag{36}$$

7

Proof. First of all, let us prove that $\psi_{\sigma,I} \in \mathrm{Dom}(H_\Theta)$.

We can write

$$\psi_{\sigma,I} = \sum_{E_j \in I} \frac{\overline{\Phi_j(p)}}{E_j - \mu} \Phi_j - \frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}} \mathcal{P}_I(g_\mathrm{i} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}}) + \frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}}(g_\mathrm{i} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}}). \tag{37}$$

Using (30) we can express this as

$$\psi_{\sigma,I} = \hat{\psi}_{\sigma,I} + \frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}} \left(g_\mathrm{i} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}} \right), \tag{38}$$

where

$$\hat{\psi}_{\sigma,I}(x) := \sum_{E_j \in I} \left(\frac{1}{E_j - \mu} - \frac{\sigma E_j}{1 + E_j^2} + \frac{\sigma \sin\Theta}{1 - \cos\Theta} \frac{1}{1 + E_j^2} \right) \overline{\Phi_j(p)} \Phi_j(x). \tag{39}$$

Now observe that due to the definition (35) of μ, $\hat{\psi}_{\sigma,I}(p) = 0$, so $\hat{\psi}_{\sigma,I} \in \mathcal{D}_p$. Thus (38) justifies the assertion $\psi_{\sigma,I} \in \mathrm{Dom}(H_\Theta)$.

Since H_0^* is an extension of H_Θ, we have

$$(H_\Theta - \mu)\psi_{\sigma,I} = (H_0 - \mu)\hat{\psi}_{\sigma,I} + \frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}} \left((\mathrm{i} - \mu)g_\mathrm{i} + \mathrm{e}^{\mathrm{i}\Theta}(\mathrm{i} + \mu)g_{-\mathrm{i}} \right). \tag{40}$$

Now,

$$(H_0 - \mu)\hat{\psi}_{\sigma,I} = \sum_{E_j \in I} \left(1 - \frac{\sigma E_j(E_j - \mu)}{1 + E_j^2} + \frac{\sigma \sin\Theta}{1 - \cos\Theta} \frac{E_j - \mu}{1 + E_j^2} \right) \overline{\Phi_j(p)} \Phi_j$$

$$= \sum_{E_j \in I} \left(1 + (1 - \sigma)E_j^2 + \sigma E_j \mu + \frac{\sigma \sin\Theta}{1 - \cos\Theta}(E_j - \mu) \right) \frac{\overline{\Phi_j(p)}}{1 + E_j^2} \Phi_j. \tag{41}$$

Using (16) we find

$$\langle \Phi_j, (\mathrm{i} - \mu)g_\mathrm{i} + \mathrm{e}^{\mathrm{i}\Theta}(\mathrm{i} + \mu)g_{-\mathrm{i}} \rangle = \left(-\frac{\mathrm{i} + \mu}{E_j + \mathrm{i}} \Phi_j(p) + \mathrm{e}^{-\mathrm{i}\Theta} \frac{-\mathrm{i} + \mu}{E_j - \mathrm{i}} \Phi_j(p) \right) \tag{42}$$

$$= \frac{1 - \mathrm{e}^{-\mathrm{i}\Theta}}{1 + E_j^2} \left(-(1 + E_j \mu) - \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu) \right) \Phi_j(p),$$

again using (31). This leads to

$$\frac{\sigma}{1 - \mathrm{e}^{\mathrm{i}\Theta}}(H_\Theta - \mu)(g_\mathrm{i} - \mathrm{e}^{\mathrm{i}\Theta} g_{-\mathrm{i}}) = -\sigma \sum_{j=1}^{\infty} \left(1 + E_j \mu + \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu) \right) \frac{\overline{\Phi_j(p)}}{1 + E_j^2} \Phi_j, \tag{43}$$

and combining this with (41), we get

$$(H_\Theta - \mu)\psi_{\sigma,I} = (1 - \sigma) \sum_{E_j \in I} \overline{\Phi_j(p)} \Phi_j$$

$$- \sigma \sum_{E_j \notin I} \left(1 + E_j \mu + \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu) \right) \frac{\overline{\Phi_j(p)}}{1 + E_j^2} \Phi_j. \tag{44}$$

8

Since the summations in (44) are over disjoint sets it is easy to calculate the norm:

$$\|(H_\Theta - \mu)\psi_{\sigma,I}\|^2 = (1 - \sigma)^2 \sum_{E_j \in I} |\Phi_j(p)|^2$$

$$+ \sigma^2 \sum_{E_j \notin I} \left(1 + E_j\mu + \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu)\right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}. \quad (45)$$

$\square$

3.2.1 Proof of theorem 1.1

The first part of the theorem follows from lemma 3.2 and proposition 3.3 and the definition of a quasimode.

For the second part, let $\mu_1 \neq \mu_2$ be two solutions of (35). We have

$$\langle \psi_{\sigma,I,\mu_1}, \psi_{\sigma,I,\mu_2} \rangle = \left\langle \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - \mu_1}\Phi_j, \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - \mu_2}\Phi_j \right\rangle + \frac{\sigma^2}{|1 - e^{i\Theta}|^2} \left\| \mathcal{P}_{I^c}(g_i - e^{i\Theta}g_{-i}) \right\|^2$$

$$= \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \mu_1)(E_j - \mu_2)} + \sigma^2 \sum_{E_j \notin I} \left(E_j - \frac{\sin\Theta}{1 - \cos\Theta}\right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}, \quad (46)$$

using (32). By elementary algebra,

$$\frac{1}{(E_j - \mu_1)(E_j - \mu_2)} = \frac{1}{\mu_1 - \mu_2}\left(\frac{1}{E_j - \mu_1} - \frac{1}{E_j - \mu_2}\right), \quad (47)$$

so, since by (35)

$$\sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{E_j - \mu_1} = \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{E_j - \mu_2}, \quad (48)$$

we get

$$\sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \mu_1)(E_j - \mu_2)} = 0. \quad (49)$$

$\square$

3.2.2 Controlling the discrepancy of quasimodes

By tuning the parameter σ, and choosing the interval I accordingly, we can find fix quasimodes with particular properties. In the previous section we have seen that sets of quasimodes with $\sigma = 0$ are orthogonal. We are particularly interested in when the discrepancy is small. In this subsubsection we prove corollary 1.2 that quasimodes with $\sigma = 1$ can be made arbitrarily precise, and that quasimodes with $\sigma = 0$ also can have a simple bound for the discrepancy.

Proof of corollary 1.2. Choosing $I = [0, T]$ for $T > E_1$ with $\sigma = 1$ gives, by theorem 1.1, that the discrepancy of $\psi_{1,I}$ satisfies

$$d^2\|\psi_{1,I}\|^2 = \sum_{E_j \geqslant T} \left(1 + E_j\mu + \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu)\right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}. \tag{50}$$

By lemma 3.2 we see that the norm of $\psi_{1,I}$ is bounded away from 0 by a constant, so that the asymptotics for d are given by the term on the right-hand side of (50). Using Weyl's law, we can estimate

$$\sum_{E_j \geqslant T} \left(1 + E_j\mu + \frac{\sin\Theta}{1 - \cos\Theta}(E_j - \mu)\right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2} \ll \frac{\mu^2}{T}, \tag{51}$$

which can be made arbitrarily small by increasing T.

For the second part with $\sigma = 0$, we have

$$d^2\|\psi_{0,I}\|^2 = \sum_{E_j \in I} |\Phi_j(p)|^2. \tag{52}$$

We observe that splitting the sum in (35) leads to

$$\sum_{\substack{E_j \in I \\ E_j > \mu}} \frac{|\Phi_j(p)|^2}{E_j - \mu} = \sum_{\substack{E_j \in I \\ E_j < \mu}} \frac{|\Phi_j(p)|^2}{\mu - E_j}. \tag{53}$$

Denote by E_+ and E_- the largest and smallest points of the spectrum $(E_j)_{j=1}^\infty$ lying in the interval I. Then

$$\sum_{\substack{E_j \in I \\ E_j > \mu}} |\Phi_j(p)|^2 \leqslant (E_+ - \mu) \sum_{\substack{E_j \in I \\ E_j > \mu}} \frac{|\Phi_j(p)|^2}{E_j - \mu} \tag{54}$$

and

$$\sum_{\substack{E_j \in I \\ E_j < \mu}} |\Phi_j(p)|^2 \leqslant (\mu - E_-) \sum_{\substack{E_j \in I \\ E_j < \mu}} \frac{|\Phi_j(p)|^2}{\mu - E_j}. \tag{55}$$

Adding these inequalities, and using (53), we get

$$\sum_{E_j \in I} |\Phi_j(p)|^2 \leqslant (E_+ - E_-) \sum_{\substack{E_j \in I \\ E_j > \mu}} \frac{|\Phi_j(p)|^2}{E_j - \mu} = (E_+ - E_-) \sum_{\substack{E_j \in I \\ E_j < \mu}} \frac{|\Phi_j(p)|^2}{\mu - E_j}. \tag{56}$$

Since $E_+ - E_- \leqslant \ell(I)$ we get

$$2d^2\|\psi_{0,I}\|^2 \leqslant \ell(I) \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{|E_j - \mu|}. \tag{57}$$

Finally,

$$\ell(I) \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{|E_j - \mu|} \leqslant \ell(I)^2 \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \mu)^2} = \ell(I)^2\|\psi_{0,I}\|^2, \tag{58}$$

noting that $|E_j - \mu| \leqslant \ell(I)$ for $\mu \in I$.

We now consider the case with $\sigma = 0$, and I containing only the two levels E_j, E_{j+1}. We can solve (35) directly to get

$$\mu = \frac{|\Phi_{j+1}(p)|^2 E_j + |\Phi_j(p)|^2 E_{j+1}}{|\Phi_{j+1}(p)|^2 + |\Phi_j(p)|^2}. \tag{59}$$

Substituting this value of μ into the definition of $\psi_{0,I}$ we get

$$\psi_{0,I} = \frac{\overline{\Phi_j(p)}}{E_j - \mu}\Phi_j + \frac{\overline{\Phi_{j+1}(p)}}{E_{j+1} - \mu}\Phi_{j+1} \tag{60}$$

$$= \frac{|\Phi_{j+1}(p)|^2 + |\Phi_j(p)|^2}{E_{j+1} - E_j}\left(\frac{-1}{\Phi_j(p)}\Phi_j + \frac{1}{\Phi_{j+1}(p)}\Phi_{j+1}\right). \tag{61}$$

So,

$$\|\psi_{0,I}\|^2 = d^2\|\psi_{0,I}\|^2\frac{|\Phi_{j+1}(p)|^2 + |\Phi_j(p)|^2}{(E_{j+1} - E_j)^2}\left(\frac{1}{|\Phi_j(p)|^2} + \frac{1}{|\Phi_{j+1}(p)|^2}\right) \tag{62}$$

$$\geqslant \frac{4d^2\|\psi_{0,I}\|^2}{\ell(I)^2}, \tag{63}$$

using the fact that

$$|\Phi_{j+1}(p)|^2 + |\Phi_j(p)|^2\left(\frac{1}{|\Phi_j(p)|^2} + \frac{1}{|\Phi_{j+1}(p)|^2}\right) = 2 + \left|\frac{\Phi_{j+1}(p)}{\Phi_j(p)}\right|^2 + \left|\frac{\Phi_j(p)}{\Phi_{j+1}(p)}\right|^2 \geqslant 4. \tag{64}$$

$\square$

The existence of arbitrarily precise quasimodes can be used to give a new proof of the often-used representation for eigenvalues and eigenfunctions of rank-one perturbations (see e.g. [18, 28, 24, 33])

Theorem 3.4. *The solutions λ to the equation*

$$\sum_{j=1}^{\infty}\left(\frac{1}{E_j - \lambda} - \frac{E_j}{1 + E_j^2}\right)|\Phi_j(p)|^2 = \frac{\sin\Theta}{1 - \cos\Theta}\sum_{j=1}^{\infty}\frac{|\Phi_j(p)|^2}{1 + E_j^2}, \tag{65}$$

are eigenvalues of H_Θ with corresponding eigenfunctions given by

$$\phi(x) = \sum_{j=1}^{\infty}\frac{\overline{\Phi_j(p)}}{E_j - \lambda}\Phi_j(x). \tag{66}$$

Note that the left-hand side of (65) converges pointwise, and (66) converges in $L^2(\Omega)$.

By analysing the resolvent, it is possible to extend theorem 3.4 to get the following [28, Theoreme 2],

Theorem 3.5. *Apart from the solutions to (65), there are no other points of the spectrum of H_Θ in any of the intervals (E_M, E_{M+1}).*

4   Localisation results

In this section we will consider the extent to which eigenfunctions of H_Θ can be approximated by quasimodes. In particular we will focus on the quasimodes with $\sigma = 0$. First we shall prove proposition 1.3, which is straightforward. Then we shall show that strengthening the assumptions made on the spectrum of Δ leads to a proof of full convergence.

Proof of proposition 1.3. The length of the interval I is $\ell(I) = E_\mathrm{c} - E_\mathrm{b}$. Let $M = \min\{E_\mathrm{d} - \mu, \mu - E_\mathrm{a}\} \geqslant \min\{E_\mathrm{d} - E_\mathrm{c}, E_\mathrm{b} - E_\mathrm{a}\}$. By applying (24) with this M we get,

$$\sum_{\lambda_j \in [E_\mathrm{a}, E_\mathrm{d}]} |\langle \psi_{0,I}, \phi_j \rangle|^2 \geqslant \|\psi_{0,I}\|^2 \left(1 - \frac{\ell(I)^2}{4 \min\{E_\mathrm{d} - E_\mathrm{c}, E_\mathrm{b} - E_\mathrm{a}\}^2} \right). \qquad (67)$$

From theorem 3.5 there are only three eigenvalues of H_Θ in the interval $[E_\mathrm{a}, E_\mathrm{d}]$. It therefore follows that for at least one of these three eigenfunctions its inner-product squared with $\psi_{0,I}$ is at least $\frac{1}{3}$ of the right-hand side of (67). $\qquad \square$

We now consider how to improve proposition 1.3 at the expense of making further assumptions about the spectrum of $-\Delta$. For simplicity we will focus henceforth on the choice of parameter $\Theta = \pi$.
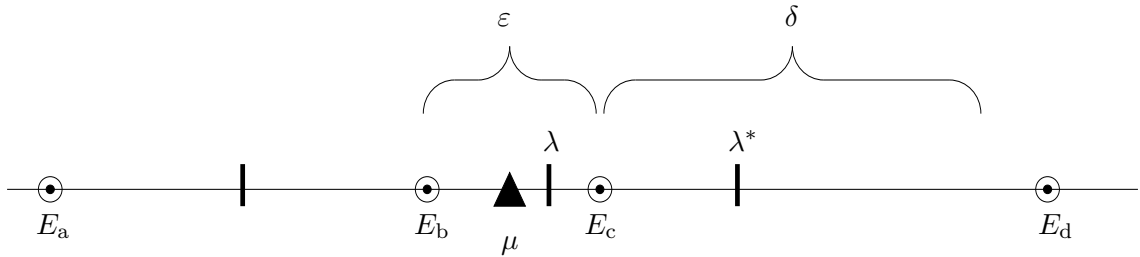

Figure 1: Part of the spectrum of $-\Delta$ and H_π. Vertical bars denote eigenvalues of H_π, and circles denote eigenvalues of $-\Delta$. The triangle μ is a quasi-eigenvalue approximating λ. See main text for further explanation.

In figure 1 a cartoon of part of the spectrum of H_π and $-\Delta$ is displayed. Highlighted are four consecutive eigenvalues of $-\Delta$, labelled $E_\mathrm{a}, E_\mathrm{b}, E_\mathrm{c}$ and E_d, chosen so that $E_\mathrm{c} - E_\mathrm{b} \leqslant \varepsilon$. (The positions of all points depend on ε.)

Between E_b and E_c is an eigenvalue, λ, of H_π.

We find a quasimode $\psi_{0,I}$ associated to the interval $I = [E_\mathrm{b}, E_\mathrm{c}]$ with quasi-eigenvalue μ approximating λ. By corollary 1.2 the discrepancy of this quasimode is no greater than $\varepsilon/2$. Between E_c and E_d is another eigenvalue λ^* of H_π. In order to be able to apply (25), we need to be sure that λ^* is not too close to E_c. An argument to show that this is the case is given below.

The eigenvalue between E_a and E_b can be handled with a similar method.

We shall make the following assumption on the spectral sequence of Δ:

12

Assumption 4.1. *For some $0 < q < 1/2$ and $1 < \rho < 2(1-q)$, there exists a sequence $(\varepsilon_n)_{n=1}^{\infty}$, $\varepsilon_n \downarrow 0$ such that for each n there are four consecutive eigenvalues, $E_{\mathrm{a}}(n) < E_{\mathrm{b}}(n) < E_{\mathrm{c}}(n) < E_{\mathrm{d}}(n) \ll \varepsilon_n^{-\rho}$, satisfying*

$$
\begin{aligned}
E_{\mathrm{c}} - E_{\mathrm{b}} &\ll \varepsilon_n \\
E_{\mathrm{d}} - E_{\mathrm{c}} &\gg \varepsilon_n^q \\
E_{\mathrm{b}} - E_{\mathrm{a}} &\gg \varepsilon_n^q.
\end{aligned}
\tag{68}
$$

as $n \to \infty$.

Assumption 4.1 asserts that the positions of eigenvalues of $-\Delta$ occur with the spacings as described above, and furthermore that this does not happen too high up in the spectrum. This upper bound is necessary as a consequence of the non-uniform convergence in λ of the series in (65). In appendix B we show that assumption 4.1 is satisfied almost surely if the sequence (E_j) comes from a Poisson process. In this sense, assumption 4.1 is consistent with the Berry-Tabor conjecture, if $-\Delta$ is the Hamiltonian corresponding to an integrable dynamical system.

We shall also assume a lower bound for the absolute values of the eigenfunctions Φ_j at the point p.

Assumption 4.2. *There exists a constant $c_0 > 0$ independent of j such that*

$$
|\Phi_j(p)| \geqslant c_0.
\tag{69}
$$

Remark 4.3. *In fact we require only that assumption 4.2 holds for (possibly a subsequence of) the sequence of pairs $\Phi_{\mathrm{b}}(p)$ and $\Phi_{\mathrm{c}}(p)$ for eigenfunctions associated with the sequences of energy levels E_{b} and E_{c} defined in assumption 4.1.*

We recall that the spectral sequence is defined in such a way that $\Phi_j(p) \neq 0$ for all j. Thus assumption 4.2 disqualifies subsequences of eigenfunctions converging to 0 at the point p.

This assumption reflects the fact that if $|\Phi_j(p)|$ becomes small, two eigenvalues of H_π will approach E_j. Then we would only be able to prove that the quasi-eigenfunction approximates a certain linear combination of these eigenfunctions of $-\Delta$, rather than an actual eigenfunction. Assumption 4.2 can be relaxed slightly—see remark 4.5 below.

Theorem 4.4. *Assume that the spectrum of $-\Delta$ satisfies assumptions 4.1 and 4.2. Then the sequence of quasimodes $\psi_{0,I}$ associated to the sequence of intervals $I = [E_{\mathrm{b}}, E_{\mathrm{c}}]$ and $\mu \in I$, with $E_{\mathrm{a}}, \ldots, E_{\mathrm{d}}$ as described in assumption 4.1, after normalisation, converge in L^2 to a subsequence of true eigenfunctions of H_π.*

Let us fix a point n of the sequence (ε_n) with $\varepsilon_n = \varepsilon$, and I fixed as described in the statement of theorem 4.4.

Proof of theorem 4.4. In order to use (25) we will employ partial summation, to estimate the position of eigenvalues of H_π. If g is a smooth function, then

$$
\sum_{X \leqslant E_j \leqslant Y} g(E_j) |\Phi_j(p)|^2 = g(Y) N(Y) - g(X) N(X) - \int_X^Y g'(t) N(t) \, \mathrm{d}t,
\tag{70}
$$

13

where $N(t)$ has been defined in (13). Equation (70) may be proved by Riemann-Stieltjes integration. Let λ^* be the solution of (65) lying between E_c and E_d. Let

$$g(t) := \frac{1}{t - \lambda^*} - \frac{t}{1 + t^2} = \frac{1 + t\lambda^*}{(t - \lambda^*)(1 + t^2)}, \tag{71}$$

and observe that $g(t) > 0$ if $t > \lambda^*$ and $g(t) < 0$ if $t < \lambda^*$. By (65) we have

$$0 = \sum_{j=1}^{\infty} g(E_j)|\Phi_j(p)|^2 \leqslant g(E_\mathrm{c})|\Phi_\mathrm{c}(p)|^2 + \sum_{E_j \geqslant E_\mathrm{d}} g(E_j)|\Phi_j(p)|^2. \tag{72}$$

Now, by (70),

$$\sum_{E_j \geqslant E_\mathrm{d}} g(E_j)|\Phi_j(p)|^2 = -g(E_\mathrm{d})N(E_\mathrm{d}) - \int_{E_\mathrm{d}}^{\infty} g'(t)N(t)\,\mathrm{d}t \tag{73}$$

$$= \frac{1}{4\pi} \int_{E_\mathrm{d}}^{\infty} g(t)\,\mathrm{d}t + \mathrm{O}\left(g(E_\mathrm{d})E_\mathrm{d}^{1/2} + \int_{E_\mathrm{d}}^{\infty} |g'(t)|t^{1/2}\,\mathrm{d}t\right),$$

using (13).

Since

$$g'(t) = \frac{-1}{(t - \lambda^*)^2} - \frac{1}{1 + t^2} + \frac{2t^2}{(1 + t^2)^2}, \tag{74}$$

we get

$$\int_{E_\mathrm{d}}^{\infty} |g'(t)|t^{1/2}\,\mathrm{d}t \sim \int_{E_\mathrm{d}}^{\infty} \frac{t^{1/2}}{(t - \lambda^*)^2}\,\mathrm{d}t$$

$$\leqslant \left(\frac{E_\mathrm{d}}{E_\mathrm{d} - \lambda^*}\right)^{1/2} \int_{E_\mathrm{d}}^{\infty} \frac{1}{(t - \lambda^*)^{3/2}}\,\mathrm{d}t$$

$$\ll \left(\frac{E_\mathrm{d}}{E_\mathrm{d} - \lambda^*}\right)^{1/2} \frac{1}{(E_\mathrm{d} - \lambda^*)^{1/2}}$$

$$= \frac{E_\mathrm{d}^{1/2}}{E_\mathrm{d} - \lambda^*}. \tag{75}$$

We can also calculate

$$\int_{E_\mathrm{d}}^{\infty} g(t)\,\mathrm{d}t = \int_{E_\mathrm{d}}^{\infty} \frac{1}{t - \lambda^*} - \frac{t}{1 + t^2}\,\mathrm{d}t = -\ln\left(\frac{E_\mathrm{d} - \lambda^*}{\sqrt{1 + E_\mathrm{d}^2}}\right). \tag{76}$$

So we have

$$\sum_{E_j \geqslant E_\mathrm{d}} g(E_j)|\Phi_j(p)|^2 = \frac{-1}{4\pi}\ln\left(\frac{E_\mathrm{d} - \lambda^*}{\sqrt{1 + E_\mathrm{d}^2}}\right) + \mathrm{O}\left(\frac{E_\mathrm{d}^{1/2}}{E_\mathrm{d} - \lambda^*}\right), \tag{77}$$

in which the dominant term on the RHS is actually the error term. We have, from (72)

$$g(E_\mathrm{c})|\Phi_\mathrm{c}(p)|^2 \ll \frac{E_\mathrm{d}^{1/2}}{E_\mathrm{d} - \lambda^*} \tag{78}$$

$$\Rightarrow \quad \frac{|\Phi_\mathrm{c}(p)|^2}{E_\mathrm{c} - \lambda^*} \ll \frac{\varepsilon^{-\rho/2}}{E_\mathrm{d} - E_\mathrm{c} - (\lambda^* - E_\mathrm{c})}$$

$$\leqslant \frac{\varepsilon^{-\rho/2}}{\varepsilon^q - (\lambda^* - E_\mathrm{c})}, \tag{79}$$

14

implying the lower bound

$$\lambda^* - E_{\mathrm{c}} \gg \varepsilon^{\rho/2+q}. \tag{80}$$

To see this, observe that if $\lambda^* = \mathrm{o}(\varepsilon^{q+\rho/2})$ then we would have from (79)

$$\frac{|\Phi_{\mathrm{c}}(p)|^2}{E_{\mathrm{c}} - \lambda^*} \ll \varepsilon^{-\rho/2-q}, \tag{81}$$

a contradiction.

By the same method, we can establish the same bound for the solution to (65) between E_{a} and E_{b}, and by theorem 3.5 we deduce that there is an interval of size $M \asymp \varepsilon^{q+\rho/2}$ about μ such that $[\mu - M, \mu + M]$ contains only one eigenvalue of H_π. Since $q + \rho/2 < 1$, and since the discrepancy of $\psi_{0,I}$ is $\mathrm{O}(\varepsilon)$, equation (25) allows us to conclude that the normalised quasimode differs from the true eigenfunction associated to μ by an amount which converges to 0 as $\varepsilon \to 0$. $\qquad\square$

Remark 4.5. *From the proof of theorem 4.4 we see that we can relax assumption 4.2 to demanding only that $|\Phi_j(p)| \gg \varepsilon^{r/2}$ with $0 < r < 1 - q - \rho/2$. However, in a generic situation this is unlikely to be achieved. In appendix A we show that for a badly-approximable position of the point p in a rectangle, the best possible bound is*

$$|\Phi_j(p)| \gg \frac{1}{E_j}, \tag{82}$$

which is not sufficiently slow.

5 Application to rectangular Šeba billiards

In this section we will apply theorem 4.4 to the original Šeba billiard [12]. We consider a rectangular billiard $\Omega = (0, 2a) \times (0, 2b) \subseteq \mathbb{R}^2$ and point $p = (a, b)$ at the centre of the billiard. However, we remark that we could position p at any point with co-ordinates that are rational multiples of the side lengths without significant changes to the forthcoming analysis.

The eigenvalues of $-\Delta$, the Laplacian with Dirichlet boundary conditions, are given by

$$E_{n,m} = \frac{\pi^2}{4}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right), \tag{83}$$

where $n, m \in \mathbb{N}$, and the corresponding eigenfunctions are

$$\Phi_{n,m}(x, y) = \frac{1}{\sqrt{ab}} \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{m\pi y}{2b}\right). \tag{84}$$

If either n or m are even, then the symmetry of the problem forces $\Phi_{n,m}(p) = 0$. So for these values of n and m, $\Phi_{n,m} \in \mathcal{D}_p$, and are automatically eigenfunctions of the extended operator H_π. We exclude these eigenvalues from the spectrum, as discussed in section 2.

Instead, we concentrate on the more interesting subsequence where n and m are both odd, e.g. $n = 2s + 1$ and $m = 2t + 1$ with $s, t = 0, 1, 2, \ldots$ Then we have

$$\Phi_{s,t}(p) = \frac{1}{\sqrt{ab}}(-1)^{s+t}, \tag{85}$$

so that along this sequence assumption 4.2 is satisfied. The corresponding set of eigenvalues is given by

$$E_{s,t} = \pi^2 \left(\frac{(s+\frac{1}{2})^2}{a^2} + \frac{(t+\frac{1}{2})^2}{b^2} \right), \qquad s,t = 0,1,2,\dots \tag{86}$$

For generic choices of a and b it is conjectured that the set of values given by (86) behave statistically like the event times of a Poisson process [17, 34, 35, 36]. Under assumption 4.1 for the set of values (86), theorem 4.4 asserts the existence of a subsequence $(j_n) \subseteq \mathbb{N}$ such that

$$\|\phi_{j_n} - \psi_n\| \to 0 \qquad \text{as } n \to \infty, \tag{87}$$

where ϕ_j are eigenfunctions of H_π, and ψ_n are of the form

$$\psi_n = \frac{1}{\sqrt{2}} \left(\Phi_{j_n} + (-1)^{\beta_n} \Phi_{j_n+1} \right), \tag{88}$$

where β_n can be 0 or 1 and depends on the relative signs of $\Phi_{j_n}(p)$ and $\Phi_{j_n+1}(p)$. So the subsequence (ϕ_{j_n}) converges to a superposition of two consecutive unperturbed eigenfunctions of $-\Delta$.

The consequences for this subsequence are most striking when one considers the momentum representation. This is is given by the Fourier transform;

$$\hat{\phi}_j(p_x, p_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xp_x - \mathrm{i}yp_y} \phi_j(x,y) \,\mathrm{d}x\mathrm{d}y. \tag{89}$$

For an ergodic system, the quantum ergodicity theorem of Šnirel'man, Zelditch and Colin de Verdière [37, 38, 39] would imply that the momentum representation of almost all eigenfunctions equi-distributes around the circle of radius $\sqrt{E_j}$ as $j \to \infty$;

$$|\hat{\phi}_j(x)|^2 \to \frac{1}{\pi} \delta(|x|^2 - E_j) \qquad \text{as } j \to \infty, \tag{90}$$

where convergence in (90) is in the weak sense. Šeba billiards are not ergodic, but we see a very different behaviour to (90) for the subsequence (ϕ_{j_n}).

From Parseval's theorem, it follows that

$$\hat{\phi}_{j_n} - \hat{\psi}_n \to 0 \qquad \text{in } L^2 \text{ norm.} \tag{91}$$

The momentum representation of the unperturbed eigenfunctions $\Phi_{n,m}$ is

$$\hat{\Phi}_{n,m}(p_x, p_y) = \frac{2\pi nm\sqrt{ab}}{(4p_x^2 a^2 - n^2\pi^2)(4p_y^2 b^2 - m^2\pi^2)} \left((-1)^n \mathrm{e}^{-2\mathrm{i}ap_x} - 1 \right) \left((-1)^m \mathrm{e}^{-2\mathrm{i}bp_y} - 1 \right). \tag{92}$$

Since

$$\frac{n\pi}{4p_x^2 a^2 - n^2\pi^2} = \frac{1}{2} \left(\frac{1}{2p_x a - n\pi} - \frac{1}{2p_x a + n\pi} \right), \tag{93}$$

we re-scale and write

$$\hat{\Phi}_{n,m}(np_x, mp_y) = \frac{\pi\sqrt{ab}}{2nm} \left(\delta_n(2p_x a + \pi) - \delta_n(2p_x a - \pi) \right) \left(\delta_m(2p_y b - \pi) - \delta_m(2p_y b + \pi) \right), \tag{94}$$

16

where δ_n is the smoothed-delta function

$$\delta_n(t) := \frac{1 - \mathrm{e}^{-\mathrm{i}nt}}{\pi \mathrm{i} t}. \tag{95}$$

The function $\delta_n(t)$ converges weakly to $\delta(t)$ as $n \to \infty$. Furthermore, it satisfies

$$|\delta_n(t)|^2 \sim \frac{2n}{\pi} \delta(t) \qquad \text{as } n \to \infty. \tag{96}$$

Hence

$$nm|\hat{\Phi}_{n,m}(np_x, mp_y)|^2 \sim ab \left(\delta(2p_x a - \pi) + \delta(2p_x a + \pi)\right)\left(\delta(2p_y b - \pi) + \delta(2p_y b + \pi)\right) \tag{97}$$

as $n, m \to \infty$. The momentum eigenfunction localises around the 4 points

$$(p_x, p_y) = \left(\pm\frac{n\pi}{2a}, \pm\frac{m\pi}{2b}\right), \tag{98}$$

which satisfy $p_x^2 + p_y^2 = E_{n,m}$. Since ψ_n is a superposition of Φ_{j_n} and Φ_{j_n+1}, the states in the subsequence $\hat{\phi}_{j_n}$ become localised around 8 points, which all lie on the circle with radius $\sqrt{E_j}$, very much in contrast to the expected equi-distribution (90) for ergodic systems. Numerical simulations illustrating this behaviour have been presented in [26]. This localisation is, in some sense, analogous to the scarring phenomenon which occurs in some chaotic systems. Since these states are not associated with an unstable periodic orbit, they do not fall into the very precise definition of a scar given in [40]. Rather they are localising around ghosts of departed tori of the unperturbed integrable system. Nevertheless they cannot be explained simply by using torus quantisation, and so they provide a further example of the already rich behaviours in systems with intermediate statistics.

Acknowledgements

A   Non-constant unperturbed eigenfunctions at the position of the scatterer

In order to consider what can happen when the value of the unperturbed eigenfunctions at the position of the scatterer can vary, let us consider the rectangular billiard Ω, with sides of length a and b, and Dirichlet boundary conditions.

The energy levels are given by

$$E = E_{n,m} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right), \tag{99}$$

for $n, m \geqslant 1$ integers.

Lemma A.1.

$$\frac{1}{n^2 m^2} \geqslant \frac{4\pi^4}{a^2 b^2} \frac{1}{E^2}. \tag{100}$$

Proof. We have

$$0 \leqslant \pi^4 \left(\frac{n^2}{a^2} - \frac{m^2}{b^2} \right)^2 = \frac{n^4 \pi^4}{a^4} - 2 \frac{n^2 m^2}{a^2 b^2} \pi^4 + \frac{m^4 \pi^4}{b^4}$$

$$= E^2 - 4 \frac{n^2 m^2}{a^2 b^2} \pi^4, \tag{101}$$

and then re-arrange to get the required estimate. $\qquad\square$

The eigenfunctions themselves are proportional to

$$\sin\left(\frac{n\pi x}{a} \right) \sin\left(\frac{m\pi y}{b} \right). \tag{102}$$

Let us choose the point $p = (x_p, y_p) \in \Omega$ so that x_p/a and y_p/b are badly-approximable, in the sense that

$$\left| n \frac{x_p}{a} - r \right| \geqslant \frac{C}{n} \qquad \forall n, r \in \mathbb{Z}, \tag{103}$$

(this is the best we can hope to do if we want to bound the eigenfunctions away from 0). Then

$$n \frac{x_p}{a} = r + \vartheta(n) \tag{104}$$

where ϑ can depend on x_p and r and satisfies

$$|\vartheta(n)| \gg \frac{1}{n} \tag{105}$$

uniformly. Furthermore this bound is achieved if r/n is a continued fraction approximant to x_p/a. We get

$$\sin^2\left(\frac{n\pi x_p}{a} \right) \gg \frac{1}{n^2}. \tag{106}$$

With a similar bound for the contribution of the y-coordinate, we find that the best bound we can obtain is

$$|\Phi_{n,m}(p)|^2 \gg \frac{1}{n^2 m^2} \gg \frac{1}{E^2} \tag{107}$$

and this bound is sharp.

B Assumption 4.1 for the event times of a Poisson process

The purpose of this appendix is to prove the following result. Let $0 < q < 1/2$ and $1 < \rho < 2(1 - q)$ be fixed throughout.

Proposition B.1. *Let $P = (E_j)_{j=1}^\infty$ be the sequence of event times for a Poisson process with parameter 1. There is, almost surely, a sequence $(\varepsilon_n)_{n=1}^\infty$, $\varepsilon_n \downarrow 0$ such that for each n there are four consecutive members of P, $E_a < E_b < E_c < E_d < \varepsilon_n^{-\rho}$, satisfying*

$$
\begin{aligned}
E_c - E_b &< \varepsilon_n \\
E_d - E_c &> \varepsilon_n^q \\
E_b - E_a &> \varepsilon_n^q.
\end{aligned}
\tag{108}
$$

Thus, assumption 4.1 is almost surely satisfied for a Poisson process.

As a model for a Poisson process, we will let $\xi_1, \xi_2, \ldots$ be a sequence of independent exponentially distributed random variables with parameter 1. Then, defining

$$
\begin{aligned}
E_1 &= \xi_1 \\
E_2 &= \xi_1 + \xi_2 \\
&\;\;\vdots
\end{aligned}
\tag{109}
$$

The sequence $P = (E_j)_{j=1}^\infty$ so-formed is a Poisson process.

Proposition B.2. *Let $\varepsilon > 0$. The probability that there are four consecutive members of P, $E_a < E_b < E_c < E_d < \varepsilon^{-\rho}$ satisfying*

$$
\begin{aligned}
E_c - E_b &< \varepsilon \\
E_d - E_c &> \varepsilon^q \\
E_b - E_a &> \varepsilon^q,
\end{aligned}
\tag{110}
$$

is $1 - \mathrm{O}(\varepsilon^\infty)$.

The notation $\mathrm{O}(\varepsilon^\infty)$ refers to a quantity which goes to zero faster than any power of ε. One can say that the event described in proposition B.2 occurs with overwhelming probability.

Let us fix $1 < \rho' < \rho$, and chose $N = 3M \sim \varepsilon^{-\rho'}$ where $M \in \mathbb{N}$. Let us define the events $\mathcal{S}_j$, $j = 0, \ldots, M-1$, by

$$
\mathcal{S}_j = \{\xi_{3j+1} > \varepsilon^q, \; \xi_{3j+2} < \varepsilon, \; \xi_{3j+3} > \varepsilon^q\}.
\tag{111}
$$

Lemma B.3. *The events $\mathcal{S}_j$, $j = 0, \ldots, M-1$ are independent, and the probability that at least one of them occurs is $1 - \mathrm{O}(\varepsilon^\infty)$.*

Proof. The independence of the events $\mathcal{S}_j$ clearly follows because they are defined on independent random variables. We first calculate the probability of one of them. By independence of ξ_1, ξ_2, ξ_3,

$$
\begin{aligned}
\mathbb{P}(\mathcal{S}_0) &= \mathbb{P}(\xi_2 < \varepsilon)\mathbb{P}(\xi_1 > \varepsilon^q)\mathbb{P}(\xi_3 > \varepsilon^q) \\
&= \left(\int_0^\varepsilon \mathrm{e}^{-x}\,\mathrm{d}x\right)\left(\int_{\varepsilon^q}^\infty \mathrm{e}^{-x}\,\mathrm{d}x\right)^2 \\
&= (1 - \mathrm{e}^{-\varepsilon})\left(\mathrm{e}^{-\varepsilon^q}\right)^2 \\
&= \varepsilon + \mathrm{O}(\varepsilon^{1+q}).
\end{aligned}
\tag{112}
$$

Then, by independence of the $\mathcal{S}_j$s,

$$p_1 := \mathbb{P}(\text{at least one } \mathcal{S}_j \text{ occurs}) = 1 - (1 - \mathbb{P}(\mathcal{S}_0))^M$$
$$= 1 - \left(1 - \varepsilon + \mathrm{O}(\varepsilon^{1+q})\right)^M. \qquad (113)$$

So, we have

$$\log(1 - p_1) = -M\varepsilon + \mathrm{O}(M\varepsilon^{1+q}) \sim -\frac{1}{3}\varepsilon^{1-\rho'}. \qquad (114)$$

For ε sufficiently small, this yields

$$1 - p_1 \leqslant \exp\left(-\tfrac{1}{6}\varepsilon^{-(\rho'-1)}\right) = \mathrm{O}(\varepsilon^\infty). \qquad (115)$$

$\square$

The probability that the upper bound of $\varepsilon^{-\rho}$ is met is given in the following lemma

Lemma B.4. *The probability that $E_N < \varepsilon^{-\rho}$ is $1 - \mathrm{O}(\varepsilon^\infty)$.*

Proof. Let $\alpha > 0$. The probability density for E_N is $\Gamma(N)^{-1}x^{N-1}\mathrm{e}^{-x}$. So

$$p_2 := \mathbb{P}(E_N < N^{1+\alpha}) = 1 - \frac{1}{\Gamma(N)}\int_{N^{1+\alpha}}^\infty x^{N-1}\mathrm{e}^{-x}\,\mathrm{d}x$$
$$= 1 - \frac{\exp(-N^{1+\alpha})}{\Gamma(N)}\int_0^\infty (x + N^{1+\alpha})^{N-1}\mathrm{e}^{-x}\,\mathrm{d}x$$
$$= 1 - \exp(-N^{1+\alpha})N^{(1+\alpha)(N-1)}\sum_{j=0}^{N-1}\frac{1}{\Gamma(N-j)N^{(1+\alpha)j}}, \qquad (116)$$

expanding the binomial. Using $\dfrac{1}{\Gamma(N-j)} \leqslant \dfrac{N^j}{\Gamma(N)}$ we can estimate

$$\sum_{j=0}^{N-1}\frac{1}{\Gamma(N-j)N^{(1+\alpha)j}} \leqslant \frac{1}{\Gamma(N)}\sum_{j=0}^{N-1}\frac{1}{N^{\alpha j}} \ll \frac{1}{\Gamma(N)}, \qquad (117)$$

where the implied constant could depend on α. This leads to

$$1 - p_2 \ll \frac{\exp(-N^{1+\alpha})N^{(1+\alpha)(N-1)}}{\Gamma(N)}$$
$$\sim \frac{\exp(-N^{1+\alpha} + N)N^{\alpha(N-1)}}{\sqrt{2\pi(N-1)}}$$
$$\ll \frac{1}{N^\infty}, \qquad (118)$$

where Stirling's formula has been used. This last line is $\mathrm{O}(\varepsilon^\infty)$ since $N^{-1} \sim \varepsilon^{\rho'}$. Finally setting

$$\alpha = \frac{\rho}{\rho'} - 1 > 0 \qquad (119)$$

gives the required estimate. $\square$

Proof of proposition B.2. We are interested in the events corresponding to lemmata B.3 and B.4 happening simultaneously. By the inclusion-exclusion principle, the probability that this happens is at least $p_1 + p_2 - 1 = 1 - O(\varepsilon^\infty)$. $\qquad\square$

Proof of proposition B.1. Let $\mathcal{E}_n$, $n \in \mathbb{N}$ be the event that there are found four consecutive members of P, $E_a < E_b < E_c < E_d < n^\rho$ satisfying

$$
\begin{aligned}
E_c - E_b &< \frac{1}{n} \\
E_d - E_c &> \frac{1}{n^q} \\
E_b - E_a &> \frac{1}{n^q}.
\end{aligned}
\tag{120}
$$

By proposition B.2, $\mathbb{P}(\mathcal{E}_n^c) \ll \frac{1}{n^2}$. Hence, by the Borel-Cantelli lemma, the probability that infinitely many $\mathcal{E}_n^c$ occur is zero. Equivalently, only finitely many $\mathcal{E}_n^c$ occur, almost surely. So, almost surely, there is an infinite subsequence of $n \in \mathbb{N}$ such that $\mathcal{E}_n$ occurs. $\qquad\square$

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